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Cosmological perturbations in modified teleparallel gravity models: boundary term extension

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Abstract Teleparallel gravity offers a new avenue in which to construct gravitational models beyond general relativity. While teleparallel gravity can be framed in a way to be dynamically equivalent to general relativity, its modifications are mostly not equivalent to the traditional route to modified gravity. $f(T, B)$ gravity is one such gravitational theory where the second and fourth order contributions to the field equations are decoupled. In this work, we explore the all important cosmological perturbations of this new framework of gravity. We derive the gravitational propagation equation, its vector perturbation stability conditions, and its scalar perturbations. Together with the matter perturbations, we derive the effective gravitational constant in this framework, and find an interesting branching behaviour that depends on the particular gravitational models being probed. We close with a discussion on the relation of these results with other gravitational theories.

1 Introduction

Cosmological perturbations have shown the possibility of opening a pathway to revealing the cosmological evolution of the Universe in General Relativity (GR) and crucially in theories beyond GR [1–3]. The results of perturbations analysis can then be used in the confrontation with observational data to better understand which models fair better against data related to cosmic evolution [4–6]. On the other hand, the ΛCDM cosmological model is supported by an abundance of evidence in describing the evolution of the Universe at all cosmological scales [7,8] when matter beyond the standard model of particle physics is included. This takes the form of dark matter as a stabilizing ingredient in galactic structures [9,10], while dark energy is represented by the cosmological constant [11,12] and is the agent responsible for producing late-time accelerated cosmic expansion [13,14] in this picture of the Universe. Nevertheless, even though great efforts have been directed at this part of the theory, internal problems persist with the concept of a cosmological constant [15], and direct evidence for dark matter particles remains elusive [16].

The performance of the ΛCDM model has also become an open problem in recent years. In essence, the ΛCDM model was realised as a confrontation with Hubble expansion data but the so-called H_0 tension calls this feature into question, where the observational discrepancy between model independent measurements in the late Universe [17,18] are in a meaning disagreement with the predicted value from the early Universe [19,20]. This tension has only grown in recent years [19,21]. Saying that the problem still appears to be open with measurements from the tip of the red giant branch (TRGB, Carnegie-Chicago Hubble Program) pointing to a lower H_0 tension, the issue may ultimately be resolved by novel future observations such as measurements using gravitational wave astronomy standard candles [22,23] which may

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be accelerated once the LISA mission [24,25] starts taking data.

There is now an abundance of theories beyond GR which aim to produce viable models of gravity that can agree with the new regime of precision measurements which have only became available in recent decades [6,26,27]. It is not enough for these theories to agree with cosmological observations at background level such as with the value of H_0. Theories beyond GR must also produce observable quantities from their perturbed dynamical equation that agree with current observations to be seriously considered. One such observable that is gaining increased interest is that of $f\sigma_8$ which also hosts a growing but weak tension with the ΛCDM model of cosmology. It was in Refs. [28,29] that cosmological perturbation theory was first developed in a consistent way, where a gauge-invariant approach was first developed. This approach has been used to analyze numerous models of gravity [6] with various successes. These theories mainly appear as an extension to GR [26,30,31] and build on corrections designed for various purposes that may have a cosmological effect at different epochs. However, these approaches can be collectively grouped by their common expression of gravitation through the use of the Levi-Civita connection, i.e. they communicate gravity by means of geometric curvature of spacetime [7,32]. This is not the only choice where torsion, through teleparallel gravity, has become an increasingly popular replacement for the curvature associated with the Levi-Civita connection [33–35].

Teleparallel Gravity (TG) refers to the collection of theories that express gravity through the torsion of the teleparallel connection [36]. The general linear teleparallel connection [37] is only required to be flat (curvature-less), but in this work we further restrict to the case of metric-compatible teleparallel connections. Given these properties, all curvature based measures of gravity will naturally vanish identically. A consequence of this is that the Einstein–Hilbert action, as determined with the teleparallel connection, will also vanish, i.e., $R = 0$, while its regular Levi-Civita connection version will remain the same, i.e., $\mathring{R} \neq 0$ (where overcircles will refer to quantities determined using the Levi-Civita connection throughout). By replacing the Ricci scalar in the Einstein–Hilbert action with its torsion scalar analog will produce identical dynamical equations. This is called the Teleparallel equivalent of General Relativity (TEGR), and differs from GR by a boundary B term in the gravitational Lagrangian.

The TEGR boundary term embodies the fourth order contributions to the field equations which is an important aspect of many theories beyond GR. In TG, the second and fourth order field equation contributions become decoupled from each other unlike in standard gravity where the Levi-Civita connection is employed. Using this rationale, modifications of TEGR will have a meaningful and impactful difference as compared with regular modified theories of gravity. The most prescient of these properties will be the realisation of producing generically second order theories of gravity in some generalizations of TEGR. This is to be contrasted with GR where by the Lovelock theorem [38], second order field equations are only produced by the Einstein–Hilbert action (with the addition of a constant) unless extra assumptions are included such as scalar fields or extra dimensions. In TG, the Lovelock theorem is weakened [39,40] allowing for a plethora of additional theories beyond TEGR that continue to produce second order field equations. TG also has a number of other attractive properties such as its similarity to Yang-mills theory [33] which gives it features of particle physics theory, as well as the possibility of giving a well-defined energy–momentum tensor for gravitation [41,42], and that it does not require an associated Gibbons–Hawking–York boundary term giving a more structured form to its Hamiltonian formalism, in addition to others.

One of the best studied modification to GR is that of $f(\mathring{R})$ gravity [26,30,31], and TEGR can similarly be generalized to produce $f(T)$ gravity [43–48]. This has several key properties chief among which is that it produced second order field equations and has shown promise in its confrontation with observations at various scales [34,49–55]. TG can also offer a path in which $f(\mathring{R})$ gravity is dynamically generalized by considering $f(T, B)$ gravity where the different order contributions are decoupled from one another [56–61]. This limits to $f(\mathring{R})$ gravity in the limit where $f(\mathring{R}) = f(-T + B)$. $f(T, B)$ gravity has shown promise as being a viable at various scales ranging from solar system tests in the weak field regime [59,62–64], as well as its cosmological theoretical structure [56,56,58,60] and confrontation with observational data [65].

In $f(T)$ gravity, cosmological perturbations have been considered in a number of works [49,50,66–68] which has been extended to a number of other extensions to TEGR such as Ref. [69] where matter perturbations are considered in $f(T, \mathcal{T})$ gravity and Ref. [70] in which the perturbations in teleparallel loop quantum cosmology are performed. In the present work, we determine the cosmological perturbations about a flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric. Together with the perturbations associated with the matter contribution, we form the linear perturbation equations in order to produce probes that can be used in observational cosmology. In Sect. 2, we briefly review $f(T, B)$ gravity and its associated cosmology. In Sect. 3, we develop the gravitational perturbations while in Sect. 4 we form the perturbations equations with the perturbations about a perfect fluid. Finally in Sect. 5, we conclude our work with a discussion of the core results. In this work we use the $(+, -, -, -)$ signature.

2 Modified teleparallel theories of gravity

The curvature associated with the Levi-Civita connection $\mathring{\Gamma}^{\sigma}_{\mu\nu}$ (we use over-circles to denote quantities calculated with the Levi-Civita connection throughout) is torsion-less and satisfies the metricity condition [7,32]. TG is distinct from GR in that it supplants this connection with a torsion-ful teleparallel connection $\Gamma^{\sigma}_{\mu\nu}$ that has vanishing curvature and continues to satisfy the metricity condition [33–35,71]. In GR, many quantities are built on the Riemann tensor since it gives a measure of curvature on a manifold, it is for this reason that many modified theories of gravity feature implementations of this tensor [6]. However, in replacing this connection with its flat counterpart in teleparallel gravity, renders these quantities null irrelevant of the entries of the metric tensor. It is in this context that TG theory requires a novel approach to constructing tensorial quantities in order to build gravitational models.

GR is built on the metric tensor $g_{\mu\nu}$ being the fundamental dynamical object, as are the modifications of GR. In TG, the metric tensor becomes a derived object with the tetrad $e^{A}{}_{\mu}$ replacing it as the fundamental gravitational variable of the theory [33]. In this context, Latin indices refer to the Minkowski space while Greek indices point to the general manifold, where the tetrad acts as a soldering agent between the two. Thus, the tetrad (and its inverses $E_{A}{}^{\mu}$) can transform between the general manifold and its associated Minkowski space through

$$g_{\mu\nu} = e^{A}{}_{\mu} e^{B}{}_{\nu} \eta_{AB}, \quad \eta_{AB} = E_{A}{}^{\mu} E_{B}{}^{\nu} g_{\mu\nu}, \tag{1}$$

where the tetrads observe orthogonality conditions

$$e^{A}{}_{\mu} E_{B}{}^{\mu} = \delta^{B}_{A}, \quad e^{A}{}_{\mu} E_{A}{}^{\nu} = \delta^{\nu}_{\mu}, \tag{2}$$

for consistency's sake. The teleparallel connection can then be defined as [36]

$$\Gamma^{\sigma}_{\nu\mu} := E_{A}{}^{\sigma} \partial_{\mu} e^{A}{}_{\nu} + E_{A}{}^{\sigma} \omega^{A}{}_{B\mu} e^{B}{}_{\nu}, \tag{3}$$

where $\omega^{A}{}_{B\mu}$ represents the spin connection. The teleparallel connection represents the most general linear affine connection that is flat and satisfies the metricity condition [33,72]. The spin connection $\omega^{A}{}_{B\mu}$ acts as a balance to retain the general covariance of the ensuing field equations due to the freedom in the choice of the tetrad components in Eq. (1) [73]. Levi-Civita based theories (such as GR) hide this feature in its inertial structure and does not play an active role for most expressions of the theory [7,32]. The spin connection in TG is totally inertial and incorporates the effects of the local Lorentz transformations (LLTs) thus producing LLT invariant theories. Naturally, there will always exist a frame where the spin connection is vanishing as in the original formulation in Ref. [36], and this choice of frame is called the Weitzenböck gauge.

The spin connection can be fully represented as $\omega^{A}{}_{B\mu} = \Lambda^{A}{}_{C} \partial_{\mu} \Lambda_{B}{}^{C}$ [33], where the full breadth of the LLTs (Lorentz boosts and rotations) are represented by $\Lambda^{A}{}_{B}$. Through this perspective, there exist an infinite number of tetrads that satisfy Eq. (1), each of which produces an independent spin connection which counter-balances each other. It is therefore the tetrads together with its associated spin connection that renders a covariant formulation of TG.

Building on rationale of the Riemann tensor, the teleparallel connection can be straightforwardly used to build a meaningful measure of torsion through an antisymmetric operation on its lower indices. Thus, torsion can be represented as an expression of antisymmetry through the torsion tensor defined as [34,35]

$$T^{\sigma}{}_{\mu\nu} := -2\Gamma^{\sigma}_{[\mu\nu]}, \tag{4}$$

where square brackets denote the usual antisymmetric operator. The field strength of TG is represented by the torsion tensor [33], which transforms covariantly under both diffeomorphisms and LLTs. As in theories of gravity based on the Levi-Civita connection, we can also construct other gravitational tensors that reveal general features of TG. Firstly, take the contorsion tensor that emerges as the difference between the teleparallel and Levi-Civita connections, and can be written purely in terms of the torsion tensor as

$$K^{\sigma}{}_{\mu\nu} := \Gamma^{\sigma}_{\mu\nu} - \mathring{\Gamma}^{\sigma}_{\mu\nu} = \frac{1}{2}\left(T_{\mu}{}^{\sigma}{}_{\nu} + T_{\nu}{}^{\sigma}{}_{\mu} - T^{\sigma}{}_{\mu\nu}\right). \tag{5}$$

This has an important role to play in relating TG with GR and its modifications, as will become apparent later on. Another core component of TG is the superpotential defined as [35]

$$S_{A}{}^{\mu\nu} := \frac{1}{2}\left(K^{\mu\nu}{}_{A} + e_{A}{}^{\nu} T^{\mu} - e_{A}{}^{\mu} T^{\nu}\right), \tag{6}$$

where $T^{\nu} := T^{\alpha}{}_{\alpha}{}^{\nu} = -T^{\alpha\nu}{}_{\alpha}$. This has been shown to have a potential relationship to the energy–momentum tensor for gravitation [74] but the issue remains open [75]. By contracting the torsion tensor together with its superpotential, the torsion scale emerges [34]

$$T := S_{A}{}^{\mu\nu} T^{A}{}_{\mu\nu}, \tag{7}$$

as being purely the product of the teleparallel connection, in an analogous way to the Ricci scalars dependence purely on the Levi-Civita connection. The standard Ricci scalar $\mathring{R}$ (computed with the Levi-Civita connection) clearly will not vanish but its TG analog will, $R = 0$. Using the contorsion tensor, it can be shown that the teleparallel Ricci scalar, which vanishes, is equal to the sum of the Ricci and torsion scalars (up to a boundary term) through [76,77]

$$R = \mathring{R} + T - \frac{2}{e}\partial_{\mu}\left(eT^{\mu}\right) = 0. \tag{8}$$

This directly leads to an equivalency relation between the standard Ricci and torsion scalars given by

$$\mathring{R} = -T + \frac{2}{e}\partial_\mu\left(eT^\mu\right) = -T + 2\mathring{\nabla}_\mu\left(T^\mu\right) = -T + B, \tag{9}$$

where we define the boundary term as

$$B := 2\mathring{\nabla}_\mu\left(T^\mu\right), \tag{10}$$

called the TEGR boundary term, and where $e = \det(e^A{}_\mu) = \sqrt{-g}$ is the tetrad determinant. The ensuing dynamical equations will thus be guaranteed to be identical since these scalars differ by a boundary when expressed linearly. In this way, we can define the Teleparallel Gravity equivalent of general relativity (TEGR) as

$$\mathcal{S}_{\rm TEGR} = -\frac{1}{2\kappa^2}\int d^4x\, eT + \int d^4x\, e\mathcal{L}_m, \tag{11}$$

where $\kappa^2 = 8\pi G$ and $\mathcal{L}_m$ is the regular matter Lagrangian. The boundary term difference at the level of the Lagrangians can have an important impact when modifications of TEGR are considered which can lead to novel approaches to gravity that not recoverable in GR. In fact, the boundary term embodies the fourth order derivative contributions to the GR field equations thus decoupling these contributions that are incorporated in the Ricci scalar in standard gravity.

In standard gravity, one of the most popular approaches to gravity beyond GR is that of $f(\mathring{R})$ gravity [26,30] in which the Ricci scalar is straightforwardly generalized to an arbitrary function therefore. Another is Horndeski theory in which a single scalar field is added with the proviso of producing second order equations of motion [78] which was recently formulated in TG [40,79,80]. In TEGR, two scalars play an important role in producing the equivalency with GR in standard gravity. The torsion scalar produces the same second order dynamics, while the boundary term absorbs the divergence quantities. The T and B scalars embody the second and fourth order contributions respectively. It is for these reasons that to fully embody the rationale of many theories beyond GR we must consider an arbitrary generalization with both scalars. This will also suitably incorporate $f(\mathring{R})$ gravity as a subcase of the broader $f(T, B)$ framework.

$f(T, B)$ gravity [56–61,81] is a novel approach to modifying gravity and limits to $f(\mathring{R})$ gravity in the limits where $f(T, B) = f(-T + B) = f(\mathring{R})$. This is expressed as a generalization of TEGR through the action

$$\mathcal{S}_{f(T,B)} = \frac{1}{2\kappa^2}\int d^4x\, ef(T, B) + \int d^4x\, e\mathcal{L}_m. \tag{12}$$

Taking a variation of the action with respect to the tetrad gives [56,59]

$$\begin{aligned} &2\delta^\lambda_\nu\mathring{\Box}f_B - 2\mathring{\nabla}^\lambda\mathring{\nabla}_\nu f_B + Bf_B\delta^\lambda_\nu + 4\Big[(\partial_\mu f_B) + (\partial_\mu f_T)\Big]S_\nu{}^{\mu\lambda} \\ &+4e^{-1}e^A{}_\nu\partial_\mu(eS_A{}^{\mu\lambda})f_T - 4f_T T^\sigma{}_{\mu\nu}S_\sigma{}^{\lambda\mu} \\ &-f\delta^\lambda_\nu = 2\kappa^2\Theta^\nu{}_\lambda, \end{aligned} \tag{13}$$

where subscripts denote derivatives, and $\Theta^\nu{}_\lambda$ is the regular energy–momentum tensor for matter. The dynamical equations here have been derived for a vanishing spin connection (Weitzenböck gauge) scenario which has been shown to be compatible with a flat FLRW metric [56–59,82] which is what we develop here.

The spectrum of $f(T, B)$ gravity in Minkowski spacetime [83] includes the usual massless graviton with a $\sim -f_T$ modulation of the propagator, and an additional "scalaron" with a mass $\sim 1/\sqrt{-f_{BB}}$. Thus, to avoid a ghost one requires that $f_T < 0$ and to avoid a tachyon that $f_{BB} < 0$. A feature of general $f(T, B)$ gravity thus is that it expresses the same gravitational wave polarization signature as $f(\mathring{R})$ gravity [57,59].

The field equations in Eq. (13) can straightforwardly be rewritten as

$$\begin{aligned} &-f_T\mathring{G}_{\mu\nu} + g_{\mu\nu}\mathring{\Box}f_B - \mathring{\nabla}_\mu\mathring{\nabla}_\nu f_B \\ &+\frac{1}{2}(Bf_B + Tf_T - f)g_{\mu\nu} \\ &+2\Big[\mathring{\nabla}^\lambda f_T + \mathring{\nabla}^\lambda f_B\Big]S_{\nu\lambda\mu} = \kappa^2\Theta_{\mu\nu}, \end{aligned} \tag{14}$$

where the Einstein tensor $\mathring{G}_{\mu\nu}$ explicitly emerges due to the close relationship between curvature and torsion. It is important to point out that while this represents the field equations of the teleparallel $f(T, B)$ gravity, the Einstein tensor and the covariant derivatives are dependent on the Levi-Civita connection. It is useful to separate these equations to its symmetric and antisymmetric parts. To do this, let us introduce the following tensor

$$Q_{\nu\mu} := \frac{1}{2}\Big[\mathring{\nabla}^\lambda f_T + \mathring{\nabla}^\lambda f_B\Big]S_{\nu\lambda\mu}, \tag{15}$$

and then the antisymmetric field part as Eq. (14) becomes

$$Q_{[\alpha\nu]} = (\partial^\lambda f_T + \partial^\lambda f_B)(T_{\alpha\lambda\nu} + g_{\alpha\lambda}T_\nu - g_{\alpha\nu}T_\lambda) \tag{16}$$

$$= \Big[(f_{TT} + f_{TB})\partial^\lambda T + (f_{BB} + f_{TB})\partial^\lambda B\Big] \times(T_{\alpha\lambda\nu} + g_{\alpha\lambda}T_\nu - g_{\alpha\nu}T_\lambda) = 0, \tag{17}$$

where we have used the condition that the energy–momentum tensor is symmetric.

Now, in this work we probe the cosmology of $f(T, B)$ gravity through the tetrad

$$e^A{}_\mu = \text{diag}(1, a(t), a(t), a(t)), \tag{18}$$

where $a(t)$ is the scale factor, and which reproduces the flat homogeneous isotropic FLRW metric

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2), \tag{19}$$

through Eq. (1). This diagonal tetrad is compatible with a flat spin connection, $\omega^{A}{}_{B\mu} = 0$ [73,84]. Through Eq. (7), the torsion scalar turns out to be

$$T = -6H^2, \tag{20}$$

and the boundary term is given by

$$B = -6(3H^2 + \dot{H}), \tag{21}$$

which together reproduce the Ricci scalar, i.e. $\mathring{R} = -T + B = -6(\dot{H} + 2H^2)$. Using the field equations in Eq. (13) together with the FLRW tetrad in Eq. (18) produces the Friedmann equations

$$3H(\dot{f}_B - 2Hf_T) + \frac{1}{2}(Bf_B - f) = \kappa^2\rho, \tag{22}$$

$$-\ddot{f}_B + 2f_T\dot{H} + 2H(3Hf_T + \dot{f}_T) + \frac{1}{2}(f - Bf_B) = \kappa^2 P, \tag{23}$$

where overdots refer to derivatives with respect to cosmic time t, and where ρ and P respectively represent the energy density and pressure of matter.

At background level, we can write the Friedmann equations for $f(T, B)$ gravity as an effective fluid equation as an addition to the TEGR Lagrangian through $f(T, B) \to -T + \tilde{f}(T, B)$. Evaluating the dynamical equations in Eq. (13) for the FLRW setting gives the Friedmann equations

$$3H^2 = \kappa^2 (\rho + \rho_{\text{eff}}), \tag{24}$$

$$3H^2 + 2\dot{H} = -\kappa^2 (P + P_{\text{eff}}). \tag{25}$$

Through the effective fluid description, this means that the fluid properties are represented by

$$\kappa^2\rho_{\text{eff}} := 3H^2(3\tilde{f}_B + 2\tilde{f}_T) - 3H\dot{\tilde{f}}_B + 3\dot{H}\tilde{f}_B + \frac{1}{2}\tilde{f}, \tag{26}$$

$$\kappa^2 P_{\text{eff}} := -\frac{1}{2}\tilde{f} - (3H^2 + \dot{H})(3\tilde{f}_B + 2\tilde{f}_T) - 2H\dot{\tilde{f}}_T + \ddot{\tilde{f}}_B. \tag{27}$$

The $\tilde{f}(T, B)$ gravity effective fluid description also satisfies the fluid equation [58]

$$\dot{\rho}_{\text{eff}} + 3H(\rho_{\text{eff}} + P_{\text{eff}}) = 0, \tag{28}$$

and leads directly to an effective fluid equation of state (EoS)

$$\omega_{\text{eff}} := \frac{P_{\text{eff}}}{\rho_{\text{eff}}} \tag{29}$$

$$= -1 + \frac{\ddot{\tilde{f}}_B - 3H\dot{\tilde{f}}_B - 2\dot{H}\tilde{f}_T - 2H\dot{\tilde{f}}_T}{3H^2\left(3\tilde{f}_B + 2\tilde{f}_T\right) - 3H\dot{\tilde{f}}_B + 3\dot{H}\tilde{f}_B - \frac{1}{2}\tilde{f}}. \tag{30}$$

In the ΛCDM limit, this EoS approaches an effective cosmological constant behaviour where $\omega_{\text{eff}} = -1$, as expected. In the next section we consider the cosmological perturbations within $\tilde{f}(T, B)$ gravity. In that context, it is more convenient to work with a pure $f(T, B)$ gravity representation.

3 Cosmological perturbations of $f(T, B)$ gravity

Cosmological perturbations can reveal an incredible amount of information about the Universe that is not immediately clear from the background cosmology such as the formation of cosmic structures and the gravitational wave background universe. Cosmic perturbations were investigated in $f(T)$ gravity several times such as Ref. [47] where the tetrad is only in the correct Weitzenböck gauge in terms of the tensor perturbations and thus results in an overly restrictive set of scalar perturbations, which is later clarified in Ref. [73]. It was only in Ref. [66] that the situation was fully resolved, which was also applied to the $f(T, \mathcal{T})$ gravity scenario in Ref. [69]. The core results have since been confirmed and widened in Refs. [85–87]. In what follows, we explore the tensor and scalar cosmological perturbations within the sub-horizon limit. This is achieved by taking the scalar-vector-tensor (SVT) decomposition of the cosmological perturbations using [87]

$$\left[\delta e^{A}{}_{\mu}\right] := \begin{bmatrix} \varphi & a(\partial_i\beta + \beta_i) \\ \delta^{I}{}_{i}\left(\partial^i b + b^i\right) & a\delta^{Ii}\left(-\psi\delta_{ij} + \partial_i\partial_j h + 2\partial_{(i}h_{j)} + \frac{1}{2}h_{ij} + \epsilon_{ijk}\left(\partial^k\sigma + \sigma^k\right)\right) \end{bmatrix}, \tag{31}$$

which inherits its symmetries from the metric and retains the Weitzenböck gauge even at perturbative level. It is important to emphasize that this tetrad remains a good tetrad even at perturbative level in that the associated spin connection components are compatible with the case where they vanish. This is crucial to producing a consistent cosmological perturbation analysis. A note on the use of indices, A, B, C, D, ... and Greek lowercase letters $\mu, \nu, \rho, \sigma, \ldots$ are used as 4-D indices on the Minkowski and general manifold respectively. The middle range Latin indices I, J, K, ... and i, i, k, ... refer to spacial 3-D indices in Minkowski and general manifold respectively. In fact, this produces the regular perturbed metric

$$\left[\delta g_{\mu\nu}\right] = \begin{bmatrix} -2\varphi & a\left(\partial_i(b-\beta)+(b_i-\beta_i)\right) \\ a\left(\partial_i(b-\beta)+(b_i-\beta_i)\right) & 2a^2\left(-\psi\delta_{ij}+\partial_i\partial_j h+2\partial_{(i}h_{j)}+\frac{1}{2}h_{ij}\right) \end{bmatrix}, \tag{32}$$

due to Eq. (1), where h_{ij} is symmetric, traceless $h_{ij}\delta^{ij}=0$, and transverse $\partial^i h_{ij}=0$, while all the vectors are solenoidal $\partial_i b^i = 0$.

Now, in our convention, the Fourier transform of a perturbation X will be given by

$$X(t,x) = \int \frac{d^3k}{(2\pi)^{2/3}}[X(t,k)e^{ikx} + X^\dagger(t,k)e^{-ikx}], \tag{33}$$

which is used throughout to transform the cosmological perturbations. In the appendices we include all important calculations of each perturbation.

Also, the matter perturbation of the energy–momentum tensor for a perfect fluid $\delta\Theta_{\mu\nu}$ is

$$\begin{aligned}\delta\Theta_{\mu\nu} = {} & (\delta\rho+\delta P)u_\mu u_\nu + (\rho+P)\delta u_\mu u_\nu + (\rho+P)u_\mu\delta u_\nu \\ & +\delta P g_{\mu\nu} + P\delta g_{\mu\nu},\end{aligned} \tag{34}$$

where the 4-velocity is represented by u_μ and 3-velocity by $v_i = \partial_i v$ with components

$$\delta\Theta_{00} = \delta\rho + 2\rho\phi, \tag{35}$$

$$\delta\Theta_{0i} = \delta\Theta_{i0} = -a\rho\partial_i(b-\beta) - a(\rho+P)\partial_i v, \tag{36}$$

$$\delta\Theta_{ij} = a^2\delta P\delta_{ij} - 2a^2 P\psi\delta_{ij} + 2a^2 P\partial_i\partial_j h. \tag{37}$$

Together, this forms the basis for the matter perturbation equations to be explored later on after the scalar perturbations. In the following computations the xAct packages [88–94] were used.

3.1 Tensor perturbations

Considering the tensor perturbation part of the cosmological perturbations in the tetrad in Eq. (31) which are

$$\delta e^I{}_j = \frac{a}{2}\delta^{Ii}h_{ij}, \tag{38}$$

we can determine the tensor perturbations within the $f(T,B)$ action. The tensor modes are determined by considering perturbations up to second order in the Lagrangian density, which in Fourier space results in the gravitational wave propagation equation

$$\ddot{h}_{ij} + (3+\nu)H\dot{h}_{ij} + \frac{k^2}{a^2}h_{ij} = 0, \tag{39}$$

which governs propagation of tensor perturbations. The background equations were used in these derivations to simplify the perturbation results. Here, the Planck mass run rate turns out to be

$$\nu = \frac{1}{H}\frac{\dot{f}_T}{f_T}, \tag{40}$$

which is a frictional term in the propagation of gravitational waves, as evidenced through the gravitational wave propagation equation [95–97]. Immediately, a stability condition in which $f_T < 0$ can be read off (which depends on the convention being used for the torsion scalar). Another crucial point that can be read of the tensor perturbations is the speed of propagation of gravitational waves, which turns out to be exactly that of light [98]

$$c_T^2 = 1, \tag{41}$$

and in total agreement with the multimessenger events of GW170817 [99] and GRB170817A [100].

In this context, $f(T,B)$ gravity is not strongly constrained by present observations since it predicts speed of light propagation of gravitational waves and no constraints exist for the Planck run rate. However, the stability conditions in Eq. (40) will be crucial to forming stable models and have an impactful effect on the other perturbations that follow. In terms of the propagation speed, this turns out to be identical to the $f(\mathring{R})$ gravity case where gravitational waves propagation at the speed of light [101].

3.2 Vector (and pseudovector) perturbations

The vector perturbations in the cosmological perturbations in Eq. (31) are represented by

$$\left[\delta e^A{}_\mu\right] = \begin{bmatrix} 0 & a\beta_i \\ \delta^I{}_i b^i & a\delta^{Ii}\epsilon_{ijk}\sigma^k \end{bmatrix}, \tag{42}$$

where the gauge freedom is fixed by the choice $h_i \equiv 0$. Using the field equations, we directly obtain the perturbation equations for the β_i and the pseudovector σ_i

$$W_{[0i]}: \quad 0 = \sigma_i(\dot{f}_B + \dot{f}_T), \tag{43}$$

$$W_{[ij]}(i\neq j): \quad 0 = \beta_i(\dot{f}_B + \dot{f}_T), \tag{44}$$

which for $\dot{f}_B + \dot{f}_T \neq 0$ give $\sigma_i = 0$ and $\beta_i = 0$. We are left with two equations that govern the evolution of b_i and v_i which are embodied through

$$W_{ij}(i\neq j): \quad 0 = \dot{b}_j + b_j\left(2H + \frac{\dot{f}_T}{f_T}\right), \tag{45}$$

$$\begin{aligned}W_{0i}: \quad 0 = {} & b_i\Big(a(H(6\dot{f}_B+4\dot{f}_T)+4f_T\dot{H}-2\ddot{f}_B) \\ & -\frac{k^2 f_T}{a}\Big) + av_i(6H(\dot{f}_B-2Hf_T)+Bf_B-f),\end{aligned} \tag{46}$$

which involves only those two components, and where v_i represents the 3-velocity (discussed further in the appendix). Immediately, it is clear that if this is solved for b_i, then it is

solvable for v_i as well. At this stage one can directly see that the vector perturbations are not propagating since Eq. (45) is just a constraint equation and can further read off the stability condition $2H > -\log \dot{f}_T$. Another important observation is that , which has exactly the same form as that reported in Ref. [87] for $f(T)$ gravity with the exception that in our case $f(T) \to f(T, B)$ (and the impact of this on derivative terms).

If $\dot{f}_B + \dot{f}_T \equiv 0$ it implies that $f_T = -f_B = -f_R$ which is the case of $f(\mathring{R})$ gravity, where all antisymmetric field equations are trivialised with $W_{[\mu\nu]} \equiv 0$. By introducing $Y_i := b_i - \beta_i$, we end up with the following nonvanishing field equations

$$W_{ij}(i \neq j): \quad 0 = \dot{Y}_j + Y_j \left(2H + \frac{\dot{f}_R}{f_R}\right), \tag{47}$$

$$W_{0i}: \quad 0 = a v_i \left(6H(2Hf_R + \dot{f}_R) + Bf_R - f\right) + Y_i \left(2a(-2f_R\dot{H} + H\dot{f}_R - \ddot{f}_R) + \frac{k^2 f_R}{a}\right), \tag{48}$$

where $f_R = df/d\mathring{R}$. In this equation we notice that, again, there are not propagating vector perturbations which is a well known result in $f(\mathring{R})$ theories [101].

3.3 Scalar perturbations

Selecting the scalar perturbations of Eq. (31) gives the following linear perturbations

$$\left[\delta e^A{}_\mu\right] = \begin{bmatrix} \varphi & a\partial_i\beta \\ \delta^I{}_i \partial^i b & a\delta^{Ii}\left(-\psi\delta_{ij} + \partial_i\partial_j h + \epsilon_{ijk}\partial^k\sigma\right) \end{bmatrix}, \tag{49}$$

in which we will adopt the Newtonian gauge where $b = \beta$ and $h = 0$. In the following we report the final field equations but in the appendix, the component calculations that build up to these results are presented. The symmetric field equations of the scalar perturbations are given by

$$W_{00}: \quad \kappa^2\delta\rho = 3H\delta\dot{f}_B + \left(\frac{k^2}{a^2} + \frac{B}{2}\right)\delta f_B - 6H^2\delta f_T - \frac{1}{2}f_T\delta T - \frac{2Hk^2 f_T}{a}b + \dot{\psi}(12Hf_T - 3\dot{f}_B) + \frac{2k^2 f_T}{a^2}\psi + 6H\phi(2Hf_T - \dot{f}_B), \tag{50}$$

$$W_{ij}(i \neq j): \quad \psi - \phi = \frac{1}{f_T}(a(\dot{f}_T + \dot{f}_B)b - \delta f_B), \tag{51}$$

$$W^i_i: \quad -\kappa^2\delta P = \delta\ddot{f}_B + \delta f_B\left(\frac{2k^2}{3a^2} + \frac{B}{2}\right) - 2H\delta\dot{f}_T - 2(3H^2 + \dot{H})\delta f_T - \frac{1}{2}f_T\delta T + 2f_T\ddot{\psi} + 2\dot{\psi}(6Hf_T + \dot{f}_T) + \frac{2k^2 f_T}{3a^2}\psi - \frac{2k^2}{3a}(\dot{f}_B + 3Hf_T + \dot{f}_T)b + \dot{\phi}(2Hf_T - \dot{f}_B) + \phi\left(4f_T\left(-\frac{2k^2 f_T}{3a^2} + 3H^2 + \dot{H} - 2\ddot{f}_B\right) + 4H\dot{f}_T\right), \tag{52}$$

where $\delta f_T = f_{TT}\delta T + f_{TB}\delta B$ and $\delta f_B = f_{BT}\delta T + f_{BB}\delta B$, while the antisymmetric contributions are

$$W_{0i}: \quad \kappa^2 a v(P + \rho) = \delta\dot{f}_B - 3H\delta f_B + 2f_T\dot{\psi} - 2H\delta f_T + (2f_T H - \dot{f}_B)\phi, \tag{53}$$

$$W_{i0}: \quad \kappa^2 a v(P + \rho) = \delta\dot{f}_B - H\delta f_B + 2f_T\dot{\psi} + 2(\dot{f}_T + \dot{f}_B)\psi + (2f_T H - \dot{f}_B)\phi, \tag{54}$$

$$W_{i0} - W_{0i}: \quad 0 = H(\delta f_T + \delta f_B) + \psi(\dot{f}_T + \dot{f}_B), \tag{55}$$

and where the energy–momentum conservation in the case of dust (for the general case see the Appendix A4) is given by

$$\nabla_\mu \Theta_0{}^\mu: \quad \delta\dot{\rho} + 3H\delta\rho = \frac{\rho}{a}k^2 v + 3\dot{\psi}\rho, \tag{56}$$

$$\nabla_\mu \Theta_i{}^\mu: \quad a\dot{v} + aHv = -\phi. \tag{57}$$

The scalar perturbations are coupled with the perturbations of the energy–momentum components and so this is not enough information to determine the impact of these cosmological perturbations on observational parameters. In the next section, we will study the matter perturbation equations to determine the role of $f(T, B)$ gravity on the growth of structure in the Universe.

4 Matter perturbation equations in $f(T, B)$ gravity

In this section, we consider dust for the perfect fluid, and derive the corresponding matter perturbation equations. Following Refs. [101,102], we introduce the variable $V := av$ and start by defining the density contrast δ_m as

$$\delta_m := \frac{\delta\rho}{\rho} + 3HV. \tag{58}$$

In order to determine the time derivative of this parameter, we need to utilize the continuity equation to obtain the density parameter time derivative, which is

$$\delta\dot{\rho} + 3H\delta\rho = \frac{k^2\rho V}{a^2} + 3\rho\dot{\psi}. \tag{59}$$

The time derivative of the density contrast parameter can then be written as

$$\dot{\delta}_m = -\frac{\nabla^2 V}{a^2} + 3\dot{\psi} + 3(\dot{H}V), \tag{60}$$

$$\dot{V} = -\phi, \tag{61}$$

where the time derivative of V is also presented. By combining both derivatives, we obtain

$$\ddot{\delta}_m + 2H\dot{\delta}_m = \frac{\nabla^2\phi}{a^2} + 3\ddot{\psi} + 3(\dot{H}\ddot{V}) + 6H\dot{\psi} + 6H(\dot{H}V). \tag{62}$$

In the sub-horizon approximation $k >> aH$, k being well inside the Hubble radius, the dominant terms are k and $\delta\rho$. Now that we have all the prerequisites we need to proceed, let us first summarize the dominant terms in this limit

$$\left\{\frac{k^2}{a^2}|\phi|, \frac{k^2}{a^2}|\psi|, \frac{k^2}{a^2}|\beta|, \frac{k^2}{a^2}|\delta f_T|, \frac{k^2}{a^2}|\delta f_B|\right\} \gg \left\{H^2|\phi|, H^2|\psi|, H^2|\beta|, H^2|\delta f_T|, H^2|\delta f_B|\right\} \tag{63}$$

and

$$|\dot{X}| \precsim |HX| \quad \text{where} \quad X \in \left\{\phi, \psi, \beta, \delta f_T, \delta f_B, \dot{\phi}, \dot{\psi}, \dot{\beta}, \delta\dot{f}_T, \delta\dot{f}_B\right\}. \tag{64}$$

Thus, it follows directly that in Fourier space of the sub-horizon limit of Eq. (62)

$$\ddot{\delta}_m + 2H\dot{\delta}_m \simeq -\frac{k^2\phi}{a^2} = 4\pi\rho G_{\text{eff}}\delta_{\text{m}} = \frac{\kappa^2}{2}\rho G_{\text{eff}}\delta_{\text{m}}, \tag{65}$$

from which it follows that the only contributing scalar is ϕ. Along a similar vein, Σ_{def} is a parameter sensitive to weak lensing which appears when we write the lensing potential $-(\phi+\psi)$ in terms of the matter density contrast δ_m, so Σ_{def} plays a similar role to G_{eff} but between the lensing potential and δ_m specifically. This parameter is defined as

$$\Sigma := \frac{1}{2}\frac{G_{\text{eff}}}{G}\left(1 + \frac{\psi}{\phi}\right), \tag{66}$$

which we will also calculate in conjunction with G_{eff} in what follows. We start from the sub-horizon approximation of the field equations in Eqs. (50) and (56)

$$W_{00}: \quad \kappa^2\delta\rho \simeq \left(\frac{2k^2 f_T}{a^2} - 3H\dot{f}_B\right)\psi + \left(\frac{k^2}{a^2} - 3\dot{H}\right)\delta f_B + 6H(Hf_T - \dot{f}_B)\phi - 6H^2\delta f_T, \tag{67}$$

$$W_{[0i]}: \quad 0 \simeq \psi(\dot{f}_B + \dot{f}_T) + H\delta f_B + H\delta f_T, \tag{68}$$

$$W_{ij}(i \neq j): \quad 0 = -ab(\dot{f}_B + \dot{f}_T) + \delta f_B + f_T\psi - f_T\phi, \tag{69}$$

$$W_i^i: \quad 0 \simeq \delta f_B(18a^2\dot{H} - 4k^2) + 12a^2(4H^2 + \dot{H})\delta f_T + 4ak^2(\dot{f}_B + \dot{f}_T)b + (6a^2(H(\dot{f}_B - 4\dot{f}_T) + 2\ddot{f}_B) + 4f_T(k^2 - 6a^2\dot{H}))\phi - 4\psi(f_T k^2 + 3a^2 H\dot{f}_T), \tag{70}$$

from which we present the fully expanded form of the $W_{[0i]}$ component

$$W_{[0i]}: \quad 0 \simeq -\frac{4H^2k^2(f_{BB} + 2f_{TB} + f_{TT})}{a}b + \frac{\left(a^2((\dot{f}_B + \dot{f}_T) + 12H^3(f_{TT} + f_{TB})) + 4Hk^2(f_{BB} + f_{TB})\right)}{a^2}\psi \tag{71}$$

$$- \frac{2H\left((f_{BB} + f_{TB})(k^2 - 6a^2\dot{H}) - 6a^2H^2(f_{TT} + f_{TB})\right)}{a^2}\phi. \tag{72}$$

Note that $W_{[0i]}$ is actually a constraint equation and so must be used in the solution process. Consequentially in order to have a closed system we only need one more equation from $\{W_{ij}, W_i^i\}$, which we choose to be W_{ij}. Henceforth our system will be comprised of $\{W_{00}, W_{[0i]}, W_{ij}\}$. We checked in every case that the fourth equation W_i^i was always satisfied. Before proceeding we define the useful parameters

$$\Pi := f_B + f_T, \tag{73}$$

$$\Upsilon := f_{BB} + 2f_{TB} + f_{TT} = \Pi_T + \Pi_B, \tag{74}$$

$$\Xi := f_{TB}^2 - f_{TT}f_{BB} = -\Pi_T\Pi_B + f_{TB}\Upsilon. \tag{75}$$

One could think of Π as the deviation from $f(\mathring{R})$ gravity where $\Pi|_{f(\mathring{R})} \equiv 0$. These quantities will help us classify the $f(T, B)$ models in three branches

1. $\{\Pi \neq \text{const}, \Upsilon \neq 0\}$
   Which can further be classified using $\Xi = -\Pi_T\Pi_B + f_{TB}\Upsilon$
   (a) $\{\Pi \neq \text{const}, \Upsilon \neq 0, \Xi \neq 0\}$ most general case of $f(T, B)$
   (b) $\{\Pi \neq \text{const}, \Upsilon \neq 0, \Xi = 0\}$ includes $f(T)$
2. $\{\Pi \neq \text{const}, \Upsilon \equiv 0\}$
   Which can further be classified using $\Upsilon \equiv 0 \Rightarrow \Pi_B \equiv -\Pi_T$ into Eq. (75) as $\Xi = \Pi_T^2 = \Pi_B^2$
   (a) $\{\Pi \neq \text{const}, \Upsilon = 0, \Xi \neq 0\}$
   (b) $\{\Pi = \text{const}, \Upsilon = 0, \Xi = 0\}$ the unique $f(\mathring{R})$ case.

The above branches may also be indicators of variable degrees of freedom (dof), since we know for sure that $f(\mathring{R})$ has 3 dof. We also know that $f(T)$ "varies" in between 3–5 maximum dof [103–106].

We will elaborate a bit on the two major conditions $\Xi \equiv 0$ and $\Upsilon \equiv 0$. Starting off with $\Xi \equiv 0$, it can be solved using separation of variables if one assumes $f(T, B) = f_1(T)f_2(B)$, then one finds

$$f(T, B) = f_0((B + Bm - C_2)(T + mT - C_3m)^m)^{\frac{1}{m+1}}, \quad m \neq -1, \tag{76}$$

$$f(T, B) = f_0 e^{C_1T + C_2B}, \quad m = -1, \tag{77}$$

where f_0, C_1, C_2, C_3, m are constants. Another family of solutions are of the form $f(T, B) = f(\Phi)$ where $\Phi = \Phi(T, B)$ i.e single variable dependence. Popular models of this type are where $\Phi \equiv \mathring{R} = -T + B$ and $\Phi \equiv T$ which represent $f(\mathring{R})$ and $f(T)$ theories of gravity respectively. Another form of single variable dependence is $f(TB) = c\sqrt{TB}$ which is the only acceptable model of the form $f(TB) = c\,(TB)^m$. Finally, a less known model of importance here is

$$f(T, B) = -T + F(B), \tag{78}$$

which will be used later on the analysis.

As for the condition $\Upsilon \equiv 0$, it is satisfied by a family of solutions of the form

$$f(T, B) = f_1(\mathring{R})X + f_2(\mathring{R}), \tag{79}$$

where $X = X(T, B)$ is any function such that $X_T + X_B \neq 0$ and $\Upsilon \equiv 0$. The condition $X_T + X_B \neq 0$ practically means that $X \neq X(\mathring{R})$ so that the total solution in Eq. (79) is not reduced to just $f(\mathring{R})$. The most intuitive form would be $X = \left(c_1 T^p + c_2 B^q + c_3\,(TB)^r\right)^m$ and upon enforcing the aforementioned conditions, the form is reduced to just $X = c_1 T + c_2 B$ where $c_1, c_2 \in \mathbb{R}$ and $c_1 \neq -c_2$. One can easily see that a solution compatible with both $\Xi \equiv 0$ and $\Upsilon \equiv 0$ is $f(\mathring{R})$.

4.1 Branch $\{\Pi \neq \text{const}, \Upsilon \neq 0, \Xi \neq 0\}$

We will start with the most complex case that includes the full totally non-linear $f(T, B)$ models meaning those which will allow us to solve the constraint field in Eq. (72) for b as

$$b = \frac{(a^2(12H^3 f_{TT} + \dot{\Pi}) + 4Hk^2\Pi_B)}{4aH^2k^2\Upsilon}\psi + \frac{(6a^2(\Pi_B\dot{H} + H^2 f_{TT}) - k^2\Pi_B)}{2aHk^2\Upsilon}\phi, \tag{80}$$

which we replace into Eq. (69) in order to find

$$\frac{\psi}{\phi} = \frac{2H\left(6a^4\dot{\Pi}(\Pi_B\dot{H} + H^2 f_{TT})\right)}{-a^4\dot{\Pi}(12H^3 f_{TT} + \dot{\Pi}) + 4a^2Hk^2(-2\Pi_B\dot{\Pi} + 12H^3 f_{BB} f_{TB} + H\Upsilon f_T) - 16H^2k^4\Xi} + \frac{2H\left(a^2k^2(-\Pi_B\dot{\Pi} - 24H^3 f_{BB} f_{TB} + 2H\Upsilon f_T + 24H\Xi\dot{H}) - 4Hk^4\Xi\right)}{-a^4\dot{\Pi}(12H^3 f_{TT} + \dot{\Pi}) + 4a^2Hk^2(-2\Pi_B\dot{\Pi} + 12H^3 f_{BB} f_{TB} + H\Upsilon f_T) - 16H^2k^4\Xi}, \tag{81}$$

that we then substitute into Eqs. (67) so that we finally end up with

$$G_{\text{eff}} = G\frac{A_1k^2 + A_2k^4 + A_3k^6}{A_4 + A_5k^2 + A_6k^4 + A_7k^6}, \tag{82}$$

$$\Sigma = \frac{\Delta_1k^2 + \Delta_2k^4 + \Delta_3k^6 + \Delta_4k^8 + \Delta_5k^{10}}{\Delta_6 + \Delta_7k^2 + \Delta_8k^4 + \Delta_9k^6 + \Delta_{10}k^8 + \Delta_{11}k^{10}}, \tag{83}$$

where all the coefficients A_i and Δ_i are presented in the Appendix B. One can further calculate the leading order terms of the above quantities by noticing that $A_3 \propto \Xi$, $A_7 \propto \Xi$ are the only coefficients, proportional to Ξ and the same happens with the coefficients $\Delta_5 \propto A_3$ and $\Delta_{11} \propto A_7$. This clarifies our choice for using Ξ as an extra layer in branching. Hence the leading order parts read respectively

$$G_{\text{eff}} = G\frac{A_3}{A_7} = -G\frac{4\Upsilon}{36H^2(f_{BB}f_{TT} + 2\Xi) + 3\Upsilon f_T}, \tag{84}$$

$$\Sigma = -\frac{\Upsilon}{\Upsilon f_T + 12H^2\,(f_{BB}f_{TT} + 2\Xi)}. \tag{85}$$

The models in this case assume the most possible general form they can from the class of $f(T, B)$, for example $f(T, B) = f_1(T) + f_2(T)f_3(B) + f_4(B)$.

4.2 Branch $\{\Pi \neq \text{const}, \Upsilon \neq 0, \Xi = 0\}$

A special case arises if $A_3 = A_7 \equiv 0$ which means that $\Xi \equiv 0$, giving that the leading order term for the gravitational effective constant is

$$G_{\text{eff}} = G\frac{A_2}{A_6}, \tag{86}$$

and for the deflection parameter, we get

$$\Sigma = \frac{\Delta_4}{\Delta_{10}} = -\frac{A_2}{A_6} = -\frac{-A_3(4H\Upsilon f_T - 5\Pi_B\dot{\Pi})}{4A_7\left(-(H\Upsilon f_T - 2\Pi_B\dot{\Pi}) + 24H^3 f_{BB}f_{TT} + 12H^3 f_{TB}(f_{TT} - \Upsilon)\right)}. \tag{87}$$

One can notice that G_{eff} becomes significantly more complicated since it depends on A_2 and A_6 (see Appendix B), and for that reason we explicitly calculate it for only two simple such models. The first one, is the popular $f(T)$ grav-

ity models which up to next to leading order we find from Eq. (86)

$$G_{\rm eff} = \frac{a^2\dot{f}_T(12H^3 f_{TT} + \dot{f}_T) - 4H^2k^2 f_T f_{TT}}{4H^2 f_T f_{TT}(6a^2 H\dot{f}_T + k^2 f_T)}, \tag{88}$$

$$\Sigma = \frac{3a^2\dot{f}_T(8H^3 f_{TT} + \dot{f}_T) - 8H^2k^2 f_T f_{\rm TT}}{2f_T\left(a^2\dot{f}_T(12H^3 f_{TT} - \dot{f}_T) + 4H^2k^2 f_T f_{TT}\right)}, \tag{89}$$

that correctly reproduce the usual leading order result $G_{\rm eff} = -G/f_T$ reported in Refs. [49,66,107]. The other, less known, model is Eq. (78) for which (86) gives

$$G_{\rm eff} = G\frac{4H(2\dot{f}_B + H)}{(\dot{f}_B + 2H)^2}, \tag{90}$$

$$\Sigma = \frac{a^2\dot{f}_B{}^2 + 4Hk^2 f_{BB}(2\dot{f}_B + H)}{3a^2 f_{BB}\dot{f}_B\left(H^2(9\dot{f}_B + 14H) - 3\dot{H}(\dot{f}_B + 2H)\right) + k^2 f_{BB}(\dot{f}_B + 2H)^2}. \tag{91}$$

4.3 Branch $\left\{\Pi \neq \text{const}, \Sigma = 0, \Xi = \Pi_T^2 = \Pi_B^2 \neq 0\right\}$

In this branch b completely drops out from Eq. (72) and we can solve for ψ as

$$\frac{\psi}{\phi} = \frac{2H\Pi_T(k^2 - 6a^2\dot{H})}{4Hk^2\Pi_T - a^2\dot{\Pi}}, \tag{92}$$

where we replace this solution into Eq. (69) and solve for b as

$$b = \frac{f_T\left(2H\Pi_T(6a^2\dot{H} + k^2) - a^2\dot{\Pi}\right) + 2\dot{\Pi}(k^2 - 6a^2\dot{H})(f_{TB} + \Pi_T)}{(4Hk^2\Pi_T - a^2\dot{\Pi})^2} a\,\phi. \tag{93}$$

Next we substitute both in Eq. (67) so that we can proceed and find $G_{\rm eff}$ as

$$G_{\rm eff} = G\frac{Z_1k^2 + Z_2k^4 + Z_3k^6}{Z_4 + Z_5k^2 + Z_6k^4 + Z_7k^6}, \tag{94}$$

where again we omitted the rest of the cumbersome coefficients. The leading order contribution is then

$$G_{\rm eff} = G\frac{Z_3}{Z_7} = -G\frac{4}{3(f_T + 12H^2 f_{TB})}. \tag{95}$$

In the same manner, we also calculate the deflection parameter

$$\Sigma = \frac{Y_1k^2 + Y_2k^4 + Y_3k^6 + Y_4k^8}{Y_5 + Y_6k^2 + Y_7k^4 + Y_8k^6 + Y_9k^8}, \tag{96}$$

where and to leading order

$$\Sigma = \frac{Y_4}{Y_9} = -\frac{1}{f_T + 12H^2 f_{TB}} \tag{97}$$

which is a much simpler form than (87).

4.4 Branch $\left\{\Pi = \text{const}, \Sigma = 0, \Xi = \Pi_T^2 = \Pi_B^2 \equiv 0\right\}$

The condition $\Pi_T^2 = \Pi_B^2 \equiv 0$ means exactly that $\Pi = f_T + f_B \equiv c$ which is the condition to obtain $f(\mathring{R})$ gravity (while not precisely $f(\mathring{R})$ gravity when $c \neq 0$, it is dynamically equivalent). This is a pivotal branch because it is the only one where the antisymmetric part of the field equations is trivialised $W_{[0i]} \equiv 0$ and also b completely drops out the field equations. We solve W_{ij} for ψ

$$\frac{\psi}{\phi} = \frac{a^2(12f_{RR}\dot{H} + f_R) - 2k^2 f_{RR}}{a^2 f_R - 4k^2 f_{RR}}, \tag{98}$$

next we substitute this in Eq. (67) so that we can proceed and find as per usual to find

$$G_{\rm eff} = \frac{8k^4 f_{RR} - 2a^2k^2 f_R}{-9a^4(f_R(4f_{RR}\dot{H}^2 + H\dot{f}_R) + 4Hf_{RR}\dot{f}_R\dot{H}) - 2a^2k^2(-15Hf_{RR}\dot{f}_R + 9f_R f_{RR}\dot{H} + f_R^2) + 6k^4 f_R f_{RR}}, \tag{99}$$

$$\Sigma = \frac{6k^4 f_{RR} - 2a^2k^2(6f_{RR}\dot{H} + f_R)}{-9a^4(f_R(4f_{RR}\dot{H}^2 + H\dot{f}_R) + 4Hf_{RR}\dot{f}_R\dot{H}) - 2a^2k^2(-15Hf_{RR}\dot{f}_R + 9f_R f_{RR}\dot{H} + f_R^2) + 6k^4 f_R f_{RR}}, \tag{100}$$

If one further employs the approximation $|\dot{X}| \sim H|X|$ where X denotes background quantities, in conjunction with the matter dominated approximation $|f_R/(H^2 f_{RR})| >> 0$ then one will straightforwardly recover

$$G_{\rm eff} \sim G\left(\frac{4}{3f_R} + \frac{1}{3(-f_R + 3\frac{k^2}{a^2}f_{RR})}\right), \tag{101}$$

$$\Sigma \sim \frac{1}{f_R}. \tag{102}$$

which are the typical $f(\mathring{R})$ results [101,102] for $G_{\rm eff}$ and Σ.

5 Conclusion and discussion

TG offers a novel approach to gravitation where curvature is replaced by teleparallel torsion giving a new framework in which to produce gravitational models. $f(T, B)$ gravity is a particularly interesting expression of TG in which the second and fourth order contributions to the Ricci scalar are separated. This offers a new perspective on modified theories of gravity such as $f(\mathring{R})$ gravity which now become a subclass of this more general approach to modifying gravity.

One of the core exhibitions of any modified theory of gravity is in its cosmological perturbations which expression the linear perturbation degrees of freedom of the metric tensor. Despite TG being based on the tetrad, the degrees of freedom are inherited from the metric due to the close relationship they share. The result is that the tetrad has ten degrees of freedom at linear perturbation and produces the regular decoupling of scalar, vector and tensor perturbations. Another potential obstacle to obtaining the cosmological perturbations appears when forming the correct tetrad at perturbative level since this must be a good tetrad both at background level and linear perturbative level. The full SVT perturbation appears in Eq. (31) which produces perturbation equations which satisfy the antisymmetric conditions of Eq. (16) while reproducing the metric through Eq. (1).

In this work, we explore these cosmological perturbations in the context of $f(T, B)$ gravity by first exploring the tensor perturbations in Eq. (38). The associated gravitational wave propagation equation in Eq. (39) results as the generic wave equation for gravitational waves in the $f(T, B)$ gravity context. The immediate result of this propagation equation is that gravitational waves propagate at the speed of light which is in good agreement with recent multimessenger measurements. The other property that emerges out of this propagation equation is the amplitude modulation by the frictional term ν. This remains outside of present observations but something interesting is that the expression that results is very similar to the $f(T)$ result, as one can observe by taking this limit in Eq. (40).

The Planck mass run rate turns out to be present in the vector perturbation equations as given in Eq. (46). Vector perturbations are not expressed in observations and so offer a consistency check on the particular choice of models that are viable in $f(T, B)$ gravity. This would favor a low Planck mass run rate. Finally, we explored the scalar perturbations in Sect. 3.3 within the Newtonian gauge. In this subsection, the evolution equations of the gravitational perturbations are presented. In Sect. 4 the matter perturbations are fully developed in order to arrive at the matter perturbation equation of Eq. (62). Our interest lies in the subhorizon limit where the limits of Eqs. (63) and (64) apply. These limits produce the Meszaros equation in Eq. (65) which reflects the growth of matter perturbations which is shown in Fourier space.

An important property of the Meszaros equation is that it produces an effective gravitational constant G_{eff} which governs the growth of structures in the Universe. In $f(T, B)$ gravity, it turns out that this effective gravitational constant is expressed through 3 branches that depend on whether Π and Υ vanish (defined in Eqs. (73) and (74)). It is interesting to note that these branches correspond to separating separable terms and mixed terms, as well as the pure $f(\mathring{R})$ gravity scenario where $f(\mathring{R}) = f(-T + B)$. The appearance of mixed terms has been shown to play an important role in the cosmology of $f(T, B)$ gravity [58,60,65,108]. In this light, the branching of $f(T, B)$ gravity is not entirely unexpected. The core results for these branches are given by Eqs. (82), (86) and (94) which are also respectively given in their leading order subhorizon limit. Through this prism, the differences between the various $f(T, B)$ gravity literature models can be better interpreted through this branching behaviour.

The next generation of cosmology surveys from upcoming observatories (such as the Euclid Mission, Square Kilometre Array project and the Large Synoptic Survey Telescope, among others) will shed further light on the evolution of structure formation over the history of the Universe and may offer new signatures of modified gravity. $f(T, B)$ gravity offers a rich landscape in which to study observational cosmology and may resolve some of the tensions in present day cosmology.

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Appendix A: Cosmological perturbations

This section is devoted in presenting the most important quantities needed for the cosmological perturbations.

1 Background

The non-zero components of the torsion tensor and superpotential, and the torsion and boundary term in the background (flat FLRW) are

$$T^{i}{}_{0j} = H\delta^{i}{}_{j}, \tag{A1}$$
$$S_{i}{}^{0j} = -H\delta^{i}{}_{j}, \tag{A2}$$
$$T = -6H^2, \tag{A3}$$
$$B = -6(3H^2 + \dot{H}). \tag{A4}$$

The matter content is fully conserved giving the standard conservation equation for a perfect fluid

$$\mathring{\nabla}_{\nu}\Theta_{\mu}{}^{\nu}: \quad \dot{\rho} + 3(\rho + P) = 0. \tag{A5}$$

2 Tensor perturbations

The non-zero components of the torsion tensor and the superpotential are

$$\delta T^{i}{}_{0j} = \frac{1}{2}\dot{h}_{ij}, \tag{A6}$$
$$\delta T^{i}{}_{jk} = \frac{1}{2}(\partial_j h_{ik} - \partial_k h_{ij}), \tag{A7}$$
$$\delta S_0{}^{0i} = 0, \tag{A8}$$
$$\delta S_i{}^{0j} = \frac{1}{4}\dot{h}_{ij}, \tag{A9}$$
$$\delta S_i{}^{jk} = -\frac{1}{4a^2}(\partial_j h_{ik} - \partial_k h_{ij}), \tag{A10}$$

while the scalars are

$$\delta T = 0, \quad \delta B = 0. \tag{A11}$$

3 Vector and pseudovector perturbations

The non-zero components of the vectorial and pseudo vectorial perturbations for the torsion tensor and the superpotential are

$$\delta T^{0}{}_{0i} = a\dot{\beta}_i, \tag{A12}$$
$$\delta T^{i}{}_{0j} = 2\partial_i\dot{h}_j - \frac{1}{a}\partial_j b_i - \epsilon_{kij}\dot{\sigma}_k, \tag{A13}$$
$$\delta T^{0}{}_{ij} = a(\partial_i\beta_j - \partial_j\beta_i), \tag{A14}$$
$$\delta T^{i}{}_{jk} = 2(\partial_i\partial_j h_k - \partial_i\partial_k h_j) + \epsilon_{ijl}\partial_k\sigma_l - \epsilon_{ikl}\partial_j\sigma_l, \tag{A15}$$
$$\delta S_0{}^{0i} = -\frac{1}{2a^2}[2aH(b_i - \beta_i) + \epsilon_{ilk}\partial_k\sigma_l], \tag{A16}$$
$$\delta S_i{}^{0j} = -\frac{1}{2a}\left[\frac{1}{2}(\partial_i(b_j + \beta_j - a\dot{h}_j) + \partial_j(b_i - \beta_i - a\dot{h}_i))\right], \tag{A17}$$
$$\delta S_0{}^{ij} = -\frac{1}{4a^3}[\partial_i(b_j - \beta_j + 2a\dot{h}_j) - \partial_j(b_i - \beta_i + 2a\dot{h}_i) - 2a\epsilon_{lij}\dot{\sigma}_l], \tag{A18}$$
$$\delta S_i{}^{jk} = -\frac{1}{2a^2}[\delta_{im}\epsilon_{kjl}\partial_l\sigma_m + \delta_{ij}(2aH(b_k - \beta_k) - a\dot{\beta}_k - 2\partial^2 h_k) - \delta_{ik}(2aH(b_j - \beta_j) - a\dot{\beta}_j - 2\partial^2 h_j) - 2\delta_{il}\partial_k\partial_l h_j + 2\delta_{kl}\partial_i\partial_j h_l], \tag{A19}$$

and the perturbations related to the torsion and boundary term scalars are

$$\delta T = 0, \tag{A20}$$
$$\delta B = 0. \tag{A21}$$

4 Scalar and pseudo scalar perturbations

The components of the torsion tensor and the superpotential for scalar and pseudo scalar perturbations up to first order are

$$\delta T^{0}{}_{0i} = \partial_i(a\dot{\beta} - \phi), \tag{A22}$$
$$\delta T^{i}{}_{0j} = \partial_i\partial_j(\dot{h} - a^{-1}b) - \epsilon_{lij}\partial_l\dot{\sigma} - \dot{\psi}\delta_{ij}, \tag{A23}$$
$$\delta T^{0}{}_{ij} = 0, \tag{A24}$$
$$\delta T^{i}{}_{jk} = \delta_{ij}\partial_k\psi - \delta_{ik}\partial_j\psi + \delta_{il}(\epsilon_{klm}\partial_j\partial_m\sigma - \epsilon_{jlm}\partial_k\partial_m\sigma), \tag{A25}$$
$$\delta S_0{}^{0i} = -\frac{H}{a}\partial_i\Big(b - \beta - (aH)^{-1}\psi\Big), \tag{A26}$$
$$\delta S_i{}^{0j} = \Big[(2H\phi + \dot{\psi})\delta_{ij} + \frac{1}{2}\partial_i\partial_j(\dot{h} - a^{-1}b) - \frac{1}{2}\partial^2(\dot{h} - a^{-1}b)\delta_{ij}\Big], \tag{A27}$$
$$\delta S_0{}^{ij} = \frac{1}{2a^2}\epsilon_{ijk}\partial_k\dot{\sigma}, \tag{A28}$$
$$\delta S_i{}^{jk} = \frac{1}{2a^2}\Big[\delta_{ik}\partial_j\Big(2aH(b - \beta) + \phi - \psi - a\dot{\beta}\Big) - \delta_{ij}\partial_k\Big(2aH(b - \beta) + \phi - \psi - a\dot{\beta}\Big)\Big], \tag{A29}$$

and the perturbations up to first order to the scalar torsion and boundary term become

$$\delta T = 4H\Big(3H\phi + 3\dot{\psi} + \frac{1}{a}\partial^2 b - \partial^2\dot{h}\Big), \tag{A30}$$
$$\delta B = -\Big[H\left(\frac{1}{a}\partial^2(6\beta - 10b) - 6(6\dot{\psi} + \dot{\phi} - 2\partial^2\dot{h} + 6H\phi)\right) + \frac{2}{a}\partial^2(\dot{\beta} - \dot{b}) + \frac{2}{a^2}\partial^2(2\psi - \phi) + 2(\partial^2\ddot{h} - 6\dot{H}\phi - 3\ddot{\psi})\Big]. \tag{A31}$$

Then, the perturbation conservation equations become

$$\mathring{\nabla}_\mu \Theta_0{}^\mu = \delta\dot{\rho} + 3H(\delta P + \delta\rho) + \frac{\partial^2 v(P+\rho)}{a} - 3\dot{\psi}(P+\rho) + \partial^2\dot{h}(P+\rho) = 0, \tag{A32}$$

$$\mathring{\nabla}_\mu \Theta_i{}^\mu = \partial_i\Big[\delta P + (\rho+P)\Big(4aH(b+v-\beta) + \phi + a(\dot{b}-\dot{\beta}+\dot{v})\Big) + a(\dot{\rho}+\dot{P})(v+b-\beta)\Big] = 0. \tag{A33}$$

5 Sub-horizon limit in the Newtonian gauge

$$\delta T \simeq -\frac{4H}{a}\left(k^2 b - 3aH(\psi+\phi)\right) \tag{A34}$$

$$\delta B \simeq -\frac{2k^2}{a^2}(2abH - 2\psi + \phi) \tag{A35}$$

$$\delta f_T \simeq -\frac{2k^2}{a^2}\left(2abH\,(f_{TB}+f_{TT}) + f_{TB}(\phi - 2\psi)\right) \tag{A36}$$

$$\delta f_B \simeq -\frac{2k^2}{a^2}\left(2abH(f_{BB}+f_{TB}) + f_{BB}(\phi - 2\psi)\right). \tag{A37}$$

Appendix B: G_{eff} calculations

1 Branch A

$$A_1 = -a^4\Upsilon\dot{\Pi}(\dot{\Pi} + 12H^3 f_{TT}), \tag{B1}$$

$$A_2 = -4a^2H\Upsilon(2\Pi_B\dot{\Pi} - 12H^3 f_{BB} f_{TB} - H\Upsilon f_T), \tag{B2}$$

$$A_3 = -16H^2\Xi\Upsilon, \tag{B3}$$

$$A_4 = -3a^6\dot{\Pi}(\Pi_B+\Pi_T)[\tag{B4}$$

$$\times \Pi'(6\dot{H}^2(\Pi_B - f_{TB}) + 6H^2\dot{H}(3f_{TB} - \Pi_B) + 18H^4(\Pi_T - f_{TB}) - H^2 f_T - H\dot{f}_T) \tag{B5}$$

$$+ 6H^2[\dot{H}(-24H^3(f_{TB}-\Pi_B) \times (f_{TB}-\Pi_T) + \Pi_B\dot{f}_T) + H\dot{\Pi}^2 \tag{B6}$$

$$- 12H\dot{H}^2(f_{TB}(\Pi_B - f_{TB}) + \Xi) + H^2(2Hf_T + \dot{f}_T)(f_{TB}-\Pi_T)]] \tag{B7}$$

$$A_7 = 12H^2\Xi(12H^2(f_{BB}f_{TT} + 2\Xi) + \Upsilon f_T), \tag{B8}$$

$$\Delta_1 = -a^4 A_1\dot{\Pi}(12H\Pi_B\dot{H} - \dot{\Pi}), \tag{B9}$$

$$\Delta_2 = a^2(-2H\Pi_B\dot{\Pi}(6a^2A_2\dot{H} - 5A_1) + a^2A_2\dot{\Pi}^2 + 8A_1H^2(-\Upsilon f_T - 6\Xi\dot{H})), \tag{B10}$$

$$\Delta_3 = a^4A_3\dot{\Pi}^2 - 2a^2H\Pi_B\dot{\Pi}(6a^2A_3\dot{H} - 5A_2) + 8H^2(a^2A_2(-\Upsilon f_T - 6\Xi\dot{H}) + 3A_1\Xi), \tag{B11}$$

$$\Delta_4 = 2H(4H(a^2A_3(-\Upsilon f_T - 6\Xi\dot{H}) + 3A_2\Xi) + 5a^2A_3\Pi_B\dot{\Pi}), \tag{B12}$$

$$\Delta_5 = 24A_3H^2\Xi, \tag{B13}$$

$$\Delta_6 = 2a^4A_4\dot{\Pi}(\dot{\Pi} + 12H^3 f_{TT}), \tag{B14}$$

$$\Delta_7 = 2a^2(\dot{\Pi}(a^2A_5(\dot{\Pi} + 12H^3 f_{TT}) + 8A_4H\Pi_B)) \tag{B15}$$

$$+ 96A_4H^4(f_{BB}f_{TT} + \Xi) + 48A_4H^4 f_{TB}(f_{TT} - \Upsilon) - 4A_4H^2\Upsilon f_T, \tag{B16}$$

$$\Delta_8 = -2a^4A_6\dot{\Pi}(-\dot{\Pi} - 12H^3 f_{TT}) + 8a^2A_5H \tag{B17}$$

$$+ (-(H\Upsilon f_T - 2\Pi_B\dot{\Pi}) + 24H^3(f_{BB}f_{TT} + \Xi) + 12H^3 f_{TB}(f_{TT} - \Upsilon)) + 32A_4H^2\Xi, \tag{B18}$$

$$\Delta_9 = 2a^4A_7 - \dot{\Pi}(-\dot{\Pi} - 12H^3 f_{TT}) + 32A_5H^2\Xi \tag{B19}$$

$$+ 8a^2A_6H(-(H\Upsilon f_T - 2\Pi_B\dot{\Pi}) + 24H^3 \times (f_{BB}f_{TT} + \Xi) + 12H^3 f_{TB}(f_{TT} - \Upsilon)), \tag{B20}$$

$$\Delta_{10} = 8H(a^2A_7(-(H\Upsilon f_T - 2\Pi_B\dot{\Pi}) + 24H^3 \times (f_{BB}f_{TT} + \Xi) + 12H^3 f_{TB}(f_{TT} - \Upsilon)) + 4A_6H\Xi), \tag{B21}$$

$$\Delta_{11} = 32A_7H^2\Xi. \tag{B22}$$

2 Branch C

$$Z_1 = a^4\dot{\Pi}^2, \tag{B23}$$

$$Z_2 = -8a^2H\dot{\Pi}\Pi_T, \tag{B24}$$

$$Z_3 = 16H^2\Xi, \tag{B25}$$

$$Z_6 = -a^2\dot{\Pi}^2(f_{TB} + \Pi_T) \tag{B26}$$

$$\times 12a^2H^3\Xi(5\dot{f}_B + 72H^3 f_{TT} + 60a^2Hf_{TB}\dot{H}) - 12a^2H^3\dot{\Pi}(f_{TB}(2f_{TT} - 7\Pi_T) + 2a^2 f_{TT}\Pi_T) + 4a^2Hf_T\Pi_T(-6H^3(f_{TT}+2\Pi_T) + 9a^2H\dot{H}\Pi_T + \dot{\Pi}), \tag{B27}$$

$$Z_7 = -12H^2\Xi(f_T + 12H^2 f_{TB}), \tag{B28}$$

$$Y_1 = -a^2A_1(\dot{\Pi} + 12H\dot{H}\Pi_T), \tag{B29}$$

$$Y_2 = 6H\Pi_T(A_1 - 2a^2A_2\dot{H}) - a^2A_2\dot{\Pi}, \tag{B30}$$

$$Y_3 = 6H\Pi_T(A_2 - 2a^2A_3\dot{H}) - a^2A_3\dot{\Pi}, \tag{B31}$$

$$Y_4 = 6A_3H\Pi_T, \tag{B32}$$

$$Y_5 = -2a^2A_4\dot{\Pi}, \tag{B33}$$

$$Y_6 = -2a^2A_5\dot{\Pi} + 8A_4H\Pi_T, \tag{B34}$$

$$Y_7 = -2a^2A_6\dot{\Pi} + 8A_5H\Pi_T, \tag{B35}$$

$$Y_8 = -2a^2A_7\dot{\Pi} + 8A_6H\Pi_T, \tag{B36}$$

$$Y_9 = 8A_7H\Pi_T. \tag{B37}$$

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