# Outline of new orthonormal bases of wavelets with arbitrary real dilation and their Hilbert transform pair of orthonormal bases (Time-frequency frames and their applications to image processing)



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# Outline of new orthonormal bases of wavelets with arbitrary real dilation and their Hilbert transform pair of orthonormal bases

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## 1 Introduction

We already proposed the orthonormal wavelet bases with an arbitrary real dilation[3, 7, 8] and the orthonormal bases of wavelets with customizable frequency bands[4, 5, 6, 9]. In this paper, we propose a new type of orthonormal bases of wavelets with an arbitrary real dilation. These bases have a unique characteristic that all the centers of the number 0 wavelets $\psi_{j,0}(t)$ of all the levels $j \in \mathbb{Z}$ are at the origin.

In this paper, we propose two types of these orthonormal bases of wavelets as Type A and Type B, and using the Type A and Type B, we construct a Hilbert transform pair of orthonormal bases. This Hilbert transform pair of orthonormal bases constructs a wavelet frame having the following two characteristics: 1) this frame achieves the perfect translation invariance[2, 10, 11, 12, 13, 14, 15, 16]; 2) for any dilation, this frame is unified to only two types of wavelet shapes, that is, the Hilbert transform pair of symmetric and anti symmetric wavelet shapes (in general, an orthonormal wavelet basis with an irrational dilation is constructed from infinite number of wavelet shapes[3, 7, 8]).

In this paper, because of space limitations, all the proofs of lemmas and theorems are omitted, however, we introduce all the calculation methods of the transforms and inverse transforms of the above bases.

## 2 Preliminaries

$\mathbb{R}$ denotes the set of real numbers, and $\mathbb{Z}$ denotes the set of integers, and $\mathbb{N}$ denotes the set of natural numbers. $L^1(\mathbb{R})$ denotes the space of integrable functions, and $L^2(\mathbb{R})$ denotes the space of square integrable functions. We use the following notation for the inner product of $f(t),\ g(t) \in L^2(\mathbb{R})$:

$$\langle f\,,\, g\rangle = \int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt.$$

Note that $\overline{g(t)}$ is the complex conjugate of $g(t)$. The Fourier transform $\hat{f}(\omega)$ of $f(t) \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\,\omega\, t}\,dt.$$

The inverse Fourier transform $f(t)$ of $\hat{f}(\omega) \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}^{-1}(\hat{f})(t) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\,\omega\, t}\,d\omega.$$

The Kronecker delta $\delta_{k,l}$ is defined by

$$\delta_{k,l} = \begin{cases} 1\,, & k = l, \\ 0\,, & k \neq l, \end{cases} \qquad k, l \in \mathbb{Z}.$$

## 3 The orthonormal basis of wavelet Type A

The orthonormal basis of wavelet Type A with an arbitrary real dilation $a > 1$ is constructed based on the following scaling function $\phi^a(t)$:

$$\hat{\phi}^a(\omega) = \begin{cases} 1\,, & |\omega| \leq \dfrac{2\pi}{a+1}, \\ \cos\left(\dfrac{\pi}{2}\nu\left(\dfrac{(a+1)|\omega| - 2\pi}{2\pi(a-1)}\right)\right), & \dfrac{2\pi}{a+1} < |\omega| < \dfrac{2\pi a}{a+1}, \\ 0\,, & \text{otherwise}, \end{cases} \tag{1}$$

where

$$\nu(x) = \begin{cases} 0, & x \leq 0, \\ x^4(35 - 84x + 70x^2 - 20x^3), & 0 < x < 1, \\ 1, & x \geq 1. \end{cases} \tag{2}$$

With the dilation $a > 1$, the wavelets $\{\psi^A_{j,n}(t) : n \in \mathbb{Z}\}$ and the scaling functions $\{\phi^A_{j,n}(t) : n \in \mathbb{Z}\}$ of the level $j \in \mathbb{Z}$ are defined by

$$\psi^A_{j,n}(t) = \begin{cases} \sqrt{a^j}\psi^{H_1}_n\left(a^j t - \dfrac{n}{a-1}\right), & j = 2m, \\ \sqrt{a^j}\psi^{H_0}_n\left(a^j t - \dfrac{n}{a-1}\right), & j = 2m+1, \end{cases} \quad m \in \mathbb{Z}, \tag{3}$$

$$\phi^A_{j,n}(t) = \begin{cases} \sqrt{a^j}\phi^a\left(a^j t - n\right), & j = 2m, \\ \sqrt{a^j}\phi^a\left(a^j t - (n+1/2)\right), & j = 2m+1, \end{cases} \quad m \in \mathbb{Z}, \tag{4}$$

where

$$\hat{\psi}^{H_0}_n(\omega) = \begin{cases} e^{-i\theta_n}\hat{\psi}^{Org}(\omega), & \omega < 0, \\ e^{i\theta_n}\hat{\psi}^{Org}(\omega), & \omega \geq 0, \end{cases} \tag{5}$$

$$\hat{\psi}^{H_1}_n(\omega) = \begin{cases} e^{-i\theta_n + i\frac{\pi}{2}}\hat{\psi}^{Org}(\omega), & \omega < 0, \\ e^{i\theta_n - i\frac{\pi}{2}}\hat{\psi}^{Org}(\omega), & \omega \geq 0, \end{cases} \tag{6}$$

$$\hat{\psi}^{Org}(\omega) = \sqrt{\frac{1}{a-1}\left\{\left|\hat{\phi}^a\left(a^{-1}\omega\right)\right|^2 - \left|\hat{\phi}^a(\omega)\right|^2\right\}}, \tag{7}$$

$$\theta_n = \frac{\pi\, n}{a-1}. \tag{8}$$

The wavelet set $\{\psi^A_{j,n}(t) : j, n \in \mathbb{Z}\}$ obtained from (3) constructs an orthonormal basis, and with $j_1, j_2, n_1, n_2 \in \mathbb{Z}$, the following equations hold:

$$\begin{aligned} \left\langle \psi^A_{j_1,n_1},\ \psi^A_{j_2,n_2}\right\rangle &= \delta_{j_1,j_2}\delta_{n_1,n_2}, \\ \left\langle \phi^A_{j_1,n_1},\ \phi^A_{j_1,n_2}\right\rangle &= \delta_{n_1,n_2}, \\ \left\langle \psi^A_{j_1,n_1},\ \phi^A_{j_2,n_2}\right\rangle &= 0, \qquad j_1 \geq j_2. \end{aligned}$$

The following equations hold for $f(t) \in L^2(\mathbb{R})$ :

$$f(t) = \sum_{j,n\in\mathbb{Z}} \left\langle f,\ \psi^A_{j,n}\right\rangle \psi^A_{j,n}(t), \tag{9}$$

$$\sum_{n\in\mathbb{Z}} \left\langle f, \phi^A_{j,n}\right\rangle \phi^A_{j,n}(t) = \sum_{n\in\mathbb{Z}} \left\langle f, \phi^A_{j-1,n}\right\rangle \phi^A_{j-1,n}(t) + \sum_{n\in\mathbb{Z}} \left\langle f, \psi^A_{j-1,n}\right\rangle \psi^A_{j-1,n}(t), \quad j \in \mathbb{Z}.$$

## 3.1 The calculation method of Type A

For the transform and the inverse transform, we need the following infinite sequences $\{g^A_{j,n,k} : j,n,k \in \mathbb{Z}\}$ and $\{h^A_{j,n,k} : j,n,k \in \mathbb{Z}\}$ :

$$g^A_{j,n,k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G^A_{j,n}(\omega)\, e^{i\,k\omega}\, d\omega, \tag{10}$$

$$h^A_{j,n,k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H^A_{j,n}(\omega)\, e^{i\,k\omega}\, d\omega, \tag{11}$$

where

$$G^A_{j,n}(\omega) = \frac{\hat{\psi}^A_{j-1,n}(a^j\omega)}{\hat{\phi}^A_{j,0}(a^j\omega)}, \quad H^A_{j,n}(\omega) = \frac{\hat{\phi}^A_{j-1,n}(a^j\omega)}{\hat{\phi}^A_{j,0}(a^j\omega)}. \tag{12}$$

These sequences satisfy the following equations:

$$\psi^A_{j-1,n}(t) = \sum_{k\in\mathbb{Z}} g^A_{j,n,k}\phi^A_{j,k}(t), \qquad j,n\in\mathbb{Z},$$

$$\phi^A_{j-1,n}(t) = \sum_{k\in\mathbb{Z}} h^A_{j,n,k}\phi^A_{j,k}(t), \qquad j,n\in\mathbb{Z}.$$

It is impossible to calculate all the infinite number $j,n \in \mathbb{Z}$ of the infinite sequences $\{g^A_{j,n,k} : k \in \mathbb{Z}\}$ and $\{h^A_{j,n,k} : k \in \mathbb{Z}\}$, however, it is possible to design a fast calculation method to obtain these sequences. From (1)–(8) and (10)–(12), $\{g^A_{j,n,k} : j,n,k \in \mathbb{Z}\}$ can be calculated by

$$g^A_{j,n,k} = \begin{cases} (-1)^{n+k}\mathcal{G}^{\,a}\left(k - \dfrac{a}{a-1}n\right), & j = 2m, \\ (-1)^{n+k}\mathcal{G}^{\,a}\left(k + \dfrac{1}{2} - \dfrac{a}{a-1}n\right), & j = 2m+1, \end{cases} \qquad m\in\mathbb{Z},$$

where

$$\hat{\mathcal{G}}^{\,a}(\omega) = \begin{cases} \sqrt{\dfrac{a}{a-1}}, & |\omega| \le \dfrac{\pi(a-1)}{a+1}, \\ \sqrt{\dfrac{a}{a-1}}\cos\left(\dfrac{\pi}{2}\nu\left(\dfrac{(a+1)|\omega| - \pi(a-1)}{2\pi(1-a^{-1})}\right)\right), & \dfrac{\pi(a-1)}{a+1} < |\omega| < \dfrac{\pi(a+1-2a^{-1})}{a+1}, \\ 0\,, & \text{otherwise}. \end{cases} \tag{13}$$

From (13), $\mathcal{G}^a(t)$ is symmetric with respect to $t = 0$ and decay rapidly. Therefore, with an appropriate integer constant $S > 0$ (the recommended value $S = \lfloor 40/(a-1) \rfloor$), we calculate the function value $\mathcal{G}^a(t)$ within $-S \leq t \leq S$, and in any other area, we approximately regard $\mathcal{G}^a(t) = 0$. Based on this policy, as a preliminary calculation, we first calculate the function values $\{\mathcal{G}^a(t_n)\}$ at small equally spaced points $\{t_n = \lambda n : -S \leq \lambda n \leq S, n \in \mathbb{Z}\}$ (note that, the constant $\lambda > 0$ is an equally spaced distance such that $\lambda = 1/256$, $1/512$ etc.). Next, by the linear interpolation using these function values, we can fast calculate the function value $\mathcal{G}^a(t)$. For example, with the equally spaced distance $\lambda = 1/512$ and the preliminary calculation of $\{\mathcal{G}^a(t_n)\}$ at $\{t_n = n/512 : -S \leq n/512 \leq S,\ n \in \mathbb{Z}\}$, $\mathcal{G}^a(t)$ is obtained by the following linear interpolation:

1. Any $t$, $-S \leq t \leq S$ can be represented as

$$t = t_\ell + \delta, \tag{14}$$

where

$$\ell = \lfloor 512 \times t \rfloor, \qquad t_\ell = \frac{\ell}{512}, \qquad \delta = t - \frac{\ell}{512}, \qquad 0 \leq \delta < \frac{1}{512}. \tag{15}$$

Note that, $\lfloor x \rfloor$ $(x \in \mathbb{R})$ is the largest integer lower than or equal to $x$.

2. $\mathcal{G}^a(t)$ can be calculated by the following interpolation:

$$\mathcal{G}^a(t) \approx \begin{cases} \dfrac{\left(\dfrac{1}{512} - \delta\right) \mathcal{G}^a(t_\ell) + \delta\, \mathcal{G}^a(t_{\ell+1})}{\dfrac{1}{512}}, & -S \leq t \leq S, \\ 0\,, & \text{otherwise.} \end{cases} \tag{16}$$

In the same manner as the above, from (1)–(8) and (10)–(12), $\{h^A_{j,n,k} : j, n, k \in \mathbb{Z}\}$ can be calculated by

$$h^A_{j,n,k} = \begin{cases} \mathcal{H}^a\left(k - a\left(n + \dfrac{1}{2}\right)\right), & j = 2m, \\ \mathcal{H}^a\left(k + \dfrac{1}{2} - a\,n\right), & j = 2m+1, \end{cases} \quad m \in \mathbb{Z},$$

where

$$\mathcal{H}^a(t) = \sqrt{a^{-1}}\phi^a\left(a^{-1}t\right). \tag{17}$$

$\phi^a(t)$ of (17) is obtained from (1), and in the same manner as $\mathcal{G}^a(t)$, with appropriate integer constants $S > 0$ (the recommended value $S = \lfloor 40/(a-1) \rfloor$) and $\lambda = 1/512$, we can fast calculate the function value $\mathcal{H}^a(t)$.

The transform is calculated by the following decomposition algorithm[1]:

$$d^A_{j-1,n} = \sum_{k\in\mathbb{Z}} \overline{g^A_{j,n,k}} c^A_{j,k}, \quad j, n \in \mathbb{Z}, \tag{18}$$

$$c^A_{j-1,n} = \sum_{k\in\mathbb{Z}} \overline{h^A_{j,n,k}} c^A_{j,k}, \quad j, n \in \mathbb{Z}. \tag{19}$$

Note that, $c^A_{j,n}$ is the scaling coefficient of the scaling function $\phi^A_{j,n}(t)$, and $d^A_{j,n}$ is the wavelet coefficient of the wavelet $\psi^A_{j,n}(t)$. Next, the inverse transform is calculated by the following reconstruction algorithm[1]:

$$c^A_{j,n} = \sum_{k\in\mathbb{Z}} h^A_{j,k,n} c^A_{j-1,k} + \sum_{k\in\mathbb{Z}} g^A_{j,k,n} d^A_{j-1,k}, \quad j, n \in \mathbb{Z}. \tag{20}$$

# 4 The orthonormal basis of wavelet Type B

In the same manner as Type A, the orthonormal basis of wavelet Type B with an arbitrary real dilation $a > 1$ is constructed based on the scaling function $\phi^a(t)$ of (1). With the dilation $a > 1$, the wavelets $\{\psi^B_{j,n}(t) : n \in \mathbb{Z}\}$ and the scaling functions $\{\phi^B_{j,n}(t) : n \in \mathbb{Z}\}$ of the level $j \in \mathbb{Z}$ are defined by

$$\psi^B_{j,n}(t) = \begin{cases} \sqrt{a^j}\psi^{H_0}_n\left(a^j t - \dfrac{n}{a-1}\right), & j = 2m, \\ \sqrt{a^j}\psi^{H_1}_n\left(a^j t - \dfrac{n}{a-1}\right), & j = 2m+1, \end{cases} \quad m \in \mathbb{Z}, \tag{21}$$

$$\phi^B_{j,n}(t) = \begin{cases} \sqrt{a^j}\phi^a\left(a^j t - (n+1/2)\right), & j = 2m, \\ \sqrt{a^j}\phi^a\left(a^j t - n\right), & j = 2m+1, \end{cases} \quad m \in \mathbb{Z}. \tag{22}$$

Note that, $\psi^{H_0}_n(t)$, $\psi^{H_1}_n(t)$ of (21) are obtained from (5), (6), and $\phi^a(t)$ of (22) is obtained from (1). The wavelet set $\{\psi^B_{j,n}(t) : j, n \in \mathbb{Z}\}$ obtained from (21) constructs an orthonormal basis, and with $j_1, j_2, n_1, n_2 \in \mathbb{Z}$, the following equations hold:

$$\begin{aligned}\left\langle \psi^B_{j_1,n_1},\ \psi^B_{j_2,n_2} \right\rangle &= \delta_{j_1,j_2}\delta_{n_1,n_2},\\ \left\langle \phi^B_{j_1,n_1},\ \phi^B_{j_1,n_2} \right\rangle &= \delta_{n_1,n_2},\\ \left\langle \psi^B_{j_1,n_1},\ \phi^B_{j_2,n_2} \right\rangle &= 0, \qquad j_1 \geq j_2.\end{aligned}$$

The following equations hold for $f(t) \in L^2(\mathbb{R})$ :

$$f(t) = \sum_{j,n\in\mathbb{Z}} \left\langle f,\ \psi^B_{j,n} \right\rangle \psi^B_{j,n}(t), \tag{23}$$

$$\sum_{n\in\mathbb{Z}} \left\langle f, \phi^B_{j,n} \right\rangle \phi^B_{j,n}(t) = \sum_{n\in\mathbb{Z}} \left\langle f, \phi^B_{j-1,n} \right\rangle \phi^B_{j-1,n}(t) + \sum_{n\in\mathbb{Z}} \left\langle f, \psi^B_{j-1,n} \right\rangle \psi^B_{j-1,n}(t), \quad j \in \mathbb{Z}.$$

## 4.1 The calculation method of Type B

In the same manner as Type A, for the transform and the inverse transform, we need the following infinite sequences $\{g^B_{j,n,k} : j,n,k \in \mathbb{Z}\}$ and $\{h^B_{j,n,k} : j,n,k \in \mathbb{Z}\}$ :

$$\begin{aligned} g^B_{j,n,k} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} G^B_{j,n}(\omega)\, e^{i\,k\omega}\, d\omega,\\ h^B_{j,n,k} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H^B_{j,n}(\omega)\, e^{i\,k\omega}\, d\omega,\end{aligned}$$

where

$$G^B_{j,n}(\omega) = \frac{\hat{\psi}^B_{j-1,n}(a^j\omega)}{\hat{\phi}^B_{j,0}(a^j\omega)}, \quad H^B_{j,n}(\omega) = \frac{\hat{\phi}^B_{j-1,n}(a^j\omega)}{\hat{\phi}^B_{j,0}(a^j\omega)}.$$

These sequences satisfy the following equations:

$$\begin{aligned}\psi^B_{j-1,n}(t) &= \sum_{k\in\mathbb{Z}} g^B_{j,n,k}\phi^B_{j,k}(t), \qquad j,n \in \mathbb{Z},\\ \phi^B_{j-1,n}(t) &= \sum_{k\in\mathbb{Z}} h^B_{j,n,k}\phi^B_{j,k}(t), \qquad j,n \in \mathbb{Z}.\end{aligned}$$

In the same manner as Type A, $\{g^B_{j,n,k} : j,n,k \in \mathbb{Z}\}$, $\{h^B_{j,n,k} : j,n,k \in \mathbb{Z}\}$ can be calculated by

$$g^B_{j,n,k} = \begin{cases} (-1)^{n+k}\mathcal{G}^a\left(k + \frac{1}{2} - \frac{a}{a-1}n\right), & j = 2m, \\ (-1)^{n+k}\mathcal{G}^a\left(k - \frac{a}{a-1}n\right), & j = 2m+1 \end{cases} \quad m \in \mathbb{Z}, \tag{24}$$

$$h^B_{j,n,k} = \begin{cases} \mathcal{H}^a\left(k + \frac{1}{2} - a\,n\right), & j = 2m, \\ \mathcal{H}^a\left(k - a\left(n + \frac{1}{2}\right)\right), & j = 2m+1, \end{cases} \quad m \in \mathbb{Z}. \tag{25}$$

Note that, $\mathcal{G}^a(t)$ of (24) is obtained from (13), and $\mathcal{H}^a(t)$ of (25) is obtained from (17), and additionally, in the same manner as (14)–(16), we can fast calculate these functions.

The transform is calculated by the following decomposition algorithm[1]:

$$d^B_{j-1,n} = \sum_{k\in\mathbb{Z}} \overline{g^B_{j,n,k}} c^B_{j,k}, \quad j,n \in \mathbb{Z}, \tag{26}$$

$$c^B_{j-1,n} = \sum_{k\in\mathbb{Z}} \overline{h^B_{j,n,k}} c^B_{j,k}, \quad j,n \in \mathbb{Z}. \tag{27}$$

Note that, $c^B_{j,n}$ is the scaling coefficient of the scaling function $\phi^B_{j,n}(t)$, and $d^B_{j,n}$ is the wavelet coefficient of the wavelet $\psi^B_{j,n}(t)$. Next, the inverse transform is calculated by the following reconstruction algorithm[1]:

$$c^B_{j,n} = \sum_{k\in\mathbb{Z}} h^B_{j,k,n} c^B_{j-1,k} + \sum_{k\in\mathbb{Z}} g^B_{j,k,n} d^B_{j-1,k}, \quad j,n \in \mathbb{Z}. \tag{28}$$

# 5 The Hilbert transform pair of orthonormal bases

With the dilation $a > 1$ and using the transform equation (9) of Type A and the transform equation (23) of Type B, we can construct the following transform equation:

$$f(t) = \frac{1}{2}\left\{ \sum_{j,n\in\mathbb{Z}} \left\langle f,\ \psi^A_{j,n}\right\rangle \psi^A_{j,n}(t) + \sum_{j,n\in\mathbb{Z}} \left\langle f,\ \psi^B_{j,n}\right\rangle \psi^B_{j,n}(t) \right\}. \tag{29}$$

It is well known that the following two functions $\psi^A(t)$, $\psi^B(t)$ construct a Hilbert transform pair:

$$\hat{\psi}^A(\omega) = \begin{cases} i\hat{\psi}^B(\omega), & \omega < 0, \\ -i\hat{\psi}^B(\omega), & \omega > 0. \end{cases}$$

From the wavelet Type A $\psi^A_{j,n}(t)$ of (3), the wavelet Type B $\psi^B_{j,n}(t)$ of (21) and (5)–(8), with $j, n \in \mathbb{Z}$, $\psi^A_{j,n}(t)$ and $\psi^B_{j,n}(t)$ construct a Hilbert transform pair. Therefore, (29) is a transform equation of a Hilbert transform pair of orthonormal bases, and we can treat $\{\psi^A_{j,n}(t), \psi^B_{j,n}(t) : j, n \in \mathbb{Z}\}$ as a wavelet frame.

## 5.1 The calculation method of the Hilbert transform pair of orthonormal bases

1. In this session, we introduce the calculation method of the Hilbert transform pair of orthonormal bases of (29). First, from the target digital signal $\{f_n : n \in \mathbb{Z}\}$, with the scaling functions $\phi^A_{j,n}(t)$ of (4) and $\phi^B_{j,n}(t)$ of (22), the scaling coefficients $\{c^A_{0,n} : n \in \mathbb{Z}\}$ and $\{c^B_{0,n} : n \in \mathbb{Z}\}$ of the level 0 are obtained by

$$c^A_{0,n} = \sum_{k\in\mathbb{Z}} f_k \overline{\phi^A_{0,n}(k)}, \qquad c^B_{0,n} = \sum_{k\in\mathbb{Z}} f_k \overline{\phi^B_{0,n}(k)}.$$

   With these scaling coefficients $\{c^A_{0,n} : n \in \mathbb{Z}\}$ and $\{c^B_{0,n} : n \in \mathbb{Z}\}$, we define the following function $f(t)$:

$$f(t) = \frac{1}{2}\left\{\sum_{n\in\mathbb{Z}} c^A_{0,n}\phi^A_{0,n}(t) + \sum_{n\in\mathbb{Z}} c^B_{0,n}\phi^B_{0,n}(t)\right\}. \tag{30}$$

   Then, we have

$$f_n = f(n), \qquad n \in \mathbb{Z}. \tag{31}$$

   That is, the function $f(t)$ defined by (30) interpolates the digital signal $\{f_n : n \in \mathbb{Z}\}$.

2. Next, we introduce the transform from the level $-1$ to the level $J$ ($J < -1$ is an integer constant). Using the decomposition algorithm of

Type A of (18), (19), the scaling coefficients $\{c^A_{0,n} : n \in \mathbb{Z}\}$ are transformed from the level $-1$ to the level $J$, and using the decomposition algorithm of Type B of (26), (27), the scaling coefficients $\{c^B_{0,n} : n \in \mathbb{Z}\}$ are transformed from the level $-1$ to the level $J$. Then, we have the following transform equation:

$$f(t) = \frac{1}{2}\left\{ \sum_{j=J}^{-1}\sum_{n\in\mathbb{Z}} d^A_{j,n}\psi^A_{j,n}(t) + \sum_{j=J}^{-1}\sum_{n\in\mathbb{Z}} d^B_{j,n}\psi^B_{j,n}(t) \right.$$
$$\left. + \sum_{n\in\mathbb{Z}} c^A_{J,n}\phi^A_{J,n}(t) + \sum_{n\in\mathbb{Z}} c^B_{J,n}\phi^B_{J,n}(t) \right\}. \qquad (32)$$

3. Each wavelet pair $\psi^A_{j,n}(t)$, $\psi^B_{j,n}(t)$ $(J \le j \le -1,\ j, n \in \mathbb{Z})$ of (32) constructs a Hilbert transform pair, however, when the dilation $a > 1$ is an irrational number, each wavelet set of each level $j$ is constructed from infinite number of different Hilbert transform pairs of different wavelet shapes. Therefore, we propose the following transform, which unifies these infinite number of pairs to only one pair, that is, only one Hilbert transform pair of the symmetric wavelet $\psi^R_{j,n}(t)$ and anti symmetric wavelet $\psi^I_{j,n}(t)$ $(J \le j \le -1,\ j, n \in \mathbb{Z})$:

   1) When level $j = 2m$ $(m \in \mathbb{Z})$,

$$\begin{bmatrix} d^R_{j,n} \\ d^I_{j,n} \end{bmatrix} = \begin{bmatrix} \cos(\theta_n) & \sin(\theta_n) \\ -\sin(\theta_n) & \cos(\theta_n) \end{bmatrix} \begin{bmatrix} d^B_{j,n} \\ d^A_{j,n} \end{bmatrix}, \qquad n \in \mathbb{Z}. \qquad (33)$$

   2) When level $j = 2m + 1$ $(m \in \mathbb{Z})$,

$$\begin{bmatrix} d^R_{j,n} \\ d^I_{j,n} \end{bmatrix} = \begin{bmatrix} \cos(\theta_n) & \sin(\theta_n) \\ -\sin(\theta_n) & \cos(\theta_n) \end{bmatrix} \begin{bmatrix} d^A_{j,n} \\ d^B_{j,n} \end{bmatrix}, \qquad n \in \mathbb{Z}. \qquad (34)$$

   Note that, $\theta_n$ is obtained from (8), and we have

$$d^R_{j,n} = \left\langle f,\ \psi^R_{j,n} \right\rangle, \quad d^I_{j,n} = \left\langle f,\ \psi^I_{j,n} \right\rangle, \qquad J \le j \le -1, \quad j, n \in \mathbb{Z},$$

   and the following transform equation holds:

$$f(t) = \frac{1}{2}\left\{ \sum_{j=J}^{-1}\sum_{n\in\mathbb{Z}} d^R_{j,n}\psi^R_{j,n}(t) + \sum_{j=J}^{-1}\sum_{n\in\mathbb{Z}} d^I_{j,n}\psi^I_{j,n}(t) \right.$$
$$\left. + \sum_{n\in\mathbb{Z}} c^A_{J,n}\phi^A_{J,n}(t) + \sum_{n\in\mathbb{Z}} c^B_{J,n}\phi^B_{J,n}(t) \right\}. \qquad (35)$$

The symmetric wavelet $\psi^{R}_{j,n}(t)$ and anti symmetric wavelet $\psi^{I}_{j,n}(t)$ of (35) construct a Hilbert transform pair, and with the dilation $a > 1$, these wavelets are defined by

$$\psi^{R}_{j,n}(t) = \sqrt{a^j}\psi^{H_R}\left(a^j t - \frac{n}{a-1}\right), \qquad j, n \in \mathbb{Z}, \tag{36}$$

$$\psi^{I}_{j,n}(t) = \sqrt{a^j}\psi^{H_I}\left(a^j t - \frac{n}{a-1}\right), \qquad j, n \in \mathbb{Z}, \tag{37}$$

where

$$\hat{\psi}^{H_R}(\omega) = \sqrt{\frac{1}{a-1}\left\{\left|\hat{\phi}^a\left(a^{-1}\omega\right)\right|^2 - \left|\hat{\phi}^a\left(\omega\right)\right|^2\right\}}, \tag{38}$$

$$\hat{\psi}^{H_I}(\omega) = \begin{cases} i\hat{\psi}^{H_R}\left(\omega\right), & \omega < 0, \\ -i\hat{\psi}^{H_R}\left(\omega\right), & \omega \geq 0. \end{cases} \tag{39}$$

Note that, $\hat{\phi}^a(\omega)$ of (38) is obtained from (1).

4. Next, we introduce the inverse transform. First, we need to transform the wavelet coefficients $\{d^R_{j,n}, d^I_{j,n} : J \leq j \leq -1,\ j, n \in \mathbb{Z}\}$ to $\{d^A_{j,n}, d^B_{j,n} : J \leq j \leq -1,\ j, n \in \mathbb{Z}\}$. This transform is the following inverse transform of (33), (34):

   1) When level $j = 2m$ $(m \in \mathbb{Z})$,

$$\begin{bmatrix} d^B_{j,n} \\ d^A_{j,n} \end{bmatrix} = \begin{bmatrix} \cos(\theta_n) & -\sin(\theta_n) \\ \sin(\theta_n) & \cos(\theta_n) \end{bmatrix} \begin{bmatrix} d^R_{j,n} \\ d^I_{j,n} \end{bmatrix}, \qquad n \in \mathbb{Z}. \tag{40}$$

   2) When level $j = 2m + 1$ $(m \in \mathbb{Z})$,

$$\begin{bmatrix} d^A_{j,n} \\ d^B_{j,n} \end{bmatrix} = \begin{bmatrix} \cos(\theta_n) & -\sin(\theta_n) \\ \sin(\theta_n) & \cos(\theta_n) \end{bmatrix} \begin{bmatrix} d^R_{j,n} \\ d^I_{j,n} \end{bmatrix}, \qquad n \in \mathbb{Z}. \tag{41}$$

   Note that, $\theta_n$ is obtained from (8).

5. The reconstruction algorithms are done as follows:

   1) Using the reconstruction algorithm of Type A of (20), the scaling coefficients $\{c^A_{J,n} : n \in \mathbb{Z}\}$ and the wavelet coefficients $\{d^A_{j,n} : J \leq j \leq$

$-1,\ j, n \in \mathbb{Z}\}$ are transformed to the scaling coefficients $\{c^A_{0,n} : n \in \mathbb{Z}\}$ of the level 0.

2) Using the reconstruction algorithm of Type B of (28), the scaling coefficients $\{c^B_{J,n} : n \in \mathbb{Z}\}$ and the wavelet coefficients $\{d^B_{j,n} : J \leq j \leq -1,\ j, n \in \mathbb{Z}\}$ are transformed to the scaling coefficients $\{c^B_{0,n} : n \in \mathbb{Z}\}$ of the level 0.

6. (30) holds for the scaling coefficients $\{c^A_{0,n} : n \in \mathbb{Z}\}$ and $\{c^B_{0,n} : n \in \mathbb{Z}\}$. Therefore, from (30), (31), we can obtain the digital signal $\{f_n : n \in \mathbb{Z}\}$.

## 5.2 An example of the analysis

In this session, we introduce an example of the analysis by the Hilbert transform pair of orthonormal bases with the following irrational dilation:

$$a = 2^{\frac{1}{12}}. \tag{42}$$

Figure 1 shows the Hilbert transform pair of the wavelets $\psi^R_{-1,0}(t)$, $\psi^I_{-1,0}(t)$ defined by (36)–(39) with (42). We analyze the following sample signal $\{f_n : 0 \leq n < N,\ n \in \mathbb{Z}\}$, $N \in \mathbb{N}$ :

$$f_n = \sin\left(\, 2\pi n^2/(4N)\,\right), \quad 0 \leq n < N. \tag{43}$$

When $N = 1024$, (43) is the sweep signal from DC to the frequency $\pi$ in $0 \leq n < 1024$ as shown in Figure 2.

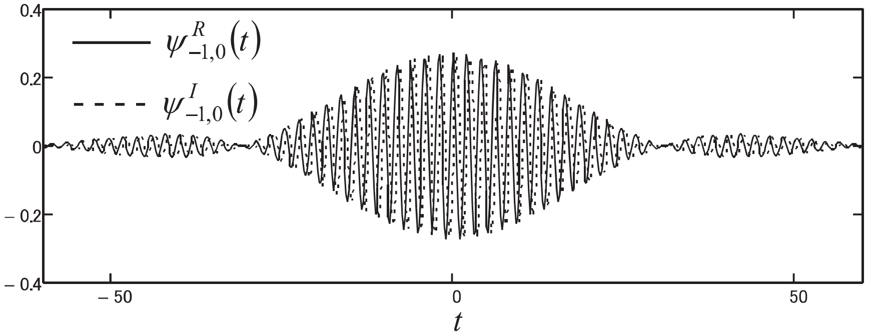


Figure 1: The Hilbert transform pair of the wavelets $\psi^R_{-1,0}(t)$, $\psi^I_{-1,0}(t)$ with the dilation $a = 2^{\frac{1}{12}}$.

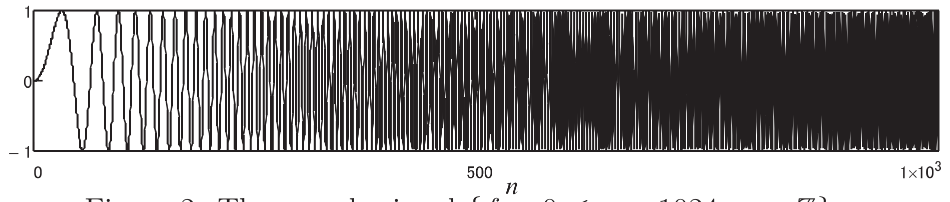

Figure 2: The sample signal $\{f_n : 0 \le n < 1024,\ n \in \mathbb{Z}\}$.

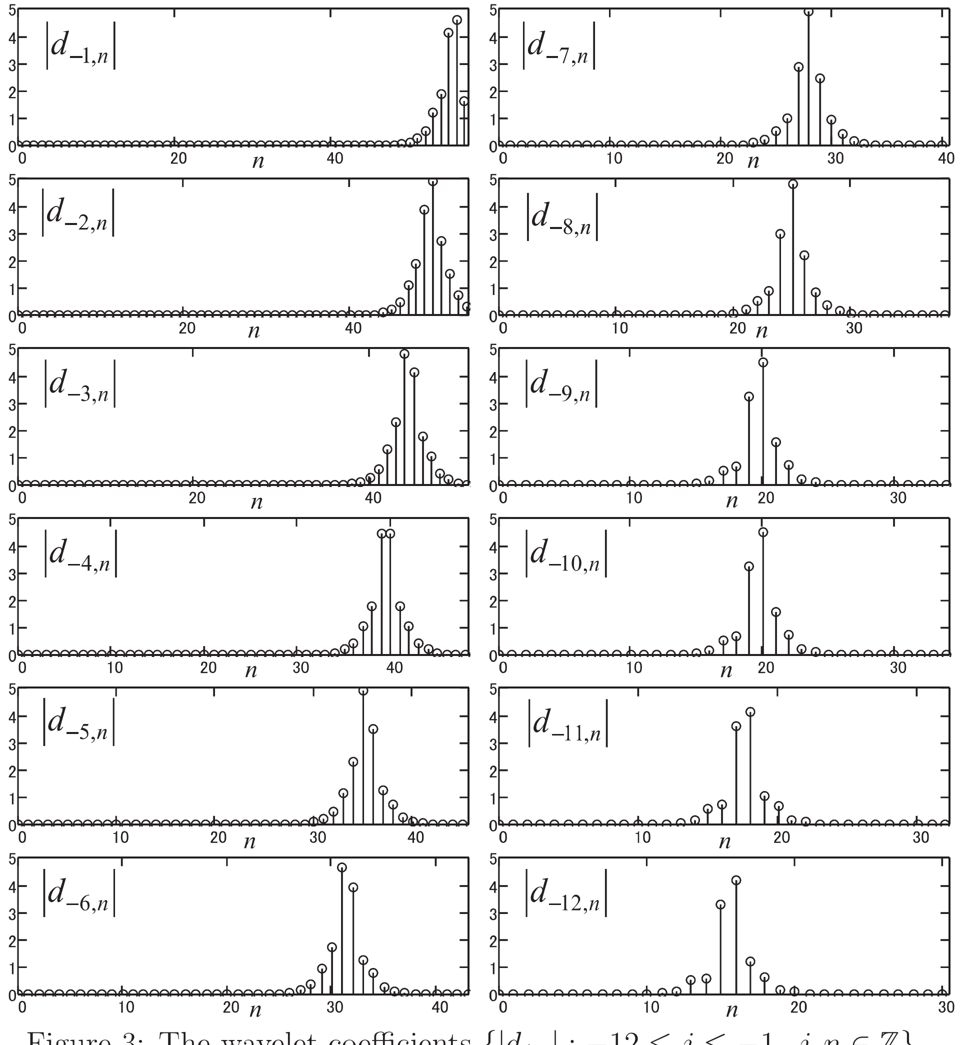

Figure 3: The wavelet coefficients $\{|d_{j,n}| : -12 \le j \le -1,\ j, n \in \mathbb{Z}\}$.

With the dilation of (42), using the Hilbert transform pair of orthonormal bases, we analyze the sample signal of (43) from level $-1$ to level $-12$. Additionally, we transform the obtained wavelet coefficients $\{d^R_{j,n}, d^I_{j,n} : -12 \leq j \leq -1,\ j,n \in \mathbb{Z}\}$ to the complex wavelet coefficients $\{d_{j,n} : -12 \leq j \leq -1,\ j,n \in \mathbb{Z}\}$ by

$$d_{j,n} = d^R_{j,n} - i d^I_{j,n}. \tag{44}$$

Note that, from the complex wavelet coefficients of (44), we can obtain some important information of the signal (for example, the information related to the amplitude, phase and frequency etc.[15]). Figure 3 shows $\{|d_{j,n}| : -12 \leq j \leq -1,\ j,n \in \mathbb{Z}\}$.

# 6 Conclusions

1. We proposed a new orthonormal basis of wavelet Type A with arbitrary real dilation.
2. We proposed another new orthonormal basis of wavelet Type B with arbitrary real dilation.
3. Using the above Type A and Type B, we constructed a Hilbert transform pair of orthonormal bases.
4. We introduced the calculation methods of the Type A, Type B and the Hilbert transform pair of orthonormal bases.
5. We introduced an example of the analysis by the Hilbert transform pair of orthonormal bases.

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