

The effect of pore shape on the Poisson ratio of porous materials

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Abstract

A brief review is given of the effect of porosity on the Poisson ratio of a porous material. In contrast to elastic moduli such as K , G , or E , which always decrease with the addition of pores into a matrix, the Poisson ratio ν may increase, decrease, or remain the same, depending on the shape of the pores, and on the Poisson ratio of the matrix phase, ν_o . In general, for a given pore shape, there is a unique critical Poisson ratio, ν_c , such that the addition of pores into the matrix will cause the Poisson ratio to increase if $\nu_o < \nu_c$, decrease if $\nu_o > \nu_c$, and remain unchanged if $\nu_o = \nu_c$. The critical Poisson ratio for spherical pores is 0.2, for prolate spheroidal pores is close to 0.2, and tends toward zero for thin cracks. For two-dimensional materials, $\nu_c = 1/3$ for circular pores, 0.306 for squares, 0.227 for equilateral triangles, and again approaches 0 for thin cracks. The presence of a “trapped” fluid in the pore space tends to cause ν_c to increase, and for the range of parameters that may occur in rocks or concrete, this increase is more pronounced for thin crack-like pores than for equi-dimensional pores. Measurements of the Poisson ratio therefore may allow insight into pore geometry and pore fluid. If the matrix phase is strongly auxetic, small amounts of porosity will generally not cause the Poisson ratio to become positive.

Keywords

Porous materials, Poisson ratio, effective medium theory, auxetic materials

1. Introduction

The problem of estimating the effective elastic moduli of a two-phase material is one of the most important and well-studied topics in the mechanics of materials. The specific case of porous materials is of great relevance to the study of geological media (Mavko et al. [1]), ceramics (Rice [2]), and cementitious materials (Lutz et al. [3], Mehta et al. [4]). An increase in porosity will of course cause the elastic moduli, such as K , G , and E , to decrease, at a rate that mainly depends on the pore geometry. But the influence of porosity on the effective Poisson ratio is more subtle, and in fact the Poisson ratio may increase or decrease with the addition of porosity, depending on the pore geometry and the Poisson ratio of the matrix phase.

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The topic of the effect of porosity on the Poisson ratio has received a fair amount of study over the past three decades. Among the papers that are devoted to this topic are Zimmerman [5], Dunn et al. [6], Brantut et al. [7], and Uhlířová et al. [8]. The present paper aims to present a brief review of the effect of porosity on the Poisson ratio, with a focus on discussing the topic within a unified framework. In addition to reviewing some of the main known results, several new results will be presented, particularly with regard to two-dimensional materials, materials having an auxetic matrix phase, and materials containing “trapped” fluids in their pore space.

2. General theory of the effective Poisson ratio

This paper will consider the situation in which the pores in a given porous material each have the same shape. In order to focus only on macroscopically isotropic materials, it will be further assumed that the orientation of the pores is random. To fully investigate the effect that these pores have on the macroscopic elastic moduli, and in particular on the Poisson ratio, the porosity will be allowed to vary.

It is obvious and “trivial” (although not easy to prove; see Gol’dshtein et al. [9]) that, for a given pore shape, both the bulk (K) and shear (G) moduli will be monotonically decreasing functions of porosity. The same will of course then be true for the other elastic moduli, such as Young’s modulus (E), and the Lamé modulus (λ). Since $3K$ and $2G$ are the eigenvalues of the elasticity tensor (Gurtin [10]), K and G play a special role, particularly with regard to upper and lower bounds and effective medium theories, and so the following discussion will generally refer to K and G , rather than to E or λ .

For small values of the porosity, the effective elastic moduli of a macroscopically isotropic material containing a dilute dispersion of randomly oriented pores of a given shape are given by (David et al. [11]):

$$\frac{K}{K_o} \approx \frac{1}{1 + P\phi} \approx 1 - P\phi \quad (1)$$

$$\frac{G}{G_o} \approx \frac{1}{1 + Q\phi} \approx 1 - Q\phi \quad (2)$$

where ϕ is the porosity, subscript o denotes the non-porous matrix material, and P and Q are the bulk and shear pore compliances, respectively. As pointed out by Kachanov et al. [12], the expression $1/(1 + P\phi)$ in some cases has a greater range of validity than the linearized form $1 - P\phi$, and similarly for the shear modulus, but for the present purposes the latter forms are more convenient. In general, the two pore compliance factors depend on pore shape, and on the Poisson ratio of the matrix material. For ellipsoidal or spheroidal pores, explicit expressions for P and Q are known, and are related to certain contractions of the Eshelby–Wu tensor (Wu [13]). Equations (1) and (2) are fully equivalent to the H tensor formalism (Kachanov et al. [14]; Sevostianov et al. [15]). Expressions for P and Q can be found in numerous books and papers (Wu [13]; Kachanov et al. [14]; Mavko et al. [1]; David et al. [16]), and so will not be repeated here, except for some simple special cases.

It follows from equations (1) and (2), and the identity $\nu = (3K - 2G)/(6K + 2G)$, that for small porosities, the effective Poisson ratio is given by

$$\nu \approx \nu_o - \frac{(1 + \nu_o)(1 - 2\nu_o)}{3} [P(\nu_o) - Q(\nu_o)]\phi \quad (3)$$

As the pre-factor $(1 + \nu_o)(1 - 2\nu_o)/3$ is inherently positive, the behavior of Poisson’s ratio therefore depends on the relative magnitudes of P and Q . For spheroidal pores, it is always the case that $P > Q$ as ν_o approaches 0.5, and $P < Q$ as ν_o approaches -1 . Moreover, both P and Q are monotonic functions of ν_o . Hence, there is always exactly one critical value of ν_o , which can be denoted by ν_c , for which $P = Q$. Therefore: if $\nu_o < \nu_c$, the effective Poisson ratio will increase with porosity; if $\nu_o > \nu_c$, the effective Poisson ratio will decrease with porosity; and if $\nu_o = \nu_c$, the effective Poisson ratio will remain unchanged by the addition of pores into the matrix.

The above general statements are known to be true for ellipsoidal or spheroidal pores. For three-dimensional pores of other shapes, exact analytical expressions for P and Q are not known. However, the numerical results of Roberts et al. [17] and Uhlířová et al. [8] for materials containing concave pores

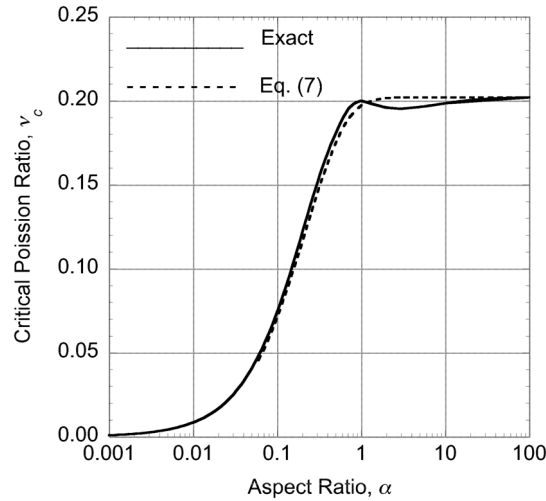


Figure 1. The critical Poisson ratio for spheroidal pores, over the entire range of aspect ratios, from thin cracks to needle-like cylinders (after Dunn et al. [6]). The exact results are plotted as a solid line; the approximate expression given by equation (7) is plotted as a dashed line.

formed by the interstices between contacting solid spheres, also indicate the existence of a single “critical” value of the matrix Poisson ratio.

3. Critical value of the Poisson ratio for (dry) materials containing spheroidal pores

The expressions for P and Q for the general case of ellipsoidal or spheroidal pores are quite cumbersome (Kachanov et al. [14]; Mavko et al. [1]; David et al. [11]), and in general the critical point ν_c at which $P = Q$ can only be computed numerically. However, for the three special cases of spheres, cylinders, and thin cracks, the expressions for P and Q are relatively simple, and ν_c can be found by simple algebra.

For spherical pores, expressions for P and Q were first derived by Dewey [18], although they can also be found from the general expressions for spheroids, by taking the limit as the aspect ratio goes to 1 (Berryman [19]). The critical Poisson ratio, is then found by setting $P = Q$:

$$P = \frac{3(1 - \nu_o)}{2(1 - 2\nu_o)}, \quad Q = \frac{15(1 - \nu_o)}{7 - 5\nu_o} \quad \rightarrow \quad \nu_c = 0.2. \quad (4)$$

For cylindrical pores, P and Q can again be found from the general expressions for spheroids, by taking the limit as the aspect ratio goes to infinity (Berryman [19]):

$$P = \frac{5 - 4\nu_o}{3(1 - 2\nu_o)}, \quad Q = \frac{8(5 - 3\nu_o)}{15} \quad \rightarrow \quad \nu_c = \frac{7 - \sqrt{29}}{8} \approx 0.202. \quad (5)$$

For thin crack-like pores, modeled as thin oblate spheroids of vanishingly small aspect ratio α , P and Q are essentially given by the results of Sneddon [20] and Segedin [21], respectively, and can also be found from the general expressions for spheroids, by extracting the singular term in each expression as the aspect ratio goes to zero (Berryman [19]):

$$P = \frac{4(1 - \nu_o^2)}{3\pi\alpha(1 - 2\nu_o)}, \quad Q = \frac{8(1 - \nu_o)(5 - \nu_o)}{15\pi\alpha(2 - \nu_o)} \quad \rightarrow \quad \nu_c = 0. \quad (6)$$

Brantut et al. [7] computed the first three non-zero terms in the Taylor series for ν_c as a function of α , and found that the leading non-zero term is $\nu_c \approx 0.861\alpha$.

The critical Poisson ratio ν_c is plotted in Figure 1 for spheroidal pores over the entire range of aspect ratios, from thin oblate cracks to needle-like prolate cylinders (see also Dunn et al. [6]). The variation of ν_c with α in the oblate range is somewhat unremarkable, as the curve varies smoothly and monotonically from 0, as $\alpha \rightarrow 0$, to 0.2, as $\alpha \rightarrow 1$. However, the behavior in the prolate range is non-monotonic, and certainly would not be intuitively expected. The critical Poisson ratio has a *local* maximum at $\alpha = 1$, where $\nu_c = 0.2$. The curve then dips down to about 0.195 when $\alpha = 3$, rises gradually above 0.2 when α reaches about 10, and finally reaches its maximum value of 0.202 as α becomes infinite.

David et al. [16] found that this entire curve could be approximated reasonably well by the simple function $\nu_c \approx 0.2(1 - e^{-5\alpha})$. Although this expression is not asymptotically exact in any limit, it provides a surprisingly good fit over the entire range of aspect ratios. A different approximation, which reduces to the correct asymptotic result $\nu_c \approx 0.861\alpha$ for small aspect ratios, and approaches the correct limit of 0.202 for large aspect ratios, is the following slight variation of the expression originally proposed by David and Zimmerman (see Figure 1):

$$\nu_c \approx 0.202(1 - e^{-4.26\alpha}). \quad (7)$$

Needless to say, neither of these approximate expressions is capable of capturing the subtle non-monotonic behavior that occurs in the prolate spheroidal range.

4. Evolution of the effective Poisson ratio with increasing porosity

As shown by equation (3), small amounts of porosity will cause the effective Poisson ratio to move toward the critical Poisson ratio at a rate that is linear in the porosity. However, predicting the rate at which the effective Poisson ratio approaches ν_c as the porosity increases further would require the use of an effective medium theorem to extend relations such as equations (1) and (2) to finite porosities. Two of the more widely used such effective medium theories, both of which have some rational basis, and which have been shown to be reasonably accurate, are the differential effective medium scheme (McLaughlin [22]; Norris [23]; Zimmerman [24]) and the Mori–Tanaka scheme (Mori et al. [25]; Benveniste [26]).

According to the differential scheme, pores are introduced into the matrix incrementally, and the new effective moduli are computed at each step using the dilute-concentration approximations given by equations (1) and (2). In the limit, as the incremental addition of new porosity becomes infinitesimal, this thought experiment leads to a pair of coupled ordinary differential equations for K and G :

$$\frac{(1 - \phi) dK}{K d\phi} = -P(\nu), \quad \frac{(1 - \phi) dG}{G d\phi} = -Q(\nu), \quad (8)$$

which must be solved subject to the initial conditions $K(\phi = 0) = K_o$ and $G(\phi = 0) = G_o$. In these two equations, $P(\nu)$ and $Q(\nu)$ must be evaluated at the local value of ν that varies with porosity, and not at the initial value ν_o .

From the identity $\nu = (3K - 2G)/(6K + 2G)$, the following ordinary differential equation for the evolution of Poisson's ratio with porosity can be derived:

$$(1 - \phi) \frac{d\nu}{d\phi} = \frac{(1 + \nu)(1 - 2\nu)}{3} [Q(\nu) - P(\nu)]. \quad (9)$$

In general, $P(\nu)$ will contain a factor $(1 - 2\nu)$ in its denominator, whereas $Q(\nu)$ will not contain an analogous factor $(1 + \nu)$; see, for example, equations (4) to (6). Hence, the right-hand side of equation (9) will vanish only at $\nu = -1$ and at $\nu = \nu_c$, the latter of which is defined as the unique point at which $P(\nu) = Q(\nu)$. As mentioned above, it is known, at least for the family of spheroidal pores, that $P(\nu) < Q(\nu)$ as ν approaches -1 . Hence, $\nu = -1$ will always be an *unstable* critical point of equation (9), and so the effective Poisson ratio will never converge toward -1 as the porosity increases. On the other hand, since $P(\nu) > Q(\nu)$ for $\nu > \nu_c$, and $P(\nu) < Q(\nu)$ for $\nu < \nu_c$, the value $\nu = \nu_c$ will always be a *stable* critical point of differential equation (9).

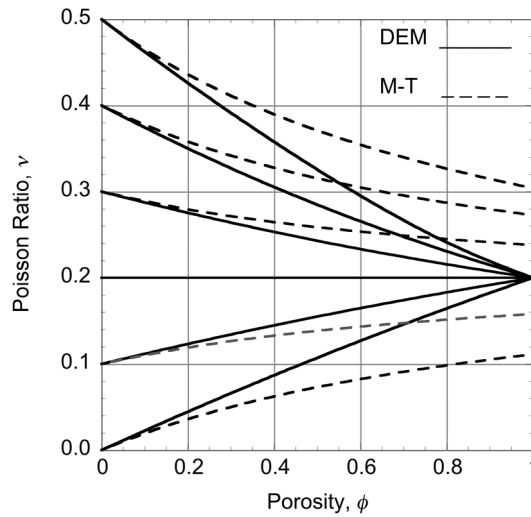


Figure 2. Predicted evolution of the effective Poisson ratio for a material containing spherical pores, as a function of porosity, according to the differential and Mori–Tanaka effective medium theories.

Specifically, in the neighborhood of $\nu = \nu_c$, the term $Q(\nu) - P(\nu)$ will be of the form $-c(\nu - \nu_c)$, where c is a positive constant. Setting $t = -\ln(1 - \phi)$, equation (9) takes the following form near the critical point:

$$\frac{d\nu}{dt} = -c(\nu - \nu_c), \quad \text{where } c > 0. \quad (10)$$

In analogy with the theory of first-order dynamical systems, ν_c is a stable critical point (Strogatz [27]), and consequently, ν will approach ν_c as $t \rightarrow \infty$, which is to say, ν will approach ν_c as $\phi \rightarrow 1$. Of course, it is not reasonable to expect any effective medium theory to be very accurate in this high-porosity limit, if only because it is difficult to imagine the porosity approaching 1 if the pore space consists of isolated pores of the same shape. Nevertheless, the above result does indicate the expected trend.

For the specific case of a material containing spherical pores, equation (9) takes the form (David et al. [16]):

$$(1 - \phi) \frac{d\nu}{d\phi} = \frac{3(1 + \nu)(1 - \nu)(1 - 5\nu)}{2(7 - 5\nu)}. \quad (11)$$

This equation can be integrated analytically to give an equation for the porosity as a function of the effective Poisson ratio (Zimmerman [28]; David et al. [16]), but the resulting expression is complicated and not very informative, and will not be repeated here. The results are more instructive in graphical form (Figure 2). Roughly speaking, when the Poisson ratio of the matrix is non-negative, the Poisson ratio of the porous material varies, more or less linearly, from the matrix value ν_o to the critical value ν_c , as the porosity increases from 0 to 1.

Since $\nu_c = 0.2$ for spherical pores, the constant c that appears in equation (10) can be shown to equal 1.2. Approximating equation (11) by equation (10), and integrating, leads to the following expression, $\nu = 0.2 + (\nu_o - 0.2)(1 - \phi)^{1.2}$. This approximate expression matches the exact solution to equation (11) quite closely when $\nu_o \geq 0$, but is much easier to interpret.

For materials containing prolate spheroidal pores, or oblate spheroidal pores having aspect ratios not too close to 0, the Poisson ratio trajectories predicted by the differential scheme are qualitatively similar to the case of spherical pores, in that ν approaches ν_o at a nearly linear rate, and reaches ν_c as $\phi \rightarrow 1$ (see David et al. [16]). However, as pointed out by Kachanov et al. [12], the porosity is not a convenient parameter to use when discussing the effective elastic parameters for materials containing cracks of very small aspect ratio. This can be seen by the following simple considerations. The volume of an

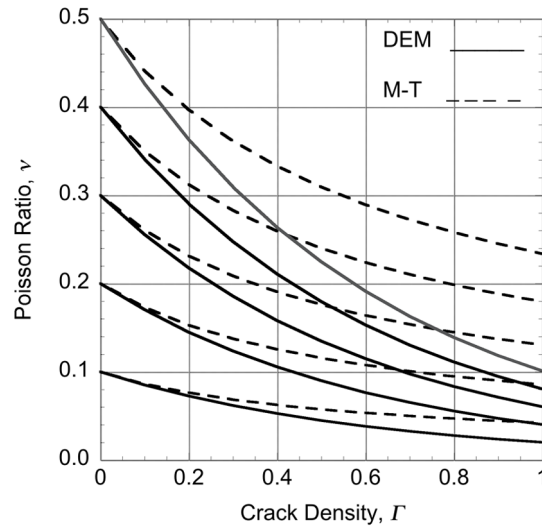


Figure 3. Predicted evolution of the effective Poisson ratio for a material containing thin circular cracks, as a function of the crack density parameter, according to the differential and Mori–Tanaka effective medium theories.

oblate spheroidal pore having semi-major axes a , and semi-minor axis $a\alpha$, is $4\pi a^3\alpha/3$. It follows that the porosity is given by $\phi = 4n\pi a^3\alpha/3$, where n is the number of pores per unit bulk volume. But for thin oblate pores, the pore compliances P and Q each contain a factor of α in the denominator, as seen in equation (6). The products $P\phi$ and $Q\phi$ will therefore each contain a factor na^3 , as well as dimensionless factors that depend only on the Poisson ratio, but will not depend explicitly on α or ϕ . Hence, the natural parameter to use to quantify the effect of cracks on the elastic moduli is actually the “crack density parameter”, defined as $\Gamma = na^3$ (Walsh [29]; Zimmerman [28]). Expressions for the elastic moduli in terms of Γ retain their validity even in the limit as $\alpha \rightarrow 0$.

The evolution of the effective Poisson ratio of a cracked body, as a function of the crack density parameter, is shown in Figure 3. It should be noted that, unlike the porosity, the crack density is not restricted to being less than 1. Nevertheless, values of $\Gamma > 1$ are probably of limited practical interest. The percolation threshold of disk-like cracks is 0.244 (Ebigbo et al. [30]), and the pore space of cracked materials having crack densities much greater than this will generally be highly interconnected, and will not consist of isolated cracks.

Another widely used effective medium theory is the Mori–Tanaka scheme (Benveniste [26]). The key idea behind this scheme is based on the fact that if a porous body is subjected to a uniform stress σ on its outer boundary, the mean stress in the matrix phase will be $\sigma/(1 - \phi)$. The usual energy considerations that are invoked in effective medium calculations then lead to the following expressions for the effective bulk and shear moduli:

$$\frac{K}{K_o} = \frac{1 - \phi}{(1 - \phi) + P(\nu_o)\phi}, \quad \frac{G}{G_o} = \frac{1 - \phi}{(1 - \phi) + Q(\nu_o)\phi}. \quad (12)$$

Explicit expressions for the predicted Poisson ratio, as a function of porosity, have been given by Dunn et al. [6] and Uhlřřova et al. [8]. The behavior of ν in the general case can easily be understood by noting that since ν is a unique function of the ratio K/G , the Poisson ratio will not change with porosity if and only if G and K change by the same relative amounts as ϕ increases. This only occurs if $P(\nu_o) = Q(\nu_o)$, which defines the critical value, ν_c . Moreover, since ν is an increasing function of K/G , it follows from equation (12) that ν increases with ϕ if $P(\nu_o) < Q(\nu_o)$, decreases with ϕ if $P(\nu_o) > Q(\nu_o)$, and does not vary with ϕ if $P(\nu_o) = Q(\nu_o)$. This is of course qualitatively the same behavior as is predicted by the differential scheme. However, according to the Mori–Tanaka scheme, the effective Poisson ratio does not reach ν_c as $\phi \rightarrow 1$. Instead (David et al. [16]),

$$\nu(\phi \rightarrow 1) = \frac{Q(\nu_o)(1 + \nu_o) - P(\nu_o)(1 - 2\nu_o)}{2Q(\nu_o)(1 + \nu_o) + P(\nu_o)(1 - 2\nu_o)}. \quad (13)$$

This expression is not easy to interpret, and does not readily show that the effective Poisson ratio does not reach the critical value. For the three special cases of spheres, cylinders, and thin cracks, the asymptotic values of the effective Poisson ratio, as predicted by the Mori–Tanaka effective medium scheme, are (for spheres: Zimmerman [5]; for cylinders and cracks: Dunn et al. [6]):

$$\text{spheres : } \nu(\phi \rightarrow 1) = \frac{1 + 5\nu_o}{9 + 5\nu_o}, \quad (14)$$

$$\text{cylinders : } \nu(\phi \rightarrow 1) = \frac{5 + 12\nu_o + 8\nu_o^2}{35 + 4\nu_o - 16\nu_o^2}, \quad (15)$$

$$\text{cracks : } \nu(\Gamma \rightarrow \infty) = \frac{\nu_o}{10 - 3\nu_o}. \quad (16)$$

Since all reasonable effective medium theories agree to first-order in porosity, the trajectory of the Poisson ratio as predicted by the Mori–Tanaka scheme (see Figures 2 and 3) initially follows the same slope as for the differential scheme, i.e., as given by equation (3). However, the predicted rate of change becomes less severe, as compared with the differential scheme, as the porosity (or crack density) increases, and the Poisson ratio never reaches the critical value, ν_c .

5. Influence of fluid saturation on the effective Poisson ratio

The previous discussion focused on dry porous materials that contain no pore fluids in their pore space. Particularly with regard to porous rocks, there is great interest in understanding the effect of pore shape, porosity, and *pore fluid compressibility*, on Poisson's ratio. In seismic exploration, the ratio of compressional wave speed to shear wave speed is often used to discriminate between different rock types, and to infer information regarding pore fluid saturation (Brantut et al. [7]; Mavko et al. [1]). The ratio V_p/V_s is essentially a measure of the effective Poisson ratio, since this ratio is exactly related, in a one-to-one and monotonic manner, to the Poisson ratio (Jaeger et al. [31]), as follows:

$$\frac{V_p}{V_s} = \sqrt{\frac{2(1 - \nu)}{1 - 2\nu}}. \quad (17)$$

Gassmann [32] derived an exact (albeit implicit) expression for the “undrained” bulk modulus, K_u , of a fluid-saturated rock in terms of the drained/dry bulk modulus, K_d , the bulk modulus of the fluid, K_f , the bulk modulus of the solid phase, K_o , and the porosity, ϕ :

$$\frac{K_u}{K_o - K_u} = \frac{K_d}{K_o - K_d} + \frac{1}{\phi} \frac{K_f}{K_o - K_f}. \quad (18)$$

This relation is valid for any value of the porosity, regardless of pore shape, although its derivation implicitly assumes that the pores are all interconnected. (The seeming contradiction of assuming a completely interconnected pore space, but using P and Q factors derived for discrete, isolated pores, is ubiquitous in rock physics, and its validity will be assumed in the present discussion.) Furthermore, according to Gassmann's theory, pore fluids will have no effect on the shear modulus, and so the shear modulus of the rock under fluid-saturated conditions will be the same as under dry conditions. This result is also exact for a material whose pores are all interconnected.

In the limit of small values of the porosity, and using the formalism of equations (1) and (2) for the dry effective moduli, Gassmann's results can be expressed as

$$K(\phi) \approx K_o \left[1 - \frac{P(\nu_o)}{1 + \xi P(\nu_o)} \phi \right], \quad G(\phi) \approx G_o [1 - Q(\nu_o) \phi], \quad \text{where } \xi = \frac{K_f}{K_o - K_f}. \quad (19)$$

Hence, the mathematical consequence of having pore fluid “trapped” in the pore space is to replace $P(\nu_o)$ with $P(\nu_o)/[1 + \xi P(\nu_o)]$. In a qualitative sense, most of the previous discussion continues to apply, in that there is always a unique critical value of the Poisson ratio, which initially serves as an “attractor” for the effective Poisson ratio. For a given pore shape, this critical Poisson ratio will always be greater than for the case of dry pores. However, as porosity increases, the effective Poisson ratio veers away from the critical Poisson ratio, and in fact approaches 0.5, which is the Poisson ratio of the pore fluid; see Brantut et al. [7] for a detailed discussion of this phenomenon.

The critical Poisson ratio for a fluid-saturated rock is therefore found by solving the equation $P(\nu_o)/[1 + \xi P(\nu_o)] = Q(\nu_o)$. For a material containing spherical pores, the following exact expression can be obtained:

$$\nu_c(\text{undrained}) = 0.2 \left(\frac{1 + 5\xi}{1 + \xi} \right). \quad (20)$$

For fluid-saturated rocks, the parameter ξ is usually small, in which case this expression reduces to the expression found by Brantut et al. [7], which was a first-order expansion in ξ :

$$\nu_c(\text{undrained}) = 0.2 + 0.8\xi. \quad (21)$$

However, ξ is not necessarily small for some soft biological materials, and so the more general expression (20) may be useful for those cases.

For materials containing cracks or needle-like pores, and arbitrary values of the parameter ξ , the solution to the equation $P(\nu_o)/[1 + \xi P(\nu_o)] = Q(\nu_o)$ can only be obtained as a very awkward quadratic equation that is not easy to interpret. For these pore shapes, the following approximate expressions that are first-order in ξ are more instructive (Brantut et al. [7]):

$$\text{cylinders : } \nu_c(\text{undrained}) = 0.202 + 0.760\xi, \quad (22)$$

$$\text{cracks : } \nu_c(\text{undrained}) = 0.157\xi/\alpha. \quad (23)$$

This latter expression assumes that, although both ξ and α are small, the condition $\alpha \ll \xi$, which is the situation of most relevance to fluid-saturated rocks, is satisfied. Note that under the condition $\alpha \ll \xi$, the critical value of Poisson’s ratio for a fluid-saturated cracked material will be quite large, and may in fact violate the constraint $\nu \leq 0.5$. This merely indicates that the *initial* slope of the curve of ν as a function of ϕ will be large; in all cases, the asymptotic value of ν at high porosities will be 0.5, rather than ν_c .

6. Effective Poisson ratio of two-dimensional materials

A similar analysis can be given for “two-dimensional” materials. Consider the case of plane strain, in which the displacement is zero in one direction, which can be denoted as the z -direction. The material is then assumed to be permeated with long prismatic pores that are aligned parallel to the z -axis, so that each plane normal to the z -axis has exactly the same microstructure. These pores may have any shape in the x - y plane. In the subsequent discussion, the term “pore shape” will refer to the shape of the pore in the x - y plane.

Before discussing the behavior of the Poisson ratio in the two-dimensional case, it is worthwhile to review the relationships between the “2D” and “3D” elastic moduli. Hooke’s law for plane strain takes the form (Jaeger et al. [31])

$$\varepsilon_{xx} = \frac{(1 - \nu^2)}{E} \left[\tau_{xx} - \frac{\nu}{1 - \nu} \tau_{yy} \right], \varepsilon_{xy} = \frac{(1 + \nu)}{E} \tau_{xy} = \frac{1}{2G} \tau_{xy}, \text{ etc.} \quad (24)$$

If the “2D” Young’s modulus is defined as $E_{2D} = E/(1 - \nu^2)$, the “2D” shear modulus is defined as $G_{2D} = G$, and the “2D” Poisson ratio is defined as $\nu_{2D} = \nu/(1 - \nu)$, the plane strain version of Hooke’s law can be written in a form that resembles the 3D version, i.e.,

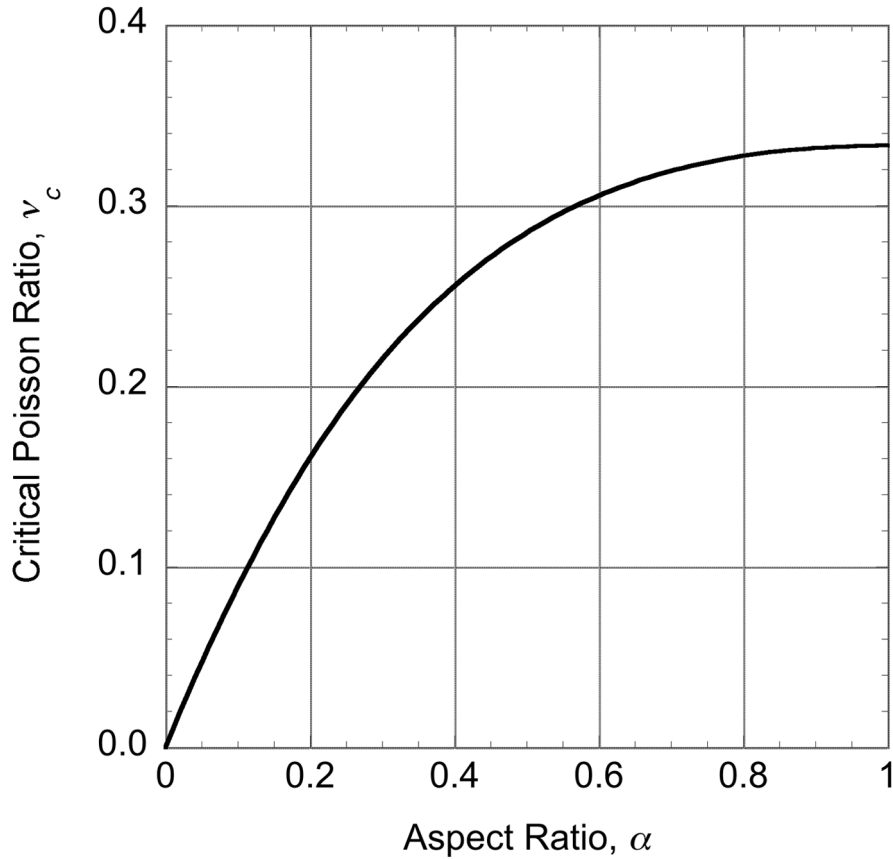


Figure 4. The critical Poisson ratio for two-dimensional elliptical pores, over the entire range of aspect ratios, from thin cracks to circles. Note that in contrast to Figure 1, in this graph the aspect ratio axis is linear, not logarithmic.

$$\varepsilon_{xx} = \frac{1}{E_{2D}} [\tau_{xx} - \nu_{2D}\tau_{yy}], \varepsilon_{xy} = \frac{1}{2G_{2D}} \tau_{xy}, \text{ etc.} \quad (25)$$

The advantage of writing the equations in this form is that the 2D Young's modulus and 2D Poisson ratio now have the same geometrical interpretation as in 3D, i.e., the Poisson ratio is the negative of the ratio of the transverse strain to axial strain under uniaxial stress, etc. Note that the 2D "areal bulk modulus" will be given by $K_{2D} = E_{2D}/2(1 - \nu_{2D})$.

The case of a 2D material permeated with elliptical pores of aspect ratio α has been treated in detail by Thorpe et al. [33], and Kachanov et al. [12]. After correcting the incorrect minus sign in the expression for Q given in equation (18) of Thorpe et al. [33], the 2D versions of P and Q can be written as





$$P = \frac{(1 + \alpha^2)}{\alpha(1 - \nu_{2Do})}, \quad Q = \frac{(1 + \alpha)^2}{\alpha(1 + \nu_{2Do})}. \quad (26)$$

The critical Poisson ratio, found as usual by setting $P = Q$, is given by (Figure 4)

$$\nu_{2Dc} = \frac{\alpha}{1 + \alpha + \alpha^2}. \quad (27)$$

The critical Poisson ratio goes to zero for infinitely thin cracks, as in the 3D case. For thin cracks of "finite" aspect ratio, $\nu_{2Dc} \approx \alpha$, which is not very different to the asymptote $\nu_c \approx 0.861\alpha$ in 3D. For circular holes, $\nu_{2Dc} = 1/3$. Unlike the 3D case, the expression for the critical Poisson ratio is sufficiently simple in form so as to render an approximate expression unnecessary.

Table 1. Bulk (P) and shear (Q) compliance factors for several two-dimensional pores, along with the critical Poisson ratio.

Pore shape	P	Q	ν_{2Dc}
	$\frac{2}{1 - \nu_{2D_0}}$	$\frac{4}{1 + \nu_{2D_0}}$	0.333
	$\frac{2.414}{1 - \nu_{2D_0}}$	$\frac{4.547}{1 + \nu_{2D_0}}$	0.306
	$\frac{3.25}{1 - \nu_{2D_0}}$	$\frac{5.186}{1 + \nu_{2D_0}}$	0.227
	$\frac{6}{1 - \nu_{2D_0}}$	$\frac{8}{1 + \nu_{2D_0}}$	0.143

Whereas analytical expressions for three-dimensional pores are available only for ellipsoids and spheroids, two-dimensional pores of essentially any shape can be analyzed using conformal mapping and the Muskhelishvili–Kolosov displacement potentials. These methods have been used to study polygonal pores (Jasiuk et al. [34]; Kachanov et al. [12]; Jasiuk [35]) and cusp-shaped pores that resemble the interstices between circular “grains” (Zimmerman [36]; Kachanov et al. [12]). Table 1 shows some values of P and Q , along with the computed critical Poisson ratio, for a few different two-dimensional pore shapes, such as circles, taken from Thorpe et al. [33], and squares, triangles, and one cusp-shaped pore, taken from Ekneligoda et al. [37, 38]. The expressions for the square and the equilateral triangle were obtained by sequentially including additional terms in the Schwarz–Christoffel mapping function, and extrapolating the results to $1/N^2 \rightarrow 0$, where N is the number of terms in the mapping function. The results of Ekneligoda et al. [37, 38], which were expressed in terms of the 3D elastic moduli, have been converted to the 2D moduli in Table 1. When expressed in terms of the 2D moduli, these results apply equally well to plane strain or plane stress.

7. Porous materials having an auxetic matrix phase

Auxetic materials are isotropic materials that have a negative Poisson ratio (Yang et al. [39]). Most known auxetic materials are highly porous materials with foam-like structures; non-porous auxetic materials are very rare (Dagdelen et al. [40]). None of the theories discussed above, which were based on the conceptual model of a solid phase permeated with discrete, non-intersecting pores of a given shape, predict that a material with a non-auxetic matrix phase will become auxetic due to the addition of pores, even at very high porosities. In fact, auxetic foam-like materials require special structural features, such as re-entrant angles, etc., (Uhlířová et al. [8]), that are very different geometrically to the spheroidal shapes discussed in previous sections.

Despite the lack of solid auxetic materials to serve as a matrix phase, it may nevertheless be of interest to examine the behavior of porous materials that have an auxetic matrix phase, for the following reason. As pointed out by Uhlířová et al. [8], such materials could in principle be created by taking a highly porous auxetic foam, and introducing pores into those materials that have a length scale that is much larger than the characteristic length scale of the foam.

The “general theory” discussed above in Sections 2 to 4 continues to hold for such materials, particularly if one accepts that the differential effective medium scheme and the Mori–Tanaka scheme provide reasonably accurate predictions of the effective moduli. The critical Poisson ratio ν_c , defined by the

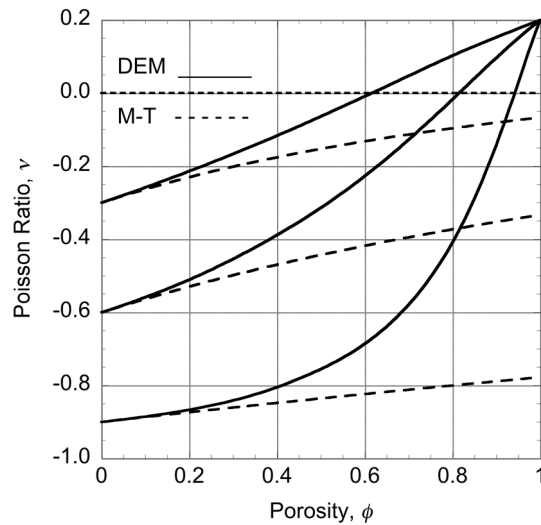


Figure 5. Predicted evolution of the effective Poisson ratio for an initially auxetic material containing spherical pores, as a function of porosity, according to the differential and Mori–Tanaka effective medium theories.

condition that $P(\nu_o) = Q(\nu_o)$, will again be expected to be non-negative, so that small amounts of porosity will cause an initially auxetic material to become “less auxetic”, i.e., its Poisson ratio will increase. However, due to the factor $1 + \nu_o$ that appears in equation (3), the rate of increase of ν with porosity will be low for highly auxetic materials having initial Poisson ratios close to -1 . Although the differential effective medium scheme predicts that ν will always eventually reach ν_c , this might not occur for porosity values in the range usually observed in rocks or concrete (see Figure 5). As mentioned previously, the Mori–Tanaka theory predicts a less severe change in ν than does the differential scheme. According to either theory, small to moderate amounts of porosity will not cause an auxetic material to become non-auxetic, unless the matrix phase is only weakly auxetic itself.

8. Summary and conclusions

This paper has presented a review of the influence of pore shape on the effective Poisson ratio of porous materials. The intention has not been to present an exhaustive review of all previous work that has been done on this topic, but rather to focus on discussing this topic within a unified framework. In addition to reviewing known results, several new results have been presented, particularly with regard to the effect of pore fluids, two-dimensional materials, and porous materials having auxetic matrices. The results can be summarized as follows.

- a) Pores of any given shape cause the bulk and shear moduli to decrease toward zero as the porosity increases, but the trend of Poisson’s ratio depends on the pore shape.
- b) For any given pore shape, increasing the porosity will drive Poisson’s ratio toward a critical value ν_c that does not depend on the elastic moduli of the matrix.
- c) For spheroidal pores, the critical value varies from 0 for thin cracks, to 0.2 for spheres, to 0.202 (but not monotonically!) for long needle-like pores.
- d) In 2D plane strain or plane stress, the critical (2D) Poisson ratio is 0.333 for circles, 0.227 for equilateral triangles, and 0 for thin elliptical cracks.
- e) It is not yet known if any pore shapes exist that have values of ν_c that are appreciably higher than those for spheres (in 3D) or circles (in 2D).
- f) If the pore space contains a “trapped” pore fluid, as is the case for rapid processes such as seismic wave propagation, as the porosity increases, Poisson’s ratio first approaches a critical value that

depends on pore shape and fluid compressibility, but then approaches 0.5 as the porosity increases toward 1.

- g) If the matrix phase is auxetic (i.e., has a negative Poisson ratio), the effective Poisson ratio initially increases toward ν_c at a slow rate, and will generally not become positive for the moderate values of porosity that are typically found in rocks or cementitious materials.


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