Real Options in Stochastic SIR Model (Financial Modeling and Analysis)

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1 Introduction

Vaccination is mostly used for controlling the diffusion of an infectious disease. This paper attempts to bridge a gap between real options and epidemiological models to analyze the optimal mitigation/vaccination strategy when the diffusion of pandemic disease follows a stochastic process. A real options model under stochastic Susceptible-Infected-Susceptible (SIS) environment is developed to examine the optimal mitigation/vaccination threshold (in terms of the herd immunities and vaccination thresholds) when the social costs and benefits of vaccination efforts are considered. A numerical illustration is provided for the case of COVID-19 in Japan to show the herd immunity level as a policy rule to suppress epidemic.

As well known, the primary objective of vaccination policy is to minimize the social impacts of pandemic by protecting susceptible individuals from possible infection. Policy implications are discussed regarding the vaccination as a countermeasure to epidemic diffusion. We intend to provide simple decision rules concerning vaccination management problems using minimal information. As reflected in an adage, an ounce of prevention is worth a pound of cure, the resulting policy rule suggested a more aggressive and precautionary strategy on the scale of vaccination and vaccine stockpile compared to the traditional epidemiological model.

2 Literature Review

2.1 Epidemiology

Originally, mathematical models of infectious diseases are traced back to 1776 when Daniel Bernoulli used a smallpox model to discuss effectiveness of mass vaccination (inoculation). His fundamental question in regards to the smallpox problem rampant in his era is whether the general population needs to be vaccinated against smallpox when there is a fatal risk of such inoculation measures. Afterwards, many pivotal research including Kermack and McKendrick (1927) have contributed to establish a field of mathematical epidemiology and their works are comprehensively summarized in Anderson and May (1982). Numerous models for infectious diseases have been developed to lead to a better understanding of how vaccination programs affect the eradication of the disease, particularly since the path-breaking works by Anderson and

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May (1982). For epidemiological models, however, economic considerations are largely ignored in finding control methods against an infectious disease.

2.2 Economics-epidemiology

Acemoglu et al. (2020) develop deterministic MG-SIR model (MG: multi-group), where infection, hospitalization and fatality rates vary between groups, in particular between the "young," "the middle-aged" and the "old." The trade-off between lives lost and economic damages improves substantially with targeted policies. Optimal policies differentially targeting risk/age groups significantly outperform optimal uniform policies and most of the gains can be realized by having stricter lockdown policies on the oldest group. Both epidemiologists and economists recognize deterministic models are potentially crude approximations of stochastic epidemic dynamics. While epidemiologists recognize such stochasticity in fitting statistical models (see, e.g., Kucharski et al. (2020), Li et al. (2020)), it would be valuable to explicitly model how transmission volatility influence infection dynamics and optimal mitigation/vaccination strategies.

2.3 Stochastic economics-epidemiology

Gray et al. (2011) introduce a Brownian motion to SIS model. Zhao and Jiang (2014) extend the baseline model set-up in Gray et al. (2011) by allowing for a third compartment for the vaccinated. Park (2016) develops a real options model for the optimal impulsive vaccination (in contrast to the continuous vaccinations), taking into account economic benefits and costs of vaccine application. More aggressive and precautionary strategy on the scale of vaccination and vaccine stockpile compared to the traditional epidemiological model. Hong et al. (2020) attempts to incorporate economic aspects into epidemiological model in a stochastic environment. Real option approach enables us to capture the stochastic nature of disease diffusion, irreversible vaccination costs and flexible mitigation/vaccination strategy in a unified framework (Dixit and Pindyck, 1994).

3 Model

Following Gray et al. (2011), we model the COVID-19 aggregate transmission shocks via a stochastic transmission rate, $\tilde{\beta}$. This key input is modeled as a random variable with constant mean (predictable transmission captured by parameter β) and transmission shocks (mean zero but with volatility captured by parameter σ). The exit rate from the infected state back into the susceptible state is assumed to be a constant γ. The resulting dynamics of the fraction of infected then follows a three-parameter non-linear diffusion process. We develop a model for the optimal impulsive vaccination (in contrast to the continuous vaccinations), taking into account economic benefits and costs of vaccine application. Our contribution is to show how aggregate transmission shocks significantly influence optimal mitigation/vaccination strategies in an SIS setting.

The herd immunity is a popular measure in epidemiological model when controlling a disease and to describe a form of threshold where the vaccination of a significant portion of a population provides immunity protection in society. In addition to the development of the herd immunity, the presented economic epidemiological model allows us to introduce other interesting measure, i.e., the vaccination threshold which cannot be derived from the conventional epidemiological approach. The vaccination threshold is the point at which it is optimal to implement the impulsive vaccination while taking into account the vaccination costs and benefits.

The presented model is based on the SIS model that is also called as a compartmental model in epidemiology literature. The other popular model is the SIR model which is not considered here because the SIR model is suitable for the disease control when vaccination provides (pseudo) permanent immunity effects. In order to transparently highlight the importance of transmission volatility and for tractability purposes, we focus on modeling just the infected population It, via a susceptible-infected-susceptible (SIS) as opposed to a susceptible-infected-recovered (SIR) setting. For tractability, we work with an SIS set-up rather than an SIR model as doing so yields an ODE rather than a PDE. This SIS setting is useful for modeling viruses where recovery does not grant long-lasting immunity.

Generally, deterministic SIS model is given by

$$\begin{aligned} \mathrm{d}S_t &= (-\beta S_t + \gamma) I_t \mathrm{d}t, && (3.1)\\ \mathrm{d}I_t &= (\beta S_t - \gamma) I_t \mathrm{d}t, && (3.2)\\ 1 &= S_t + I_t, && (3.3)\\ \mathrm{d}I_t &= (\beta(1 - I_t) - \gamma) I_t \mathrm{d}t. && (3.4) \end{aligned}$$

Then, we assume the transmission rate is stochastic and (3.4) is replaced by

$$\mathrm{d}I_t = (\beta(1 - I_t) - \gamma) I_t \mathrm{d}t + \sigma(\beta(1 - I_t) - \gamma) I_t \mathrm{d}W_t, \tag{3.5}$$

where W_t is the standard Brownian motion. Analogous to logistic equation, we can solve (3.5):

$$\begin{aligned} \frac{\mathrm{d}I_t}{(\beta(1 - I_t) - \gamma) I_t} &= \mathrm{d}t + \sigma \mathrm{d}W_t,\\ \frac{1}{\beta - \gamma}\left(\frac{1}{I_t} + \frac{\beta}{\beta(1 - I_t) - \gamma}\right) \mathrm{d}I_t &= \mathrm{d}t + \sigma \mathrm{d}W_t,\\ [\log I_s]_0^t + [-\log(\beta(1 - I_s) - \gamma)]_0^t &= (\beta - \gamma)(t + \sigma W_t),\\ \frac{(\beta(1 - I_0) - \gamma) I_t}{(\beta(1 - I_t) - \gamma) I_0} &= \mathrm{e}^{(\beta-\gamma)(t+\sigma W_t)},\\ \left((\beta(1 - I_0) - \gamma) + \beta I_0 \mathrm{e}^{(\beta-\gamma)(t+\sigma W_t)}\right) I_t &= (\beta - \gamma) I_0 \mathrm{e}^{(\beta-\gamma)(t+\sigma W_t)}, \end{aligned}$$

As a result, we have

$$\begin{aligned} I_t &= \frac{(\beta - \gamma) I_0}{\beta I_0 + (\beta(1 - I_0) - \gamma)\mathrm{e}^{-(\beta-\gamma)(t+\sigma W_t)}}, && (3.6)\\ \mathbb{E}[I_t] &= \frac{(\beta - \gamma) I_0}{\beta I_0 + (\beta(1 - I_0) - \gamma)\mathrm{e}^{-(\beta-\gamma+(\beta-\gamma)^2\sigma^2/2)t}}. && (3.7) \end{aligned}$$

Next, we consider the situation where policymaker observes the number of infections I_t and decides when to execute impulsive mass vaccination. After vaccination, the vaccine effect reduces the transmission rate $\beta_0 \to \beta_1$. Policymaker's goal is to maximize the social welfare $G(I)$, therefore, policymaker's option is given by

$$F(I) = \max\{G(I), (1-\rho \mathrm{d}t)\mathbb{E}[F(I+\mathrm{d}I)]\}. \tag{3.8}$$

By Ito's formula,

$$\mathrm{d}F = F'\mathrm{d}I + \frac{1}{2}F''(\mathrm{d}I)^2 \tag{3.9}$$
$$= F'((\beta_0(1-I)-\gamma)I\mathrm{d}t + \sigma I\mathrm{d}W) + \frac{1}{2}F''\sigma^2(\beta_0(1-I)-\gamma)^2 I^2\mathrm{d}t, \tag{3.10}$$

and we have the ODE

$$\frac{1}{2}\sigma^2(\beta_0(1-I)-\gamma)^2 I^2 F'' + (\beta_0(1-I)-\gamma)IF' - \rho F = 0. \tag{3.11}$$

Given the optimal threshold I^*, the corresponding boundary conditions are

$$F(I^*) = G(I^*), \tag{3.12}$$
$$F'(I^*) = G'(I^*). \tag{3.13}$$

To solve (3.11)–(3.13), numerical calculation is required.

We define social welfare as reduction of medical cost

$$C(I;\beta) = \mathbb{E}\left[\int_0^T \mathrm{e}^{-\rho t} c I_t \mathrm{d}t\right], \tag{3.14}$$

where ρ denotes discount rate, c individual medical cost and T vaccine efficacy period. And let v and K denote vaccination rate and unit cost of vaccination respectively, then vaccination cost is denoted by vK. Given the vaccine efficacy w, the vaccine effect is computed by reduced transmission rate

$$\beta_1 = (1+(w-1)v)\beta_0. \tag{3.15}$$

Then, social welfare is defined by

$$G(I) = C(I;\beta_0) - C(I;\beta_1). \tag{3.16}$$

At last, we discuss the herd immunity threshold (HIT). HIT is defined as

$$H_0 = 1 - \frac{1}{R_0} \tag{3.17}$$

by conventional epidemiological model. This is interpreted as the smallest proportion of a population for extinction of the disease. Park (2016) defined the optimal HIT as

$$H^* = v(1-I^*), \tag{3.18}$$

associated with the optimal threshold of vaccination. This is interpreted as the proportion of a population when vaccination is executed optimally, and give important insights on the level of vaccine stockpiles.

4 Numerical Results

We choose parameter values as follows. Mitarai and Yanagi (2021) estimated

$$\beta_0 = 0.1653, \quad \gamma = 0.057 \quad \Rightarrow \quad R_0 = 2.9, \quad H_0 = 0.6552,$$

using data for the 1st wave in Tokyo (2020/03/01–04/18). National Institute of Infectious Diseases (NIID) reported the vaccine effect has 60% reduction of infection rate:

$$w = 0.4.$$

World Health Organization (WHO) declared vaccination rate is aimed at least 40% by the end of the year:

$$v = 0.4.$$

We calculate the weighted average individual hospitalization cost per day 65,972 yen by using data of Japan Municipal Hospital Association (JMHA):

$$c = 6.6.$$

Ministry of Health, Labour and Welfare (MHLW) reports the effective period of vaccine is about 6 months:

$$T = 180.$$

We set

$$K = 1,$$

because the price of one shot of Pfizer vaccine is 39 dollar. We choose

$$\sigma = 0.2, \quad \rho = 0.1$$

for the base case.

The optimal threshold and HIT for the base case are

$$I^* = 0.0063 \ \Rightarrow \ H^* = 0.3975 < H_0,$$

and value functions are illustrated in Figure 1. Policymaker executes vaccination when the number of infection reaches 0.63%. The optimal threshold and HIT for $\sigma = 0.4$ are

$$I^* = 0.0061 \ \Rightarrow \ H^* = 0.3976 < H_0,$$

and value functions are illustrated in Figure 2. Results are almost same as the base case. The optimal threshold and HIT for $v = 0.7$ are

$$I^* = 0.0099 \ \Rightarrow \ H^* = 0.6931 > H_0,$$

and value functions are illustrated in Figure 3. High vaccination rate results in high social welfare and high vaccination cost. In this case, the impact of cost is dominant. The optimal

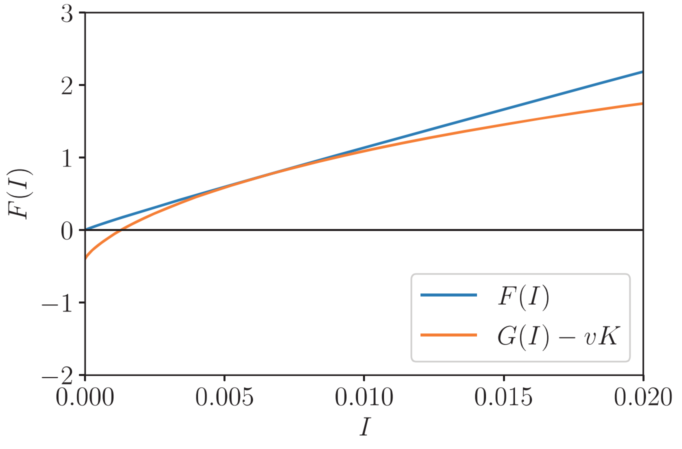

Figure 1: Value functions for the base case.

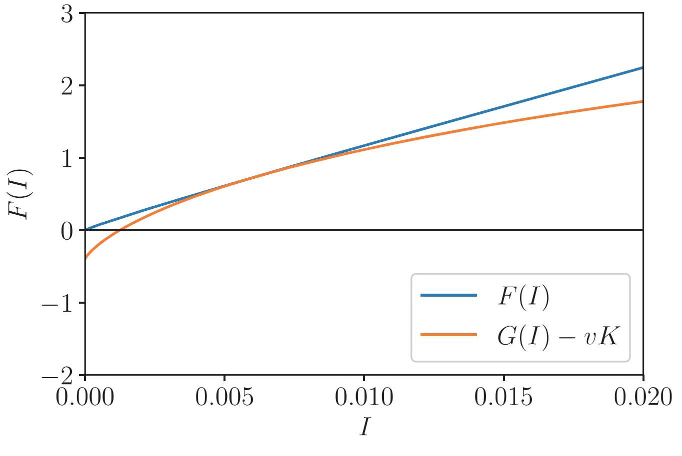

Figure 2: Value functions for $\sigma = 0.4$.

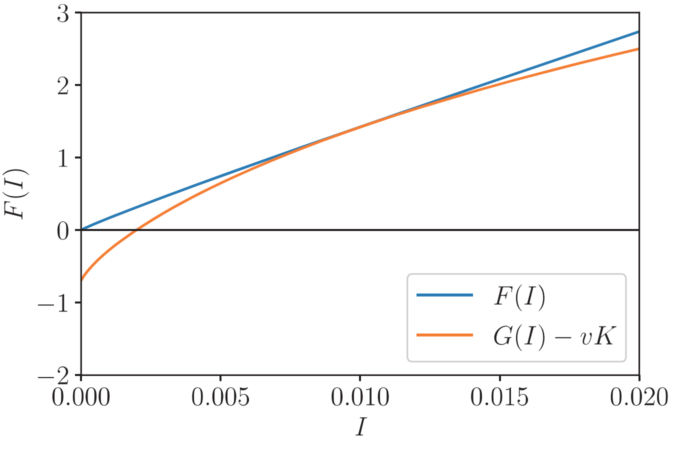

Figure 3: Value functions for $v = 0.7$.

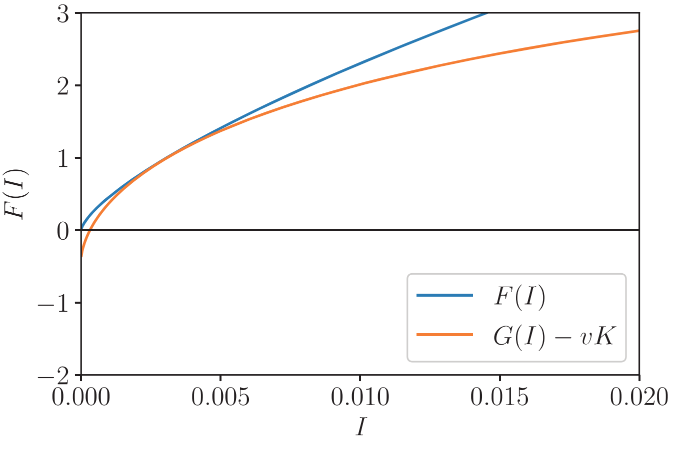

Figure 4: Value functions for $\beta_0 = 0.2$.

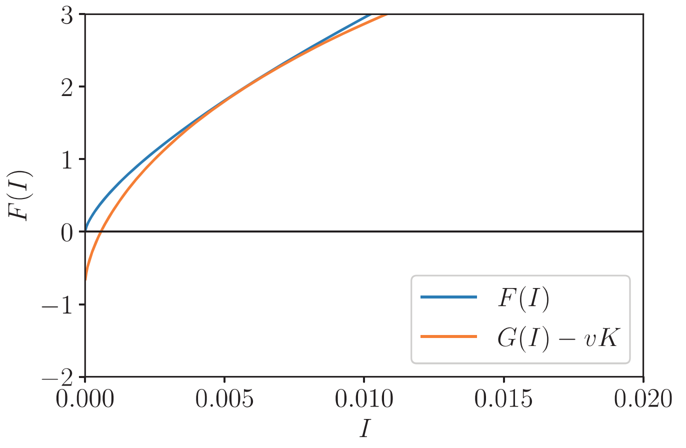

Figure 5: Value functions for $\beta_0 = 0.2$, $v = 0.7$.

threshold and HIT for $\beta_0 = 0.2$ are

$$I^* = 0.0030 \;\Rightarrow\; H^* = 0.3988 < H_0 = 0.7150,$$

and value functions are illustrated in Figure 4. High infection rate result in high social welfare, therefore, the optimal threshold is lower than the base case. The optimal threshold and HIT for $\beta_0 = 0.2$, $v = 0.7$ are

$$I^* = 0.0059 \;\Rightarrow\; H^* = 0.6959 < H_0 = 0.7150$$

and value functions are illustrated in Figure 5. There are opposing impacts between high infection rate and high vaccination rate.

5 Conclusion

We have developed a real options model under stochastic SIS environment, while the diffusion term is tricky for the closed-form solution. We have solved the optimal threshold of impulsive mass vaccination for maximization of social welfare that consists of medical cost reduction and vaccination cost. As numerical results, we have chosen rational parameters from literature and government agencies. Uncertainty has few impact on HIT while the infection rate has large impact. For future plan, we consider SIRD model, random arrival of new variants and heterogeneity.

References

Acemoglu, D., Chernozhukov, V., Werning, I. and Whinston, M. D. (2020): Optimal targeted lockdowns in a multi-group SIR model, NBER Working Paper, 27102.

Anderson, R. M. and May, R. M. (1982): Coevolution of hosts and parasites, *Parasitology*, **85**, 411–426.

Dixit, A. K. and Pindyck, R. S. (1994): *Investment under Uncertainty*, Princeton University Press.

Gray, A., Greenhalgh, D., Hu, L., Mao, X. and Pan, J. (2011):: A stochastic differential equation SIS epidemic model, *SIAM Journal on Applied Mathematics*, **71**, 876–902.

Hong, H., Wang, N. and Yang, J. (2020): Implications of stochastic transmission rates for managing pandemic risks, Working Paper.

Kermack, W. O. and McKendrick, A. G. (1927): A contribution to the mathematical theory of epidemics, *Proceedings of the Royal Society of London, Series A*, **115**, 700–721.

Kucharski, A. J., Russell, T. W., Diamond, C., Liu, Y., Edmunds, J., Funk, S., Eggo, R. M., Sun, F., Jit, M., Munday, J. D. and Davies, N. (2020): Early dynamics of transmission and control of COVID-19: A mathematical modeling study, *Lancet Infectious Diseases*, **20**, 553–558.

Li, R., Pei, S., Chen, B., Song, Y., Zhang, T., Yang, W. and Shaman, J. (2020): Substantial undocumented infection facilitates the rapid dissemination of novel coronavirus (SARS-CoV-2), *Science*, **368**, 489–493.

Mitarai, O. and Yanagi, N. (2021): To achieve both economic activity and suppression of COVID-19: Based on the analogy between the SIR model and the controlled fusion, *Journal of Plasma and Fusion Research*, **97**, 539–550.

Park, H. (2016): A real option analysis for stochastic disease control and vaccine stockpile policy: An application to H1N1 in Korea, *Economic Modelling*, **53**, 187–194.

Zhao, Y. and Jiang, D. (2014): The threshold of a stochastic SIS epidemic model with vaccination, *Applied Mathematics and Computation*, **243**, 718–727.