

Regularized Diffusion Adaptation via Conjugate Smoothing

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Abstract—The purpose of this work is to develop and study a decentralized strategy for Pareto optimization of an aggregate cost consisting of regularized risks. Each risk is modeled as the expectation of some loss function with unknown probability distribution while the regularizers are assumed deterministic, but are not required to be differentiable or even continuous. The individual, regularized, cost functions are distributed across a strongly-connected network of agents and the Pareto optimal solution is sought by appealing to a multi-agent diffusion strategy. To this end, the regularizers are smoothed by means of infimal convolution and it is shown that the Pareto solution of the approximate, smooth problem can be made arbitrarily close to the solution of the original, non-smooth problem. Performance bounds are established under conditions that are weaker than assumed before in the literature, and hence applicable to a broader class of adaptation and learning problems.

Index Terms—Distributed optimization, diffusion strategy, smoothing, proximal operator, non-smooth regularizer, proximal diffusion, regularized diffusion.

I. INTRODUCTION

The objective of decentralized learning is the solution of global, stochastic optimization problems across networks of agents through localized interactions and without information about the statistical properties of the data. Using streaming data, the resulting strategies are adaptive in nature and able to track drifts in the location of the minimizers due to variations in the statistical properties of the data. Regularization is one useful technique to encourage or enforce structural properties on the sought after minimizer, such as sparsity or constraints. A substantial number of regularizers are inherently non-smooth, while many cost functions are differentiable. This article proposes a *fully-decentralized* and *adaptive* strategy that is able to minimize an aggregate sum of regularized costs. To do so, we fully exploit the structure of the individual objectives as sums of differentiable costs and non-differentiable regularizers.

Notation: Throughout the manuscript, random quantities are denoted in boldface. Matrices are denoted in capital letters while vectors and scalars are denoted in small-case letters. The symbol \leq denotes a regular inequality, while \preceq denotes an *element-wise* inequality. Unless specified otherwise, $\|\cdot\|$ denotes the Euclidean norm. We utilize the notation $f(\mu) = O(g(\mu))$ to state that $\limsup_{\mu \rightarrow 0} \frac{|f(\mu)|}{g(\mu)} < \infty$, while

$f(\mu) = o(g(\mu))$ denotes $\limsup_{\mu \rightarrow 0} \frac{|f(\mu)|}{g(\mu)} = 0$. The relation $f(\mu) = O(1)$ states that $|f(\mu)|$ is bounded independent of μ for small μ , while $f(\delta) = O(1)$ states that $|f(\delta)|$ is bounded independent of δ for small δ .

A. Problem Formulation

We consider a strongly-connected network consisting of N agents. For any two agents k and ℓ , we attach a pair of non-negative coefficients $\{a_{\ell k}, a_{k\ell}\}$ to the edge linking them. The scalar $a_{\ell k}$ is used to scale data moving from agent ℓ to k ; likewise, for $a_{k\ell}$. Strong-connectivity means that it is always possible to find a path in each direction with nonzero scaling weights linking any two agents (either directly if they are neighbors or indirectly through other agents). In addition, at least one agent k in the network possesses a self-loop with $a_{kk} > 0$. This condition ensures that at least one agent in the network has some confidence in its local information. Let \mathcal{N}_k denote the set of neighbors of agent k , i.e., the set of agents to which agent k assigns positive weight. The coefficients $\{a_{\ell k}\}$ are convex combination weights that satisfy

$$a_{\ell k} > 0 \text{ if } \ell \in \mathcal{N}_k, \quad \sum_{\ell \in \mathcal{N}_k} a_{\ell k} = 1, \quad a_{\ell k} = 0 \text{ if } \ell \notin \mathcal{N}_k. \quad (1)$$

If we introduce the combination matrix $A = [a_{\ell k}]$, it then follows from (1) and the strong-connectivity property that A is a left-stochastic primitive matrix. In view of the Perron-Frobenius Theorem [2]–[4], this ensures that A has a single eigenvalue at one while all other eigenvalues are inside the unit circle, so its spectral radius is given by $\rho(A) = 1$. Moreover, if we let p denote the right-eigenvector of A that is associated with the eigenvalue at one, and if we normalize the entries of p to add up to one, then it also holds that all entries of p are strictly positive, i.e.,

$$Ap = p, \quad \mathbb{1}^\top p = 1, \quad p_k > 0 \quad (2)$$

where $\mathbb{1} \triangleq \text{col}\{1, \dots, 1\}$ and the $\{p_k\}$ denote the individual entries of the Perron vector, p .¹

¹If a particular Perron vector p is desired, the combination matrix A can be designed accordingly in a decentralized manner. For example, as long as the communication graph is symmetric, i.e. $k \in \mathcal{N}_\ell \Leftrightarrow \ell \in \mathcal{N}_k$, relation (2) can be verified for the choice:

$$a_{\ell k} = \begin{cases} 0, & \text{if } \ell \notin \mathcal{N}_k, \\ p_\ell, & \text{if } \ell \in \mathcal{N}_k \setminus \ell, \\ 1 - \sum_{m \in \mathcal{N}_k \setminus \ell} a_{mk}, & \text{when } \ell = k. \end{cases} \quad (3)$$

Other design choices and a discussion on the advantages of appropriately designed combination policies can be found in [4, Chapter 14].

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We associate with each agent k a risk function $J_k(w) : \mathbb{R}^M \rightarrow \mathbb{R}$, assumed differentiable. In most adaptation and learning problems, risk functions are expressed as the expectation of loss functions. Hence, we assume that each risk function is of the form $J_k(w) = \mathbb{E} Q(w; \mathbf{x})$, where $Q(\cdot)$ is the loss function and \mathbf{x} denotes random data. The expectation is computed over the distribution of this data (note that, in our notation, we use boldface letters for random quantities and normal letters for deterministic quantities or data realizations). We also associate with agent k a regularization term, $R_k(w) : \mathbb{R}^M \rightarrow \mathbb{R}$, which is a known deterministic function although possibly non-differentiable. Regularization factors of this form can, for example, help induce sparsity properties (such as using ℓ_1 or elastic-net regularizers) [5]–[7].

The objective we are interested in is to devise a fully decentralized strategy to seek the unique minimizer of the following strongly-convex, weighted, aggregate cost, denoted by w° :

$$w^\circ = \arg \min_{w \in \mathbb{R}^M} \sum_{k=1}^N p_k \{J_k(w) + R_k(w)\} \quad (4)$$

The weights $\{p_k\}$ indicate that the resulting minimizer w° can be interpreted as a Pareto solution for the collection of regularized risks $\{J_k(w) + R_k(w)\}$ [4], [8] and will depend on the entries of the Perron eigenvector in a manner specified further below. We are particularly interested in determining this Pareto solution in the *stochastic* setting when the distribution of the data \mathbf{x} is unknown. This means that the risks $J_k(w)$, or their gradient vectors, are also unknown. As such, *approximate* gradient vectors will need to be employed. A common construction in stochastic approximation theory is to employ the following choice at each iteration i :

$$\widehat{\nabla} J_k(w) = \nabla Q_k(w; \mathbf{x}_{k,i}) \quad (5)$$

where $\mathbf{x}_{k,i}$ represents the data that is available (observed) at time i . The difference between the true gradient vector and its approximation is called gradient noise. This noise will seep into the operation of the decentralized algorithm and one main challenge is to show that, despite its presence, the proposed solution is able to approach w° asymptotically. A second challenge we face in constructing an effective decentralized solution is the non-smoothness (non-differentiability) of the regularizers. Motivated by a technique proposed in [9] in the context of *single* agent optimization, we will address this difficulty in the multi-agent case by introducing a smoothed version of the regularizers and then showing that the solution w° can still be recovered under this substitution as the size of the smoothing parameter is reduced. We adopt a general formulation that will be shown to include proximal iterations as a special case.

B. Related Works in the Literature

The literature on decentralized optimization is extensive. Some early strategies include incremental [10], consensus or decentralized gradient descent [11]–[14], and the diffusion algorithm [4], [8], [15]–[17]. When exact gradients are employed, these strategies converge to a small area around the minimizer

of the aggregate cost at a linear rate [8], [14]. Exact convergence requires diminishing step-sizes, resulting in sublinear rates of convergence. A number of more recent works focusing primarily on deterministic optimization, have proposed variations yielding linear rates of convergence pursued either in the primal [18]–[25] or dual domain [26]–[35] where [25], [27], [34] allow for stochastic gradient approximations and [30] considers empirical risk minimization problems for a linear model.

One common method for handling non-differentiable cost functions is the utilization of subgradient recursions, where the ordinary gradient is replaced by subgradients [11]–[13], [27], [28], [34]. Most often, these works assume the subgradients are *bounded*. This condition is not satisfied in many important cases of interest, for example, even when $J_k(w)$ is simply quadratic in w (as happens in mean-square-error designs) or when the $R_k(w)$ are indicator functions used to encode constraints. Variations for specific choices of costs functions are examined in [36]–[39] where only the subgradients of $R_k(\cdot)$ are required to be bounded. The work [40] generalized these conditions to allow for (sub-)gradients that are “affine-Lipschitz”, which holds for many, but not all costs and regularizers of interest, such as indicator functions. For the case when the $R_k(w)$ are chosen as indicator functions in constrained problem formulations, as an alternative to projection based schemes [12], [13], a distributed diffusion strategy based on the use of suitable penalty functions was proposed and studied in [41].

Some other studies pursue distributed solutions by relying instead on the use of proximal iterations (as opposed to subgradient iterations); an accessible survey on the proximal operator and its properties appears in [42]. For example, for purely deterministic costs, distributed proximal strategies are developed in [18], [20]–[22], [43]. Stochastic variations for mean-square error costs with bounded regularizer subgradients are proposed in [44], [45] for single-task problems and in [46] for multi-task environments. A strategy for general stochastic costs with *small*, Lipschitz continuous regularizers is studied in [47].

C. Contributions

The purpose of this work is to propose a general distributed strategy and a line of analysis that is applicable to a wide class of stochastic costs and non-differentiable regularizers. The first step in the solution will involve replacing each non-differentiable component, $R_k(w)$, by a differentiable approximation $R_k^\delta(w)$, parametrized by $\delta > 0$, such that

$$\|w^\circ - w_\delta^\circ\|^2 \leq O(\delta). \quad (6)$$

The accuracy of the approximation is controlled through the smoothing parameter δ . Subsequently, we will solve for the minimizer:

$$w_\delta^\circ = \arg \min_w \sum_{k=1}^N p_k \{J_k(w) + R_k^\delta(w)\} \quad (7)$$

Smoothing non-differentiable costs via infimal convolution [9], [48], [49] is a popular technique in the deterministic optimization literature, and it can be used to motivate some

known algorithms, such as the proximal point algorithm [42]. The technique has been mainly developed for deterministic optimization by *single* stand-alone agents. In this work, we pursue an extension in two non-trivial directions. First, we consider networked agents (rather than a single agent) working together to solve the aggregate optimization problem (4) (or (7)) and, second, the risk functions involved are a combination of stochastic costs defined as the expectations of certain loss functions and deterministic regularizers. Moreover, the probability distribution of the data is assumed unknown and, therefore, the aggregate risks themselves are not known but can only be approximated. The challenge is to devise a distributed strategy that is able to converge to the desired Pareto solution despite these difficulties.

We note that an alternative smoothing procedure by means of adding small stochastic perturbations is considered in [50] and extended to decentralized stochastic optimization in [51], requiring bounded subgradients. In contrast, our focus is on smooth stochastic risks regularized by non-smooth, deterministic risks. Splitting the smooth stochastic part from the non-differentiable deterministic risk, and smoothing only the deterministic risk via a deterministic procedure will allow us to only require looser bounds on both components.

In the next sections we will explain how to construct the smooth approximation, $R_k^\delta(w)$, by appealing to conjugate functions and will show that the distance $\|w^o - w_\delta^o\|$ can be made arbitrarily small for $\delta \rightarrow 0$. We then present an algorithm to solve for the minimizer of (7) in a distributed manner and derive bounds on its performance. The analysis in future sections will rely on the following common assumptions [4], [16], [17]:

Assumption 1 (Lipschitz gradients). *For each k , the gradient $\nabla J_k(\cdot)$ is Lipschitz, namely, there exists $\lambda_U \geq 0$ such that for any $x, y \in \mathbb{R}^M$:*

$$\|\nabla J_k(x) - \nabla J_k(y)\| \leq \lambda_U \|x - y\| \quad (8)$$

Assumption 2 (Strong Convexity). *The weighted aggregate of the differentiable risks is strongly convex, namely, there exists $\lambda_L \geq 0$ such that for any $x, y \in \mathbb{R}^M$:*

$$(x - y)^\top \cdot \sum_{k=1}^N p_k (\nabla_w J_k(x) - \nabla_w J_k(y)) \geq \lambda_L \|x - y\|^2 \quad (9)$$

Assumption 3 (Regularizers). *For each k , $R_k(\cdot)$ is closed convex. In other words, $R_k(\cdot)$ is convex and $\{w \in \text{dom } R_k(\cdot) \mid R_k(w) \leq x\}$ is a closed set for all x .*

II. ALGORITHM FORMULATION

A. Construction of Smooth Approximation

To begin with, following the works [9], [48], we explain how smoothing of the regularizers is performed. Thus, recall that the conjugate function, denoted by $R_k^*(w)$, of a regularizer $R_k(w)$ is defined as

$$R_k^*(w) \triangleq \sup_{u \in \text{dom } R_k} \{w^\top u - R_k(u)\}. \quad (10)$$

A useful property of conjugate functions is that $R_k^*(w)$ is always closed convex regardless of whether $R_k(w)$ is convex or not.

Definition 1 (Proximity function [9]). *A proximity function $d(\cdot)$ for a closed convex set C is a continuous, strongly-convex function with $C \subseteq \text{dom } d(\cdot)$. We center the function so that*

$$\min_{w \in C} d(w) = 0 \quad (11)$$

and

$$\arg \min_{w \in C} d(w) = 0 \quad (12)$$

Furthermore, the proximity function is scaled to satisfy:

$$d(w) \geq \frac{1}{2} \|w\|^2. \quad (13)$$

Existence and uniqueness of the minimum in (12) is guaranteed since $d(w)$ is strongly convex. Relation (13) on the other hand normalizes the strong-convexity constant of $d(w)$ to be equal to one.

Definition 2 (Smooth approximation). *We choose a proximity function over $C = \text{dom } R_k^*(w)$ and define the smooth approximation of $R_k(\cdot)$ as:*

$$\begin{aligned} R_k^\delta(w) &\triangleq \max_{u \in \text{dom } R_k^*} \{w^\top u - R_k^*(u) - \delta \cdot d(u)\} \\ &= (R_k^* + \delta \cdot d)^*(w) \end{aligned} \quad (14)$$

The maximum in (14) is attained for all w since $R_k^*(u) + \delta \cdot d(u)$ is strongly convex. Thus, observe that the smooth approximation for $R_k(w)$, which we are denoting by $R_k^\delta(w)$, is obtained by first perturbing the conjugate function $R_k^*(u)$ by $\delta \cdot d(u)$ and then conjugating the result again. The perturbation makes the sum $R_k^*(u) + \delta \cdot d(u)$ a strongly-convex function. The motivation behind this construction is the fact that the conjugate of a strongly-convex function is *differentiable* everywhere and, therefore, $R_k^\delta(w)$ is differentiable everywhere. This intuition is formalized in the following known theorem [9], preceded by an elementary lemma [52].

Lemma 1 (Conjugate subgradients [52, Theorem 23.5]). *If $G(\cdot)$ is some closed and convex function, the subgradients of $G(\cdot)$ and its conjugate $G^*(\cdot)$ are related as:*

$$v \in \partial G(w) \iff w \in \partial G^*(v) \quad (15)$$

Theorem 1 (Gradient of smooth approximation [9]). *Any $R_k^\delta(w)$ constructed according to (14) is differentiable with gradient vector*

$$\nabla R_k^\delta(w) = \arg \max_{u \in \text{dom } R_k^*} \{w^\top u - R_k^*(u) - \delta \cdot d(u)\}. \quad (16)$$

Furthermore, the gradient is co-coercive, i.e., it satisfies:

$$(x - y)^\top (\nabla R_k^\delta(x) - \nabla R_k^\delta(y)) \geq \delta \|\nabla R_k^\delta(x) - \nabla R_k^\delta(y)\|^2 \quad (17)$$

By Cauchy-Schwarz, this implies Lipschitz continuity, i.e.,

$$\|\nabla R_k^\delta(x) - \nabla R_k^\delta(y)\| \leq \frac{1}{\delta} \|x - y\|. \quad (18)$$

Proof: The theorem is from [9]. ■

The feasibility of stochastic-gradient algorithms for the minimization of (7) hinges on the assumption that (16) can be evaluated in closed form or at least easily. Fortunately, this is the case for a large class of regularizers of interest — see [53] for an overview of closed form solutions in the special case $d(\cdot) = \frac{1}{2} \|\cdot\|^2$ and [9], [48] for other distance choices. For example, for every function where the proximal operator [42, Eq. (3.2)]

$$\text{prox}_{\delta R_k}(w) = \arg \min_u \left(R_k(u) + \frac{1}{2\delta} \|w - u\|^2 \right) \quad (19)$$

can be evaluated in closed form, we can let $d(\cdot) \triangleq \frac{1}{2} \|\cdot\|^2$ and obtain [42]:

$$\nabla R_k^\delta(w) = \frac{1}{\delta} (w - \text{prox}_{\delta R_k}(w)) \quad (20)$$

Depending on the regularizers $R_k(\cdot)$, other proximity functions may be more appropriate [9]. We point out that the smooth approximation (14) can equivalently be written as [48]:

$$R_k^\delta(w) = \min_{u \in \text{dom } R_k} \left\{ R_k(u) + \delta \cdot d^* \left(\frac{w - u}{\delta} \right) \right\} \quad (21)$$

To verify this, observe that (following the argument of [48])

$$\begin{aligned} & R_k^\delta(w) \\ &= \min_{u \in \text{dom } R_k} \left\{ R_k(u) + \delta \cdot d^* \left(\frac{w - u}{\delta} \right) \right\} \\ &\stackrel{(a)}{=} \min_{u \in \text{dom } R_k} \{ R_k(u) - d'(u) \} \\ &\stackrel{(b)}{=} \max_{z \in \text{dom } R_k^*} \left\{ -R_k^*(z) + \inf_v (v^\top z - d'(v)) \right\} \\ &\stackrel{(c)}{=} \max_{z \in \text{dom } R_k^*} \left\{ -R_k^*(z) + \inf_v \left(v^\top z + \delta \cdot d^* \left(\frac{w - v}{\delta} \right) \right) \right\} \\ &= \max_{z \in \text{dom } R_k^*} \left\{ -R_k^*(z) - \sup_v \left(-v^\top z - \delta \cdot d^* \left(\frac{w - v}{\delta} \right) \right) \right\} \\ &\stackrel{(d)}{=} \max_{z \in \text{dom } R_k^*} \left\{ -R_k^*(z) - \sup_{v'} \left((\delta v' - w)^\top z - \delta \cdot d^*(v') \right) \right\} \\ &= \max_{z \in \text{dom } R_k^*} \left\{ z^\top w - R_k^*(z) - \delta \cdot \sup_{v'} \left(v'^\top z - d^*(v') \right) \right\} \\ &= \max_{z \in \text{dom } R_k^*} \{ z^\top w - R_k^*(z) - \delta \cdot d(z) \} \quad (22) \end{aligned}$$

where in (a) we introduced $d'(u) \triangleq -\delta \cdot d^* \left(\frac{w - u}{\delta} \right)$, (b) follows from Fenchel's duality theorem [52, Theorem 31.1], (c) substitutes back $d'(v) = \delta \cdot d^* \left(\frac{w - v}{\delta} \right)$ and (d) makes the change of variables $v' \triangleq \frac{w - v}{\delta} \Leftrightarrow v \triangleq w - \delta v'$. Expression (21) is known as the infimal convolution.

B. Accuracy of the Smooth Approximation

Replacing the original optimization problem (4) by the smoothed cost (7) naturally results in a bias, since the new minimizer w_δ^o will generally be different from the original minimizer w^o . This bias, when not properly controlled, can

degrade the performance of the algorithm. For this reason, a number of works have examined the smoothing bias introduced through conjugate smoothing under various conditions on the cost functions. In the centralized setting, when $N = 1$, it has been established that $R_k^\delta(w) \rightarrow R_k(w)$ both pointwise and epigraphically, which implies $w_\delta^o \rightarrow w^o$ as $\delta \rightarrow 0$ [54], while [55] showed a sum of costs $\sum_{k=1}^N p_k R_k(w)$, when smoothed individually, will continue to converge epigraphically. While encouraging, these results do not guarantee a rate at which $w_\delta^o \rightarrow w^o$, complicating the choice of the smoothing parameter δ . Pointwise convergence has been strengthened to uniform convergence, i.e., $|R_k(w) - R_k^\delta(w)| \leq O(\delta)$ for costs with bounded subgradients for $N = 1$ [9], [48] and for a collection of costs, each with bounded subgradients in [49].

We present here a variation of these results by restricting ourselves to *strongly*-convex costs, but allowing for regularizers with *unbounded* subgradients and establishing $\|w^o - w_\delta^o\|^2 \leq O(\delta)$ rather than simply $w_\delta^o \rightarrow w^o$.

Theorem 2 (Accuracy of smooth approximation). *The bias introduced by smoothing the original problem diminishes linearly with δ , i.e.,*

$$\|w^o - w_\delta^o\|^2 \leq \delta \cdot \frac{2}{\lambda_L} \sum_{k=1}^N p_k d(r_k^o) = O(\delta) \quad (23)$$

where $r_k^o \in \partial R_k(w^o)$ such that

$$\sum_{k=1}^N p_k \{ \nabla J_k(w^o) + r_k^o \} = 0. \quad (24)$$

This collection of $\{r_k^o\}$ is guaranteed to exist, since $w^o \triangleq \arg \min \sum_{k=1}^N p_k \{ J_k(w) + R_k(w) \}$. Furthermore, the gradients of the smooth regularizers are bounded independently of δ at the minimizer w_δ^o :

$$\|\nabla R_k^\delta(w_\delta^o)\|^2 \leq \frac{2}{p_k} \sum_{\ell=1}^N p_\ell d(r_\ell^o) = O(1) \quad (25)$$

Proof: Appendix A. ■

C. Regularized Diffusion Strategy

Now that we have established a method for constructing a differentiable approximation for each regularizer, we can solve for the minimizer of (7) by resorting to the following (adapt-then-combine form of the) diffusion strategy [4], [16], [17]:

$$\phi_{k,i} = \mathbf{w}_{k,i-1} - \mu \widehat{\nabla J}_k(\mathbf{w}_{k,i-1}) - \mu \nabla R_k^\delta(\mathbf{w}_{k,i-1}) \quad (26)$$

$$\mathbf{w}_{k,i} = \sum_{\ell=1}^N a_{\ell k} \phi_{\ell,i} \quad (27)$$

where $\mu > 0$ is a small step-size parameter and $a_{\ell k}$ are the entries of a combination matrix A with Perron eigenvector p , i.e. $Ap = p$. In this implementation, each agent k first performs the stochastic-gradient update (26), starting from its existing iterate value $\mathbf{w}_{k,i-1}$, and obtains an intermediate iterate $\phi_{k,i}$. Subsequently, agent k consults with its neighbors and combines their intermediate iterates into $\mathbf{w}_{k,i}$ according to (27).

Motivated by the construction in [41], we can refine (26)–(27) further as follows. We first introduce an auxiliary variable $\psi_{k,i}$ and rewrite (26) in the equivalent form:

$$\phi_{k,i} = \mathbf{w}_{k,i-1} - \mu \widehat{\nabla} J_k(\mathbf{w}_{k,i-1}) \quad (28)$$

$$\psi_{k,i} = \phi_{k,i} - \mu \nabla R_k^\delta(\mathbf{w}_{k,i-1}) \quad (29)$$

$$\mathbf{w}_{k,i} = \sum_{\ell=1}^N a_{\ell k} \psi_{\ell,i} \quad (30)$$

We can now appeal to an incremental-type argument [10], [56] by noting that it is reasonable to expect $\phi_{k,i}$ to be an improved estimate for w_δ^0 compared to $\mathbf{w}_{k,i-1}$. Therefore, we replace $\mathbf{w}_{k,i-1}$ in (29) by $\phi_{k,i}$ and arrive at the following regularized diffusion implementation.

Algorithm: Regularized Diffusion Strategy

$$\phi_{k,i} = \mathbf{w}_{k,i-1} - \mu \widehat{\nabla} J_k(\mathbf{w}_{k,i-1}) \quad (31)$$

$$\psi_{k,i} = \phi_{k,i} - \mu \nabla R_k^\delta(\phi_{k,i}) \quad (32)$$

$$\mathbf{w}_{k,i} = \sum_{\ell=1}^N a_{\ell k} \psi_{\ell,i} \quad (33)$$

Example 1 (Proximal Diffusion Learning). Choosing $d(w) = \frac{1}{2}\|w\|^2$ turns the smooth approximation (14) into

$$R_k^\delta(w) = \left(R_k^*(w) + \frac{\delta}{2}\|w\|^2 \right)^* \quad (34)$$

which is the well-known Moreau envelope [42, p. 136]. It can be rewritten equivalently as

$$R_k^\delta(w) = \min_u \left(R_k(u) + \frac{1}{2\delta}\|w - u\|^2 \right) \quad (35)$$

where the minimizing argument is identified as the proximal operator:

$$\text{prox}_{\delta R_k}(w) = \arg \min_u \left(R_k(u) + \frac{1}{2\delta}\|w - u\|^2 \right) \quad (36)$$

For many costs $R_k(w)$, the proximal operator can be evaluated in closed form. The gradient of the Moreau envelope can also be written as

$$\nabla R_k^\delta(w) = \frac{1}{\delta} (w - \text{prox}_{\delta R_k}(w)). \quad (37)$$

This allows us to rewrite iterations (31)–(33) as

$$\phi_{k,i} = \mathbf{w}_{k,i-1} - \mu \widehat{\nabla} J_k(\mathbf{w}_{k,i-1}) \quad (38)$$

$$\psi_{k,i} = \left(1 - \frac{\mu}{\delta} \right) \phi_{k,i} + \frac{\mu}{\delta} \text{prox}_{\delta R_k}(\phi_{k,i}) \quad (39)$$

$$\mathbf{w}_{k,i} = \sum_{\ell=1}^N a_{\ell k} \psi_{\ell,i} \quad (40)$$

which is a damped variation of the proximal diffusion algorithm studied in [47] under the stronger assumption of small Lipschitz continuous regularizers. ■

III. CONVERGENCE ANALYSIS

A. Centralized Recursion

We now examine the convergence properties of the diffusion strategy (31)–(33). To do so, and motivated by the approach introduced in [17], it is useful to introduce the following centralized recursion to serve as a frame of reference:

$$w_i = w_{i-1} - \mu \sum_{k=1}^N p_k \nabla J_k(w_{i-1}) - \mu \sum_{k=1}^N p_k \nabla R_k^\delta(w_{i-1}) \quad (41)$$

This recursion amounts to a gradient-descent iteration applied to the smoothed aggregate cost in (7) under the assumption that the risk functions (and therefore their gradients) are known. For convenience of presentation, we introduce the central operator $T_c(x) : \mathbb{R}^M \rightarrow \mathbb{R}^M$ defined as follows:

$$T_c(x) \triangleq x - \mu \sum_{k=1}^N p_k \nabla J_k(x) - \mu \sum_{k=1}^N p_k \nabla R_k^\delta(x) \quad (42)$$

so that the reference recursion (41) becomes $w_i = T_c(w_{i-1})$.

Lemma 2 (Contraction mapping). *Assume $\mu < 2\delta$. Then, the centralized recursion operator (42) satisfies*

$$\|T_c(x) - T_c(y)\| \leq \gamma_c \|x - y\| \quad (43)$$

where $\gamma_c > 0$ can be made strictly less than one by selecting sufficiently small μ and is given by:

$$\gamma_c = 1 - \mu \lambda_L + \mu^2 \left(\frac{\lambda_U^2}{2 - \frac{\mu}{\delta}} \right). \quad (44)$$

From Banach's fixed point theorem [57, Theorem 5.1-2] and (42), we conclude that for sufficiently small μ , $w_i = T_c(w_{i-1})$ converges exponentially to the unique fixed-point w_δ^0 , the minimizer of (7).

Proof: Appendix B. ■

B. Network Basis Transformation

We are now ready to examine the behavior of the diffusion strategy (31)–(33), which employs stochastic gradients. We begin by introducing the following extended vectors and matrices, which collect quantities of interest from across all agents in the network:

$$\mathbf{w}_i \triangleq \text{col} \{ \mathbf{w}_{1,i}, \dots, \mathbf{w}_{N,i} \} \quad (45)$$

$$\mathcal{A} \triangleq \mathcal{A} \otimes I_M \quad (46)$$

$$g(\mathbf{w}_i) \triangleq \text{col} \{ \nabla_w J_1(\mathbf{w}_{1,i}), \dots, \nabla_w J_N(\mathbf{w}_{N,i}) \} \quad (47)$$

$$\widehat{g}(\mathbf{w}_i) \triangleq \text{col} \{ \widehat{\nabla}_w J_1(\mathbf{w}_{1,i}), \dots, \widehat{\nabla}_w J_N(\mathbf{w}_{N,i}) \} \quad (48)$$

$$r(\mathbf{w}_i) \triangleq \text{col} \{ \nabla_w R_1^\delta(\mathbf{w}_{1,i}), \dots, \nabla_w R_N^\delta(\mathbf{w}_{N,i}) \} \quad (49)$$

$$q(\mathbf{w}_i) \triangleq r(\mathbf{w}_i - \mu g(\mathbf{w}_i)) \quad (50)$$

$$\widehat{q}(\mathbf{w}_i) \triangleq r(\mathbf{w}_i - \mu \widehat{g}(\mathbf{w}_i)) \quad (51)$$

Using these definitions, after substituting (31) and (32) into (33) to obtain a single recursion, we find that \mathbf{w}_i evolves according to the following dynamics:

$$\mathbf{w}_i = \mathcal{A}^\top \mathbf{w}_{i-1} - \mu \mathcal{A}^\top (\widehat{g}(\mathbf{w}_{i-1}) + \widehat{q}(\mathbf{w}_{i-1})) \quad (52)$$

By construction, the combination matrix A is left-stochastic and primitive and hence admits a Jordan decomposition of the form $A = V_\epsilon J V_\epsilon^{-1}$ with [4], [17]:

$$V_\epsilon = \begin{bmatrix} p & V_R \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & J_\epsilon \end{bmatrix}, \quad V_\epsilon^{-1} = \begin{bmatrix} \mathbf{1}^\top \\ V_L^\top \end{bmatrix} \quad (53)$$

where J_ϵ is a block Jordan matrix with the eigenvalues $\lambda_2(A)$ through $\lambda_N(A)$ on the diagonal and ϵ on the first lower sub-diagonal. The extended matrix \mathcal{A} then satisfies $\mathcal{A} = V_\epsilon \mathcal{J} V_\epsilon^{-1}$ with $V_\epsilon = V_\epsilon \otimes I_N$, $\mathcal{J} = J \otimes I_N$, $V_\epsilon^{-1} = V_\epsilon^{-1} \otimes I_N$. Multiplying both sides of (52) by V_ϵ^\top and introducing the transformed iterate vector $\mathcal{w}'_i \triangleq V_\epsilon^\top \mathcal{w}_i$, we obtain:

$$\mathcal{w}'_i = \mathcal{J}^\top \mathcal{w}'_{i-1} - \mu \mathcal{J}^\top V_\epsilon^\top (\widehat{g}(\mathcal{w}_{i-1}) + \widehat{q}(\mathcal{w}_{i-1})) \quad (54)$$

Following [4], [17], we can exploit the structure of the decomposition (53) to provide further insight into this transformed recursion. Let $\mathcal{w}'_i = \text{col}\{\mathcal{w}_{c,i}, \mathcal{w}_{e,i}\}$, where $\mathcal{w}_{c,i} \in \mathbb{R}^{N \times 1}$ and $\mathcal{w}_{e,i} \in \mathbb{R}^{(N-1)M \times 1}$. Then, recursion (54) can be decomposed as:

$$\mathcal{w}_{c,i} = \mathcal{w}_{c,i-1} - \mu (p^\top \otimes I_N) (\widehat{g}(\mathcal{w}_{i-1}) + \widehat{q}(\mathcal{w}_{i-1})) \quad (55)$$

$$\mathcal{w}_{e,i} = \mathcal{J}_\epsilon^\top \mathcal{w}_{e,i-1} - \mu \mathcal{J}_\epsilon^\top V_R^\top (\widehat{g}(\mathcal{w}_{i-1}) + \widehat{q}(\mathcal{w}_{i-1})) \quad (56)$$

Note from $\mathcal{w}'_i = V_\epsilon^\top \mathcal{w}_i$, that [17]:

$$\mathcal{w}_{c,i} = (p^\top \otimes I_M) \mathcal{w}_i = \sum_{k=1}^N p_k \mathcal{w}_{k,i} \quad (57)$$

Hence, $\mathcal{w}_{c,i}$ is the weighted centroid vector of all iterates $\mathcal{w}_{k,i}$ across the network. From $\mathcal{w}_i = (V_\epsilon^{-1})^\top \mathcal{w}'_i$ on the other hand, one obtains [17]:

$$\mathcal{w}_i = \mathbf{1} \otimes \mathcal{w}_{c,i} + V_L \mathcal{w}_{e,i} \quad (58)$$

so that $\mathcal{w}_{e,i}$ can be interpreted as the deviation of individual estimates from the weighted centroid vector $\mathcal{w}_{c,i}$ across the network.

We examine the centroid recursion (55) in greater detail. Thus, note that

$$\begin{aligned} & \mathcal{w}_{c,i} \\ &= \mathcal{w}_{c,i-1} - \mu (p^\top \otimes I_M) (\widehat{g}(\mathcal{w}_{i-1}) + \widehat{q}(\mathcal{w}_{i-1})) \\ &= \mathcal{w}_{c,i-1} - \mu (p^\top \otimes I_M) (g(\mathbf{1} \otimes \mathcal{w}_{c,i-1}) + r(\mathbf{1} \otimes \mathcal{w}_{c,i-1})) \\ &\quad - \mu (p^\top \otimes I_M) (g(\mathcal{w}_{i-1}) + q(\mathcal{w}_{i-1}) \\ &\quad\quad - g(\mathbf{1} \otimes \mathcal{w}_{c,i-1}) - q(\mathbf{1} \otimes \mathcal{w}_{c,i-1})) \\ &\quad - \mu (p^\top \otimes I_M) (\widehat{g}(\mathcal{w}_{i-1}) + \widehat{q}(\mathcal{w}_{i-1}) \\ &\quad\quad - g(\mathcal{w}_{i-1}) - q(\mathcal{w}_{i-1})) \\ &\quad - \mu (p^\top \otimes I_M) (q(\mathcal{w}_{i-1}) - r(\mathcal{w}_{i-1})) \\ &= T_c(\mathcal{w}_{c,i-1}) - \mu (p^\top \otimes I_M) (\mathbf{t}_{i-1} + \mathbf{s}_i + \mathbf{u}_{i-1}) \end{aligned} \quad (59)$$

where we replaced

$$\begin{aligned} & \mathcal{w}_{c,i-1} - \mu (p^\top \otimes I_M) (g(\mathbf{1} \otimes \mathcal{w}_{c,i-1}) + r(\mathbf{1} \otimes \mathcal{w}_{c,i-1})) \\ &= \mathcal{w}_{c,i-1} - \mu \sum_{k=1}^N p_k \nabla J_k(\mathcal{w}_{c,i-1}) - \mu \sum_{k=1}^N p_k \nabla R_k^\delta(\mathcal{w}_{c,i-1}) \\ &\stackrel{(42)}{=} T_c(\mathcal{w}_{c,i-1}) \end{aligned} \quad (60)$$

and introduced the perturbation terms:

$$\mathbf{t}_{i-1} = g(\mathcal{w}_{i-1}) + q(\mathcal{w}_{i-1}) - g(\mathbf{1} \otimes \mathcal{w}_{c,i-1}) - q(\mathbf{1} \otimes \mathcal{w}_{c,i-1}) \quad (61)$$

$$\mathbf{s}_i = \widehat{g}(\mathcal{w}_{i-1}) + \widehat{q}(\mathcal{w}_{i-1}) - g(\mathcal{w}_{i-1}) - q(\mathcal{w}_{i-1}) \quad (62)$$

$$\mathbf{u}_{i-1} = q(\mathcal{w}_{i-1}) - r(\mathcal{w}_{i-1}) \quad (63)$$

It follows from (59) that the centroid recursion is a perturbed version of the central recursion introduced earlier in (42). The perturbation arising from disagreement across agents in the network is captured in \mathbf{t}_{i-1} , while stochastic perturbations due to instantaneous gradient approximations is captured in \mathbf{s}_i . The incremental implementation causes \mathbf{u}_{i-1} . It is therefore reasonable to expect that $\mathcal{w}_{c,i}$ will evolve close to the central variable w_i from (41), which was already shown to converge to w_δ^o in Lemma 2. To formalize this intuition, we define $\widetilde{\mathcal{w}}_{c,i-1} = w_\delta^o - \mathcal{w}_{c,i-1}$. Since w_δ^o is a fixed-point of $T_c(\cdot)$, i.e., $w_\delta^o = T_c(w_\delta^o)$, the error $\widetilde{\mathcal{w}}_{c,i-1}$ satisfies the recursion:

$$\begin{aligned} \widetilde{\mathcal{w}}_{c,i-1} &= T_c(w_\delta^o) - T_c(\mathcal{w}_{c,i-1}) \\ &\quad + \mu (p^\top \otimes I_M) (\mathbf{t}_{i-1} + \mathbf{s}_i + \mathbf{u}_{i-1}) \end{aligned} \quad (64)$$

With the same perturbation terms, expression (56) turns into:

$$\begin{aligned} \mathcal{w}_{e,i} &= \mathcal{J}_\epsilon^\top \mathcal{w}_{e,i-1} - \mu \mathcal{J}_\epsilon^\top V_R^\top (\mathbf{t}_{i-1} + \mathbf{s}_i + \mathbf{u}_{i-1} \\ &\quad - g(\mathbf{1} \otimes \mathcal{w}_{c,i-1}) - r(\mathbf{1} \otimes \mathcal{w}_{c,i-1})) \end{aligned} \quad (65)$$

We employ the following common assumption on the perturbations caused by the gradient noise [4], [16], [17].

Assumption 4 (Gradient noise process). *For each k , the gradient noise process is defined as*

$$\mathbf{s}_{k,i}(\mathcal{w}_{k,i-1}) = \widehat{\nabla} J_k(\mathcal{w}_{k,i-1}) - \nabla J_k(\mathcal{w}_{k,i-1}) \quad (66)$$

and satisfies

$$\mathbb{E}[\mathbf{s}_{k,i}(\mathcal{w}_{k,i-1}) | \mathcal{F}_{i-1}] = 0 \quad (67a)$$

$$\mathbb{E}[\|\mathbf{s}_{k,i}(\mathcal{w}_{k,i-1})\|^2 | \mathcal{F}_{i-1}] \leq \beta^2 \|\mathcal{w}_{k,i-1}\|^2 + \sigma^2 \quad (67b)$$

for some non-negative constants $\{\beta^2, \sigma^2\}$, and where \mathcal{F}_{i-1} denotes the filtration generated by the random processes $\{\mathcal{w}_{\ell,j}\}$ for all $\ell = 1, 2, \dots, N$ and $j \leq i-1$, i.e., \mathcal{F}_{i-1} represents the information that is available about the random processes $\{\mathcal{w}_{\ell,j}\}$ up to time $i-1$. ■

For a block-vector $\mathbf{x} \in \mathbb{R}^{MN \times 1}$ consisting of N blocks of size $M \times 1$, let $P[\mathbf{x}] = \text{col}\{\mathbb{E}\|\mathbf{x}_1\|^2, \dots, \mathbb{E}\|\mathbf{x}_N\|^2\} \in \mathbb{R}^{N \times 1}$ [17]. Note that $\mathbf{1}^\top P[\mathbf{x}] = \mathbb{E}\|\mathbf{x}\|^2$. Furthermore, let $v_{L,k}$ denote the k -th row of V_L and let $\nu = \max_k \|v_{L,k} \otimes I_M\|$, which is independent of μ and δ .

Lemma 3 (Bounds on perturbation terms). *The perturbation terms in (64) satisfy the following bounds:*

$$P[\mathbf{t}_{i-1}] \preceq \left(2\lambda_U^2 + 4\frac{1+\mu^2}{\delta^2}\right) \nu^2 \mathbf{1}\mathbf{1}^\top P[\mathbf{w}_{e,i-1}] \quad (68)$$

$$P[\mathbf{u}_{i-1}] \preceq \frac{\mu^2}{\delta^2} (3\lambda_U^2 \nu^2 \mathbf{1}\mathbf{1}^\top P[\mathbf{w}_{e,i-1}] + 3\lambda_U^2 P[\mathbf{1} \otimes \tilde{\mathbf{w}}_{c,i-1}] + 3P[g(\mathbf{1} \otimes w_\delta^o)]) \quad (69)$$

$$P[\mathbf{s}_i - \mathbb{E} \mathbf{s}_i] \preceq 3\beta^2 P[\mathbf{1} \otimes \tilde{\mathbf{w}}_{c,i-1}] + 3\beta^2 \nu^2 \mathbf{1}\mathbf{1}^\top P[\mathbf{w}_{e,i-1}] + 3\beta^2 P[\mathbf{1} \otimes w_\delta^o] + \sigma^2 \mathbf{1} \quad (70)$$

$$P[\mathbb{E} \mathbf{s}_i] \preceq 3\beta^2 \frac{\mu^2}{\delta^2} P[\mathbf{1} \otimes \tilde{\mathbf{w}}_{c,i-1}] + 3\beta^2 \frac{\mu^2}{\delta^2} \nu^2 \mathbf{1}\mathbf{1}^\top P[\mathbf{w}_{e,i-1}] + 3\beta^2 \frac{\mu^2}{\delta^2} P[\mathbf{1} \otimes w_\delta^o] + \frac{\mu^2}{\delta^2} \sigma^2 \mathbf{1} \quad (71)$$

$$P[g(\mathbf{1} \otimes \mathbf{w}_{c,i-1})] \preceq 2\lambda_U^2 P[\mathbf{1} \otimes \tilde{\mathbf{w}}_{c,i-1}] + 2P[g(\mathbf{1} \otimes w_\delta^o)] \quad (72)$$

$$P[r(\mathbf{1} \otimes \mathbf{w}_{c,i-1})] \preceq \frac{2}{\delta^2} P[\mathbf{1} \otimes \tilde{\mathbf{w}}_{c,i-1}] + 2P[r(\mathbf{1} \otimes w_\delta^o)] \quad (73)$$

Proof: Appendix C. ■

C. Mean-Square-Error Bounds

Using the bounds on the perturbation terms obtained in Lemma 3, we can formulate a recursive bound on the mean-square error.

Lemma 4 (Mean-Square-Error Recursion). *Suppose $\mu \leq \delta$. Then, the variances of $\tilde{\mathbf{w}}_{c,i}$ and $\mathbf{w}_{e,i}$ are coupled and recursively bounded as*

$$\begin{bmatrix} \mathbb{E} \|\tilde{\mathbf{w}}_{c,i}\|^2 \\ \mathbb{E} \|\mathbf{w}_{e,i}\|^2 \end{bmatrix} \preceq \Gamma \begin{bmatrix} \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 \\ \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 \end{bmatrix} + \begin{bmatrix} \frac{\mu^3}{\delta^2} b_1 + \frac{\mu^3}{\delta^2} b_2 + \mu^2 b_3 \\ \mu^2 b_4 + \mu^2 b_5 + \frac{\mu^4}{\delta^2} b_6 \end{bmatrix} \quad (74)$$

where

$$\Gamma = \begin{bmatrix} \gamma_c + \frac{\mu^3}{\delta^2} h_1 + \mu^2 h_2 & \frac{\mu}{\delta^2} h_3 + \mu h_4 + \frac{\mu^3}{\delta^2} h_5 + \mu^2 h_6 \\ \frac{\mu^2}{\delta^2} h_7 + \mu^2 h_8 + \frac{\mu^4}{\delta^2} h_9 & \|\mathcal{J}_\epsilon\| + \frac{\mu^2}{\delta^2} h_{10} + \mu^2 h_{11} + \frac{\mu^4}{\delta^2} h_{12} \end{bmatrix} \quad (75)$$

$$\gamma_c \triangleq 1 - \mu\lambda_L + \mu^2 \left(\frac{\lambda_U^2}{2 - \frac{\mu}{\delta}} \right) \quad (76)$$

$$a_1 \triangleq \frac{1}{\lambda_L - \mu \frac{\lambda_U^2}{2 - \frac{\mu}{\delta}}} \stackrel{\mu \leq \delta}{\preceq} O(1) \quad (77)$$

$$a_2 \triangleq \frac{25N \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{1 - \|\mathcal{J}_\epsilon\|} = O(1) \quad (78)$$

$$h_1 \triangleq 9(\beta^2 + \lambda_U^2) a_1 = O(1) \quad (79)$$

$$h_2 \triangleq 3\beta^2 = O(1) \quad (80)$$

$$h_3 \triangleq 3\nu^2 a_1 = O(1) \quad (81)$$

$$h_4 \triangleq 6\nu^2 \lambda_U^2 a_1 = O(1) \quad (82)$$

$$h_5 \triangleq 9\nu^2 (\lambda_U^2 + \beta^2) a_1 = O(1) \quad (83)$$

$$h_6 \triangleq 3\nu^2 \beta^2 = O(1) \quad (84)$$

$$h_7 \triangleq 2a_2 = O(1) \quad (85)$$

$$h_8 \triangleq \left(2\lambda_U^2 + \frac{1 - \|\mathcal{J}_\epsilon\|}{25} 3\beta^2 \right) a_2 = O(1) \quad (86)$$

$$h_9 \triangleq 3(\lambda_U^2 + \beta^2) a_2 = O(1) \quad (87)$$

$$h_{10} \triangleq \nu^2 a_2 = O(1) \quad (88)$$

$$h_{11} \triangleq \nu^2 \left(2\lambda_U^2 + \frac{1 - \|\mathcal{J}_\epsilon\|}{25} 3\beta^2 \right) a_2 = O(1) \quad (89)$$

$$h_{12} \triangleq \nu^2 (1 + 3\lambda_U^2 + 3\beta^2) a_2 = O(1) \quad (90)$$

$$b_1 \triangleq 9a_1 \|g(\mathbf{1} \otimes w_\delta^o)\|^2 \stackrel{(23)}{\preceq} O(1) \quad (91)$$

$$b_2 \triangleq 3a_1 (3\beta^2 \|w_\delta^o\|^2 + \sigma^2) \stackrel{(23)}{\preceq} O(1) \quad (92)$$

$$b_3 \triangleq 3\beta^2 \|w_\delta^o\|^2 + \sigma^2 \stackrel{(23)}{\preceq} O(1) \quad (93)$$

$$b_4 \triangleq 2a_2 (\|r(\mathbf{1} \otimes w_\delta^o)\|^2) \stackrel{(25)}{\preceq} O(1) \quad (94)$$

$$b_5 \triangleq 2a_2 \|g(\mathbf{1} \otimes w_\delta^o)\|^2 + \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2 N (3\beta^2 \|w_\delta^o\|^2 + \sigma^2) \stackrel{(23)}{\preceq} O(1) \quad (95)$$

$$b_6 \triangleq a_2 (3\|g(\mathbf{1} \otimes w_\delta^o)\|^2 + 3\beta^2 N \|w_\delta^o\|^2 + N\sigma^2) \stackrel{(23)}{\preceq} O(1) \quad (96)$$

Proof: Appendix D. ■

Note that while the constants b_1 through b_6 exhibit dependence on δ through w_δ^o , they are nevertheless bounded independently of δ for small δ in light of (23) and (25) in Theorem 2. It is evident from expression (75) that the stability of the driving matrix Γ depends critically on the fraction between the step-size μ and the smoothing parameter δ . Motivated by this observation, let us set, for a small $\kappa > 0$:

$$\delta = \mu^{\frac{1}{2} - \kappa} \quad (97)$$

so that

$$\frac{\mu}{\delta^2} = \mu^{2\kappa} \rightarrow 0 \text{ as } \mu \rightarrow 0 \quad (98)$$

Under this construction, the driving matrix satisfies

$$\Gamma = \begin{bmatrix} \gamma_c + O(\mu^2) & O(\mu^{2\kappa}) \\ O(\mu^{1+2\kappa}) & \|\mathcal{J}_\epsilon\| + O(\mu^{1+2\kappa}) \end{bmatrix} \quad (99)$$

which ensures that the off-diagonal coupling terms diminish as $\mu, \delta \rightarrow 0$.

Lemma 5. *Let $\delta = \mu^{\frac{1}{2} - \kappa}$, $\frac{1}{2} > \kappa > 0$. Then there exists a small enough μ , such that $\rho(\Gamma) < 1$. Furthermore:*

$$\limsup_{i \rightarrow \infty} \begin{bmatrix} \mathbb{E} \|\tilde{\mathbf{w}}_{c,i}\|^2 \\ \mathbb{E} \|\mathbf{w}_{e,i}\|^2 \end{bmatrix} \preceq \begin{bmatrix} O(\mu b_3) + o(\mu) \\ O(\mu^2 b_4) + O(\mu^2 b_5) + o(\mu^2) \end{bmatrix} \quad (100)$$

The rate of convergence is given by:

$$\rho(\Gamma) \leq \max \{1 - \mu\lambda_L + O(\mu^{1+2\kappa}), \|\mathcal{J}_\epsilon\|^2 + O(\mu^{2\kappa})\} \quad (101)$$

Proof: See Appendix E. ■

Theorem 3. *Let $\delta = \mu^{\frac{1}{2} - \kappa}$, $\frac{1}{2} > \kappa > 0$. Then it holds that for sufficiently small μ ,*

$$\limsup_{i \rightarrow \infty} \mathbb{E} \|\mathbf{w}_\delta^o - \mathbf{w}_{k,i}\|^2 \leq O(\mu b_3) + o(\mu). \quad (102)$$

Proof: We have

$$\begin{aligned} \mathbb{E} \|\mathbf{w}_\delta^o - \mathbf{w}_{k,i}\|^2 &= \mathbb{E} \|\tilde{\mathbf{w}}_{c,i} + (v_{L,k} \otimes I_M) \mathbf{w}_{e,i}\|^2 \\ &\leq 2 \mathbb{E} \|\tilde{\mathbf{w}}_{c,i}\|^2 + 2\nu^2 \mathbb{E} \|\mathbf{w}_{e,i}\|^2 \end{aligned} \quad (103)$$

so that the theorem follows after taking the limit and applying Lemma 5. ■

IV. APPLICATION: DIVISION OF LABOR IN MACHINE LEARNING

We illustrate the performance of the algorithm in an online machine learning problem over a heterogeneous network. Given random binary class variables $\gamma = \pm 1$ and feature vectors $\mathbf{h} \in \mathbb{R}^M$, the general objective in single-agent machine learning is to find a classifier $c^*(\mathbf{h})$, such that:

$$c^* \triangleq \arg \min_c \text{Prob} \{c(\mathbf{h}) \neq \gamma\}. \quad (104)$$

We restrict the class of permissible classifiers to linear classifiers of the form $c(\mathbf{h}) = \mathbf{h}^\top w$ with $w \in \mathbb{R}^M$ and approximate (104) by the logistic cost to obtain:

$$w^* \triangleq \arg \min_w \mathbb{E} \ln [1 + e^{-\gamma \mathbf{h}^\top w}] \quad (105)$$

A. Group Lasso

Regularization is an effective technique to incorporate prior structural knowledge about the classifier into the optimization problem as a means to avoiding overfitting and improving generalization ability. For example, when the linear classifier is known to be sparse, regularization through the ℓ_1 -norm, also known as Lasso-regularization, has been shown to encourage sparse solutions [7]. When there is further knowledge about the structure of the sparsity, the group-Lasso has been proposed [58], [59]. It takes the form

$$R(w) = \sum_k \lambda_k \|D_k w\|_2 = \sum_k \lambda_k \|w_g^k\|_2 \quad (106)$$

where

$$w_g^k \triangleq D_k w \quad (107)$$

and D_k denotes a diagonal selection matrix with entries 0 or 1 where 1's appear for entries of w belonging to a group. Relation (106) is in the form of a sum-of-costs and hence immediately decomposable.

B. Network Structure

We consider a network consisting of 3 types of agents: fully-informed (\mathcal{F}), data-informed (\mathcal{D}), and structure-informed (\mathcal{S}) agents. Fully-informed agents have access to streaming realizations $\{\gamma_k(i), \mathbf{h}_{k,i}\}$ as well as knowledge about a subset of covariates of w which are likely to be sparse, collected in w_g^k . These agents are equipped with the regularized cost $J_k(w) + R_k(w)$, where

$$J_k(w) = \mathbb{E} \ln [1 + e^{-\gamma_k \mathbf{h}_{k,i}^\top w}] + \rho_2 \|w\|_2^2 \quad (108)$$

$$R_k(w) = \rho_1 \|w_g^k\|_2 \quad (109)$$

for $k \in \mathcal{F}$. Data-informed agents have access to streaming realizations $\{\gamma_k(i), \mathbf{h}_{k,i}\}$, but no knowledge about the structure of sparsity in w . They are equipped with

$$J_k(w) = \mathbb{E} \ln [1 + e^{-\gamma_k \mathbf{h}_{k,i}^\top w}] + \rho_2 \|w\|_2^2 \quad (110)$$

$$R_k(w) = 0 \quad (111)$$

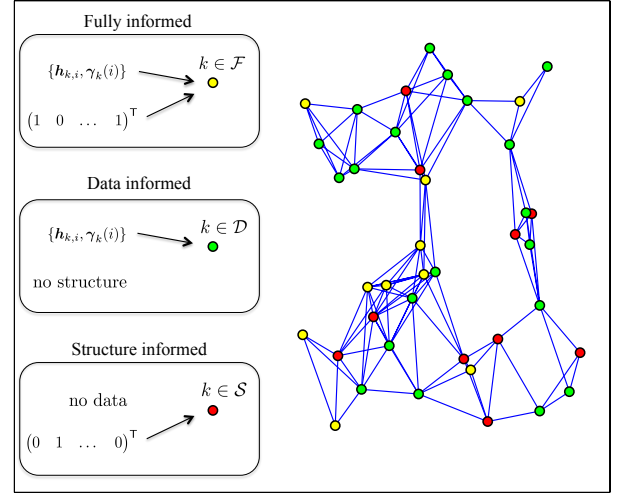


Fig. 1. Sample network consisting of $N = 40$ agents, $\text{card}(\mathcal{F}) = 10$, $\text{card}(\mathcal{D}) = 20$, $\text{card}(\mathcal{S}) = 10$. Fully-informed agents have access to data as well as partial structural information. Data-informed agents observe realizations of the feature vector along with class-labels, but have no information on the structure of the classifier. Structure-informed agents do not have access to data, but do have partial information on sparse elements.

for $k \in \mathcal{D}$. Structure-informed agents have information about the sparsity of w , but no access to realizations of feature vectors. They are equipped with

$$J_k(w) = 0 \quad (112)$$

$$R_k(w) = \rho_1 \|w_g^k\|_2 \quad (113)$$

for $k \in \mathcal{S}$. Similar to ordinary ℓ_2 -norm regularization, the proximal operator of $\rho_1 \|w_g^k\|_2$ is available in closed form. Note that $\|w_g^k\|_2 = \|D_k w\|_2$, where D_k is a diagonal matrix with $D_{(ii)} = 1$, if the i -th element of w is likely to be sparse and 0 otherwise. We then obtain:

$$\text{prox}_{\delta \rho_1 \|w_g^k\|_2}(w) = (I - D_k)w + D_k \text{prox}_{\delta \rho_1 \|w\|_2}(D_k w). \quad (114)$$

It is hence possible for each agent k to run (31)–(33). As long as at least one agent in the network is either fully-informed or data-informed, the weighted sum of costs across the network is strongly convex and assumptions 1 through 3 are satisfied. We conclude from Theorem 3 that all agents in the network will converge to the neighborhood of:

$$w^o = \arg \min_w \sum_{k \in \mathcal{F} \cup \mathcal{D}} p_k \left\{ \mathbb{E} \ln [1 + e^{-\gamma_k \mathbf{h}_{k,i}^\top w}] \right\} + \rho_2 \cdot \sum_{k \in \mathcal{F} \cup \mathcal{D}} p_k \|w\|_2^2 + \sum_{k \in \mathcal{F} \cup \mathcal{S}} p_k \|w_g^k\|_2 \quad (115)$$

This classifier minimizes the weighted average logistic cost across the network, hence incorporating data from all agents, regularized by the ℓ_2 -norm and weighted group Lasso. Through local interactions, both data and structural information is diffused across the entire network, allowing all agents, irrespective of their type and available information, to arrive at an accurate classification decision.

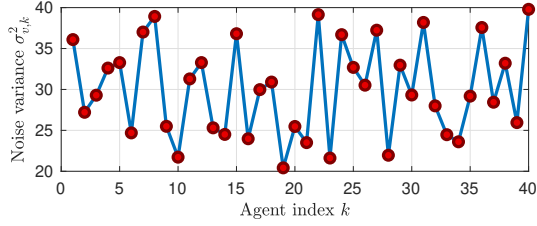


Fig. 2. Noise profile across the network for training (if $k \in \mathcal{F} \cup \mathcal{D}$) and testing.

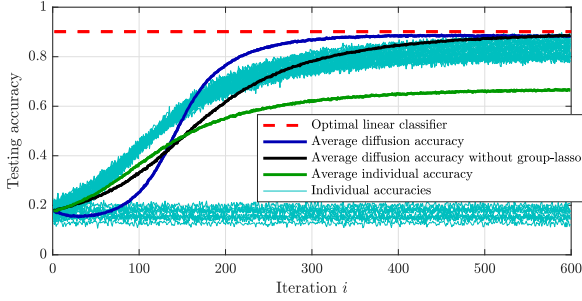


Fig. 3. Testing accuracy is evaluated at every iteration for each of the strategies under consideration by generating a separate test set and evaluating $\text{Prob}\{\text{sign}\{\mathbf{h}^\top \mathbf{w}_{k,i}\} \neq \gamma\}$. Parameters are set to $\rho_1 = 100$, $\rho_2 = 0.2$, $\mu = \delta = 0.01$.

C. Numerical Results

Performance is illustrated on the network depicted in Fig. 1, consisting of a total of $N = 40$ agents, 20 of which are data-informed and 10 each of which are fully and structure informed respectively. The network is heterogeneous in both the types of available information and the noise profile of feature realizations, when data is available. Features are generated as

$$\mathbf{h}_{k,i} = \gamma_k(i) (1 \ 1 \ \cdots \ 0 \ 0)^\top + \mathbf{v}_k(i) \quad (116)$$

where $\mathbf{v}_k(i) \sim \mathcal{N}(0, \sigma_{v,k}^2)$ and $(1 \ 1 \ \cdots \ 0 \ 0)^\top$ consists of 50 leading 1's followed by 50 trailing 0's. It is evident, that all class information is contained in the first half of the feature vector. This information is dispersed across the network as follows. The noise profile across the network is depicted in Fig. 2.

Each agent with $k \in \mathcal{F} \cup \mathcal{S}$, i.e., fully and data-informed agents, are supplied with 5 indices, chosen uniformly at random, of irrelevant feature covariates. They use this information to augment their cost by an appropriate regularization as in (109) and (113).

The evolution of performance is illustrated in Fig. 3. We observe that the diffusion strategy with structured sparsity regularization quickly approaches the performance of the optimal linear classifier. The rate of convergence is reduced in the absence of regularization. Finally, when no cooperation takes place, and hence information does not diffuse across the network, agents without access to observations, and those with noisy data, perform significantly worse than the cooperative strategy. We display the learning curve of the proposed strategy in Fig. 4 and observe improved steady-state performance when compared to a subgradient based strategy [12].

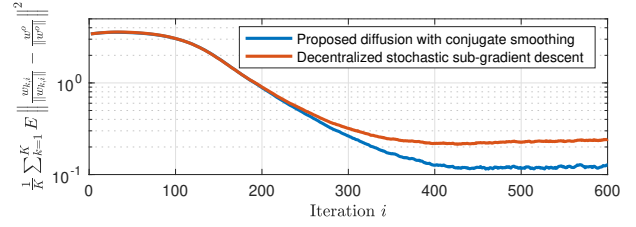


Fig. 4. Comparison between the proposed strategy and the decentralized stochastic subgradient descent algorithm [12]. Parameters are set to $\rho_1 = 100$, $\rho_2 = 0.2$, $\mu = \delta = 0.01$. Hyperplanes are normalized before comparison since $\text{sign}\{\mathbf{h}^\top \mathbf{w}\}$ is independent of the magnitude of \mathbf{w} .

V. CONCLUSION

We have proposed a decentralized strategy for stochastic optimization with non-smooth regularization terms, which are only required to be closed and convex. Using conjugate smoothing, the proposed algorithm transforms the non-smooth optimization problem into a smooth approximation, where the trade-off between smoothness and bias are controlled through the smoothing parameters δ . We proceeded to conduct a detailed performance analysis clarifying the effects of the smoothing procedure on the bias of the smoothed objective as well as the rate of convergence and limiting performance of the iterative strategy.

APPENDIX A PROOF OF THEOREM 2

For ease of exposition, let us introduce:

$$F(w) \triangleq \sum_{k=1}^N p_k \{J_k(w) + R_k(w)\} \quad (117)$$

$$F^\delta(w) \triangleq \sum_{k=1}^N p_k \{J_k(w) + R_k^\delta(w)\} \quad (118)$$

We establish a string of inequalities around the difference in function values $F(w^o) - F^\delta(w_\delta^o)$. We first express the difference in aggregate function values in terms of the differentiable and non-differentiable local components, respectively:

$$\begin{aligned} & F(w^o) - F^\delta(w_\delta^o) \\ &= \sum_{k=1}^N p_k \{J_k(w^o) + R_k(w^o)\} - \sum_{k=1}^N p_k \{J_k(w_\delta^o) + R_k^\delta(w_\delta^o)\} \\ &= \sum_{k=1}^N p_k \{J_k(w^o) - J_k(w_\delta^o)\} + \sum_{k=1}^N p_k \{R_k(w^o) - R_k^\delta(w_\delta^o)\} \\ &\stackrel{(a)}{=} \sum_{k=1}^N p_k \{J_k(w^o) - J_k(w_\delta^o)\} \\ &\quad + \sum_{k=1}^N p_k \left\{ R_k(w^o) - \max_u (u^\top w_\delta^o - R_k^*(u) - \delta d(u)) \right\} \\ &\stackrel{(b)}{=} \sum_{k=1}^N p_k \{J_k(w^o) - J_k(w_\delta^o)\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^N p_k \left\{ R_k(w^o) - \nabla R_k^\delta(w_\delta^o)^\top w_\delta^o + R_k^*(\nabla R_k^\delta(w_\delta^o)) \right. \\
& \quad \left. + \delta d(\nabla R_k^\delta(w_\delta^o)) \right\} \quad (119)
\end{aligned}$$

Here, (a) follows from the definition of the smooth approximation (14) and (b) follows from the expression for the gradient of the smooth approximation (16). We can then proceed to bound:

$$\begin{aligned}
& F(w^o) - F^\delta(w_\delta^o) \\
& \stackrel{(a)}{\geq} \sum_{k=1}^N p_k \{J_k(w^o) - J_k(w_\delta^o)\} \\
& + \sum_{k=1}^N p_k \left\{ \nabla R_k^\delta(w_\delta^o)^\top w^o - \nabla R_k^\delta(w_\delta^o)^\top w_\delta^o + \delta d(\nabla R_k^\delta(w_\delta^o)) \right\} \\
& \stackrel{(b)}{\geq} \sum_{k=1}^N p_k \nabla J_k(w_\delta^o)^\top (w^o - w_\delta^o) + \frac{\lambda_L}{2} \|w^o - w_\delta^o\|^2 \\
& + \sum_{k=1}^N p_k \left\{ \nabla R_k^\delta(w_\delta^o)^\top w^o - \nabla R_k^\delta(w_\delta^o)^\top w_\delta^o + \delta d(\nabla R_k^\delta(w_\delta^o)) \right\} \\
& = \sum_{k=1}^N p_k (\nabla J_k(w_\delta^o) + \nabla R_k^\delta(w_\delta^o))^\top (w^o - w_\delta^o) \\
& \quad + \frac{\lambda_L}{2} \|w^o - w_\delta^o\|^2 + \sum_{k=1}^N p_k \delta d(\nabla R_k^\delta(w_\delta^o)) \\
& \stackrel{(c)}{=} \frac{\lambda_L}{2} \|w^o - w_\delta^o\|^2 + \sum_{k=1}^N p_k \delta d(\nabla R_k^\delta(w_\delta^o)) \quad (120)
\end{aligned}$$

where (a) follows from the property $R^*(x) \triangleq \sup_u (u^\top x - R(u)) \geq y^\top x - R(y) \forall x, y$ with $x = \nabla R_k^\delta(w_\delta^o)$ and $y = w^o$. Step (b) follows from the aggregate strong convexity (9) and (c) follows from the definition of w_δ^o and the minimizer of the smoothed aggregate cost.

To prove the upper bound, we bound the bias for each agent individually. To begin with, note that convexity of $J_k(\cdot)$ and $R_k(\cdot)$ yields for all $r_k(w^o) \in \partial R_k(w^o)$:

$$J_k(w_\delta^o) - J_k(w^o) \geq (\nabla J_k(w^o))^\top (w_\delta^o - w^o)$$

$$\iff J_k(w^o) - J_k(w_\delta^o) \leq (\nabla J_k(w^o))^\top (w^o - w_\delta^o) \quad (121)$$

$$R_k(u) - R_k(w^o) \geq (r_k(w^o))^\top (u - w^o) \quad (122)$$

Then,

$$\begin{aligned}
& J_k(w^o) + R_k(w^o) - J_k(w_\delta^o) - R_k^\delta(w_\delta^o) \\
& = J_k(w^o) + R_k(w^o) - J_k(w_\delta^o) \\
& \quad - \min_u \left\{ R_k(u) + \delta d^* \left(\frac{w_\delta^o - u}{\delta} \right) \right\} \\
& = J_k(w^o) - J_k(w_\delta^o) \\
& \quad - \min_u \left\{ R_k(u) - R_k(w^o) + \delta d^* \left(\frac{w_\delta^o - u}{\delta} \right) \right\} \\
& \leq (\nabla J_k(w^o))^\top (w^o - w_\delta^o) \\
& \quad - \min_u \left\{ (r_k(w^o))^\top (u - w^o) + \delta d^* \left(\frac{w_\delta^o - u}{\delta} \right) \right\} \\
& = (\nabla J_k(w^o) + r_k(w^o))^\top (w^o - w_\delta^o)
\end{aligned}$$

$$\begin{aligned}
& - \min_u \left\{ (r_k(w^o))^\top (u - w_\delta^o) + \delta d^* \left(\frac{w_\delta^o - u}{\delta} \right) \right\} \\
& \stackrel{(a)}{=} (\nabla J_k(w^o) + r_k(w^o))^\top (w^o - w_\delta^o) \\
& \quad - \delta \min_v \left\{ -(r_k(w^o))^\top v + d^*(v) \right\} \\
& = (\nabla J_k(w^o) + r_k(w^o))^\top (w^o - w_\delta^o) \\
& \quad + \delta \max_v \left\{ (r_k(w^o))^\top v - d^*(v) \right\} \\
& \stackrel{(b)}{=} (\nabla J_k(w^o) + r_k(w^o))^\top (w^o - w_\delta^o) + \delta d(r_k(w^o)) \quad (123)
\end{aligned}$$

where (a) follows after a change of variables $v \triangleq \frac{w_\delta^o - u}{\delta}$ and (b) is a result of the definition of the conjugate function. Returning to the aggregate cost, we then have

$$\begin{aligned}
& \sum_{k=1}^N p_k \{J_k(w^o) + R_k(w^o)\} - \sum_{k=1}^N p_k \{J_k(w_\delta^o) + R_k^\delta(w_\delta^o)\} \\
& = \sum_{k=1}^N p_k \{J_k(w^o) + R_k(w^o) - J_k(w_\delta^o) + R_k^\delta(w_\delta^o)\} \\
& \leq \sum_{k=1}^N p_k \left\{ (\nabla J_k(w^o) + r_k(w^o))^\top (w^o - w_\delta^o) \right\} \\
& \quad + \sum_{k=1}^N p_k \delta d(r_k(w^o)) \\
& = \left\{ \sum_{k=1}^N p_k (\nabla J_k(w^o) + r_k(w^o)) \right\}^\top (w^o - w_\delta^o) \\
& \quad + \sum_{k=1}^N p_k \delta d(r_k(w^o)) \quad (124)
\end{aligned}$$

By definition, w^o is the minimizer of $\sum_{k=1}^N p_k \{J_k(w^o) + R_k(w^o)\}$, so there exist subgradients $r_k^o \in \partial R_k(w^o)$, such that

$$\sum_{k=1}^N p_k (\nabla J_k(w^o) + r_k^o) = 0 \quad (125)$$

Then,

$$\begin{aligned}
& \sum_{k=1}^N p_k \{J_k(w^o) + R_k(w^o)\} - \sum_{k=1}^N p_k \{J_k(w_\delta^o) + R_k^\delta(w_\delta^o)\} \\
& \leq \sum_{k=1}^N p_k \delta d(r_k^o) = O(\delta) \quad (126)
\end{aligned}$$

We conclude from (120):

$$\begin{aligned}
& \frac{\lambda_L}{2} \|w^o - w_\delta^o\|^2 + \sum_{k=1}^N p_k \delta d(\nabla R_k^\delta(w_\delta^o)) \\
& \leq F(w^o) - F^\delta(w_\delta^o) \leq \sum_{k=1}^N p_k \delta d(r_k^o) \quad (127)
\end{aligned}$$

Eq. (23) follows after rearranging. To obtain (25), note that:

$$\begin{aligned}
& \frac{\delta}{2} p_k \|\nabla R_k^\delta(w_\delta^o)\|^2 \\
& \stackrel{(13)}{\leq} p_k \delta d(\nabla R_k^\delta(w_\delta^o))
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\lambda_L}{2} \|w^o - w_\delta^o\|^2 + \sum_{\ell=1}^N p_\ell \delta d (\nabla R_\ell^\delta(w_\delta^o)) \\
&\leq \sum_{\ell=1}^N p_\ell \delta d (r_\ell^o)
\end{aligned} \tag{128}$$

After rearranging and cancelling δ , we find (25).

APPENDIX B PROOF OF LEMMA 2

Let α be an arbitrary real number such that $0 < \alpha < 1$. Then

$$\begin{aligned}
&\|T_c(x) - T_c(y)\|^2 \\
&= \left\| x - y - \mu \sum_{k=1}^N p_k \left\{ \nabla J_k(x) - \nabla J_k(y) \right. \right. \\
&\quad \left. \left. + \nabla R_k^\delta(x) - \nabla R_k^\delta(y) \right\} \right\|^2 \\
&= \|x - y\|^2 + \mu^2 \left\| \sum_{k=1}^N p_k \left\{ \nabla J_k(x) - \nabla J_k(y) \right. \right. \\
&\quad \left. \left. + \nabla R_k^\delta(x) - \nabla R_k^\delta(y) \right\} \right\|^2 \\
&\quad - 2\mu \sum_{k=1}^N p_k (x - y)^\top (\nabla J_k(x) - \nabla J_k(y)) \\
&\quad - 2\mu \sum_{k=1}^N p_k (x - y)^\top (\nabla R_k^\delta(x) - \nabla R_k^\delta(y)) \\
&\stackrel{(a)}{\leq} \|x - y\|^2 + \mu^2 \sum_{k=1}^N p_k \left\| \nabla J_k(x) - \nabla J_k(y) \right. \\
&\quad \left. + \nabla R_k^\delta(x) - \nabla R_k^\delta(y) \right\|^2 \\
&\quad - 2\mu\lambda_L \|x - y\|^2 - 2\mu\delta \sum_{k=1}^N p_k \left\| \nabla R_k^\delta(x) - \nabla R_k^\delta(y) \right\|^2 \\
&\stackrel{(b)}{\leq} \|x - y\|^2 + \mu^2 \sum_{k=1}^N p_k \frac{1}{\alpha} \left\| \nabla J_k(x) - \nabla J_k(y) \right\|^2 \\
&\quad + \mu^2 \sum_{k=1}^N p_k \frac{1}{1-\alpha} \left\| \nabla R_k^\delta(x) - \nabla R_k^\delta(y) \right\|^2 - 2\mu\lambda_L \|x - y\|^2 \\
&\quad - 2\mu\delta \sum_{k=1}^N p_k \left\| \nabla R_k^\delta(x) - \nabla R_k^\delta(y) \right\|^2
\end{aligned} \tag{129}$$

where (a) follows from Jensen's inequality, strong convexity (9), and co-coercivity (17), and (b) from $\|a + b\|^2 \leq \frac{1}{\alpha} \|a\|^2 + \frac{1}{1-\alpha} \|b\|^2$ for any $a, b \in \mathbb{R}^M$. Since, by assumption, $\mu < 2\delta$, we select $\alpha = 1 - \frac{\mu}{2\delta}$. This results in $\frac{\mu^2}{1-\alpha} = 2\mu\delta$ and allows us to cancel all terms involving $\nabla_w R_k^\delta(\cdot)$ in the above inequality. Hence,

$$\begin{aligned}
&\|T_c(x) - T_c(y)\|^2 \\
&\leq \|x - y\|^2 + \mu^2 \sum_{k=1}^N p_k \frac{1}{1 - \frac{\mu}{2\delta}} \left\| \nabla J_k(x) - \nabla J_k(y) \right\|^2 \\
&\quad - 2\mu\lambda_L \|x - y\|^2
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(a)}{\leq} \|x - y\|^2 + \frac{\mu^2 \lambda_U^2}{1 - \frac{\mu}{2\delta}} \|x - y\|^2 - 2\mu\lambda_L \|x - y\|^2 \\
&= \left(1 - 2\mu\lambda_L + \mu^2 \frac{\lambda_U^2}{1 - \frac{\mu}{2\delta}} \right) \|x - y\|^2 \\
&\stackrel{(b)}{\leq} \left(1 - \mu\lambda_L + \mu^2 \frac{\lambda_U^2}{2 - \frac{\mu}{\delta}} \right)^2 \|x - y\|^2
\end{aligned} \tag{130}$$

where (a) is due to the Lipschitz property (8) and (b) is due to $1 - a \leq (1 - \frac{1}{2}a)^2$ for all $a \in \mathbb{R}$. From Banach's fixed-point theorem, we know that as long as $\gamma_c < 1$, $w_i = T_c(w_{i-1})$ converges exponentially to a unique fixed point, which satisfies $w_\infty = T_c(w_\infty)$. From (42), we conclude that

$$\sum_{k=1}^N p_k \nabla J_k(w_\infty) + \sum_{k=1}^N p_k \nabla R_k^\delta(w_\infty) = 0 \tag{131}$$

so that from (7), $w_\infty = w_\delta^o$.

APPENDIX C PROOF OF LEMMA 3

The proof of the first three inequalities relies on the Lipschitz properties of the gradients and the decomposition (55)–(56). First, we bound the terms arising from the disagreement across the network. Denote the k -th element of $P[\cdot]$ by $P_{(k)}[\cdot]$. Then

$$\begin{aligned}
&P_{(k)}[\mathbf{t}_{i-1}] \\
&= \mathbb{E} \left\| \nabla J_k(\mathbf{w}_{c,i-1}) - \nabla J_k(\mathbf{w}_{k,i-1}) \right. \\
&\quad \left. + \nabla R_k^\delta(\mathbf{w}_{c,i-1} - \mu \nabla J_k(\mathbf{w}_{c,i-1})) \right. \\
&\quad \left. - \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \nabla J_k(\mathbf{w}_{k,i-1})) \right\|^2 \\
&\stackrel{(a)}{\leq} 2 \mathbb{E} \left\| \nabla J_k(\mathbf{w}_{c,i-1}) - \nabla J_k(\mathbf{w}_{k,i-1}) \right\|^2 \\
&\quad + 2 \mathbb{E} \left\| \nabla R_k^\delta(\mathbf{w}_{c,i-1} - \mu \nabla J_k(\mathbf{w}_{c,i-1})) \right. \\
&\quad \left. - \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \nabla J_k(\mathbf{w}_{k,i-1})) \right\|^2 \\
&\stackrel{(b)}{\leq} 2\lambda_U^2 \mathbb{E} \left\| \mathbf{w}_{c,i-1} - \mathbf{w}_{k,i-1} \right\|^2 \\
&\quad + \frac{2}{\delta^2} \mathbb{E} \left\| \mathbf{w}_{c,i-1} - \mu \nabla J_k(\mathbf{w}_{c,i-1}) \right. \\
&\quad \left. - \mathbf{w}_{k,i-1} + \mu \nabla J_k(\mathbf{w}_{k,i-1}) \right\|^2 \\
&\stackrel{(c)}{\leq} \left(2\lambda_U^2 + 4 \frac{1 + \mu^2}{\delta^2} \right) \mathbb{E} \left\| \mathbf{w}_{c,i-1} - \mathbf{w}_{k,i-1} \right\|^2 \\
&\stackrel{(d)}{=} \left(2\lambda_U^2 + 4 \frac{1 + \mu^2}{\delta^2} \right) \mathbb{E} \left\| (v_{L,k} \otimes I_M) \mathbf{w}_{e,i-1} \right\|^2 \\
&\leq \left(2\lambda_U^2 + 4 \frac{1 + \mu^2}{\delta^2} \right) \nu^2 \mathbb{E} \left\| \mathbf{w}_{e,i-1} \right\|^2 \\
&= \left(2\lambda_U^2 + 4 \frac{1 + \mu^2}{\delta^2} \right) \nu^2 \mathbf{1}^\top P[\mathbf{w}_{e,i-1}]
\end{aligned} \tag{132}$$

where (a) is due Jensen's inequality, (b) and (c) are due to Lipschitz continuity of the gradients and (d) is due to $\mathbf{w}_i = \mathbf{1} \otimes \mathbf{w}_{c,i} + \mathcal{V}_L \mathbf{w}_{e,i}$. Stacking both sides of the above inequality yields (68).

Now consider \mathbf{u}_{i-1} , which arises from the incremental implementation:

$$\begin{aligned}
&P_{(k)}[\mathbf{u}_{i-1}] \\
&= \mathbb{E} \left\| \nabla R_k^\delta(\mathbf{w}_{k,i-1}) - \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \nabla J_k(\mathbf{w}_{k,i-1})) \right\|^2
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(a)}{\leq} \frac{\mu^2}{\delta^2} \mathbb{E} \|\nabla J_k(\mathbf{w}_{k,i-1})\|^2 \\
&= \frac{\mu^2}{\delta^2} \mathbb{E} \|\nabla J_k(\mathbf{w}_{k,i-1}) - \nabla J_k(\mathbf{w}_{c,i-1}) + \nabla J_k(\mathbf{w}_{c,i-1}) \\
&\quad - \nabla J_k(w_\delta^o) + \nabla J_k(w_\delta^o)\|^2 \\
&\stackrel{(b)}{\leq} \frac{\mu^2}{\delta^2} (3\lambda_U^2 \nu^2 \mathbf{1}^\top P[\boldsymbol{\mathcal{W}}_{e,i-1}] + 3\lambda_U^2 \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 \\
&\quad + 3\|\nabla J_k(w_\delta^o)\|^2) \tag{133}
\end{aligned}$$

where (a) is due to Lipschitz continuity of $\nabla R_k^\delta(w)$ and (b) is due to Jensen's inequality and Lipschitz continuity of $\nabla J_k(w)$. Upon stacking we obtain (69).

Next, we bound the perturbations caused by the gradient noise $\mathbf{s}_{k,i}(\mathbf{w}_{k,i}) = \widehat{\nabla J}_k(\mathbf{w}_{k,i-1}) - \nabla J_k(\mathbf{w}_{k,i-1})$. While a loose upper bound can be obtained immediately from Jensen's inequality, it turns out that the incremental implementation (32) along with the co-coercivity (17) of $\nabla R_k^\delta(w)$ have a variance reducing effect on the recursion. We begin by expanding:

$$\begin{aligned}
&P_{(k)}[\mathbf{s}_i^g + \mathbf{s}_i^p - \mathbb{E} \mathbf{s}_i^p] \\
&\stackrel{(a)}{\leq} P_{(k)}[\mathbf{s}_i^g + \mathbf{s}_i^p] \\
&= \mathbb{E} \left\| \nabla J_k(\mathbf{w}_{k,i-1}) - \widehat{\nabla J}_k(\mathbf{w}_{k,i-1}) \right\|^2 \\
&\quad + \mathbb{E} \left\| \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \nabla J_k(\mathbf{w}_{k,i-1})) \right. \\
&\quad \quad \left. - \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \widehat{\nabla J}_k(\mathbf{w}_{k,i-1})) \right\|^2 \\
&\quad + 2 \mathbb{E} \left(\nabla J_k(\mathbf{w}_{k,i-1}) - \widehat{\nabla J}_k(\mathbf{w}_{k,i-1}) \right)^\top \\
&\quad \quad \times \left(\nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \nabla J_k(\mathbf{w}_{k,i-1})) \right. \\
&\quad \quad \quad \left. - \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \widehat{\nabla J}_k(\mathbf{w}_{k,i-1})) \right) \\
&= \mathbb{E} \left\| \nabla J_k(\mathbf{w}_{k,i-1}) - \widehat{\nabla J}_k(\mathbf{w}_{k,i-1}) \right\|^2 \\
&\quad + \mathbb{E} \left\| R_k^\delta(\mathbf{w}_{k,i-1} - \mu \nabla J_k(\mathbf{w}_{k,i-1})) \right. \\
&\quad \quad \left. - \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \widehat{\nabla J}_k(\mathbf{w}_{k,i-1})) \right\|^2 \\
&\quad - \frac{2}{\mu} \mathbb{E} \left(\mathbf{w}_{k,i-1} - \mu \nabla J_k(\mathbf{w}_{k,i-1}) \right. \\
&\quad \quad \left. - (\mathbf{w}_{k,i-1} - \mu \widehat{\nabla J}_k(\mathbf{w}_{k,i-1})) \right)^\top \\
&\quad \quad \times \left(\nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \nabla J_k(\mathbf{w}_{k,i-1})) \right. \\
&\quad \quad \quad \left. - \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \widehat{\nabla J}_k(\mathbf{w}_{k,i-1})) \right) \tag{134}
\end{aligned}$$

where (a) follows from $\mathbb{E} \|\mathbf{x} - \mathbb{E} \mathbf{x}\|^2 \leq \mathbb{E} \|\mathbf{x}\|^2$. We now leverage co-coercivity (17), the step-size condition $\mu < 2\delta$ and the gradient noise condition (67b) to bound:

$$\begin{aligned}
&P_{(k)}[\mathbf{s}_i^g + \mathbf{s}_i^p - \mathbb{E} \mathbf{s}_i^p] \\
&\stackrel{(a)}{\leq} \mathbb{E} \left\| \nabla J_k(\mathbf{w}_{k,i-1}) - \widehat{\nabla J}_k(\mathbf{w}_{k,i-1}) \right\|^2 \\
&\quad + \mathbb{E} \left\| R_k^\delta(\mathbf{w}_{k,i-1} - \mu \nabla J_k(\mathbf{w}_{k,i-1})) \right. \\
&\quad \quad \left. - \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \widehat{\nabla J}_k(\mathbf{w}_{k,i-1})) \right\|^2
\end{aligned}$$

$$\begin{aligned}
&- \frac{2\delta}{\mu} \mathbb{E} \left\| \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \nabla J_k(\mathbf{w}_{k,i-1})) \right. \\
&\quad \quad \left. - \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \widehat{\nabla J}_k(\mathbf{w}_{k,i-1})) \right\|^2 \\
&= \mathbb{E} \left\| \nabla J_k(\mathbf{w}_{k,i-1}) - \widehat{\nabla J}_k(\mathbf{w}_{k,i-1}) \right\|^2 \\
&\quad - \left(\frac{2\delta}{\mu} - 1 \right) \mathbb{E} \left\| R_k^\delta(\mathbf{w}_{k,i-1} - \mu \nabla J_k(\mathbf{w}_{k,i-1})) \right. \\
&\quad \quad \left. - \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \widehat{\nabla J}_k(\mathbf{w}_{k,i-1})) \right\|^2 \\
&\stackrel{(b)}{\leq} \mathbb{E} \left\| \nabla J_k(\mathbf{w}_{k,i-1}) - \widehat{\nabla J}_k(\mathbf{w}_{k,i-1}) \right\|^2 \\
&= \mathbb{E} \|\mathbf{s}_{k,i}(\mathbf{w}_{k,i-1})\|^2 \\
&\stackrel{(c)}{\leq} \beta^2 \mathbb{E} \|\mathbf{w}_{k,i-1}\|^2 + \sigma^2 \tag{135}
\end{aligned}$$

where (a) follows from co-coercivity (17), (b) follows from $\mu < 2\delta$ and (c) is due to (67b). Now from $\mathbf{w}_{k,i-1} = \mathbf{w}_{c,i-1} + (v_{L,k} \otimes I) \boldsymbol{\mathcal{W}}_{e,i-1}$, we can bound

$$\begin{aligned}
&\|\mathbf{w}_{k,i-1}\|^2 \\
&= \|\mathbf{w}_{c,i-1} + (v_{L,k} \otimes I) \boldsymbol{\mathcal{W}}_{e,i-1}\|^2 \\
&= \|\mathbf{w}_{c,i-1} - w_\delta^o + (v_{L,k} \otimes I) \boldsymbol{\mathcal{W}}_{e,i-1} + w_\delta^o\|^2 \\
&\leq 3\|\tilde{\mathbf{w}}_{c,i-1}\|^2 + 3\nu^2 \mathbf{1}^\top P[\boldsymbol{\mathcal{W}}_{e,i-1}] + 3\|w_\delta^o\|^2. \tag{136}
\end{aligned}$$

where we appealed to Jensen's inequality again. Eq. (70) follows after stacking. Next, note that because $\|\mathbb{E} \mathbf{x}\|^2 \leq \mathbb{E} \|\mathbf{x}\|^2$

$$P[\mathbb{E} \mathbf{s}_i^p] \preceq P[\mathbf{s}_i^p]. \tag{137}$$

Subsequently,

$$\begin{aligned}
P_{(k)}[\mathbf{s}_i^p] &= \mathbb{E} \left\| \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \nabla J_k(\mathbf{w}_{k,i-1})) \right. \\
&\quad \left. - \nabla R_k^\delta(\mathbf{w}_{k,i-1} - \mu \widehat{\nabla J}_k(\mathbf{w}_{k,i-1})) \right\|^2 \\
&\stackrel{(a)}{\leq} \frac{\mu^2}{\delta^2} \mathbb{E} \left\| \nabla J_k(\mathbf{w}_{k,i-1}) - \widehat{\nabla J}_k(\mathbf{w}_{k,i-1}) \right\|^2 \\
&= \frac{\mu^2}{\delta^2} \|\mathbf{s}_{k,i}(\mathbf{w}_{k,i-1})\|^2 \tag{138}
\end{aligned}$$

where (a) is due to (18), so that similarly to the above

$$\begin{aligned}
P[\mathbb{E} \mathbf{s}_i^p] &\preceq 3\beta^2 \frac{\mu^2}{\delta^2} P[\mathbf{1} \otimes \tilde{\boldsymbol{\mathcal{W}}}_{c,i-1}] + 3\beta^2 \frac{\mu^2}{\delta^2} \nu^2 \mathbf{1} \mathbf{1}^\top P[\boldsymbol{\mathcal{W}}_{e,i-1}] \\
&\quad + 3\beta^2 \frac{\mu^2}{\delta^2} P[\mathbf{1} \otimes w_\delta^o] + \frac{\mu^2}{\delta^2} \sigma^2 \mathbf{1} \tag{139}
\end{aligned}$$

which is (71). Next,

$$\begin{aligned}
&P_{(k)}[g(\mathbf{1} \otimes \mathbf{w}_{c,i-1})] \\
&= \mathbb{E} \|\nabla J_k(\mathbf{w}_{c,i-1})\|^2 \\
&= \mathbb{E} \|\nabla J_k(\mathbf{w}_{c,i-1}) - \nabla J_k(w_\delta^o) + \nabla J_k(w_\delta^o)\|^2 \\
&\leq 2\lambda_U^2 \mathbb{E} \|\mathbf{w}_{c,i-1} - w_\delta^o\|^2 + 2\|\nabla J_k(w_\delta^o)\|^2 \tag{140}
\end{aligned}$$

which implies (72) after stacking. Eq. (73) follows analogously.

APPENDIX D PROOF OF LEMMA 4

Lemma 3 provides bounds on the perturbation terms appearing in the recursive relations (64) and (65). We now leverage these relations to recursively bound the evolution of the coupled

recursions (64)–(65). We make use of Jensen's inequality $\|x+y\|^2 \leq \frac{1}{\alpha}\|x\|^2 + \frac{1}{1-\alpha}\|y\|^2$ for all x, y and $0 < \alpha < 1$ and expand (64):

$$\begin{aligned}
& \mathbb{E} \|\tilde{\mathbf{w}}_{c,i}\|^2 \\
&= \mathbb{E} \left\| T_c(\mathbf{w}_{c,i-1}) - T_c(w_\delta^o) \right. \\
&\quad \left. + \mu (p^\top \otimes I_M) (\mathbf{t}_{i-1} + \mathbf{u}_{i-1} + \mathbf{s}_i - \mathbb{E} \mathbf{s}_i + \mathbb{E} \mathbf{s}_i) \right\|^2 \\
&\stackrel{(a)}{=} \mathbb{E} \left\| T_c(\mathbf{w}_{c,i-1}) - T_c(w_\delta^o) \right. \\
&\quad \left. + \mu (p^\top \otimes I_M) (\mathbf{t}_{i-1} + \mathbf{u}_{i-1} + \mathbb{E} \mathbf{s}_i) \right\|^2 \\
&\quad + \mu^2 \mathbb{E} \left\| (p^\top \otimes I_M) (\mathbf{s}_i - \mathbb{E} \mathbf{s}_i) \right\|^2 \\
&\stackrel{(b)}{\leq} \frac{1}{\gamma_c} \mathbb{E} \left\| T_c(\mathbf{w}_{c,i-1}) - T_c(w_\delta^o) \right\|^2 \\
&\quad + \frac{\mu^2}{1-\gamma_c} \mathbb{E} \left\| (p^\top \otimes I_M) (\mathbf{t}_{i-1} + \mathbf{u}_{i-1} + \mathbb{E} \mathbf{s}_i) \right\|^2 \\
&\quad + \mu^2 \mathbb{E} \left\| (p^\top \otimes I_M) (\mathbf{s}_i - \mathbb{E} \mathbf{s}_i) \right\|^2 \\
&\stackrel{(c)}{\leq} \gamma_c \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 \\
&\quad + \frac{\mu^2}{1-\gamma_c} \mathbb{E} \left\| (p^\top \otimes I_M) (\mathbf{t}_{i-1} + \mathbf{u}_{i-1} + \mathbb{E} \mathbf{s}_i) \right\|^2 \\
&\quad + \mu^2 \mathbb{E} \left\| (p^\top \otimes I_M) (\mathbf{s}_i - \mathbb{E} \mathbf{s}_i) \right\|^2 \\
&\stackrel{(d)}{\leq} \gamma_c \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 + \frac{\mu^2}{1-\gamma_c} p^\top P [\mathbf{t}_{i-1} + \mathbf{u}_{i-1} + \mathbb{E} \mathbf{s}_i] \\
&\quad + \mu^2 p^\top P [\mathbf{s}_i - \mathbb{E} \mathbf{s}_i] \\
&\stackrel{(e)}{\leq} \gamma_c \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 + \frac{3\mu^2}{1-\gamma_c} p^\top (P[\mathbf{t}_{i-1}] + P[\mathbf{u}_{i-1}] + P[\mathbb{E} \mathbf{s}_i]) \\
&\quad + \mu^2 p^\top P [\mathbf{s}_i - \mathbb{E} \mathbf{s}_i] \tag{141}
\end{aligned}$$

In step (a), cross-terms are eliminated because $\mathbb{E} \{\mathbf{s}_i - \mathbb{E} \mathbf{s}_i\} = 0$. Step (b) is due to $\gamma_c < 1$ and Jensen's inequality, (c) is due to Lemma 2, (d) and (e) follow from Jensen's inequality. We observe that $\mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2$ contracts at a rate given by γ_c , but is subject to a number of perturbations. The effect of the additive perturbations can be bounded using the relations established in Lemma 3:

$$\begin{aligned}
& \mathbb{E} \|\tilde{\mathbf{w}}_{c,i}\|^2 \\
&\stackrel{(a)}{\leq} \gamma_c \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 \\
&\quad + \frac{3\mu^2}{1-\gamma_c} p^\top \left((2\lambda_U^2 + \frac{1+\mu^2}{\delta^2}) \nu^2 \mathbf{1} \mathbf{1}^\top P [\mathbf{w}_{e,i-1}] \right. \\
&\quad \left. + \frac{\mu^2}{\delta^2} (3\lambda_U^2 \nu^2 \mathbf{1} \mathbf{1}^\top P [\mathbf{w}_{e,i-1}] + 3\lambda_U^2 P [\mathbf{1} \otimes \tilde{\mathbf{w}}_{c,i-1}] \right. \\
&\quad \left. + 3P [g(\mathbf{1} \otimes w_\delta^o)]) \right. \\
&\quad \left. + 3\beta^2 \frac{\mu^2}{\delta^2} P [\mathbf{1} \otimes \tilde{\mathbf{w}}_{c,i-1}] + 3\beta^2 \frac{\mu^2}{\delta^2} \nu^2 \mathbf{1} \mathbf{1}^\top P [\mathbf{w}_{e,i-1}] \right. \\
&\quad \left. + 3\beta^2 \frac{\mu^2}{\delta^2} P [\mathbf{1} \otimes w_\delta^o] + \frac{\mu^2}{\delta^2} \sigma^2 \mathbf{1} \right) \\
&\quad + \mu^2 p^\top \left(3\beta^2 P [\mathbf{1} \otimes \tilde{\mathbf{w}}_{c,i-1}] + 3\beta^2 \nu^2 \mathbf{1} \mathbf{1}^\top P [\mathbf{w}_{e,i-1}] \right. \\
&\quad \left. + 3\beta^2 P [\mathbf{1} \otimes w_\delta^o] + \sigma^2 \mathbf{1} \right)
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(b)}{=} \gamma_c \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 + \frac{3\mu^2}{1-\gamma_c} \left((2\lambda_U^2 + \frac{1+\mu^2}{\delta^2}) \nu^2 \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 \right. \\
&\quad \left. + \frac{\mu^2}{\delta^2} (3\lambda_U^2 \nu^2 \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 + 3\lambda_U^2 \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 \right. \\
&\quad \left. + 3\|g(w_\delta^o)\|^2) \right. \\
&\quad \left. + 3\beta^2 \frac{\mu^2}{\delta^2} \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 + 3\beta^2 \frac{\mu^2}{\delta^2} \nu^2 \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 \right. \\
&\quad \left. + 3\beta^2 \frac{\mu^2}{\delta^2} \|w_\delta^o\|^2 + \frac{\mu^2}{\delta^2} \sigma^2) \right. \\
&\quad \left. + \mu^2 (3\beta^2 \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 + 3\beta^2 \nu^2 \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 \right. \\
&\quad \left. + 3\beta^2 \|w_\delta^o\|^2 + \sigma^2) \right) \\
&\stackrel{(c)}{=} \left(\gamma_c + \frac{9\mu^4(\beta^2 + \lambda_U^2)}{(1-\gamma_c)\delta^2} + 3\mu^2\beta^2 \right) \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 \\
&\quad + \left(\frac{3\mu^2\nu^2}{1-\gamma_c} \left(2\lambda_U^2 + \frac{1+\mu^2+3\mu^2\lambda_U^2+3\mu^2\beta^2}{\delta^2} \right) \right. \\
&\quad \left. + 3\mu^2\beta^2\nu^2 \right) \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 \\
&\quad + \frac{3\mu^4}{(1-\gamma_c)\delta^2} (3\|g(w_\delta^o)\|^2 + 3\beta^2\|w_\delta^o\|^2 + \sigma^2) \\
&\quad + \mu^2 (3\beta^2\|w_\delta^o\|^2 + \sigma^2) \\
&\stackrel{(d)}{=} \left(\gamma_c + \frac{\mu^3}{\delta^2} \frac{9(\beta^2 + \lambda_U^2)}{\lambda_L - \mu \frac{\lambda_U^2}{2-\frac{\mu}{\delta}}} + 3\mu^2\beta^2 \right) \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 \\
&\quad + \left(\frac{\mu}{\delta^2} \frac{3\nu^2}{\lambda_L - \mu \frac{\lambda_U^2}{2-\frac{\mu}{\delta}}} + \mu \frac{6\nu^2}{\lambda_L - \mu \frac{\lambda_U^2}{2-\frac{\mu}{\delta}}} \lambda_U^2 \right. \\
&\quad \left. + \frac{\mu^3}{\delta^2} \frac{9\nu^2}{\lambda_L - \mu \frac{\lambda_U^2}{2-\frac{\mu}{\delta}}} (\lambda_U^2 + \beta^2) + 3\mu^2\beta^2\nu^2 \right) \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 \\
&\quad + \frac{\mu^3}{\delta^2} \frac{9}{\lambda_L - \mu \frac{\lambda_U^2}{2-\frac{\mu}{\delta}}} \|g(w_\delta^o)\|^2 \\
&\quad + \frac{\mu^3}{\delta^2} \frac{3}{\lambda_L - \mu \frac{\lambda_U^2}{2-\frac{\mu}{\delta}}} (3\beta^2\|w_\delta^o\|^2 + \sigma^2) \\
&\quad + \mu^2 (3\beta^2\|w_\delta^o\|^2 + \sigma^2) \tag{142}
\end{aligned}$$

The bounds from Lemma 3 are used in (a) and (b) is due to $\mathbf{1}^\top P[\mathbf{x}] = \mathbb{E} \|\mathbf{x}\|^2$ for $\mathbf{x} \in \mathbb{R}^{MN}$ and $p^\top P[\mathbf{1} \otimes \mathbf{y}] = \mathbb{E} \|\mathbf{y}\|^2$ for $\mathbf{y} \in \mathbb{R}^M$. In (c) and (d), the terms are rearranged to expose the dependence on μ and δ more clearly.

Now let us turn to the mean-square recursion of $\mathbf{w}_{e,i}$ in (65). First note that $\rho(\mathcal{J}_\epsilon) = \lambda_2(A) < 1$. Since \mathcal{J}_ϵ has a Jordan structure, this means that we can choose ϵ small enough, such that $\|\mathcal{J}_\epsilon\|_2 = \rho(\mathcal{J}_\epsilon^\top \mathcal{J}_\epsilon) \leq \|\mathcal{J}_\epsilon^\top \mathcal{J}_\epsilon\|_\infty < 1$. Then,

$$\begin{aligned}
& \mathbb{E} \|\mathbf{w}_{e,i}\|^2 \\
&= \mathbb{E} \left\| \mathcal{J}_\epsilon^\top \mathbf{w}_{e,i-1} + \mu \mathcal{J}_\epsilon^\top \mathcal{V}_R^\top (\mathbf{t}_{i-1} + \mathbf{u}_{i-1} + \mathbf{s}_i - \mathbb{E} \mathbf{s}_i \right. \\
&\quad \left. + \mathbb{E} \mathbf{s}_i - g(\mathbf{1} \otimes \mathbf{w}_{c,i-1}) - r(\mathbf{1} \otimes \mathbf{w}_{c,i-1})) \right\|^2 \\
&\stackrel{(a)}{=} \mathbb{E} \left\| \mathcal{J}_\epsilon^\top \mathbf{w}_{e,i-1} + \mu \mathcal{J}_\epsilon^\top \mathcal{V}_R^\top (\mathbf{t}_{i-1} + \mathbf{u}_{i-1} + \mathbb{E} \mathbf{s}_i \right. \\
&\quad \left. - g(\mathbf{1} \otimes \mathbf{w}_{c,i-1}) - r(\mathbf{1} \otimes \mathbf{w}_{c,i-1})) \right\|^2
\end{aligned}$$

$$\begin{aligned}
& + \mu^2 \mathbb{E} \left\| \mathcal{J}_\epsilon^\top \mathcal{V}_R^\top (\mathbf{s}_i - \mathbb{E} \mathbf{s}_i) \right\|^2 \\
\stackrel{(b)}{\leq} & \frac{1}{\|\mathcal{J}_\epsilon\|} \mathbb{E} \left\| \mathcal{J}_\epsilon^\top \mathbf{w}_{e,i-1} \right\|^2 + \frac{\mu^2}{1 - \|\mathcal{J}_\epsilon\|} \\
& \times \mathbb{E} \left\| \mathcal{J}_\epsilon^\top \mathcal{V}_R^\top (\mathbf{t}_{i-1} + \mathbf{u}_{i-1} + \mathbb{E} \mathbf{s}_i \right. \\
& \quad \left. - g(\mathbf{1} \otimes \mathbf{w}_{c,i-1}) - r(\mathbf{1} \otimes \mathbf{w}_{c,i-1})) \right\|^2 \\
& + \mu^2 \mathbb{E} \left\| \mathcal{J}_\epsilon^\top \mathcal{V}_R^\top (\mathbf{s}_i - \mathbb{E} \mathbf{s}_i) \right\|^2 \\
\stackrel{(c)}{\leq} & \|\mathcal{J}_\epsilon\| \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 + \frac{\mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{1 - \|\mathcal{J}_\epsilon\|} \\
& \times \mathbb{E} \left\| \mathbf{t}_{i-1} + \mathbf{u}_{i-1} + \mathbb{E} \mathbf{s}_i \right. \\
& \quad \left. - g(\mathbf{1} \otimes \mathbf{w}_{c,i-1}) - r(\mathbf{1} \otimes \mathbf{w}_{c,i-1}) \right\|^2 \\
& + \mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2 \mathbb{E} \|\mathbf{s}_i - \mathbb{E} \mathbf{s}_i\|^2 \\
\stackrel{(d)}{\leq} & \|\mathcal{J}_\epsilon\| \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 + \frac{25\mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{1 - \|\mathcal{J}_\epsilon\|} \\
& \times \left(\mathbb{E} \|\mathbf{t}_{i-1}\|^2 + \mathbb{E} \|\mathbf{u}_{i-1}\|^2 + \mathbb{E} \|\mathbb{E} \mathbf{s}_i^p\|^2 \right. \\
& \quad \left. + \mathbb{E} \|g(\mathbf{1} \otimes \mathbf{w}_{c,i-1})\|^2 + \mathbb{E} \|r(\mathbf{1} \otimes \mathbf{w}_{c,i-1})\|^2 \right) \\
& + \mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2 \mathbb{E} \|\mathbf{s}_i - \mathbb{E} \mathbf{s}_i\|^2 \\
\stackrel{(e)}{=} & \|\mathcal{J}_\epsilon\| \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 + \frac{25\mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{1 - \|\mathcal{J}_\epsilon\|} \\
& \times \mathbf{1}^\top \left(P[\mathbf{t}_{i-1}] + P[\mathbf{u}_{i-1}] + P[\mathbb{E} \mathbf{s}_i] \right. \\
& \quad \left. + P[g(\mathbf{1} \otimes \mathbf{w}_{c,i-1})] + P[r(\mathbf{1} \otimes \mathbf{w}_{c,i-1})] \right) \\
& + \mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2 \mathbf{1}^\top P[\mathbf{s}_i - \mathbb{E} \mathbf{s}_i]
\end{aligned} \tag{143}$$

In step (a), cross-terms are eliminated because $\mathbb{E}\{\mathbf{s}_i - \mathbb{E} \mathbf{s}_i\} = 0$. Step (b) is due to $\|\mathcal{J}_\epsilon\| < 1$ and Jensen's inequality, (c) is due to the sub-multiplicative property of norms, (d) follows from Jensen's inequality, and (e) is due to $\mathbf{1}^\top P[\mathbf{x}] = \mathbb{E} \|\mathbf{x}\|^2$. Similarly to $\mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2$, we observe that $\mathbb{E} \|\mathbf{w}_{e,i-1}\|^2$ also contracts at a linear rate, now given by $\|\mathcal{J}_\epsilon\|$, and is subject to a number of additive perturbations. We appeal to Lemma 3 to bound:

$$\begin{aligned}
& \mathbb{E} \|\mathbf{w}_{e,i}\|^2 \\
\stackrel{(a)}{\leq} & \|\mathcal{J}_\epsilon\| \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 + \frac{25\mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{1 - \|\mathcal{J}_\epsilon\|} \\
& \times \mathbf{1}^\top \left(\left(2\lambda_U^2 + \frac{1 + \mu^2}{\delta^2} \right) \nu^2 \mathbf{1}^\top P[\mathbf{w}_{e,i-1}] \right. \\
& \quad \left. + \frac{\mu^2}{\delta^2} (3\lambda_U^2 \nu^2 \mathbf{1}^\top P[\mathbf{w}_{e,i-1}] + 3\lambda_U^2 P[\mathbf{1} \otimes \tilde{\mathbf{w}}_{c,i-1}] \right. \\
& \quad \left. + 3P[g(\mathbf{1} \otimes \mathbf{w}_\delta^o)]) \right. \\
& \quad \left. + 3\beta^2 \frac{\mu^2}{\delta^2} P[\mathbf{1} \otimes \tilde{\mathbf{w}}_{c,i-1}] + 3\beta^2 \frac{\mu^2}{\delta^2} \nu^2 \mathbf{1}^\top P[\mathbf{w}_{e,i-1}] \right. \\
& \quad \left. + 3\beta^2 \frac{\mu^2}{\delta^2} P[\mathbf{1} \otimes \mathbf{w}_\delta^o] + \frac{\mu^2}{\delta^2} \sigma^2 \mathbf{1} \right. \\
& \quad \left. + 2\lambda_U^2 P[\mathbf{1} \otimes \tilde{\mathbf{w}}_{c,i-1}] + 2P[g(\mathbf{1} \otimes \mathbf{w}_\delta^o)] \right. \\
& \quad \left. + \frac{2}{\delta^2} P[\mathbf{1} \otimes \tilde{\mathbf{w}}_{c,i-1}] + 2P[r(\mathbf{1} \otimes \mathbf{w}_\delta^o)] \right) \\
& + \mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2 \mathbf{1}^\top \left(3\beta^2 P[\mathbf{1} \otimes \tilde{\mathbf{w}}_{c,i-1}] \right.
\end{aligned}$$

$$\begin{aligned}
& \quad \left. + 3\beta^2 \nu^2 \mathbf{1}^\top P[\mathbf{w}_{e,i-1}] + 3\beta^2 P[\mathbf{1} \otimes \mathbf{w}_\delta^o] + \sigma^2 \mathbf{1} \right) \\
\stackrel{(b)}{=} & \|\mathcal{J}_\epsilon\| \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 + \frac{25\mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{1 - \|\mathcal{J}_\epsilon\|} \\
& \times \left(\left(2\lambda_U^2 + \frac{1 + \mu^2}{\delta^2} \right) \nu^2 N \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 \right. \\
& \quad \left. + \frac{\mu^2}{\delta^2} (3\lambda_U^2 \nu^2 N \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 + 3\lambda_U^2 N \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 \right. \\
& \quad \left. + 3\|g(\mathbf{1} \otimes \mathbf{w}_\delta^o)\|^2) \right. \\
& \quad \left. + 3\beta^2 \frac{\mu^2}{\delta^2} N \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 + 3\beta^2 \frac{\mu^2}{\delta^2} \nu^2 N \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 \right. \\
& \quad \left. + 3\beta^2 \frac{\mu^2}{\delta^2} N \|\mathbf{w}_\delta^o\|^2 + \frac{\mu^2}{\delta^2} N \sigma^2 \right. \\
& \quad \left. + 2\lambda_U^2 N \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 + 2\|g(\mathbf{1} \otimes \mathbf{w}_\delta^o)\|^2 \right. \\
& \quad \left. + \frac{2}{\delta^2} N \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 + 2\|r(\mathbf{1} \otimes \mathbf{w}_\delta^o)\|^2) \right) \\
& + \mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2 \left(3\beta^2 N \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 \right. \\
& \quad \left. + 3\beta^2 \nu^2 N \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 \right. \\
& \quad \left. + 3\beta^2 N \|\mathbf{w}_\delta^o\|^2 + N \sigma^2 \right) \\
\stackrel{(c)}{=} & \left(\|\mathcal{J}_\epsilon\| + \mu^2 \nu^2 N \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2 \left(\frac{25}{1 - \|\mathcal{J}_\epsilon\|} (2\lambda_U^2 \right. \right. \\
& \quad \left. \left. + \frac{1 + \mu^2}{\delta^2} + 3\frac{\mu^2}{\delta^2} (\lambda_U^2 + \beta^2)) + 3\beta^2 \right) \right) \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 \\
& + \mu^2 N \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2 \left(\frac{25}{1 - \|\mathcal{J}_\epsilon\|} \left(3\lambda_U^2 \frac{\mu^2}{\delta^2} + 3\beta^2 \frac{\mu^2}{\delta^2} \right. \right. \\
& \quad \left. \left. + 2\lambda_U^2 + \frac{2}{\delta^2} \right) + 3\beta^2 \right) \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2 \\
& + \frac{25\mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{1 - \|\mathcal{J}_\epsilon\|} \left(\left(2 + 3\frac{\mu^2}{\delta^2} \right) \|g(\mathbf{1} \otimes \mathbf{w}_\delta^o)\|^2 \right. \\
& \quad \left. + 3\beta^2 \frac{\mu^2}{\delta^2} N \|\mathbf{w}_\delta^o\|^2 + \frac{\mu^2}{\delta^2} N \sigma^2 + 2\|r(\mathbf{1} \otimes \mathbf{w}_\delta^o)\|^2 \right) \\
& + \mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2 \left(3\beta^2 N \|\mathbf{w}_\delta^o\|^2 + N \sigma^2 \right) \\
\stackrel{(d)}{=} & \left(\|\mathcal{J}_\epsilon\| + \frac{\mu^2 25\nu^2 N \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{\delta^2 (1 - \|\mathcal{J}_\epsilon\|)} \right. \\
& \quad \left. + \mu^2 \frac{25\nu^2 N \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{1 - \|\mathcal{J}_\epsilon\|} \left(2\lambda_U^2 + \frac{1 - \|\mathcal{J}_\epsilon\|}{25} 3\beta^2 \right) \right. \\
& \quad \left. + \frac{\mu^4 25\nu^2 N \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{\delta^2 (1 - \|\mathcal{J}_\epsilon\|)} (1 + 3\beta^2 + 3\lambda_U^2) \right) \\
& \quad \times \mathbb{E} \|\mathbf{w}_{e,i-1}\|^2 \\
& + \left(\frac{\mu^2 50N \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{\delta^2 (1 - \|\mathcal{J}_\epsilon\|)} \right. \\
& \quad \left. + \mu^2 \frac{25N \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{1 - \|\mathcal{J}_\epsilon\|} \left(2\lambda_U^2 + \frac{1 - \|\mathcal{J}_\epsilon\|}{25} 3\beta^2 \right) \right. \\
& \quad \left. + \frac{\mu^4 75N \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{\delta^2 (1 - \|\mathcal{J}_\epsilon\|)} (\lambda_U^2 + \beta^2) \right) \mathbb{E} \|\tilde{\mathbf{w}}_{c,i-1}\|^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{25\mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2}{1 - \|\mathcal{J}_\epsilon\|} \left(\left(2 + 3\frac{\mu^2}{\delta^2} \right) \|g(\mathbf{1} \otimes w_\delta^o)\|^2 \right. \\
& \quad \left. + 3\beta^2 \frac{\mu^2}{\delta^2} N \|w_\delta^o\|^2 + \frac{\mu^2}{\delta^2} N \sigma^2 + 2\|r(\mathbf{1} \otimes w_\delta^o)\|^2 \right) \\
& + \mu^2 \|\mathcal{J}_\epsilon\|^2 \|\mathcal{V}_R\|^2 \left(3\beta^2 N \|w_\delta^o\|^2 + N \sigma^2 \right) \quad (144)
\end{aligned}$$

The bounds from Lemma 3 are used in (a) and (b) is due to $\mathbf{1}^\top P[\mathbf{x}] = \mathbb{E} \|\mathbf{x}\|^2$ for $\mathbf{x} \in \mathbb{R}^{MN}$ and $\mathbf{1}^\top P[\mathbf{1} \otimes \mathbf{y}] = N \cdot \mathbb{E} \|\mathbf{y}\|^2$ for $\mathbf{y} \in \mathbb{R}^M$. Steps (c) and (d) are obtained by grouping terms.

APPENDIX E PROOF OF LEMMA 5

For $\delta = \mu^{\frac{1}{2}-\kappa}$ and small step-sizes μ ,

$$\Gamma = \begin{bmatrix} 1 - \mu\lambda_L + O(\mu^2) & O(\mu^{2\kappa}) \\ O(\mu^{1+2\kappa}) & \|\mathcal{J}_\epsilon\| + O(\mu^{1+2\kappa}) \end{bmatrix} \quad (145)$$

so that

$$\|\Gamma\|_1 = \max \{ 1 - \mu\lambda_L + O(\mu^{1+2\kappa}), \|\mathcal{J}_\epsilon\| + O(\mu^{2\kappa}) \} < 1 \quad (146)$$

for small enough μ . Since $\rho(\Gamma) \leq \|\Gamma\|_1 < 1$, Γ is stable. It is also invertible and we obtain

$$\limsup_{i \rightarrow \infty} \begin{bmatrix} \mathbb{E} \|\tilde{\mathbf{w}}_{c,i}\|^2 \\ \mathbb{E} \|\mathbf{w}_{e,i}\|^2 \end{bmatrix} \preceq (I - \Gamma)^{-1} \begin{bmatrix} \mu^2 b_3 + O(\mu^{2+2\kappa}) \\ \mu^2 b_4 + \mu^2 b_5 + O(\mu^{3+2\kappa}) \end{bmatrix} \quad (147)$$

Using the matrix inversion lemma [60, Prop. 2.8.7], we have

$$\begin{aligned}
& (I - \Gamma)^{-1} \\
& = \begin{bmatrix} \mu\lambda_L - O(\mu^2) & -O(\mu^{2\kappa}) \\ -O(\mu^{1+2\kappa}) & 1 - \|\mathcal{J}_\epsilon\| - O(\mu^{1+2\kappa}) \end{bmatrix}^{-1} \\
& = \frac{1}{(\mu\lambda_L - O(\mu^2))(1 - \|\mathcal{J}_\epsilon\| - O(\mu^{1+2\kappa})) - O(\mu^{1+4\kappa})} \\
& \quad \times \begin{bmatrix} 1 - \|\mathcal{J}_\epsilon\| - O(\mu^{1+2\kappa}) & O(\mu^{2\kappa}) \\ O(\mu^{1+2\kappa}) & \mu\lambda_L - O(\mu^2) \end{bmatrix} \\
& = \frac{1}{(\lambda_L - O(\mu))(1 - \|\mathcal{J}_\epsilon\| - O(\mu^{1+2\kappa})) - O(\mu^{4\kappa})} \\
& \quad \times \begin{bmatrix} \mu^{-1}(1 - \|\mathcal{J}_\epsilon\|) - O(\mu^{2\kappa}) & O(\mu^{-1+2\kappa}) \\ O(\mu^{2\kappa}) & \lambda_L - O(\mu) \end{bmatrix} \\
& = O(1) \cdot \begin{bmatrix} O(\mu^{-1}) & O(\mu^{-1+2\kappa}) \\ O(\mu^{2\kappa}) & O(1) \end{bmatrix} \quad (148)
\end{aligned}$$

The result follows after multiplication and cancellation.

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