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Abstract

In this paper, we obtain a (p, ν)-extension of Srivastava's triple hypergeometric function $H_B(\cdot)$, together by using the extended Beta function $B_{p,\nu}(x, y)$ introduced in [19]. We give some of the main properties of this extended function, which include several integral representations involving Exton's hypergeometric function, the Mellin transform, a differential formula, recursion formulas and a bounded inequality.

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1. Introduction and Preliminaries

In the present paper, we employ the following notations:

$$N := \{1, 2, ...\}, \quad N_0 := N \cup \{0\}, \quad Z_0^- := Z^- \cup \{0\},$$

where the symbols N and Z denote the set of integer and natural numbers; as usual, the symbols R and C denote the set of real and complex numbers, respectively.

In the available literature, the hypergeometric series and its generalizations appear in various branches of mathematics associated with applications. A large number of triple hypergeometric functions have been introduced and investigated. The work of Srivastava and Karlsson [23, Chapter 3] provides a table of 205 distinct triple hypergeometric functions. Srivastava introduced the triple hypergeometric functions H_A, H_B and H_C of the second order in [20, 21]. It is known that H_C and H_B are generalizations of Appell's hypergeometric functions F_1 and F_2, while H_A is the generalization of both F_1 and F_2.

In the present study, we confine our attention to Srivastava's triple hypergeometric function H_B given by [23, p. 43, 1.5(11) to 1.5(13)] (see also [20] and [22, p. 68])

$$H_B(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) := \sum_{m,n,k=0}^{\infty} \frac{(b_1)_{m+k}(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}$$

$$= \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{B(b_1+m+k, b_2+m+n)}{B(b_1, b_2)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}. \tag{1.1}$$

Here $(\lambda)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & (n = 0,\ \lambda \in C\backslash\{0\}) \\ \lambda(\lambda+1)...(\lambda+n-1), & (n \in N,\ \lambda \in C) \end{cases}$$

and $B(\alpha, \beta)$ denotes the classical Beta function defined by [16, (5.12.1)]

$$B(\alpha, \beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, & (\Re(\alpha) > 0, \Re(\beta) > 0) \\ \displaystyle\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, & ((\alpha, \beta) \in C\backslash Z_0^-). \end{cases} \tag{1.2}$$

The convergence region for Srivastava's triple hypergeometric series $H_B(\cdot)$ is given in [13, p.243] as $|x| < \alpha$, $|y| < \beta$, $|z| < \gamma$, where α, β, γ satisfy the relation $\alpha+\beta+\gamma+2\sqrt{\alpha\beta\gamma} = 1$.

A different type of triple hypergeometric function is Exton's function $X_4(\cdot)$, which is defined by (see [12] and [23, p. 84, Entry (45a)])

$$X_4(b_1, b_2; c_1, c_2, c_3; x, y, z) := \sum_{m,n,k=0}^{\infty} \frac{(b_1)_{2m+n+k}(b_2)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}. \tag{1.3}$$

The convergence region for this series is $2\sqrt{|x|} + (\sqrt{|y|} + \sqrt{|z|})^2 < 1$. We shall also find it convenient to introduce an additional parameter a into $H_B(\cdot)$ in the form

$$H_B^{(a)}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) :=$$

$$\sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{B(b_1+a+m+k, b_2+a+m+n)}{B(b_1, b_2)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}, \tag{1.4}$$

which reduces to (1.1) when $a = 0$.

In 1997, Chaudhry *et al.* [2, Eq.(1.7)] gave a p-extension of the Beta function $B(x, y)$ given by

$$B(x, y; p) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[\frac{-p}{t(1-t)}\right] dt, \qquad (\Re(p) > 0)$$

and they proved that this extension has connections with the Macdonald, error and Whittaker functions. Also, Chaudhry *et al.* [3] extended the Gaussian hypergeometric series ${}_2F_1(\cdot)$ and its integral representations. Recently, Parmar *et al.* [19] have given a further extension of the extended Beta function $B(x, y; p)$ by adding one more parameter ν, which we denote and define by

$$B_{p,\nu}(x, y) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}} K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt, \tag{1.5}$$

where $\Re(p) > 0$, $\nu \geq 0$ and $K_\nu(z)$ is the modified Bessel function (sometimes known as the Macdonald function) of order ν. When $\nu = 0$, (1.5) reduces to $B(x, y; p)$, since $K_{\frac{1}{2}}(z) = \sqrt{\pi/(2z)}e^{-z}$. A different generalization of the Beta function has been given in [17].

Motivated by some of the above-mentioned extensions of special functions, many authors have studied integral representations of $H_B(\cdot)$ functions; see [5, 6, 7, 8]. Our aim in this paper is to introduce a (p, v)-extension of Srivastava's triple hypergeometric function $H_B(\cdot)$ in (1.1), which we denote by $H_{B,p,v}(\cdot)$, based on the extended Beta function in (1.5), and to systematically investigate some properties of this extended function. We consider the Mellin transform, a differential formula, recursion formulas and a bounded inequality satisfied by this function.

The plan of this paper as follows. The extended Srivastava hypergeometric function $H_{B,p,\nu}(\cdot)$ is defined in Section 2 and some integral representations are presented involving the modified Bessel function and Exton's function X_4. The main properties of $H_{B,p,\nu}(\cdot)$ namely, its Mellin transform, a differential formula, a bounded inequality and recursion formulas are established in Sections 3-6. Some concluding remarks are made in Section 7.

2. The $(p, v)-$extended Srivastava triple hypergeometric function $H_{B,p,\nu}(\cdot)$

Srivastava introduced the triple hypergeometric function $H_B(\cdot)$, together with its integral representations, in [20] and [22]. Here we consider the following (p, v)-extension of this function, which we denote by $H_{B,p,\nu}(\cdot)$, based on the extended Beta function $B(x, y; p, v)$ defined in (1.5). This is given by

$$H_{B,p,\nu}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z)$$
$$= \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{B_{p,\nu}(b_1+m+k, b_2+m+n)}{B(b_1, b_2)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}, \tag{2.1}$$

where the parameters $b_1, b_2, b_3 \in C$ and $c_1, c_2, c_2 \in C\backslash Z_0^-$. The region of convergence is $|x| < r$, $|y| < s$, $|z| < t$, where $r + s + t + 2\sqrt{rst} = 1$. This definition clearly reduces to the original classical function when $\nu = 0$.

Several integral representations for $H_{B,p,\nu}(\cdot)$ involving Exton's triple hypergeometric function in (1.3) can be given. We have

Theorem 1. *Each of the following integral representations of the extended Srivastava triple hypergeometric function $H_{B,p,\nu}(\cdot)$ holds for $\Re(p) > 0$, and $\min\{\Re(b_1), \Re(b_2)\} > 0$:*

$$H_{B,p,\nu}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) = \frac{\Gamma(b_1 + b_2)}{\Gamma(b_1)\Gamma(b_2)} \sqrt{\frac{2p}{\pi}} \int_0^1 t^{b_1-\frac{3}{2}}(1-t)^{b_2-\frac{3}{2}} K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)$$
$$\times X_4(b_1 + b_2, b_3; c_1, c_2, c_3; xt(1-t), y(1-t), zt)\, dt; \tag{2.2}$$

$$H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)=\frac{(\beta-\gamma)^{b_1-\frac{1}{2}}(\alpha-\gamma)^{b_2-\frac{1}{2}}}{(\beta-\alpha)^{b_1+b_2-2}}\frac{\Gamma(b_1+b_2)}{\Gamma(b_1)\Gamma(b_2)}\sqrt{\frac{2p}{\pi}}$$

$$\times\int_\alpha^\beta \frac{(\xi-\alpha)^{b_1-\frac{3}{2}}(\beta-\xi)^{b_2-\frac{3}{2}}}{(\xi-\gamma)^{b_1+b_2-1}}K_{\nu+\frac{1}{2}}\left(\frac{p}{\sigma_1\sigma_2}\right)X_4(b_1+b_2,b_3;c_1,c_2,c_3;\sigma_1\sigma_2x,\sigma_1y,\sigma_2z)\,d\xi, \tag{2.3}$$

where

$$\sigma_1=\frac{(\alpha-\gamma)(\beta-\xi)}{(\beta-\alpha)(\xi-\gamma)},\qquad \sigma_2=\frac{(\beta-\gamma)(\xi-\alpha)}{(\beta-\alpha)(\xi-\gamma)}\quad(\gamma<\alpha<\beta);$$

$$H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)=\frac{2\Gamma(b_1+b_2)}{\Gamma(b_1)\Gamma(b_2)}\sqrt{\frac{2p}{\pi}}\int_0^{\frac{\pi}{2}}(\sin^2\xi)^{b_1-1}(\cos^2\xi)^{b_2-1}$$

$$\times K_{\nu+\frac{1}{2}}\left(\frac{p}{\sigma_1\sigma_2}\right)X_4(b_1+b_2,b_3;\;c_1,c_2,c_3;\;\sigma_1\sigma_2x,\sigma_1y,\sigma_2z)\,d\xi, \tag{2.4}$$

where

$$\sigma_1=\cos^2\xi,\qquad \sigma_2=\sin^2\xi;$$

$$H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)=\frac{2(1+\lambda)^{b_1-\frac{1}{2}}\Gamma(b_1+b_2)}{\Gamma(b_1)\Gamma(b_2)}\sqrt{\frac{2p}{\pi}}\int_0^{\frac{\pi}{2}}\frac{(\sin^2\xi)^{b_1-1}(\cos^2\xi)^{b_2-1}}{(1+\lambda\sin^2\xi)^{b_1+b_2-1}}$$

$$\times K_{\nu+\frac{1}{2}}\left(\frac{p}{\sigma_1\sigma_2}\right)X_4(b_1+b_2,b_3;c_1,c_2,c_3;\sigma_1\sigma_2x,\sigma_1y,\sigma_2z)\,d\xi, \tag{2.5}$$

where

$$\sigma_1=\frac{\cos^2\xi}{1+\lambda\sin^2\xi},\qquad \sigma_2=\frac{(1+\lambda)\sin^2\xi}{1+\lambda\sin^2\xi}\quad(\lambda>-1);$$

and

$$H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)=\frac{2\lambda^{b_1-\frac{1}{2}}\Gamma(b_1+b_2)}{\Gamma(b_1)\Gamma(b_2)}$$

$$\times\sqrt{\frac{2p}{\pi}}\int_0^{\frac{\pi}{2}}\frac{(\sin^2\xi)^{b_1-1}(\cos^2\xi)^{b_2-1}}{(\cos^2\xi+\lambda\sin^2\xi)^{b_1+b_2-1}}K_{\nu+\frac{1}{2}}\left(\frac{p}{\sigma_1\sigma_2}\right)$$

$$\times X_4(b_1+b_2,b_3;c_1,c_2,c_3;\sigma_1\sigma_2x,\sigma_1y,\sigma_2z)\,d\xi, \tag{2.6}$$

where

$$\sigma_1=\frac{\cos^2\xi}{\cos^2\xi+\lambda\sin^2\xi},\qquad \sigma_2=\frac{\lambda\sin^2\xi}{\cos^2\xi+\lambda\sin^2\xi}\quad(\lambda>0).$$

Proof: The proof of the first integral representation (2.2) follows by use of the extended beta function (1.5) in (2.1), a change the order of integration and summation (with uniform convergence of the integral) and, after simplification, use of Exton's triple hypergeometric function (1.3), to obtain the right-hand side of the result (2.2). The integral representations (2.3)-(2.6) can be proved directly by using the following transformations

$$(2.3):\qquad t=\frac{(\beta-\gamma)(\xi-\alpha)}{(\beta-\alpha)(\xi-\gamma)},\quad \frac{dt}{d\xi}=\frac{(\beta-\gamma)(\alpha-\gamma)}{(\beta-\alpha)(\xi-\gamma)^2},$$

$$(2.4):\qquad t=\sin^2\xi,\quad \frac{dt}{d\xi}=2\sin\xi\cos\xi$$

$$(2.5):\qquad t=\frac{(1+\lambda)\sin^2\xi}{1+\lambda\sin^2\xi},\quad \frac{dt}{d\xi}=\frac{2(1+\lambda)\sin\xi\cos\xi}{(1+\lambda\sin^2\xi)^2},$$

$$(2.6):\qquad t=\frac{\lambda\sin^2\xi}{\cos^2\xi+\lambda\sin^2\xi},\quad \frac{dt}{d\xi}=\frac{2\lambda\sin\xi\cos\xi}{(\cos^2\xi+\lambda\sin^2\xi)^2}$$

in turn in (2.2) to obtain the right-hand side of each result.

3. The Mellin transform of $H_{B,p,\nu}(\cdot)$

The Mellin transform of a locally integrable function $f(x)$ on $(0,\infty)$ is given by (see, for example, [15, p.193, §2.1])

$$\Phi(s)=\mathcal{M}\{f(x)\}(s)=\int_0^\infty x^{s-1}f(x)\,dx \tag{3.1}$$

which defines an analytic function in its strip of analyticity $a<\Re(s)<b$. The inverse Mellin transform of the above function (3.1) is defined by

$$f(x)=\mathcal{M}^{-1}\{\Phi(s)\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}\Phi(s)\,ds\qquad (a<c<b). \tag{3.2}$$

Theorem 2. *The following Mellin transform of the extended Srivastava triple hypergeometric function $H_{B,p,\nu}(\cdot)$ holds true:*

$$\mathcal{M}\{H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)\}(s)=\int_0^\infty p^{s-1}H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)\,dp,$$

$$=\frac{2^{s-1}}{\sqrt{\pi}}\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu+1}{2}\right)H_B^{(s)}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z), \tag{3.3}$$

where $\Re(s)>\nu>0$, $c_1,c_2,c_3\in C\backslash Z_0^-$ and $H_B^{(s)}(\cdot)$ is defined in (1.4).

Proof: Substituting the extended Srivastava function (2.1) into the integral on the left-hand side of (3.3) and changing the order of integration (by the uniform convergence of the integral), we obtain

$$\mathcal{M}\{H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)\}(s)$$

$$=\sum_{m,n,k=0}^{\infty}\frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k\,B(b_1,b_2)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}\left\{\int_0^\infty p^{s-1}B_{p,\nu}(b_1+m+k,b_2+m+n)\,dp\right\}.$$

Using the extended Beta function (1.5) then shows that

$$\mathcal{M}\{H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)\}(s)=\sqrt{\frac{2}{\pi}}\sum_{m,n,k=0}^{\infty}\frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k\,B(b_1,b_2)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$

$$\times\int_0^1 t^{b_1+m+k-\frac{3}{2}}(1-t)^{b_2+m+n-\frac{3}{2}}\left\{\int_0^\infty p^{s-\frac{1}{2}}K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)dp\right\}dt.$$

Application of the result [16, (10.43.19)]

$$\int_0^\infty w^{s-\frac{1}{2}} K_{\alpha+\frac{1}{2}}(w)dw = 2^{s-\frac{3}{2}}\Gamma\left(\frac{s-\alpha}{2}\right)\Gamma\left(\frac{s+\alpha+1}{2}\right) \qquad (|\Re(\alpha)| < \Re(s))$$

followed by the substitution $w = p/(t(1-t))$ produces

$$\mathcal{M}\left\{H_{B,p,\nu}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z)\right\}(s) = \frac{2^{s-1}}{\sqrt{\pi}}\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu+1}{2}\right)$$
$$\times \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k\ B(b_1,b_2)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}\left\{\int_0^1 t^{b_1+m+k+s-1}(1-t)^{b_2+m+n+s-1}dt\right\}.$$

Evaluation of the integral in terms of the classical Beta function then finally yields

$$\Phi(s) = \mathcal{M}\left\{H_{B,p,\nu}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z)\right\}(s) = \frac{2^{s-1}}{\sqrt{\pi}}\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu+1}{2}\right)$$
$$\times \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{B(b_1+m+k+s, b_2+m+n+s)}{B(b_1,b_2)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}.$$

Identifying the above sum as $H_B^{(s)}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z)$ in (1.4), we obtain the right-hand side of (3.3).

Corollary 1: The following inverse Mellin formula for $H_{B,p,v}(\cdot)$ holds:

$$H_{B,p,\nu}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z) = \mathcal{M}^{-1}\left\{\Phi(s)\right\}$$
$$= \frac{\pi^{-3/2}}{4i}\int_{c-i\infty}^{c+i\infty}\left(\frac{2}{p}\right)^s \Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu+1}{2}\right) H_B^{(s)}\left(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z\right) ds, \qquad (3.4)$$

where $c > \nu$.

4. A differentiation formula for $H_{B,p,\nu}(\cdot)$

Theorem 3. *The following derivative formula for $H_{B,p,\nu}(\cdot)$ holds:*

$$\frac{\partial^{M+N+K}}{\partial x^M \partial y^N \partial z^K}\ H_{B,p,\nu}\left(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z\right) = \frac{(b_1)_{M+K}(b_2)_{M+N}(b_3)_{N+K}}{(c_1)_M(c_2)_N(c_3)_K}$$
$$\times H_{B,p,\nu}(b_1+M+K, b_2+M+N, b_3+N+K; c_1+M, c_2+N, c_3+K; x, y, z), \qquad (4.1)$$

where $M,\ N,\ K \in N_0$.

Proof: If we differentiate partially the series for $\mathcal{H} \equiv H_{B,p,\nu}(b_1, b_2, b_3; c_1, c_2, c_3; x, y, z)$ in (2.1) with respect to x we obtain

$$\frac{\partial\mathcal{H}}{\partial x} = \sum_{m=1}^{\infty}\sum_{n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{B_{p,\nu}(b_1+m+k, b_2+m+n)}{B(b_1,b_2)} \frac{x^{m-1}}{(m-1)!}\frac{y^n}{n!}\frac{z^k}{k!}.$$

Making use of the fact that

$$B(b_1,b_2)=\frac{(b_1+b_2)_2}{b_1b_2}\,B(b_1+1,b_2+1) \tag{4.2}$$

and $(\lambda)_{m+n}=(\lambda)_m(\lambda+m)_n$, we have upon setting $m\to m+1$

$$\frac{\partial\mathcal{H}}{\partial x}=\frac{b_1b_2}{c_1}\sum_{m,n,k=0}^{\infty}\frac{(b_1+b_2+2)_{2m+n+k}(b_3)_{n+k}}{(c_1+1)_m(c_2)_n(c_3)_k}\,\frac{B_{p,\nu}(b_1{+}1{+}m{+}k,b_2{+}1{+}m{+}n)}{B(b_1+1,b_2+1)}\,\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$

$$=\frac{b_1b_2}{c_1}\,H_{B,p,\nu}(b_1+1,b_2+1,b_3;c_1+1,c_2,c_3;x,y,z). \tag{4.3}$$

Repeated application of (4.3) then yields for $M=1,2,\ldots$

$$\frac{\partial^M}{\partial x^M}\mathcal{H}=\frac{(b_1)_M(b_2)_M}{(c_1)_M}\,H_{B,p,\nu}(b_1+M,b_2+M,b_3;c_1+M,c_2,c_3;x,y,z).$$

A similar reasoning shows that

$$\frac{\partial^{M+1}}{\partial x^M\partial y}\mathcal{H}=\frac{(b_1)_M(b_2)_M}{(c_1)_M}\sum_{m=0}^{\infty}\sum_{n,k=1}^{\infty}\frac{(b_1{+}b_2{+}2M)_{2m+n+k}(b_3)_{n+k}}{(c_1{+}M)_m(c_2)_n(c_3)_k}$$

$$\times\frac{B_{p,\nu}(b_1{+}M{+}m{+}k,b_2{+}M{+}n{+}k)}{B(b_1{+}M,b_2{+}M)}\,\frac{x^m}{m!}\,\frac{y^{n-1}}{(n-1)!}\,\frac{z^k}{k!}$$

$$=\frac{(b_1)_M(b_2)_{M+1}b_3}{(c_1)_Mc_2}\,H_{B,p,\nu}(b_1+M,b_2+M+1,b_3+1;c_1+M,c_2+1,c_3;x,y,z) \tag{4.4}$$

upon putting $n\to n+1$ and using the property of the Beta function in (1.2). Repeated differentiation of (4.4) N times with respect to y then produces

$$\frac{\partial^{M+N}}{\partial x^M\partial y^N}\mathcal{H}=\frac{(b_1)_M(b_2)_{M+N}(b_3)_N}{(c_1)_M(c_2)_N}\,H_{B,p,\nu}(b_1{+}M,b_2{+}M{+}N,b_3{+}N;c_1{+}M,c_2{+}N,c_3;x,y,z).$$

Application of the same procedure to deal with differentiation with respect to z then yields the result stated in (4.1).

5. An upper bound for $H_{B,p,\nu}(\cdot)$

Theorem 4. *Let the parameters b_j, c_j $(1\le j\le 3)$ be positive and the variables x, y, z be complex. Then the following bounded inequality for $H_{B,p,\nu}(\cdot)$ holds:*

$$|H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)|$$

$$<\frac{2^\nu|p|^{\nu+1}}{\sqrt{\pi}(\Re(p))^{2\nu+1}}\,\Gamma(\nu+\tfrac{1}{2})\,H_B^{(\nu)}(b_1,b_2,b_3;c_1,c_2,c_3;|x|,|y|,|z|), \tag{5.1}$$

where $\Re(p)>0$, $\nu>0$ and $H_B^{(\nu)}(\cdot)$ is defined in (1.4)

The integral representation of the extension $H_{B,p,\nu}(\cdot)$ in (2.2) is associated with the modified Bessel function of the second kind, for which we have the following expression [16, (10.32.8)]

$$K_{\nu+\frac{1}{2}}(z)=\frac{\sqrt{\pi}\left(\frac{1}{2}z\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)}\int_1^\infty e^{-zt}(t^2-1)^\nu dt,\qquad (\nu>-1,\ \Re(z)>0).$$

In our problem we have $\nu>0$, $\Re(z)>0$. Further, we let $x=\Re(z)$, so that

$$|K_{\nu+\frac{1}{2}}(z)|\leq\frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)}\left|\int_1^\infty e^{-zt}(t^2-1)^\nu dt\right|<\frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)}\int_0^1 t^{2\nu}e^{-xt}dt$$

$$=\frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)}\frac{\Gamma(2\nu+1,x)}{x^{2\nu+1}},\tag{5.2}$$

where $\Gamma(a,z)$ is the upper incomplete gamma function [16, (8.2.2)]. Although this bound is numerically found to be quite sharp when z is real, it involves the incomplete gamma function which would make the integral for $F_{1,p,\nu}(b_1,b_2,b_3;c_1;x,y)$ difficult to bound. We can simplify (5.2) by making use of the simple inequality $\Gamma(2\nu+1,x)<\Gamma(2\nu+1)$ to find

$$|K_{\nu+\frac{1}{2}}(z)|<\frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)}\frac{\Gamma(2\nu+1)}{x^{2\nu+1}}=\frac{1}{2}\left(\frac{2|z|}{x^2}\right)^{\nu+\frac{1}{2}}\Gamma(\nu+\tfrac{1}{2}),\tag{5.3}$$

upon use of the duplication formula for the gamma function. The bound (5.3) is less sharp than (5.2) but has the advantage of being easier to handle in the integral for $H_{B,p,\nu}(\cdot)$.

Proof: Setting $z=p/(t(1-t))$, where $t\in(0,1)$ and $\Re(p)>0$, in (5.3) we obtain

$$\left|K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)\right|<\frac{1}{2}\left(\frac{2|p|t(1-t)}{(\Re(p))^2}\right)^{\nu+\frac{1}{2}}\Gamma(\nu+\tfrac{1}{2}).$$

For ease of presentation we shall assume that the parameters b_j, $c_j>0$ $(1\leq j\leq 3)$. Then, from (2.2),

$$|H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)|\leq\frac{\sqrt{2|p|/\pi}}{B(b_1,b_2)}\int_0^1\left|t^{b_1-\frac{3}{2}}(1-t)^{b_2-\frac{3}{2}}K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)\right.$$

$$\left.\times X_4(b_1+b_2,b_3;c_1,c_2,c_3;xt(1-t),y(1-t),zt)\right|dt$$

$$<\frac{2^\nu|p|^{\nu+1}}{\sqrt{\pi}(\Re(p))^{2\nu+1}}\frac{\Gamma(\nu+\frac{1}{2})}{B(b_1,b_2)}\sum_{m,n,k=0}^\infty\frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k}\frac{|x|^m}{m!}\frac{|y|^n}{n!}\frac{|z|^k}{k!}$$

$$\times\int_0^1 t^{b_1+\nu+m+k-1}(1-t)^{b_2+\nu+m+n-1}dt$$

$$<\frac{2^\nu|p|^{\nu+1}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}(\Re(p))^{2\nu+1}}\sum_{mn,k=0}^\infty\frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k}$$

$$\times \frac{B(b_1+\nu+m+k, b_2+\nu+m+n)}{B(b_1,b_2)} \frac{|x|^m}{m!} \frac{|y|^n}{n!} \frac{|z|^k}{k!} \tag{5.4}$$

which is the result stated in (5.1).

6. Recursion formulas for $H_{B,p,\nu}(\cdot)$

In this section, we obtain two recursion formulas for the extended Srivastava function $H_{B,p,\nu}(\cdot)$. The first formula gives a recursion with respect to the numerator parameter b_3, and the second a recursion with respect to any one of the denominator parameters c_j $(1 \le j \le 3)$.

Theorem 5. *The following recursion for* $H_{B,p,\nu}(\cdot)$ *with respect to the numerator parameter* b_3 *holds:*

$$H_{B,p,\nu}(b_1,b_2,b_3+1;c_1,c_2,c_3;x,y,z) = H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)$$

$$+\frac{yb_2}{c_2} H_{B,p,\nu}(b_1,b_2+1,b_3+1;c_1,c_2+1,c_3;x,y,z)+\frac{zb_1}{c_3} H_{B,p,\nu}(b_1+1,b_2,b_3+1;c_1,c_2,c_3+1;x,y,z). \tag{6.1}$$

Proof. From (2.1) and the result $(b_3+1)_{n+k} = (b_3)_{n+k}(1+n/b_3+k/b_3)$, we obtain

$$H_{B,p,\nu}(b_1,b_2,b_3+1;c_1,c_2,c_3;x,y,z)$$

$$= \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3+1)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{B_{p,\nu}(b_1+m+k,b_2+m+n)}{B(b_1,b_2)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}$$

$$= H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)$$

$$+\frac{y}{b_3} \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{B_{p,\nu}(b_1+m+k,b_2+m+n)}{B(b_1,b_2)} \frac{x^m}{m!} \frac{y^{n-1}}{(n-1)!} \frac{z^k}{k!}$$

$$+\frac{z}{b_3} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k} \frac{B_{p,\nu}(b_1+m+k,b_2+m+n)}{B(b_1,b_2)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^{k-1}}{(k-1)!}. \tag{6.2}$$

Consider the first sum in (6.2) which we denote by S. Put $n \to n+1$ and use the identity $(a)_{n+1} = a(a+1)_n$ to find

$$S = \frac{y}{b_3} \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2)_{2m+n+1+k}(b_3)_{n+1+k}}{(c_1)_m(c_2)_{n+1}(c_3)_k} \frac{B_{p,\nu}(b_1+m+k,b_2+1+m+n)}{B(b_1,b_2)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}$$

$$= \frac{y(b_1+b_2)}{c_2} \sum_{m,n,k=0}^{\infty} \frac{(b_1+b_2+1)_{2m+n+k}(b_3+1)_{n+k}}{(c_1)_m(c_2+1)_n(c_3)_k} \frac{B_{p,\nu}(b_1+m+k,b_2+1+m+n)}{B(b_1,b_2)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}.$$

Using the fact that

$$B(b_1,b_2) = \frac{b_1+b_2}{b_2} B(b_1,b_2+1),$$

we then obtain

$$S=\frac{yb_2}{c_2}\sum_{m,n,k=0}^{\infty}\frac{(b_1+b_2+1)_{2m+n+k}(b_3+1)_{n+k}}{(c_1)_m(c_2+1)_n(c_3)_k}\,\frac{B_{p,\nu}(b_1+m+k,b_2+1+m+n)}{B(b_1,b_2+1)}\,\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$

$$=\frac{yb_2}{c_2}H_{B,p,\nu}(b_1,b_2+1,b_3+1;c_1,c_2+1,c_3;x,y,z). \tag{6.3}$$

Proceeding in a similar manner for the second series in (6.2) with $k\to k+1$, we find that this sum can be expressed as

$$\frac{zb_1}{c_3}H_{B,p,\nu}(b_1+1,b_2,b_3+1;c_1,c_2,c_3+1;x,y,z). \tag{6.4}$$

Combination of (6.3) and (6.4) with (6.2) then produces the result stated in (6.1).

Corollary 2: From (6.1) the following recursion holds

$$H_{B,p,\nu}(b_1,b_2,b_3+N;c_1,c_2,c_3;x,y,z)=H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)$$

$$+\frac{yb_2}{c_2}\sum_{\ell=1}^{N}H_{B,p,\nu}(b_1,b_2+1,b_3+\ell;c_1,c_2+1,c_3;x,y,z)$$

$$+\frac{zb_1}{c_3}\sum_{\ell=1}^{N}H_{B,p,\nu}(b_1+1,b_2,b_3+\ell;c_1,c_2,c_3+1;x,y,z) \tag{6.5}$$

for positive integer N.

Theorem 6. *The following 3-term recursion for $H_{B,p,\nu}(\cdot)$ with respect to the denominator parameter c_1 holds:*

$$H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)=$$

$$H_{B,p,\nu}(b_1,b_2,b_3;c_1+1,c_2,c_3;x,y,z)+\frac{xb_1b_2}{c_1(c_1+1)}H_{B,p,\nu}(b_1+1,b_2+1,b_3;c_1+2,c_2,c_3;x,y,z). \tag{6.6}$$

Permutation of the c_j enables analogous recursions in the denominator parameters c_2 and c_3 to be obtained.

Proof. Consider the case when c_1 is reduced by 1, namely

$$H\equiv H_{B,p,\nu}(b_1,b_2,b_3;c_1-1,c_2,c_3;x,y,z)$$

and use $(c_1-1)_m=(c_1)_m/\{1+\frac{m}{c_1-1}\}$. Then

$$H=\sum_{m,n,k=0}^{\infty}\frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1-1)_m(c_2)_n(c_3)_k}\,\frac{B_{p,\nu}(b_1+m+k,b_2+m+n)}{B(b_1,b_2)}\,\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$

$$=\sum_{m,n,k=0}^{\infty}\frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k}\,\frac{B_{p,\nu}(b_1+m+k,b_2+m+n)}{B(b_1,b_2)}\left(1+\frac{m}{c_1-1}\right)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$

$$= H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)$$
$$+\frac{x}{c_1-1}\sum_{m=1}^{\infty}\sum_{n,k=0}^{\infty}\frac{(b_1+b_2)_{2m+n+k}(b_3)_{n+k}}{(c_1)_m(c_2)_n(c_3)_k}\,\frac{B_{p,\nu}(b_1+m+k,b_2+m+n)}{B(b_1,b_2)}\,\frac{x^{m-1}}{(m-1)!}\frac{y^n}{n!}\frac{z^k}{k!}.$$

Putting $m\to m+1$ in the above sum, we obtain

$$\frac{x}{c_1-1}\sum_{m,n,k=0}^{\infty}\frac{(b_1+b_2)_{2m+2+n+k}(b_3)_{n+k}}{(c_1)_{m+1}(c_2)_n(c_3)_k}\,\frac{B_{p,\nu}(b_1+1+m+k,b_2+1+m+n)}{B(b_1,b_2)}\,\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$

$$=\frac{x(b_1+b_2)_2}{c_1(c_1-1)}\sum_{m,n,k=0}^{\infty}\frac{(b_1+b_2+2)_{2m+n+k}(b_3)_{n+k}}{(c_1+1)_m(c_2)_n(c_3)_k}\,\frac{B_{p,\nu}(b_1+1+m+k,b_2+1+m+n)}{B(b_1,b_2)}\,\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}.$$

Using (4.2), we find that this last sum becomes

$$\frac{xb_1b_2}{c_1(c_1-1)}\sum_{m,n,k=0}^{\infty}\frac{(b_1+b_2+2)_{2m+n+k}(b_3)_{n+k}}{(c_1+1)_m(c_2)_n(c_3)_k}\,\frac{B_{p,\nu}(b_1+1+m+k,b_2+1+m+n)}{B(b_1+1,b_2+1)}\,\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$

$$=\frac{xb_1b_2}{c_1(c_1-1)}H_{B,p,\nu}(b+1,b_2+1,b_3;c_1+1,c_2,c_3;x,y,z).$$

This then yields the recurrence relation (in c_1) given by

$$H_{B,p,\nu}(b_1,b_2,b_3;c_1-1,c_2,c_3;x,y,z)=$$
$$H_{B,p,\nu}(b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)+\frac{xb_1b_2}{c_1(c_1-1)}H_{B,p,\nu}(b_1+1,b_2+1,b_3;c_1+1,c_2,c_3;x,y,z).$$

Replacement of c_1 by c_1+1 then yields the result stated in (6.6).

7. Concluding remarks

In this paper, we have introduced the (p,v)-extended Srivastava triple hypergeometric function given by $H_{B,p,v}(\cdot)$ in (2.1). We have given some integral representations of this function that involve the modified Bessel function of the second kind and Exton's triple hypergeometric function X_4. We have also established some properties of the function $H_{B,p,v}(\cdot)$, namely the Mellin transform, a differential formula, a bounded inequality and some recursion relations.

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