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A characterization of the Vickrey rule in slot allocation problems

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Abstract

We study the slot allocation problem where agents have quasi-linear single-peaked preferences over slots and identify the rules satisfying efficiency, strategy-proofness, and individual rationality. Since the quasi-linear single-peaked domain is not connected, the famous characterization of the Vickrey rule in terms of Holmström (1979)'s three properties cannot be applied. However, we are able to establish that on the quasi-linear single-peaked domain, the Vickrey rule is still the only rule satisfying efficiency, strategy-proofness, and individual rationality.

KEYWORDS

efficiency, individual rationality, single-peakedness, slot allocation problem, strategy-proofness, Vickrey rule

JEL CLASSIFICATION

C78, D44, D71

1 | INTRODUCTION

We study the slot allocation problem of deciding how to assign slots to agents and how much each agent should pay. We focus on the case where agents have quasi-linear single-peaked preferences over slots, represented by single-peaked valuations of slots. We allow the possibility that some agent cannot get a slot, due to the facility capacity. The allocation of time slots in a gym or on a golf course matches our settings. For example, each gym member or golfer may have her most preferred starting

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time, and the preferred starting time may be different across gym members or golfers. Moreover, the gym or golf club capacity may not be sufficient to accommodate all members.

We try to identify the rule satisfying *efficiency*, *strategy-proofness*, and *individual rationality*. An *allocation* specifies how slots are assigned to agents and how much each agent should pay. A *rule* is a mapping from the domain, that is, the class of valuation profiles, to the set of allocations. An allocation is *efficient* if the assignment of slots maximizes the sum of all agents' valuations. *Efficiency* describes the property of a rule that for each valuation profile, the rule always selects an efficient allocation. *Strategy-proofness* states that for each agent and each valuation profile, revealing one's true valuation is a weakly dominant strategy. *Individual rationality* states that for each agent and each valuation profile, everyone should not be worse off than receiving no slot and paying nothing. This property guarantees the agents' voluntary participation.

On the quasi-linear domain, the Vickrey rule, originated by Vickrey (1961), Clarke (1971), and Groves (1973), satisfies the above-mentioned three properties. Holmström (1979) establishes a stronger result: on the smoothly connected domain, the Vickrey rule is the only rule satisfying the three properties. Since the quasi-linear domain is smoothly connected, Holmström's characterization holds.

On the other hand, we are interested on the quasi-linear single-peaked domain, that is, the class of single-peaked valuation profiles, which is a proper subdomain of the quasi-linear domain. The smaller a domain is, the weaker the properties of rules are, such as *efficiency* and *strategy-proofness*. Thus, smaller domains imply the possibility of better rules tailored to our problems. Although the Vickrey rule is *efficient*, *strategy-proof*, and *individually rational* on the quasi-linear single-peaked domain, it is not clear whether it is the only rule satisfying the three properties. In fact, since the quasi-linear single-peaked domain is not smoothly connected, Holmström's technique cannot be applied in our setting to establish the characterization.

In this paper, we show that Holmström's result still holds on our domain: on the quasi-linear single-peaked domain, the Vickrey rule is the only rule satisfying efficiency, strategy-proofness, and individual rationality.

On the quasi-linear domain with identical objects, Ashlagi and Serizawa (2012) characterize the Vickrey rule in terms of strategy-proofness, anonymity in welfare, and individual rationality. In task-assignment settings where individual rationality may not be respected, Yengin (2012) characterizes the class of egalitarian-equivalent Groves mechanisms and studies different fairness properties of Groves mechanisms. In the queueing setting,¹ Chun *et al.* (2014) characterize the Vickrey–Clarke–Groves (VCG) mechanisms in terms of queue efficiency, strategy-proofness, and egalitarian equivalence. Kayı and Ramaekers (2010) and Chun *et al.* (2019) characterize the symmetrically balanced VCG mechanisms by outcome efficiency and strategy-proofness, together with some fairness axioms. Yengin and Chun (2020) characterize some subfamilies of the VCG mechanisms in terms of no-envy or solidarity properties.

On the non-quasi-linear domain, the minimum price (MP) rule is a natural extension of the Vickrey rule, which associates each preference profile to an MP equilibrium. Notice that on the quasi-linear domain, the MP equilibrium coincides with the Vickrey allocation (Leonard, 1983). However, on the non-quasi-linear domain, the MP equilibrium price in general has no closed-form expression as the Vickrey payment. In the settings of identical objects, Saitoh and Serizawa (2008) and Sakai (2008) characterize the MP rule by Pareto-efficiency, strategy-proofness, individual rationality, and non-negative payments. In the setting of heterogeneous objects, Morimoto and Serizawa (2015)

¹Chun (2016) gives a detailed survey of the queueing problem.

characterize the MP rule by Pareto-efficiency, strategy-proofness, individual rationality, and no subsidy for losers. In the generalized queuing settings, that is, where agents have non-quasi-linear common-object-ranking preferences, Zhou and Serizawa (2018) characterize the MP rule by using the same axioms as Morimoto and Serizawa (2015).²

It is worth mentioning that Schummer and Vohra (2013), Hougaard *et al.* (2014), and Chun and Park (2017) also study slot allocation problems. Schummer and Vohra (2013) study the reassignment of landing slots without transfers and focus on the mechanisms with incentive and property rights properties. They provide a new reassignment mechanism that respects these properties, and discuss the pros and cons between their mechanism and the reassignment algorithm currently used by the Federal Aviation Administration. Hougaard *et al.* (2014) consider the problem of assigning agents to slots in a line where each agent prefers to be served as close as possible to her target. They consider the aggregate gap-minimizing methods that minimize the total gap between targets and assigned slots, in terms of both deterministic and probabilistic assignments. Chun and Park (2017) consider a problem of assigning slots to a group of agents with identical unit cost. They form such a problem in a bipartite graph setting, and present a simple way of identifying all efficient assignments. In addition, they also introduce the leximin and the leximax rules and discuss their properties.

The rest of this paper is organized as follows. Section 2 defines the model. Section 3 defines the Vickrey rule. Section 4 shows the characterization result, together with its proof. Section 5 concludes.

2 | THE MODEL

There is a nonempty finite set of agents N and a nonempty finite set of slots M . Let n be the cardinality of N and m^* be the cardinality of M . Not getting a slot is called receiving the dummy, denoted by 0. Let L be the set of slots, together with the dummy, that is, $L \equiv M \cup \{0\}$. Each agent either receives a slot or the dummy. We denote the slot that agent $i \in N$ receives by $x_i \in L$. We denote the amount that agent i pays by $t_i \in \mathbb{R}$. A generic *bundle* for agent i is a pair $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$.

Each agent has a *quasi-linear* preference over $L \times \mathbb{R}$. There is a valuation function $v_i: L \rightarrow \mathbb{R}_+$ such that (i) $v_i(0) = 0$, (ii) for each $m \in M$, $v_i(m) > 0$, and (iii) for each pair $(m, t_i), (m', t'_i) \in L \times \mathbb{R}$, $(m, t_i) R_i (m', t'_i)$ if and only if $v_i(m) - t_i \geq v_i(m') - t'_i$. Let V_i be the class of agent i 's valuation functions. Let $V \equiv V_1 \times \dots \times V_n$ be the *quasi-linear domain*. A valuation profile is an element $v \equiv (v_i)_{i \in N} \in V$. Given $v \in V$ and $N' \subseteq N$, let $v_{N'} \equiv (v_i)_{i \in N'}$ and $v_{-N'} \equiv (v_i)_{i \in N \setminus N'}$.

Let $\pi \equiv (\pi(1), \dots, \pi(m^*))$ be a permutation of slots in M where $\pi(1)$ denotes the first slot, $\pi(2)$ denotes the second slot, and so on. For each pair $m, m' \in M$, $m \succ_\pi m'$ means that slot m has a higher rank than slot m' according to π . We assume that the slot permutation π is exogenously given and commonly known by all agents, and agents' valuation functions are private to themselves.

² In this paper we do not impose no subsidy or non-negative payment as an axiom. Instead, we assume that the payment in an allocation is always non-negative.

Definition 1. A valuation function $v_i \in V_i$ is *single-peaked according to π* if there is $k \in M$ such that:

- (i) for each pair $m, m' \in M$ such that $k \succ_{\pi} m \succ_{\pi} m'$, $v_i(k) > v_i(m) > v_i(m')$;
- (ii) for each pair $m, m' \in M$ such that $m' \succ_{\pi} m \succ_{\pi} k$, $v_i(k) > v_i(m) > v_i(m')$.

Let $V_i^{\text{SP}}(\pi)$ be the class of agent i 's valuation functions satisfying single-peakedness according to π . Let $V^{\text{SP}}(\pi) \equiv V_1^{\text{SP}}(\pi) \times \dots \times V_n^{\text{SP}}(\pi)$ be the *quasi-linear single-peaked domain according to π* . It is easily seen that $V^{\text{SP}}(\pi) \subsetneq V$.

An *assignment* is an n -tuple $(x_1, \dots, x_n) \in L^n$ such that, for each pair $i, j \in N$, if $x_i \neq 0$ and $i \neq j$, then $x_i \neq x_j$, that is, each slot is assigned to at most one agent. We denote the set of assignments by X . An *allocation* is an n -tuple $z \equiv (z_1, \dots, z_n) \equiv ((x_1, t_1), \dots, (x_n, t_n)) \in [L \times \mathbb{R}_+]^n$ such that $(x_1, \dots, x_n) \in X$. In an allocation, the payment for each slot is non-negative. We denote the set of allocations by Z . Given $z \in Z$, we denote its assignment and payment (components) at z by $x \equiv (x_1, \dots, x_n)$ and $t \equiv (t_1, \dots, t_n)$, respectively.

A *rule* on V is a mapping f from V to Z . Given a rule f and $v \in V$, we denote the bundle assigned to agent i by $f_i(v) \equiv (x_i(v), t_i(v))$, where $x_i(v)$ denotes the assigned object and $t_i(v)$ the associated payment. We write

$$f(v) \equiv (f_i(v))_{i \in N}, \quad x(v) \equiv (x_i(v))_{i \in N}, \quad t(v) \equiv (t_i(v))_{i \in N}.$$

We focus on the following three properties of a rule: efficiency, strategy-proofness, and individual rationality.

First, we define the notion of efficiency. An allocation $z \in Z$ for $v \in V$ is efficient if its assignment maximizes the sum of agents' valuations, i.e., $x \in \arg \max_{x' \in X} \sum_{i \in N} v_i(x'_i)$. The *efficiency* of a rule states that for each valuation profile, the rule chooses an efficient allocation.

Efficiency. For each $v \in V$, $x(v)$ is efficient for v .

Remark 1. An allocation $z \in Z$ for $v \in \mathcal{R}^n$ is Pareto-efficient if there is no allocation $z' = (x', t') \in [L \times \mathbb{R}]^n$ such that (i) for each $i \in N$, $v_i(x'_i) - t'_i \geq v_i(x_i) - t_i$ with at least one strict relation, and (ii) $\sum_{i \in N} t'_i \leq \sum_{i \in N} t_i$.³ A rule is *Pareto-efficient* if, for each valuation profile, it selects a Pareto-efficient allocation. On the quasi-linear domain, efficiency is independent of payments. A rule is efficient if and only if it is Pareto-efficient.⁴ This result does not hold if agents have non-quasi-linear preferences.

Second, we define the incentive property of a rule. *Strategy-proofness* states that no agent ever benefits from misrepresenting her valuation function.

³ There are two equivalent ways to define the Pareto-efficient allocation. (1) An allocation $z \in Z$ for $v \in V$ is Pareto-efficient if there is no allocation $z' = (x', t') \in [L \times \mathbb{R}]^n$ such that (i) for each $i \in N$, $v_i(x'_i) - t'_i \geq v_i(x_i) - t_i$ with at least strict inequality, and (ii) $\sum_{i \in N} t'_i = \sum_{i \in N} t_i$. (2) An allocation $z \in Z$ for $v \in V$ is Pareto-efficient if there is no allocation $z' = (x', t') \in [L \times \mathbb{R}]^n$ such that (i) for each $i \in N$, $v_i(x'_i) - t'_i = v_i(x_i) - t_i$, and (ii) $\sum_{i \in N} t'_i < \sum_{i \in N} t_i$.

⁴ Let f be a rule on V . First, we show that if f is efficient, then f is Pareto-efficient. To obtain a contradiction, suppose that f is not Pareto-efficient. By (1) in footnote 3, there is a valuation profile v and an allocation $z' \in Z$ such that (i) for each $i \in N$, $v_i(x'_i) - t'_i \geq v_i(x_i(v)) - t_i(v)$, (ii) there is $j \in N$ such that $v_j(x'_j) - t'_j > v_j(x_j(v)) - t_j(v)$, and (iii) $\sum_{i \in N} t'_i = \sum_{i \in N} t_i(v)$. If we sum up (i) side-by-side for all the agents, then by (ii) and (iii), $\sum_{i \in N} v_i(x'_i) > \sum_{i \in N} v_i(x_i(v))$, contradicting that f is efficient. Second, we show that if f is Pareto-efficient, then f is efficient. We prove the contra-positive argument: if f is not efficient, then f is not Pareto-efficient. If f is not efficient, then there is a valuation profile $v \in V$ and an assignment $x^* \in X$ such that $\sum_{i \in N} v_i(x_i^*) > \sum_{i \in N} v_i(x_i(v))$. Let $z' = (x', t') \in Z$ be such that, for each $i \in N$, $x'_i \equiv x_i^*$ and $t'_i \equiv v_i(x_i^*) - v_i(x_i(v)) + t_i(v)$. Then, for each $i \in N$, $v_i(x'_i) - t'_i = v_i(x_i(v)) - t_i(v)$. Note that $\sum_{i \in N} t'_i = \sum_{i \in N} t_i(v) + \sum_{i \in N} v_i(x_i^*) - \sum_{i \in N} v_i(x_i(v))$. Since $\sum_{i \in N} v_i(x_i^*) > \sum_{i \in N} v_i(x_i(v))$, it follows that $\sum_{i \in N} t'_i > \sum_{i \in N} t_i(v)$. Thus, by (2) in footnote 3, (x', t') is not Pareto-efficient, which implies that f is not Pareto-efficient.

Strategy-proofness. For each $v \in V$, each $i \in N$, and each $v'_i \in V_i$, $v_i(x_i(v)) - t_i(v) \geq v_i(x_i(v'_i, v_{-i})) - t_i(v'_i, v_{-i})$.

Individual rationality states that no agent is ever assigned a bundle that makes her worse off than she would be if she had received the dummy and paid nothing.

Individual rationality. For each $v \in V$ and each $i \in N$, $v_i(x_i(v)) - t_i(v) \geq 0$.

3 | THE VICKREY RULE

In this section, we define the Vickrey rule, and show its famous characterization in terms of efficiency, strategy-proofness, and individual rationality by Holmström (1979). In addition, we argue that Holmström's result cannot be applied to the quasi-linear single-peaked domain.

Definition 2. A rule f on V is called a *Vickrey rule* if

- (i) for each $v \in V$, $x(v) \in \arg \max_{x' \in X} \sum_{i \in N} v_i(x'_i)$;
- (ii) for each $i \in N$, $t_i(v) \equiv \sigma_{-i}(v) - \sigma_{-i}^*(v)$, where $\sigma_{-i}(v) \equiv \max_{x \in X} \sum_{j \in N \setminus \{i\}} v_j(x_j)$ and $\sigma_{-i}^*(v) \equiv \sum_{j \in N \setminus \{i\}} v_j(x_j(v))$.

Remark 2. Let f and f' be two Vickrey rules. Then we have (i) $\sum_{i \in N} v_i(x_i(v)) = \sum_{i \in N} v_i(x'_i(v))$, and (ii) for each $i \in N$, $v_i(x_i(v)) - t_i(v) = v_i(x'_i(v)) - t'_i(v)$. The Vickrey assignments could be different under f and f' , but all Vickrey allocations are welfare equivalent.

In the following, we define the smooth connectedness property of a domain, which was introduced by Holmström (1979).

Definition 3. A domain $V' \equiv V'_1 \times \dots \times V'_n \subseteq V$ is *smoothly connected* if, for each $i \in N$, each $v_{-i} \in V'_{-i}$, each $v_i \in V'_i$, and each $v'_i \in V'_i$, there is a one-dimensional parameterized family $V_i(v_i, v'_i) \equiv \{v_i(\cdot; y) \in V'_i : y \in [0, 1]\}$ of valuation functions such that:

- (i) for each $x \in L$, $v_i(x; 0) = v_i(x)$ and $v_i(x; 1) = v'_i(x)$;
- (ii) for each $x \in L$ and each $y \in [0, 1]$, $\partial v_i(x; y) / \partial y$ exists;
- (iii) for each $y \in [0, 1]$ and each $\hat{x} = (\hat{x}_i, \hat{x}_{-i}) \in X^*(v_i, v'_i; v_{-i})$, there is $K \in (0, +\infty)$ such that $|\partial v_i(\hat{x}_i; y) / \partial y| \leq K$, where $X^*(v_i, v'_i; v_{-i}) \equiv \{\hat{x} \in X : \exists y \in [0, 1] \text{ such that } v_i(\hat{x}_i; y) + \sum_{j \in N \setminus \{i\}} v_j(\hat{x}_j) = \max_{\tilde{x} \in X} \{v_i(\tilde{x}_i; y) + \sum_{j \in N \setminus \{i\}} v_j(\tilde{x}_j)\}\}$.

It is not hard to verify that the quasi-linear domain V is smoothly connected. Although Holmström (1979) studies the public good model, his result implies that in our model, when each agent has quasi-linear preferences and receives at most one slot, Theorem H holds.

Theorem H Holmström (1979). *Let $V' \subseteq V$ be smoothly connected. Then a rule f on V' satisfies efficiency, strategy-proofness, and individual rationality if and only if it is a Vickrey rule.*

However, the quasi-linear single-peaked domain $V^{SP}(\pi)$ is not smoothly connected. The following example details this point.

Example 1. Let $N = \{1\}$, $M = \{m_1, m_2\}$, and $\pi = (\pi(1), \pi(2)) = (m_1, m_2)$. Under our assumption on the quasi-linear domain, $V^{SP}(\pi) = V_1^{SP}(\pi) = \{v_1(\cdot) \in V_1: v_1(0) = 0 \text{ and } v_1(m_1), v_1(m_2) > 0 \text{ such that } v_1(m_1) \neq v_1(m_2)\}$.

A domain $V' \subseteq V$ is *path connected* if, for each $i \in N$, each $v_{-i} \in V'_{-i}$, each $v_i \in V'_i$, and each $v'_i \in V'_i$, there is a one-dimensional parameterized family $V_i(v_i, v'_i) \equiv \{v_i(\cdot; y) \in V'_i: y \in [0, 1]\}$ of valuation functions such that (i') condition (i) of Definition 3 holds and (ii') for each $x \in L$ and each $y \in [0, 1]$, $v_i(x; y)$ is continuous with respect to y . It is easy to see that smooth connectedness implies path connectedness. Thus, the violation of path connectedness implies the violation of smooth connectedness.

In the following, we show that $V^{SP}(\pi)$ is not path connected. To obtain a contradiction, suppose that $V^{SP}(\pi)$ is path connected. Now consider two valuation functions $v_1(\cdot)$ and $v'_1(\cdot)$ such that $v_1(0) = 0$, $v_1(m_1) = 2$, and $v_1(m_2) = 1$; $v'_1(0) = 0$, $v'_1(m_1) = 1$, and $v'_1(m_2) = 2$. Then, there is a one-dimensional parameterized family $V_1(v_1, v'_1) \equiv \{v_1(\cdot; y) \in V^{SP}(\pi): y \in [0, 1]\}$ of valuation functions satisfying (i') and (ii').

For each $y \in [0, 1]$, let $h(y) \equiv v_1(m_1; y) - v_1(m_2; y)$. Then, by (ii'), $h(\cdot)$ is continuous. Moreover, by (i'), $h(0) = 1$ and $h(1) = -1$. Thus, by the intermediate-value theorem, there is $y \in [0, 1]$ such that $h(y) = v_1(m_1; y) - v_1(m_2; y) = 0$, that is, $v_1(m_1; y) = v_1(m_2; y)$, contradicting $v_1(\cdot; y) \in V^{SP}(\pi)$. Thus, $V^{SP}(\pi)$ is not path connected, and so it is not smoothly connected.

Example 1 implies that Theorem H cannot be applied on the quasi-linear single-peaked domain. The next section provides a characterization of the Vickrey rule on the quasi-linear single-peaked domain. The following remark uses Example 1 to show that Theorem H cannot be applied to the quasi-linear single-dipped domain and the linear preference domain with two extreme peaks, either.

Remark 3.

- (i) A valuation function $v_i \in V_i$ satisfies *single-dippedness according to π* if there is $k \in M$ such that (i) for each pair $m, m' \in M$ such that $k >_\pi m >_\pi m'$, $v_i(k) < v_i(m) < v_i(m')$, and (ii) for each pair $m, m' \in M$ such that $m' >_\pi m >_\pi k$, $v_i(k) < v_i(m) < v_i(m')$. Let $V_i^{SD}(\pi)$ be the class of agent i 's valuation functions satisfying single-dippedness according to π . Let $V^{SD}(\pi) \equiv V_1^{SD}(\pi) \times \dots \times V_n^{SD}(\pi)$ be the *quasi-linear single-dipped domain according to π* . In Example 1, $V^{SD}(\pi) = V^{SP}(\pi)$ and so Theorem H cannot be applied to $V^{SD}(\pi)$.
- (ii) A single-peaked domain $V^{SP}(\pi)$ satisfies *two-common-extreme-peakedness of slots according to π* if, for each $i \in N$ and each $v_i \in V_i^{SP}(\pi)$, either $v_i(\pi(1)) >_\pi \dots >_\pi v_i(\pi(m^*))$ or $v_i(\pi(m^*)) >_\pi \dots >_\pi v_i(\pi(1))$. That is, the slots $\pi(1)$ and $\pi(m^*)$ are two peaks across all the agents. Let $V^L(\pi) \subseteq V^{SP}(\pi)$ be the *linear preference domain with two extreme peaks according to π* . In Example 1, $V^L(\pi) = V^{SP}(\pi)$ and so Theorem H cannot be applied to $V^L(\pi)$.

We end this section with two remarks. Remark 4 illustrates a proper subdomain of the quasi-linear single-peaked domains that satisfies the smooth connectedness condition.

Remark 4.

- (i) A single-peaked domain $V^{SP}(\pi)$ satisfies *common-peakedness of slots according to π* if there is $k \in M$ such that, for each $i \in N$, each $v_i \in V_i^{SP}(\pi)$, and each $m \in M \setminus \{k\}$, $v_i(k) > v_i(m)$. This means that the “peak” of valuations over slots is the same across all the agents. Moreover, the peak is prefixed and publicly known by all the agents. The single-peaked domain with common-peakedness satisfies the smooth connectedness condition. Thus Theorem H can be applied.
- (ii) A single-peaked domain $V^{SP}(\pi)$ satisfies the *common-ranking of slots according to π* if, for each $i \in N$, each $v_i \in V_i^{SP}(\pi)$, and each pair $m, m' \in M$ with $m \succ_{\pi} m'$, $v_i(m) > v_i(m')$. This means that there is a common ranking of slots across all the agents, and such a common ranking is prefixed and publicly known by all the agents. The single-peaked domain with the common-ranking property satisfies the smooth connectedness condition. Thus Theorem H can be applied.

Remark 5 illustrates a proper superdomain of the quasi-linear single-peaked domain that satisfies the smooth connectedness condition.

Remark 5. For each pair $m, m' \in M$, $m \geq_{\pi} m'$ means that slot m is ranked as high as slot m' according to π . A valuation function $v_i \in V_i$ is *single-plateaued according to π* if there are $k, k' \in M$ with $k \geq_{\pi} k'$ such that:

- (i) for each $m \in M$ such that $k \geq_{\pi} m \geq_{\pi} k'$, $v_i(k) = v_i(m) = v_i(k')$;
- (ii) for each $m, m' \in M$ such that $k' \succ_{\pi} m \succ_{\pi} m'$, $v_i(k) > v_i(m) > v_i(m')$;
- (iii) for each $m, m' \in M$ such that $m' \succ_{\pi} m \succ_{\pi} k$, $v_i(k) > v_i(m) > v_i(m')$.

Let $V_i^{SPL}(\pi)$ be the class of agent i 's valuation functions satisfying single-plateauedness according to π . Let $V^{SPL}(\pi) \equiv V_1^{SPL}(\pi) \times \cdots \times V_n^{SPL}(\pi)$ be the *quasi-linear single-plateaued domain according to π* . It is easily seen that $V^{SP}(\pi) \subsetneq V^{SPL}(\pi)$. The quasi-linear single-plateaued domain satisfies the smooth connectedness condition. Thus Theorem H can be applied. We conjecture that this is the minimal domain that includes the single-peaked domain where Theorem H can be applied.

4 | MAIN RESULT

We are ready to establish our main result.

Theorem 1. *A rule f on $V^{SP}(\pi)$ satisfies efficiency, strategy-proofness, and individual rationality if and only if it is a Vickrey rule.*

On the quasi-linear domain, a Vickrey rule satisfies the three properties of Holmström (1979). Thus the “if” part holds. We only need to show the “only if” part: a rule f on $V^{SP}(\pi)$ satisfying efficiency, strategy-proofness, and individual rationality is a Vickrey rule. We borrow the proof technique from Chew and Serizawa (2007) and apply it to our setting. We establish four lemmas to complete the proof.

Lemma 1 follows from *individual rationality*. It says that an agent who gets the dummy pays nothing. Lemma 2 follows from *efficiency*. It says that any efficient rule selects an assignment satisfying Definition 2(i). Lemma 3 follows from *strategy-proofness*. It says that if an agent truncates her valuations by following the “ z_i -favoring transformation,” the new bundle that the agent gets remains the same as her original bundle. Lemma 4 says that if a rule satisfies *efficiency*, *strategy-proofness*, and *individual rationality*, then the payment of each agent is equal to the expression given in Definition 2(ii). In consequence, Lemmas 2 and 4 together establish the “only if” part. We relegate the complete proof to Section 4.1.

A rule f on V satisfies *budget balance* if for each $v \in V$, $\sum_{i \in N} t_i(v) = 0$. In general, a Vickrey rule on the quasi-linear domain does not satisfy budget balance. It is also easy to see that a Vickrey rule on $V^{\text{SP}}(\pi)$ does not satisfy budget balance either. Therefore, from Theorem 1, we have the following result.

Corollary 1. *There is no rule f on $V^{\text{SP}}(\pi)$ satisfying efficiency, strategy-proofness, individual rationality, and budget balance.*

In the following, we discuss the indispensability of the three axioms. We show that the “only if” part of Theorem 1 fails if one of the axioms is dropped.

Example 2.

- (i) (Dropping *efficiency*) Let f be the “no-trade rule” such that for each $v \in V^{\text{SP}}(\pi)$, it assigns $(0, 0)$ to each agent. This rule satisfies strategy-proofness and individual rationality, but not efficiency.
- (ii) (Dropping *strategy-proofness*) A rule f on $V^{\text{SP}}(\pi)$ is the zero-payment rule if for each $v \in V^{\text{SP}}(\pi)$, $x(v) \in \arg \max_{x' \in X} \sum_{i \in N} v_i(x'_i)$, and for each $i \in N$, $t_i(v) = 0$. This rule satisfies efficiency and individual rationality, but not strategy-proofness.
- (iii) (Dropping *individual rationality*) Let $n = m + 1$ and $e = (e_i)_{i \in N} \in R_{++}^N$. A rule f on $V^{\text{SP}}(\pi)$ is a Vickrey rule with a positive entry fee e if, for each $v \in V$, $x(v) \in \arg \max_{x' \in X} \sum_{i \in N} v_i(x'_i)$, and for each $i \in N$, $t_i(v) \equiv \sigma_{-i}(v) - \sigma_{-i}^*(v) + e_i$. A Vickrey rule with positive entry fee e satisfies efficiency and strategy-proofness. Since $n = m + 1$, there is $i \in N$ such that $x_i = 0$. Since i pays a positive entry fee $e_i > 0$, it follows that $0 > v_i(0) - e_i = -e_i$, violating individual rationality.

4.1 | Proof of theorem 1

We state four lemmas and show their proofs to formally establish the “only if” part of Theorem 1. Since it is easy to verify that Lemma 1 and Lemma 2 hold, we omit their proofs.

Lemma 1. *Let f satisfy individual rationality. Let $v \in V^{\text{SP}}(\pi)$. Then, for each $i \in N$, if $x_i(v) = 0$, then $t_i(v) = 0$.*

Lemma 2. *Let f satisfy efficiency. Let $v \in V^{\text{SP}}(\pi)$. Then $x(v) \in \arg \max_{x' \in X} \sum_{i \in N} v_i(x'_i)$.*

Lemma 3 shows the implication of strategy-proofness via a z_i -favoring transformation.

Definition 4. Given $z_i = (x_i, t_i) \in M \times \mathbb{R}_+$ and $v_i \in V^{SP}(\pi)$, $v'_i \in V^{SP}(\pi)$ is a z_i -favoring transformation of v_i at z_i if, for each $y \in M \setminus \{x_i\}$, $v'_i(x_i) - v'_i(y) > t_i$.

Let $I(v_i, z_i)$ be the set of z_i -favoring transformations of v_i at z_i .

Lemma 3. Let f satisfy strategy-proofness. Let $v \in V^{SP}(\pi)$. Let $i \in N$, $x_i(v) \in M$, and $v'_i \in I(v_i, f_i(v))$. Then $f_i(v'_i, v_{-i}) = f_i(v)$.

Proof. Let $v' = (v'_i, v_{-i})$ and $y = x_i(v')$. Strategy-proofness implies

$$v_i(x_i(v)) - t_i(v) \geq v_i(y) - t_i(v'), \quad (1)$$

$$v'_i(y) - t_i(v') \geq v'_i(x_i(v)) - t_i(v). \quad (2)$$

In the following, we show that $x_i(v) = y$. To obtain a contradiction, suppose that $x_i(v) \neq y$. Since

$$\begin{aligned} t_i(v) - t_i(v') &\geq v'_i(x_i(v)) - v'_i(y) && \text{(by (2))} \\ &> t_i(v) && \text{(by } v'_i \in I(v_i, f_i(v)) \text{)} \end{aligned}$$

it follows that $t_i(v') < 0$, contradicting $f(v') \in Z$.

Thus $x_i(v) = y$. Then by (1), $t_i(v') \geq t_i(v)$, and by (2), $t_i(v) \geq t_i(v')$. Thus $t_i(v) = t_i(v')$ and so $f_i(v'_i, v_{-i}) = f_i(v)$. \square

Lemma 4. Let f satisfy efficiency, strategy-proofness, and individual rationality. Let $v \in V^{SP}(\pi)$. Then, for each $i \in N$, $t_i(v) = \sigma_{-i}(v) - \sigma_{-i}^*(v)$.

Proof. Let $i \in N$. If $x_i(v) = 0$, by efficiency, it is easy to see that $\sigma_{-i}(v) = \sigma_{-i}^*(v)$. By individual rationality and Lemma 1, $t_i(v) = 0 = \sigma_{-i}(v) - \sigma_{-i}^*(v)$. Let $x_i(v) \in M$. To show $t_i(v) = \sigma_{-i}(v) - \sigma_{-i}^*(v)$, we show the impossibility of $\sigma_{-i}(v) - \sigma_{-i}^*(v) < t_i(v)$ (case 1) or $0 \leq t_i(v) < \sigma_{-i}(v) - \sigma_{-i}^*(v)$ (case 2).

Case 1: $\sigma_{-i}(v) - \sigma_{-i}^*(v) < t_i(v)$

By Definition 2, $\sigma_{-i}(v) - \sigma_{-i}^*(v) \geq 0$. Let $v'_i \in V_i^{SP}$ be such that:

- (1-i) $0 \leq \sigma_{-i}(v) - \sigma_{-i}^*(v) < v'_i(x_i(v)) < t_i(v)$;
- (1-ii) for each $y \in M \setminus \{x_i(v)\}$, $0 < v'_i(y) < v'_i(x_i(v)) - \sigma_{-i}(v) + \sigma_{-i}^*(v)$.⁵

Let $v' = (v'_i, v_{-i})$ and $y = x_i(v')$. In the following, we show that $x_i(v) = y$. To obtain a contradiction, suppose that $x_i(v) \neq y$. We consider two cases.

⁵Note that by (1-i), $v'_i(x_i(v)) - \sigma_{-i}(v) + \sigma_{-i}^*(v) > 0$.

Case A: $y = 0$.

In this case, $v'_i(y) = 0$, and so $v'_i(y) + \sum_{j \in N \setminus \{i\}} v_j(x_j(v')) = \sum_{j \in N \setminus \{i\}} v_j(x_j(v'))$. Since

$$\begin{aligned} \sum_{j \in N \setminus \{i\}} v_j(x_j(v')) &\leq \sigma_{-i}(v) \\ &< \sigma_{-i}^*(v) + v'_i(x_i(v)) && \text{(by (1-i))} \\ &\leq \max_{x \in X} \left(\sum_{j \in N \setminus \{i\}} v_j(x_j) + v'_i(x_i) \right), \end{aligned}$$

it follows that $v'_i(y) + \sum_{j \in N \setminus \{i\}} v_j(x_j(v')) < \max_{x \in X} \left(\sum_{j \in N \setminus \{i\}} v_j(x_j) + v'_i(x_i) \right)$, contradicting that $x(v')$ is *efficient*.

Case B: $y \in M \setminus \{x_i(v)\}$.

In this case, we have

$$\begin{aligned} v'_i(y) + \sum_{j \in N \setminus \{i\}} v_j(x_j(v')) &\leq v'_i(y) + \sigma_{-i}(v) \\ &< v'_i(x_i(v)) + \sigma_{-i}^*(v) && \text{(by (1-ii))} \\ &\leq \max_{x \in X} \left(\sum_{j \in N \setminus \{i\}} v_j(x_j) + v'_i(x_i) \right). \end{aligned}$$

Thus $v'_i(y) + \sum_{j \in N \setminus \{i\}} v_j(x_j(v')) < \max_{x \in X} \left(\sum_{j \in N \setminus \{i\}} v_j(x_j) + v'_i(x_i) \right)$, contradicting that $x(v')$ is *efficient*.

Altogether, we have that $x_i(v) = y = x_i(v')$. By *individual rationality*, $t_i(v') \leq v'_i(x_i(v))$. By (1-i), $t_i(v') \leq v'_i(x_i(v)) < t_i(v)$. Thus $v_i(y) - t_i(v') > v_i(x_i(v)) - t_i(v)$, contradicting *strategy-proofness*.

Case 2: $0 \leq t_i(v) < \sigma_{-i}(v) - \sigma_{-i}^*(v)$.

In such a case $\delta > 0$ such that $t_i(v) + \delta < \sigma_{-i}(v) - \sigma_{-i}^*(v)$. It is easy to see that there is $v'_i \in V_i^{SP}$ such that:

- (2-i) $v'_i(x_i(v)) = t_i(v) + \delta$;
- (2-ii) for each $y \in M \setminus \{x_i(v)\}$, $0 < v'_i(y) < \delta$.

Thus, for each $y \in M \setminus \{x_i(v)\}$, we have $v'_i(x_i(v)) - v'_i(y) > t_i(v)$. Since $t_i(v) \geq 0$, $v'_i \in I(v_i, f_i(v))$. By Lemma 3, $f_i(v) = f_i(v')$, where $v' = (v'_i, v_{-i})$.

First, we show that $\sum_{j \in N \setminus \{i\}} v_j(x_j(v')) \leq \sigma_{-i}^*(v)$. To obtain a contradiction, suppose that $\sum_{j \in N \setminus \{i\}} v_j(x_j(v')) > \sigma_{-i}^*(v)$. Recall that $x_i(v) = x_i(v')$. Let $\bar{x} = (x_i(v), x_{-i}(v')) \in X$. Then we have

$$\sum_{i \in N} v_i(\bar{x}_i) = \sum_{j \in N \setminus \{i\}} v_j(x_j(v')) + v_i(x_i(v)) > \sigma_{-i}^*(v) + v_i(x_i(v)),$$

which implies that $x(v) \notin \arg \max_{x' \in X} \sum_{i \in N} v_i(x'_i)$, contradicting *efficiency*.

Thus we have

$$\begin{aligned} \sum_{j \in N \setminus \{i\}} v_j(x_j(v')) + v'_i(x_i(v)) &\leq \sigma_{-i}^*(v) + v'_i(x_i(v)) \\ &< \sigma_{-i}(v) && \text{(by (2-i))} \\ &\leq \max_{x \in X} \left(\sum_{j \in N \setminus \{i\}} v_j(x_j) + v'_i(x_i) \right) \end{aligned}$$

Thus $x(v') \notin \arg \max_{x \in X} \left(\sum_{j \in N \setminus \{i\}} v_j(x_j) + v'_i(x_i) \right)$, contradicting *efficiency*. \square

5 | CONCLUDING REMARKS

We consider the problem of deciding how to assign slots to agents and how much each agent should pay, with particular attention to the situation where agents have single-peaked valuations of slots. We aim to identify the rules satisfying efficiency, strategy-proofness, and individual rationality. Holmström (1979) shows that on the domain satisfying the smooth connectedness condition, the Vickrey rule is the unique rule that satisfies the above-mentioned three properties. Since quasi-linear domain is a smoothly connected domain, Holmström's characterization can be applied. Nevertheless, the quasi-linear single-peaked domain is not smoothly connected. Therefore it is not clear whether the above characterization still holds. We establish that on the quasi-linear single-peaked domain, the Vickrey rule is still the only rule satisfying efficiency, strategy-proofness, and individual rationality.

Demange *et al.* (1986) provide two forms of auctions which implement an allocation selected by the Vickrey rule in the quasi-linear environment. These auctions work as well for the quasi-linear single-peaked domain. It is interesting to see whether the single-peakedness helps design auctions with a simpler form. Morimoto and Serizawa (2015) show that the minimum price rule is a natural extension of the Vickrey rule to the non-quasi-linear environment, which can be implemented by the serial Vickrey mechanism of Zhou and Serizawa (2020). It is open question whether the minimum price rule is still the only rule that satisfies our three axioms on the non-quasi-linear single-peaked domain.

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