Zeta morphisms for rank two universal deformations (Automorphic forms, Automorphic representations, Galois representations, and its related topics)

NAKAMURA, KENTARO

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By

Kentaro Nakamura*

§ 1. Kato's zeta morphisms and Kato's main conjecture

Let p be a prime number. Throughout the article, we assume that p is odd for simplicity. We fix an isomorphism $\overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, by which we freely regard complex numbers as p-adic numbers. For any field F, we denote by G_F the absolute Galois group of F.

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k^{\mathrm{new}}(\Gamma_1(N_f))$ be a normalized Hecke eigen cusp newform of level $N_f \geqq 1$, weight $k \in \mathbb{Z}_{\geqq 2}$, with neben type character $\chi_f : (\mathbb{Z}/N_f\mathbb{Z})^\times \to \mathbb{C}^\times$. We set $\Sigma_f = \mathrm{prime}(N_f) \cup \{p\}$ the union of the set of prime divisors of N_f and $\{p\}$, $L_f = \mathbb{Q}_p(\{a_n\}_{n\geqq 1}) \subset \overline{\mathbb{Q}}_p$, and $\mathcal{O}_f$ its ring of integers. Let $\rho_f \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ be (a model over $\mathcal{O}$ of) the Galois representation associated to f, i.e. the representation which is odd and unramified outside Σ_f satisfying

$$\mathrm{tr}(\rho_f(\mathrm{Frob}_\ell)) = a_\ell$$

for all the primes $\ell \notin \Sigma_f$ (Frob_ℓ is the geometric Frobenius).

In his celebrated work [Ka04] on Iwasawa main conjecture for modular forms, Kato defined a non trivial Euler systems for $\rho_f^*(1) = \mathrm{Hom}_{\mathcal{O}_f}(\rho_f, \mathcal{O}_f)(1)$, i.e. a collection of elements

$$\{z_{np^k} \in \mathrm{H}^1(\mathbb{Q}(\zeta_{np^k}), \rho_f^*(1))\}_{k\geqq 0, n\geqq 1, (n,\Sigma_f)=1}$$

satisfying the Euler system norm relation, where, for a commutative ring R, we denote by $\mathrm{H}^i(R, -) = \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(R), -)$ the étale cohomology of $\mathrm{Spec}(R)$. We set $\Gamma = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$, $\chi_p : \Gamma \xrightarrow{\sim} \mathbb{Z}_p^\times$ the p-adic cyclotomic character, and $\Lambda_{\mathcal{O}_f} = \mathcal{O}_f[[\Gamma]]$ the Iwasawa algebra of Γ. In his study of Iwasawa main conjecture, the following $\Lambda_{\mathcal{O}_f}$ module $\mathbf{H}^i(\mathbb{Z}[1/p], \rho_f^*(1))$ which we call the Iwasawa cohomology of $\rho_f^*(1)$ is the fundamental object :

$$\mathbf{H}^i(\rho_f^*(1)) = \varprojlim_{k\geqq 0} \mathrm{H}^i(\mathbb{Z}[1/p, \zeta_{p^k}], (j_k)_*(\rho_f^*)(1)),$$

*Saga University, 1 Honjo-Machi, Saga, 840-8502, Japan.
e-mail:nkentaro@cc.saga-u.ac.jp

where $j_k\colon \mathrm{Spec}(\mathbb{Z}[1/\Sigma_f, \zeta_{p^k}]) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[1/p, \zeta_{p^k}])$ is the canonical inclusion and the limit is taken with respect to the corestriction $\mathrm{H}^i(\mathbb{Z}[1/p, \zeta_{p^{k+1}}], -) \to \mathrm{H}^i(\mathbb{Z}[1/p, \zeta_{p^k}], -)$ for the canonical morphism $\mathrm{Spec}(\mathbb{Z}[1/p, \zeta_{p^{k+1}}]) \to \mathrm{Spec}(\mathbb{Z}[1/p, \zeta_{p^k}])$ $(k \geqq 0)$.

As a consequence of the existence of an Euler system and the general theory of Euler systems, Kato proved the following theorem.

Theorem 1.1 ([Ka04] Theorem 12.4)**.**

(1) $\mathbf{H}^2(\rho_f^*(1))$ *is a torsion* $\Lambda_{\mathcal{O}_f}$*-module.*

(2) $\mathbf{H}^1(\rho_f^*(1))[1/p]$ *is a free* $\Lambda_{\mathcal{O}_f}[1/p]$*-module of rank one, and, if* ρ_f *is residually absolutely irreducible, then* $\mathbf{H}^1(\rho_f^*(1))$ *is a free* $\Lambda_{\mathcal{O}_f}$*-module of rank one.*

We remark that, for an Euler system $\{z_{np^k}\}_{k\geqq 0, n\geqq 1, (n,\Sigma_f)=1}$ as above, one can define an element

$$\{z_{p^k}\}_{k\geqq 0} \in \mathbf{H}^1(\mathbb{Z}[1/p], \rho_f^*(1))$$

in the Iwasawa cohomology. However, it is not canonical since $\{z_{np^k}\}_{n,k}$ depends on many choices, e.g. $c, d \geqq 2$ such that $(cd, 6pN_f) = 1$, $1 \leqq j \leqq k-1$ and $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, etc. appearing in Kato's article. Dividing its dependent factors (and the L-factors at the bad primes $\ell \neq p$), he constructed the following canonical map which we call the zeta morphism for f, or ρ_f.

Theorem 1.2 ([Ka04] Theorem 12.5)**.** *There is a canonical* $\mathcal{O}_f$*-linear map*

$$\mathbf{z}(f)\colon \rho_f^* \to \mathbf{H}^1(\rho_f^*(1))[1/p]$$

satisfying the following properties (1) *and* (2)*:*

(1) (*Relation to L-function*) *By Bloch-Kato's dual exponential and the period map, the map* $\mathbf{z}(f)$ *is related to all the critical values of the L-functions*

$$L_{\{p\}}(f, \chi, s) = \sum_{n=1, (n,p)=1}^{\infty} \frac{a_n \chi(n)}{n^s}$$

for all the finite characters $\chi\colon \Gamma(\xrightarrow{\sim} \mathbb{Z}_p^\times) \to \mathbb{C}^\times$ (*we omit to explain the precise interpolation formula*)*.*

(2) (*Relation to Selmer group*) *One has the following inclusion of characteristic ideals of torsion* $\Lambda_{\mathcal{O}_f}[1/p]$*-modules :*

$$\mathrm{Char}_{\Lambda_{\mathcal{O}_f}[1/p]}(\mathbf{H}^1(\rho_f^*(1))[1/p]/\Lambda_{\mathcal{O}_f}[1/p]\cdot \mathrm{Im}(\mathbf{z}(f))) \subset \mathrm{Char}_{\Lambda_{\mathcal{O}_f}}(\mathbf{H}^2(\rho_f^*(1)))[1/p].$$

Moreover, if ρ_f *is residually absolutely irreducible, then the image of* $\mathbf{z}(f)$ *is contained in* $\mathbf{H}^1(\rho_f^*(1))$*, and if furthermore there exists an element* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^\infty}))$

such that $\rho_f/(1-\sigma)\rho_f$ is a free of rank one $\mathcal{O}_f$-module, then one has the following inclusion :

$$\mathrm{Char}_{\Lambda_{\mathcal{O}_f}}(\mathbf{H}^1(\rho_f^*(1))/\Lambda_{\mathcal{O}_f}\cdot \mathrm{Im}(\mathbf{z}(f))) \subset \mathrm{Char}_{\Lambda_{\mathcal{O}_f}}(\mathbf{H}^2(\rho_f^*(1)))$$

(*i.e. we don't need to invert p in this case*).

We remark that the property (1) in the theorem uniquely characterizes the map $\mathbf{z}(f)$. Concerning the inclusion in the property (2), Kato formulated the following conjecture, which we call Kato's main conjecture for f, or ρ_f.

Conjecture 1.3 ([Ka04] Conjecture 12.10)**.** *Assume* (*for simplicity*) *that ρ_f is residually absolutely irreducible, then one has*

$$\mathrm{Char}_{\Lambda_{\mathcal{O}_f}}(\mathbf{H}^1(\rho_f^*(1))/\Lambda_{\mathcal{O}_f}\cdot \mathrm{Im}(\mathbf{z}(f))) = \mathrm{Char}_{\Lambda_{\mathcal{O}_f}}(\mathbf{H}^2(\rho_f^*(1))).$$

When f is a p-ordinary modular form, this conjecture is known to be equivalent to the usual Iwasawa main conjecture ([Ma72], [Gr89]), i.e. the conjecture on the equality between the p-adic L-function and the cyclotomic Selmer group associated to f. Up to now, the latter type of Iwasawa main conjecture is formulated only for f whose p-th component is not a supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, e.g. by [Ko03], [LLZ10] for non p-ordinary, but p-finite slope case. When the p-th component of f is supercuspidal, the existence of "good" p-adic L-function and cyclotomic Selmer group is not known. The advantage of Kato's main conjecture is that the zeta morphism $\mathbf{z}(f)$ exists and we can formulate the conjecture as above for all the Hecke eigen cusp new forms (even for p-supercuspidal case!).

In [Ka93] (and [FK06]), Kato (and Fukaya-Kato) formulated a conjecture which he called the generalized Iwasawa main conjecture. This conjecture unifies both Bloch-Kato conjecture [BK90] and Iwasawa main conjecture, and predicts the existence of zeta morphisms as in Theorem 1.2 for arbitrary families of global p-adic Galois representations. Conjecture 1.3 is noting but the generalized Iwasawa main conjecture for the cyclotomic deformation of ρ_f (a family over $\mathrm{Spf}(\Lambda_{\mathcal{O}_f})$). The main theme of our article [Na20] is this conjecture for the rank two case, precisely, the existence of zeta morphisms for arbitrary families of rank two (odd and residually irreducible) Galois representations.

§ 2. Zeta morphisms for rank two universal deformations

Let $L \subset \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ its integer ring, $\mathbb{F}$ its residue field. Let $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be an odd absolutely irreducible representation which is unramified outside a finite set of primes. We set $\Sigma_{\overline{\rho}}$ the union of $\{p\}$ and the set of ramified primes of $\overline{\rho}$. Let $\mathrm{Comp}_{\mathbb{F}}(\mathcal{O})$ be the category of commutative local Noetherian complete $\mathcal{O}$-algebras with residue field $\mathbb{F}$. For each finite set Σ of primes containing $\Sigma_{\overline{\rho}}$, we denote by

$$D_\Sigma\colon \mathrm{Comp}_{\mathbb{F}}(\mathcal{O}) \to \mathrm{Sets}$$

the deformation functor whose value $D_\Sigma(A)$ at $A \in \mathrm{Comp}_{\mathbb{F}}(\mathcal{O})$ is the set of all the equivalence classes of deformations of $\overline{\rho}$ over A which are unramified outside Σ (no condition above the primes in Σ).

By the assumption of absolutely irreducibility of $\overline{\rho}$, D_Σ is representable. We denote by R_Σ the universal deformation ring, and $\rho_\Sigma : G_{\mathbb{Q}} \to \mathrm{GL}_2(R_\Sigma)$ the universal deformation. We set $X_\Sigma = \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}-\mathrm{alg}}(R_\Sigma, \overline{\mathbb{Q}}_p)$ and, for $x \in X_\Sigma$, we denote by $\rho_x : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ the base change of ρ_Σ by x. We also set

$$X^{\mathrm{mod}}_\Sigma = \{x \in X_\Sigma \mid \text{there exists a Hecke eigen cusp new form } f_x \text{ with some level } N_x \geqq 1 \text{ and weight } k_x \geqq 2, \text{ such that } \rho_x \xrightarrow{\sim} \rho_{f_x}\}$$

For each prime $\ell \notin \Sigma$, we set

$$L_\ell(\rho_\Sigma)(T) = \det\nolimits_{R_\Sigma}(1 - \mathrm{Frob}_\ell \cdot T \mid \rho_\Sigma) \in R_\Sigma[T].$$

We also set

$$K_\Sigma = \{n \in \mathbb{N} \mid (n, \Sigma) = 1\},$$

and, for any $n \in K_\Sigma$, $\Sigma_n = \Sigma \cup \mathrm{prime}(n)$.

For two finite subsets $\Sigma \subset \Sigma'$ of primes containing $\Sigma_{\overline{\rho}}$ as above, one has a natural inclusion $D_\Sigma \subset D_{\Sigma'}$, which induces a canonical surjection $R_{\Sigma'} \to R_\Sigma$, inclusions $X_\Sigma \subset X_{\Sigma'}$ and $X^{\mathrm{mod}}_\Sigma \subset X^{\mathrm{mod}}_{\Sigma'}$, and an isomorphism $\rho_{\Sigma'} \otimes_{R_{\Sigma'}} R_\Sigma \xrightarrow{\sim} \rho_{\Sigma'}$.

The following is the main theorem ([Na20] Theorem 1.1).

Theorem 2.1. *Assume the following :*

(i) $\overline{\rho}$ *is absolutely irreducible,* *(ii)* $\mathrm{End}_{\mathbb{F}[G_{\mathbb{Q}_p}]}(\overline{\rho}) = \mathbb{F}$,

(iii) $\overline{\rho}|_{G_{\mathbb{Q}_p}} \not\xrightarrow{\sim} \begin{pmatrix} \overline{\chi}_p & * \\ 0 & 1 \end{pmatrix} \otimes \eta$ *for any character* $\eta : G_{\mathbb{Q}_p} \to \overline{\mathbb{F}}_p^\times$.

Then, for each finite set Σ *as above, there exists a system of* R_Σ*-linear maps*

$$\{\mathbf{z}_{\Sigma,n} : \rho^*_\Sigma \to \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma_n, \zeta_n], \rho^*_\Sigma(1))\}_{n \in K_\Sigma}$$

$(\mathrm{H}^i_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma_n, \zeta_n], -) = \varprojlim_{k \geqq 0} \mathrm{H}^i(\mathbb{Z}[1/\Sigma_n, \zeta_{np^k}], -))$ *satisfying the following properties:*

(1) (*Euler system relation*) *For every prime* $\ell \notin \Sigma$ *and* $n \in K_\Sigma$*, one has*

$$\mathrm{cor} \circ \mathbf{z}_{\Sigma,n\ell} = \begin{cases} \mathbf{z}_{\Sigma,n} & \text{if } \ell | N, \\ L_\ell(\rho_\Sigma)(\mathrm{Frob}_\ell) \cdot \mathbf{z}_{\Sigma,n} & \text{if } (\ell, N) = 1, \end{cases}$$

where

$$\mathrm{cor} : \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma_{n\ell}, \zeta_{n\ell}], \rho^*_\Sigma(1)) \to \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma_{n\ell}, \zeta_n], \rho^*_\Sigma(1))$$

is the corestriction, and we see $L_\ell(\rho_\Sigma)(\mathrm{Frob}_\ell)$ *as an element in* $R_\Sigma[\mathrm{Gal}(\mathbb{Q}(\zeta_{np^\infty})/\mathbb{Q})]$.

(2) (*Relation to Kato's zeta morphisms*) *For each* $x \in X_\Sigma$*, we set*

$$\{\mathbf{z}_{\Sigma.n}(\rho_x) : \rho^*_x \to \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma_n, \zeta_n], \rho^*_x(1))\}_{n \in K_\Sigma}$$

the base change of $\{\mathbf{z}_{\Sigma,n}\}_{n \in K_\Sigma}$ *by* x*. Then, for any* $x \in X^{\mathrm{mod}}_\Sigma$ (*and* $n = 1 \in K_\Sigma$)*, one has*

$$\mathbf{z}_{\Sigma,1}(\rho_x) = \prod_{\ell \in \Sigma \setminus \{p\}} L_\ell(\rho_{f_x})(\mathrm{Frob}_\ell) \cdot \mathbf{z}(f_x),$$

where $L_\ell(\rho_{f_x})(T) = \det_{\mathcal{O}_{f_x}}(1 - \mathrm{Frob}_\ell \cdot T \mid \rho^{I_\ell}_{f_x}) \in \mathcal{O}_x[T]$ ($I_\ell \subset G_{\mathbb{Q}_\ell}$ *the inertia subgroup*)*.*

(3) *For any finite subsets* $\Sigma \subset \Sigma'$ *as above, one has*

$$\mathbf{z}_{\Sigma',n} \otimes \mathrm{id}_{R_\Sigma} = \prod_{\ell \in \Sigma' \setminus \Sigma} L_\ell(\rho_\Sigma)(\mathrm{Frob}_\ell) \cdot \mathbf{z}_{\Sigma,n}$$

for every $n \in K_{\Sigma'}$ *under the canonical isomorphism* $\rho_{\Sigma'} \otimes_{R_{\Sigma'}} R_\Sigma \xrightarrow{\sim} \rho_\Sigma$*.*

Remark 2.2. The property (2) of our theorem shows that not the zeta morphism $\mathbf{z}(f)$ it self, but a modified one $\prod_{\ell \in \Sigma \setminus \{p\}} L_\ell(\rho_f)(\mathrm{Frob}_\ell) \cdot \mathbf{z}(f)$ ($\Sigma_f \subset \Sigma$) can be extended to the universal deformation. By (1) of Theorem 1.2, the zeta morphism $\mathbf{z}(f)$ interpolates the L-functions removing its p-th Euler factor. Therefore, a modified zeta morphism $\prod_{\ell \in \Sigma \setminus \{p\}} L_\ell(\rho_f)(\mathrm{Frob}_\ell) \cdot \mathbf{z}(f)$ interpolates the L-functions removing the Euler factors at all the primes in Σ. Since we don't assume any ramification condition at the primes in Σ for the deformation functor D_Σ, the ℓ-th (Iwasawa theoretic) Euler factor $L_\ell(\rho_f)(\mathrm{Frob}_\ell)$ in general can not be extended to the universal deformation for the primes ℓ in Σ. Hence, we can't naively extend the zeta morphism $\mathbf{z}(f)$ itself to whole the universal deformation. However, Kato [Ka93] predicted that it can be extended to the universal deformations in the level of derived category (precisely, the determinant of Iwasawa cohomology), not in the level of (Iwasawa) cohomology.

Remark 2.3. As is well known, Kato's Euler system is defined from the system of modular curves $\{Y(Np^k)\}_{k \geqq 0}$ (with full level structure). Therefore, it is important to factor out ρ_Σ-part from such systems. In Fukaya-Kato [FK12], they defined zeta morphisms for Hida families by factoring out the Hida family (ordinary part) from the systems $\{Y_1(Np^k)\}_{k\geqq 0}$ (with Γ_1 level structure) using the p-th ordinary projector. In [Na20], we defined zeta morphism for universal deformation by factoring out the ρ_Σ-part form the system $\{Y(Np^k)\}_{k\geqq 0, N\geqq 1, (N,p)=1}$ (with full level structure) using many deep results in p-adic local Langlands correspondence for GL_2. By Emerton's refined local and global compatibility [Em11], the system (we consider projective limit) of H^1 of étale cohomology of $\{Y(Np^k)\}_{k\geqq 0, N\geqq 1, (N,p)=1}$ is written as a product of ρ_Σ^* with the representations of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ ($\ell\in\Sigma$) corresponding to $\rho_\Sigma|_{G_{\mathbb{Q}_\ell}}$ by the family version of local Langlands correspondence ([Co10] for $\ell=p$, [EH14] for $\ell\neq p$). Using deep results of [Pa13] (for $\ell=p$) and [EH14] (for $\ell\neq p$), we can remove $\mathrm{GL}_2(\mathbb{Q}_\ell)$ part from the system $\{Y(Np^k)\}_{k\geqq 0, N\geqq 1, (N,p)=1}$, and can factor out ρ_Σ-part. After defining zeta morphism for the universal deformation, the most subtle task is to compare our zeta morphism with Kato's zeta morphism $\mathbf{z}(f)$. For this, we need another very deep result of Paskunas [Pa15] describing local Galois deformation ring with fixed Hodge-Tate type in terms of $\mathrm{GL}_2(\mathbb{Q}_p)$ side.

Finally, we remark that Colmez-Wang [CW21] recently obtained zeta morphism for the universal deformation in a very different method. Based on Emerton's refined local and global compatibility [Em11] and many subtle results in [Co10], they carefully study local Iwasawa cohomology $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p,\rho_\Sigma^*(1))$, and first define local zeta morphism $\rho_\Sigma^*\to\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p,\rho_\Sigma^*(1))$ interpolating local Kato's zeta morphisms $\mathrm{rec}_p\circ\mathbf{z}(f):\rho_f^*\to\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p,\rho_f^*(1))$. Then, they show that this local one extends to a global one by some arguments on Galois cohomology.

§ 3. An application to Kato's main conjecture

As an application of Theorem 2.1, one can obtain some results on Kato's main conjecture for congruent modular forms, more generally, for congruent Galois representations. Let $\overline{\rho}: G_{\mathbb{Q}}\to\mathrm{GL}_2(\mathbb{F})$ be a representation satisfying all the assumptions in Theorem 2.1.

We first generalize Kato's main conjecture for all the points in X_Σ and $\overline{\rho}$. Let $x\in X_\Sigma$ be a point, and $\rho_x: G_{\mathbb{Q}}\to\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ the base change of ρ_Σ by x. This is in fact defined over the integer ring $\mathcal{O}_x$ of a finite extension $L_x\subset\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, which we write by the same notation $\rho_x: G_{\mathbb{Q}}\to\mathrm{GL}_2(\mathcal{O}_x)$. Let $\{\mathbf{z}_{\Sigma,n}(\rho_x):\rho_x^*\to\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma_n,\zeta_n],\rho^*(1))\}_{n\in K_\Sigma}$ be the base change of $\{\mathbf{z}_{\Sigma,n}\}_{n\in K_\Sigma}$ at x, and

$$\{\mathbf{z}_{\Sigma,n}(\overline{\rho}):\overline{\rho}^*\to\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma_n,\zeta_n],\overline{\rho}^*(1))\}_{n\in K_\Sigma}$$

be the base change of $\{\mathbf{z}_{\Sigma,n}\}_{n\in K_\Sigma}$ to the residue field of R_Σ. For $n=1\in K_\Sigma$, we simply write $\mathbf{z}_\Sigma(\rho_x)=\mathbf{z}_{\Sigma,1}(\rho_x), \mathbf{z}_\Sigma(\overline{\rho})=\mathbf{z}_{\Sigma,1}(\overline{\rho})$.

Throughout this section, we assume that

$$\mathbf{z}_\Sigma(\overline{\rho}) : \overline{\rho}^* \to \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma], \overline{\rho}^*(1))$$

is non zero (at all the components in $\mathrm{Spec}(\mathbb{F}[[\Gamma]])$). As we explain later, this is a kind of $\mu = 0$ assumption, which the author believes always hold.

This assumption immediately implies that $\mathbf{z}_\Sigma(\rho_x)$ is also non zero for all the points $x \in X_\Sigma$ (even for non-modular points!). In particular, we obtain a non-zero Euler system $\{\mathbf{z}_{\Sigma,n}(\rho_x)\}_{n\in K_\Sigma}$, and the general theory of Euler system implies (under some very mild assumptions) that

(1) $\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma], \overline{\rho}^*(1))$ is a torsion $\Lambda_{\mathcal{O}_x}$-module (in fact, a finite generated $\mathcal{O}_x$-module by our $\mu = 0$ assumption).

(2) $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma], \overline{\rho}^*(1))$ is a finite free $\Lambda_{\mathcal{O}_x}$-module of rank one.

(3) One has the following inclusion of characteristic ideals of torsion $\Lambda_{\mathcal{O}_x}$-modules :

$$\mathrm{Char}_{\Lambda_{\mathcal{O}_x}}(\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma], \rho_x^*(1))/\Lambda_{\mathcal{O}_x} \cdot \mathrm{Im}(\mathbf{z}_\Sigma(\rho_x))) \subset \mathrm{Char}_{\Lambda_{\mathcal{O}_x}}(\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma], \rho_x^*(1))).$$

As a generalization of Kato's main conjecture to ρ_x, we propose the following conjecture.

Conjecture 3.1. (*Main conjecture for ρ_x*)

$$\mathrm{Char}_{\Lambda_{\mathcal{O}_x}}(\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma], \rho_x^*(1))/\Lambda_{\mathcal{O}_x} \cdot \mathrm{Im}(\mathbf{z}_\Sigma(\rho_x))) = \mathrm{Char}_{\Lambda_{\mathcal{O}_x}}(\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma], \rho_x^*(1))).$$

Remark 3.2. For every $\Sigma \subset \Sigma'$, we can compare $\mathbf{z}_\Sigma(\rho_x)$ with $\mathbf{z}_{\Sigma'}(\rho_x)$ by (3) of Theorem 2.1. We can similarly compare $\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma], \rho_x^*(1))$ with $\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma'], \rho_x^*(1))$ by some standard arguments in Iwasawa cohomology. Hence, the conjecture above is independent of the choice of Σ such that $x \in X_\Sigma$. Moreover, for every modular point $x \in X_\Sigma^{\mathrm{mod}}$, we can compare $\mathbf{z}_\Sigma(\rho_x)$ with $\mathbf{z}(f_x)$ by (2) of Theorem 2.1, and we can similarly compare $\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma], \rho_x^*(1))$ with $\mathbf{H}^2(\rho_x^*(1))$. Hence, for the modular points, the conjecture above for ρ_x is also equivalent to Kato's main conjecture (Conjecture 1.3) for f_x.

Under our $\mu = 0$ assumption, we can also show that

(1) $\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma], \overline{\rho}^*(1))$ is a torsion $\mathbb{F}[[\Gamma]]$-module, in particular, a finite dimensional $\mathbb{F}$-vector space.

(2) $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma], \overline{\rho}^*(1))$ is a free $\mathbb{F}[[\Gamma]]$-module of rank one.

As a generalization of Kato main conjecture to $\overline{\rho}$, we also propose the following conjecture.

Conjecture 3.3. (*Main conjecture for* $\overline{\rho}$)

$$\dim_{\mathbb{F}}(\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma],\overline{\rho}^*(1))/\mathbb{F}[[\Gamma]]\cdot\mathrm{Im}(\mathbf{z}_\Sigma(\overline{\rho}))) = \dim_{\mathbb{F}}(\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Z}[1/\Sigma],\overline{\rho}^*(1))).$$

As a consequence of Theorem 2.1 and some arguments in Iwasawa cohomology, we can prove the following ([Na20] Corollary 5.14).

Theorem 3.4. *The following conditions* (1), (2) *and* (3) *are equivalent.*

(1) *Conjecture* 3.1 *holds for some* $x \in \bigcup_\Sigma X_\Sigma$.

(2) *Conjecture* 3.1 *holds for all* $x \in \bigcup_\Sigma X_\Sigma$.

(3) *Conjecture* 3.4 *holds.*

Remark 3.5. Since Conjecture 3.1 is equivalent to Kato's main conjecture for modular points, this theorem implies the equivalences of Kato's main conjecture for congruent modular forms. Such results were previous known by Greenberg-Vatsal [GV00] for congruent p-ordinary rational elliptic curves, by Emerton-Pollack-Weston [EPW06] for congruent p-ordinary modular forms, under the similar assumptions (precisely, $\overline{\rho}$ is absolutely irreducible, μ-invariants of both the p-adic L-function and the cyclotomic Selmer group for one f are zero). Since the p-adic L-function for f is the image of $\mathbf{z}(f)$ by some p-integral Coleman map when f is p-ordinary, our $\mu = 0$ assumption is weaker than the $\mu = 0$ assumption for the p-adic L-function. More importantly, our theorem can treat all the modular points with arbitrary weights and levels. For example, we can compare Kato's main conjecture for p-ordinary, or p-finite slope modular forms with that for p-supercuspidal modular forms. Kato's main conjecture for p-ordinary modular forms (which is equivalent to Iwasawa main conjecture [Ma72], [Gr89]) are known in many cases by the works of Skinner-Urban [SU14] and Skinner [Sk16]. Kato's main conjecture for p-finite slope modular forms (which is equivalent to Iwasawa main conjecture [Ko03], [LLZ10]) are also recently known in many cases, e.g. by [Wa14], [Wa16], [Sp16], [CCSS18]. On the other hands, no results are known for Kato's main conjecture for p-supercuspidal forms (at least when the preprint [Na20] was submitted). By Theorem 3.4, we can prove Kato's main conjecture even for p-supercuspidal forms from the known main conjecture for p-ordinary, or p-finite slope modular forms (remark also that X_Σ always contains p-ordinary, or p-finite slope modular points).

Remark 3.6. Fouquet-Wan [FW21] recently wrote up an article, where they prove Kato's main conjecture for arbitrary f without $\mu = 0$ assumption (but the author has not checked any details of the proof). Their method seems to be essentially similar to ours, i.e. they reduce Kato's main conjecture for arbitrary f to that for p-ordinary, or p-finite slope case using our zeta morphism $\mathbf{z}_\Sigma$ for the universal deformation. Even though their result is correct, the author thinks that the problem whether $\mathbf{z}(\overline{\rho})$ is always non-zero (our $\mu = 0$ assumption) itself is an important problem.

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