International Journal of Sciences: Basic and Applied Research (IJSBAR)

ISSN 2307-4531
(Print & Online)

http://gssrr.org/index.php?journal=JournalOfBasicAndApplied

Numerical Simulation Based on First Order Difference Scheme for Three Lanes Traffic Flow Model

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Abstract

In this article performs the numerical solution of a three lanes traffic flow model based on a linear velocity-density relationship is studied. A multilane traffic flow with three lanes which is moderated by a system of nonlinear partial differential equation appended with initial and boundary conditions reads as an initial boundary value problem (IBVP). In order to compute the numerical solution, we present the discretization of the considered model which leads to the explicit upwind difference scheme. The numerical simulation of 10km highway of three lanes is performed for 6 minutes using the explicit upwind difference scheme based on artificially generated initial and boundary data. An experimental result for the stability condition of the numerical scheme is also presented by performing numerical experiments. The computed result satisfies some well known qualitative features of the solution.

Keywords: Three lanes traffic flow model; Non-linear PDE; Numerical simulation.

1. Introduction

Now traffic jams are a major problem in most of the countries. So at the core of traffic congestion, development of the traffic management is the needed of time. Therefore, an efficient traffic control and management is essential in order to get rid of such huge traffic congestion.

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Modeling and computer simulation play an increasing role in the flow management. Many scientists have been working to develop various mathematical models [1-2] in order to describe traffic flow. The most elementary continuum traffic flow model was the first order model developed by Lighthill, Whitham (1955) and Richards (1956) [10] proposed a macroscopic traffic flow model, which is very popularly known as the LWR model based on the assumption of mass density conservation, that is, the number of vehicles between any two points if there are no entrances or exits is conserved. The LWR model is a first-order model in the sense it is formulated as a scalar hyperbolic conservation law, and solved by finite difference methods [11-13]. According to this model, the traffic flow model was presented a first order non-linear partial differential equation and was based on a hyperbolic system of conversation laws. Typically, continuum models for multilane are traffic based on a system of conservation laws with source terms. Authors shows two lane traffic flow model by a finite difference scheme named Lax-Friedrichs scheme [5]. Axel [6] presented a hierarchy of multilane traffic flow models and described the derivation of macroscopic multilane traffic flow model. For elucidation of the derivation of multilane traffic model describe the balance equation of traffic flow and as well as closure relations on the balance equations from [6-8]. In [9] and [14], authors presented a macroscopic of mixed multilane freeway traffic that can be easily calibrated to empirical traffic data and the model is derived from a gas-kinetic level of description, including effects of vehicular space requirements and velocity correlations between successive vehicles. The main objective of this study is first order non-linear partial differential equation of multilane traffic flow model is appended with initial and boundary value leads to an initial boundary value problem (IBVP). It is too complex to be solved by analytical methods so we solve numerical solution of multilane traffic flow for three lanes using explicit upwind difference scheme (EUDS). We build up a computer programming code for the EUDS and implement schemes for artificially generated initial and boundary data and verify the well-known qualitative behaviors of different traffic flow variables. Some experimental results are also presented here regarding the stability and well-posed-ness condition of the numerical schemes. The rest of the article is set in the following: the mathematical model of three lane traffic model in section 2. In section 3, we study numerical solution and section 4 computed result satisfies some well known qualitative features by performing numerical experiments and finally we discuss the conclusions.

2. Multilane Traffic Flow Model for Three Lane

We focus on the macroscopic multilane traffic flow model which can be written in generalized form as flows from [6-8]:

$$
\begin{aligned}
&\frac{\partial \rho_1}{\partial t} + \frac{\partial q_1}{\partial x} = \frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2} \\
&\frac{\partial \rho_j}{\partial t} + \frac{\partial q_j}{\partial x} = \frac{\rho_{j-1}}{T_{j-1}^j} - \frac{\rho_j}{T_j^{j-1}} + \frac{\rho_{j+1}}{T_{j+1}^j} - \frac{\rho_j}{T_j^{j+1}} \\
&\frac{\partial \rho_N}{\partial t} + \frac{\partial q_N}{\partial x} = \frac{\rho_{N-1}}{T_{N-1}^N} - \frac{\rho_N}{T_N^{N-1}}
\end{aligned}
\tag{2.1}
$$

Here, the subscripts 1, $j = 2, ..., \mathrm{N}-1$ and N refer to the lane numbers. The quantities ρ_j and $q_j = \rho_j v_j$ are the vehicle density and the vehicle flux in the j-th lane respectively where as v_j is the vehicle velocity at the j-th lane for $j = 1, 2, ..., \mathrm{N}$; at last $T_j^k = T_j^k(\rho_j, \rho_k)$ is the vehicle transition rate from lane j to lane k, with $|j-k| = 1$. In particular, we choose multilane traffic flow model (1) for three lanes that is for $j = 1, 2, 3\ (\mathrm{N}=3)$:

$$\begin{aligned}
\frac{\partial \rho_1(t,x)}{\partial t} + \frac{\partial q_1(\rho_1(t,x))}{\partial x} &= \frac{\rho_2(t,x)}{T_2^1} - \frac{\rho_1(t,x)}{T_1^2} \\
\frac{\partial \rho_2(t,x)}{\partial t} + \frac{\partial q_2(\rho_2(t,x))}{\partial x} &= \frac{\rho_1(t,x)}{T_1^2} - \frac{\rho_2(t,x)}{T_2^1} \\
\frac{\partial \rho_3(t,x)}{\partial t} + \frac{\partial q_3(\rho_3(t,x))}{\partial x} &= \frac{\rho_2(t,x)}{T_2^3} - \frac{\rho_3(t,x)}{T_3^2}
\end{aligned} \qquad (2.2)$$

We consider velocity $v = v(\rho)$ as a function of density and therefore, we have the flux $q = q(\rho) = \rho v(\rho)$. In this paper multilane traffic flow model (2.2) is approximated by the Greenschield's linear density-velocity relation: $v(\rho) = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$, where $v_{\max}$ is the maximum velocity and $\rho_{\max}$ is the maximum density which is based on bumper to bumper traffic. Therefore, the flux q takes the form $q = q(\rho) = \rho v(\rho) = \rho . v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right) = v_{\max}\left(\rho - \frac{\rho^2}{\rho_{\max}}\right)$.

3. Numerical Scheme for Three Lane Traffic Flow Model

Explicit upwind difference scheme is one of the finite difference methods to find the numerical approximation of hyperbolic partial differential equation. So our considered non-linear first order partial differential equation of multilane traffic flow model for three lane as an initial boundary value problem (IBVP) as follows:

$$\left.\begin{array}{l}
\dfrac{\partial \rho_1(t,x)}{\partial t}+\dfrac{\partial q_1\left(\rho_1(t,x)\right)}{\partial x}=\dfrac{\rho_2(t,x)}{T_2^1}-\dfrac{\rho_1(t,x)}{T_1^2}\\
\dfrac{\partial \rho_2(t,x)}{\partial t}+\dfrac{\partial q_2\left(\rho_2(t,x)\right)}{\partial x}=\dfrac{\rho_1(t,x)}{T_1^2}-\dfrac{\rho_2(t,x)}{T_2^1}\\
\dfrac{\partial \rho_3(t,x)}{\partial t}+\dfrac{\partial q_3\left(\rho_3(t,x)\right)}{\partial x}=\dfrac{\rho_2(t,x)}{T_2^3}-\dfrac{\rho_3(t,x)}{T_3^2}\\
\text{with i.c. } \rho_1(0,x)=(\rho_1)_o(x),\ \rho_2(0,x)=(\rho_2)_o(x), \rho_3(0,x)=(\rho_3)_o(x); a\le x\le b\\
\text{and b.c. } \rho_1(t,a)=(\rho_1)_a(t),\ \rho_2(t,a)=(\rho_2)_a(t),\ \rho_3(t,a)=(\rho_3)_a(t); t_o\le t\le T\\
\text{where } q_1=q_1\left(\rho_1(t,x)\right)=v_{1\max}\left(\rho_1-\dfrac{\rho_1{}^2}{\rho_{1\max}}\right),\ q_2=q_2\left(\rho_2(t,x)\right)=v_{2\max}\left(\rho_2-\dfrac{\rho_2{}^2}{\rho_{2\max}}\right)\\
\qquad q_3=q_3\left(\rho_3(t,x)\right)=v_{3\max}\left(\rho_3-\dfrac{\rho_3{}^2}{\rho_{3\max}}\right)
\end{array}\right\} (3.1)$$

We discretize the time derivatives $\dfrac{\partial \rho_1(t,x)}{\partial t}$, $\dfrac{\partial \rho_2(t,x)}{\partial t}$ and $\dfrac{\partial \rho_3(t,x)}{\partial t}$ by first order forward difference in time and the space derivatives $\dfrac{\partial q_1\left(\rho_1(t,x)\right)}{\partial x}$, $\dfrac{\partial q_2\left(\rho_2(t,x)\right)}{\partial x}$ and $\dfrac{\partial q_3\left(\rho_3(t,x)\right)}{\partial x}$ by first order backward difference in space.

Forward difference in time:

From the Taylor's series expansion we can write

$$\begin{array}{l}
\rho(x,\ t+k)=\rho(x,t)+k\dfrac{\partial\rho}{\partial t}+\dfrac{k^2}{2!}\dfrac{\partial^2\rho}{\partial t^2}+\ ....\\
\Rightarrow \dfrac{\partial\rho}{\partial t}=\dfrac{\rho(x,\ t+k)-\rho(x,t)}{k}-o(k)\\
\Rightarrow \dfrac{\partial\rho}{\partial t}\approx\dfrac{\rho(x,\ t+k)-\rho(x,t)}{k}
\end{array}$$

Therefore,

$$\left.\begin{array}{l}
\dfrac{\partial\rho_1}{\partial t}\approx\dfrac{\rho_1(x,\ t+k)-\rho_1(x,t)}{k}\\
\dfrac{\partial\rho_2}{\partial t}\approx\dfrac{\rho_2(x,\ t+k)-\rho_2(x,t)}{k}\\
\dfrac{\partial\rho_3}{\partial t}\approx\dfrac{\rho_3(x,\ t+k)-\rho_3(x,t)}{k}
\end{array}\right\} \qquad (3.2)$$

Backward difference in space:

Again, from the Taylor's series expansion we can write

$$q(x-h,\, t) = q(x,t) - h\frac{\partial q}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 q}{\partial x^2} - \;..........$$
$$\Rightarrow \frac{\partial q}{\partial x} = \frac{q(x,\, t) - q(x-h,t)}{h} + o(h)$$
$$\Rightarrow \frac{\partial q}{\partial x} \approx \frac{q(x,\, t) - q(x-h,t)}{h}$$

Therefore,

$$\left.\begin{aligned} \frac{\partial q_1}{\partial x} &\approx \frac{q_1(x,\, t) - q_1(x-h,t)}{h} \\ \frac{\partial q_2}{\partial x} &\approx \frac{q_2(x,\, t) - q_2(x-h,t)}{h} \\ \frac{\partial q_3}{\partial x} &\approx \frac{q_3(x,\, t) - q_3(x-h,t)}{h} \end{aligned}\right\} \qquad (3.3)$$

We assume the uniform grid spacing with step size k and h for space and time respectively $t^{n+1} = t^n + k$ and $x_{i+1} = x_i + h.$ We write $(\rho_1)_i^n$ for $\rho_1(t,x)$, $(\rho_2)_i^n$ for $\rho_2(t,x)$ and $(\rho_3)_i^n$ for $\rho_3(t,x)$ in equations of (3.2) and also we write $(q_1)_i^n$ for $q_1(t,x)$, $(q_2)_i^n$ for $q_2(t,x)$ and $(q_3)_i^n$ for $q_3(t,x)$ in equations of (3.3).

Finally, substitute (3.2), (3.3) in (3.1), the discrete version of the non-linear PDE in the IBVP (3.1) formulates the first order explicit upwind difference scheme take the form

$$\frac{\rho_{1\,i}^{\,n+1} - \rho_{1\,i}^{\,n}}{\Delta t} + \frac{q_{1\,i}^{\,n} - q_{1\,i-1}^{\,n}}{\Delta x} = \frac{\rho_{2\,i}^{\,n}}{T_2^1} - \frac{\rho_{1\,i}^{\,n}}{T_1^2}$$
$$\Rightarrow \rho_{1\,i}^{\,n+1} = \rho_{1\,i}^{\,n} - \frac{\Delta t}{\Delta x}\left(q_{1\,i}^{\,n} - q_{1\,i-1}^{\,n}\right) + \Delta t\left(\frac{\rho_{2\,i}^{\,n}}{T_2^1} - \frac{\rho_{1\,i}^{\,n}}{T_1^2}\right) \qquad (3.4)$$

Similarly $\rho_{2\,i}^{\,n+1} = \rho_{2\,i}^{\,n} - \dfrac{\Delta t}{\Delta x}\left(q_{2\,i}^{\,n} - q_{2\,i-1}^{\,n}\right) + \Delta t\left(\dfrac{\rho_{1\,i}^{\,n}}{T_1^2} - \dfrac{\rho_{2\,i}^{\,n}}{T_2^1}\right)$ (3.5)

and $\rho_{3\,i}^{\,n+1} = \rho_{3\,i}^{\,n} - \dfrac{\Delta t}{\Delta x}\left(q_{3\,i}^{\,n} - q_{3\,i-1}^{\,n}\right) + \Delta t\left(\dfrac{\rho_{2\,i}^{\,n}}{T_2^3} - \dfrac{\rho_{3\,i}^{\,n}}{T_3^2}\right)$ (3.6)

Where $q_{1\,i}^{\,n} = v_{1\max}\left(\rho_{1\,i}^{\,n} - \frac{(\rho_{1\,i}^{\,n})^2}{\rho_{1\max}}\right)$, $q_{2\,i}^{\,n} = v_{2\max}\left(\rho_{2\,i}^{\,n} - \frac{(\rho_{2\,i}^{\,n})^2}{\rho_{2\max}}\right)$ and $q_{3\,i}^{\,n} = v_{3\max}\left(\rho_{3\,i}^{\,n} - \frac{(\rho_{3\,i}^{\,n})^2}{\rho_{3\max}}\right)$.

So, Equations (3.4), (3.5) and (3.6) are the explicit upwind difference schemes for the IBVP (3.1).

The stability condition of EUDS for single lane traffic flow model follows [3] is guaranteed by the simultaneous conditions $\frac{v_{\max}\Delta t}{\Delta x} \le 1$ and $\rho_{\max} = k \max_i \rho_o(x_i),\ k \ge 2$. In case of three lane traffic flow model, we see that the stability condition of EUDS also remain unchanged for our considered model.

4. Numerical Simulation and Results Discussion

In this section, we presents the numerical results for some specific cases of traffic flow focusing on traffic flow parameters using the explicit upwind difference scheme. We present the density profile as we as computed velocity and flux profiles of our considered model by the EUDS. For a particular case, we choose maximum velocity $v_{1\max} = v_{2\max} = v_{3\max} = 60$ km/hour. For satisfying the CFL condition we pick the unit of velocity as km/sec. We consider $\rho_{1\max} = \rho_{2\max} = \rho_{3\max} = 180/\text{km}$, and perform the numerical experiment for 6 minutes in 3600 time steps with $\Delta t = 0.1$ second for a three lanes highway of 10 km in 401 spatial grid points with step size $\Delta \text{x} = 100$ meters. We consider the initial density of multilane traffic flow model for three lanes are $\rho_1(0,x)$, $\rho_2(0,x)$ and $\rho_3(0,x)$. The transition rate from second lane to first lane 20%, first lane to second lane 10%, second lane to third lane 20% and third lane to second lane 10%. In figure 1 shows the initial position of car as well as the position of car after 6 minutes with respect to the certain points of 10 km highway. The constant boundary data are $\rho_1^a = 20/100$ meters (0.1km) for first lane, $\rho_2^a = 15/100$ meters (0.1km) for second lane and $\rho_3^a = 18/100$ meters (0.1km) for third lane.

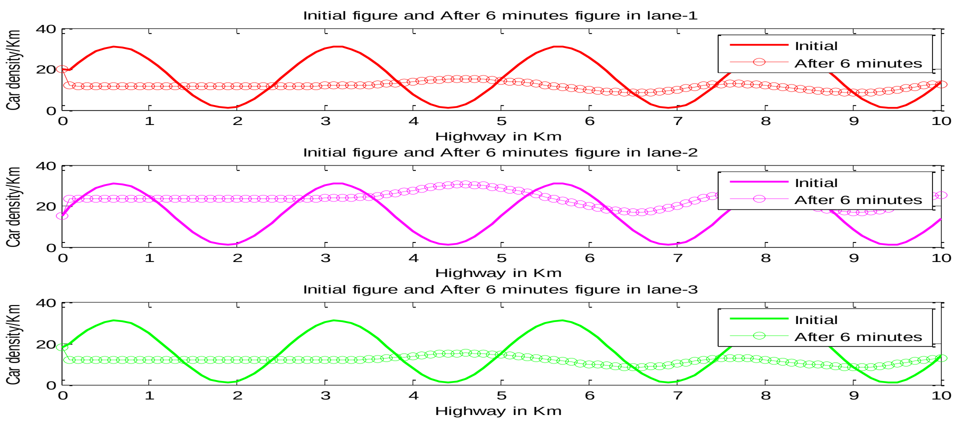

Figure 1: Comparative position of the cars between initial and six (06) minutes.

In figure 2 we run the behavior of car density with respect to highway in km for 6 minutes. The curve marked by "solid line" shows the car density at 2 minutes, the curve visible by "-o-"represents the car density profile at 4 minutes and the curve manifested as "-*-" corresponds to the density of car at 6 minutes respectively. We observed that as time goes on the traffic wave is moving forward with reducing wave height that means that the change of the density is failing in the long route.

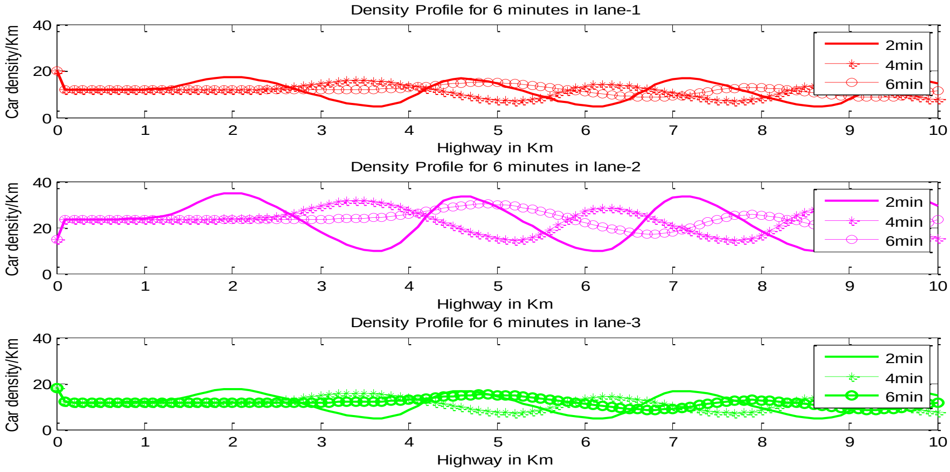

Figure 2: Density Profile for 6 minutes.

In this case, we reduce the parameter of maximum velocity for three lanes $v_{1\max} = v_{2\max} = v_{3\max} = 30\,\text{km/hour}$ but treating the same maximum density $\rho_{1\max} = \rho_{2\max} = \rho_{3\max} = 180\,/\text{km}$ with the same initial density as in the case figure 2 for 6 minutes. As maximum velocity is reduced by a factor 2 (two) from the previous case, the density is increased that is the speed is decreased. The computed density profile as shown in figure 3 and desired traffic waves are moving much slower than that in the previous case.

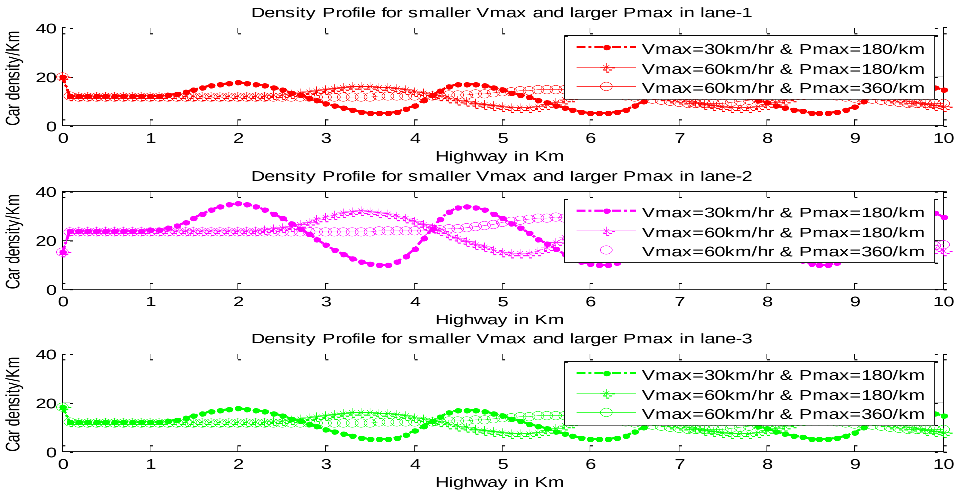

Figure 3: Density Profile for smaller $v_{\max}$ and larger $\rho_{\max}$

The numerical solution of multilane traffic flow model for different spatial and temporal grid size is depicted in figure 4. It shows that the numerical solution of the model also converges with respect to the smaller spatial grid size, Δx and temporal grid size, Δt which is also a very good agreement of the numerical solution of multilane traffic flow model for three lanes.

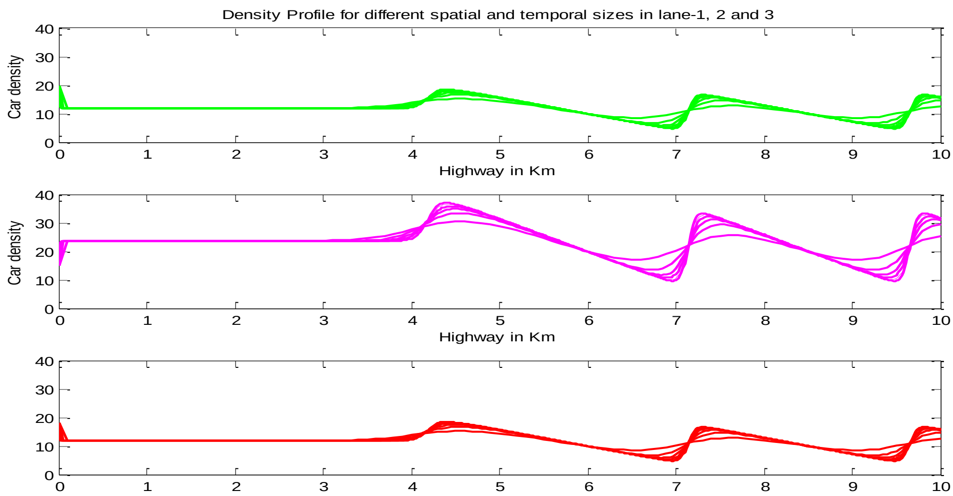

Figure 4: Density Profile for different spatial and temporal step sizes.

In figure 5 shows the corresponding velocity profile of three lanes traffic flow according to the certain points of the highway. The velocity of traffic is computed by the Greenshield's velocity-density relation; $v(\rho)=v_{\max}\left(1-\frac{\rho}{\rho_{\max}}\right)$. Here we observe that, as time goes on, the traffic wave is moving forward.

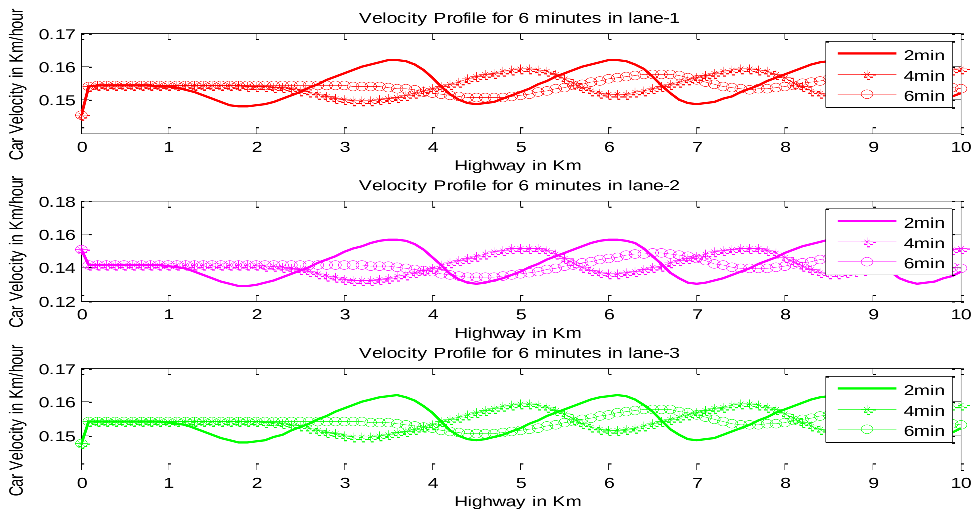

Figure 5: Velocity Profile for 6 minutes.

At this stage, we are the known about the density and velocity of multilane traffic flow for a certain highway. So now we are capable to determine the flux of multilane traffic for three lanes with the support of the relation, $q = \rho v.$ Figure 6 represents the computed flux with respect to the distance. Here we observe that, as time goes on, the traffic wave is moving forward.

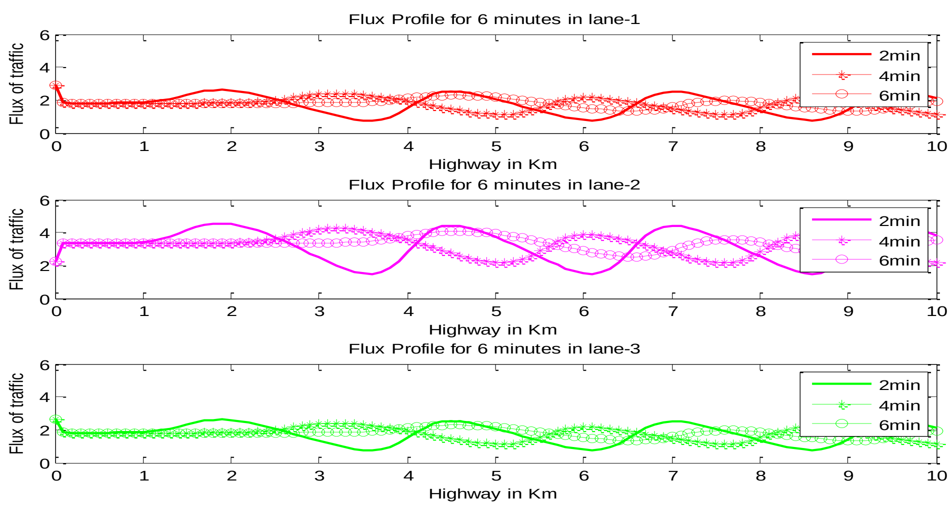

Figure 6: Flux Profile for 6 minutes.

In figure 7 we plot the computed velocity profile with respect to the computed density profile by the formula $v(\rho) = v_{\max}\left(1 - \dfrac{\rho}{\rho_{\max}}\right)$ of multilane traffic flow for three lanes. The figure 7 shows that the velocity and density relationship is linear which agrees accurately with our assumptions. In this case the elapsed time is 20.916 seconds.

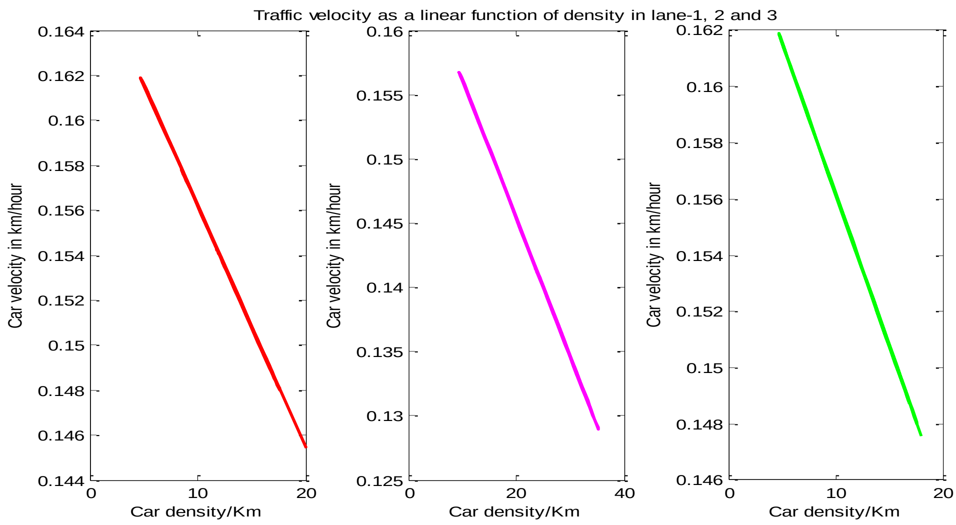

Figure 7: Traffic velocity as a function of density (linear case).

We would like to compute the total traffic flow from the traffic flow of three lanes as our consideration. In that case we add the computed density of three lanes which is presented in figure 8 according to the distance. The total density of three lanes agrees with density profile for single lane traffic flow model as in [3] and this is one of the good agreement of multilane traffic flow model for three lanes.

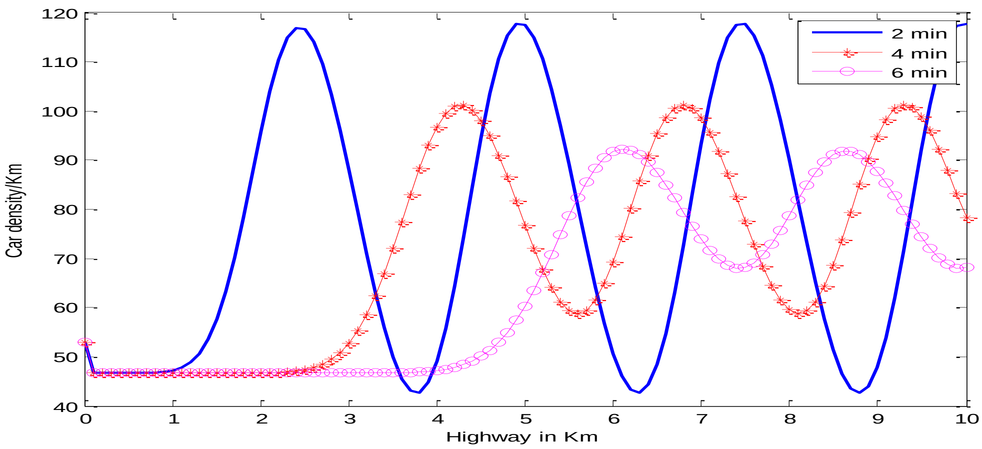

Figure 8: Total Density of three lanes.

Finally we have experimentally seen that the physical constraint ant the stability condition of the EUDS for single lane traffic flow is same as that of our considered model. In above qualitative traffic behavior we apply artificial initial and boundary data and guarantee that our considered model well defined for this data. Next also we apply real data in this model in a highway.

5. Conclusion

In this article, we study first order explicit upwind difference scheme for the numerical solution of multilane traffic flow for three lanes which reads as an initial boundary value problem. We establish a computer programming code in order to perform numerical simulation with respect to various traffic flow parameters. Performing numerical simulation, we have verified some qualitative traffic flow behavior for various traffic parameters of multilane traffic flow model for three lanes. This qualitative behavior agreement verified the implementation of the multilane traffic model for three lanes with sufficient accuracy. In our considered model is designed with no entrance and no exit. The model can be extended for multilane traffic flow model with entrance and exit of vehicles which we leave as our future work.

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