# THE MCKAY CORRESPONDENCE FOR ISOLATED SINGULARITIES VIA FLOER THEORY

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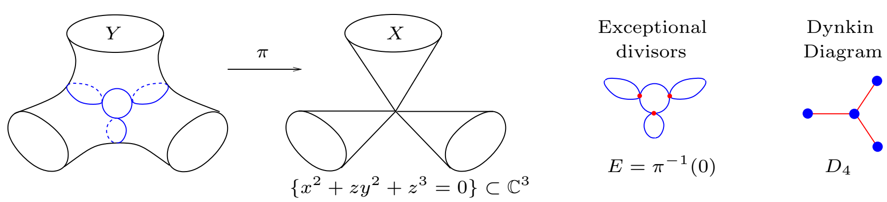
Abstract. We prove the generalised McKay correspondence for isolated singularities using Floer theory. Given an isolated singularity $\mathbb{C}^n/G$ for a finite subgroup $G \subset SL(n, \mathbb{C})$ and any crepant resolution $Y$, we prove that the rank of positive symplectic cohomology $SH^*_+(Y)$ is the number $|\mathrm{Conj}(G)|$ of conjugacy classes of $G$, and that twice the age grading on conjugacy classes is the $\mathbb{Z}$-grading on $SH^{*-1}_+(Y)$ by the Conley-Zehnder index. The generalized McKay correspondence follows as $SH^{*-1}_+(Y)$ is naturally isomorphic to ordinary cohomology $H^*(Y)$, due to a vanishing result for full symplectic cohomogy. In the Appendix we construct a novel filtration on the symplectic chain complex for any non-exact convex symplectic manifold, which yields both a Morse-Bott spectral sequence and a construction of positive symplectic cohomology.

## 1. Introduction

1.1. **The classical McKay correspondence.** The classical McKay correspondence is a description of the representation theory of finite subgroups $G \subset SL(2, \mathbb{C})$ in terms of the geometry of the minimal resolution $\pi : Y \to \mathbb{C}^2/G$. Recall a resolution consists of a non-singular quasi-projective variety $Y$ together with a proper, birational morphism $\pi$ which is a biholomorphism away from the singular locus. In the case of $\mathbb{C}^2/G$, there is only an isolated singularity at the origin. Minimality means other resolutions factor through it, and in this case it is equivalent to the absence of rational holomorphic $(-1)$-curves in $Y$. The exceptional locus $E = \pi^{-1}(0) \subset Y$ is a tree of transversely intersecting exceptional divisors $E_j$, where each $E_j$ is a rational holomorphic $(-2)$-curve. Finite subgroups $G \subset SL(2, \mathbb{C})$, up to conjugation, are in 1-to-1 correspondence with ADE Dynkin diagrams. The diagram for $G$ can be recovered by assigning a vertex to each $E_j$, and an edge between vertices whenever the corresponding divisors intersect. For example, the real picture for $D_4$ is[1]




The correspondence [39] states that the non-trivial irreducible representations $V_i$ of $G$ can be labelled by the vertices of the Dynkin diagram so that the adjacency matrix $A_{ij}$ of the diagram determines the tensor products $\mathbb{C}^2 \otimes V_i \cong \oplus\, A_{ij}\, V_j$ with the canonical representation.

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[1] $X = \mathbb{C}^2/\widetilde{\mathbb{D}_4}$. The binary dihedral group $\widetilde{\mathbb{D}_4}$ is the quaternion group; it has size 8 and double covers via $SU(2) \to SO(3)$ a size 4 dihedral group $C_2 \times C_2 \cong \mathbb{D}_4 \subset SO(3)$. Circles depicting $E$ represent copies of $\mathbb{CP}^1$. The quaternion group has four non-trivial conjugacy classes: $-1$, $\pm i$, $\pm j$, $\pm k$.

The cohomology $H^*(Y,\mathbb{C})$ consists of $H^0(Y)=\mathbb{C}\cdot 1$, $H^2(Y)=\oplus\,\mathbb{C}\cdot\mathrm{PD}[E_j]$. So the dimension of $H^*(Y)$, or the Euler characteristic $\chi(Y)$, is the number of irreducible representations. As $G$ is finite, this is the number of conjugacy classes, namely the dimension of the representation ring $\mathrm{Rep}(G)$ (although there is no natural bijection between $\mathrm{Irreps}(G)$ and $\mathrm{Conj}(G)$).

**Example 1.1.** *The simplest case $G=\mathbb{Z}/2=\{\pm I\}\subset SL(2,\mathbb{C})$ yields $Y=T^*\mathbb{CP}^1=\mathcal{O}_{\mathbb{CP}^1}(-2)$ arising as the blow-up at 0 of the Veronese variety*[1] $\mathbb{V}(XZ-Y^2)\subset\mathbb{C}^3$. *The Kähler form on $T^*\mathbb{CP}^1$ makes $\mathbb{CP}^1$ holomorphic and symplectic, unlike the canonical exact symplectic form on $T^*S^2$. Here $H^*(Y)$ has two generators $1,\omega$, and $G$ has two conjugacy classes $I,-I\in\mathrm{Conj}(G)$.*

**Remark 1.2.** *This fails for $G\subset GL(2,\mathbb{C})$, for example for $G\cong\mathbb{Z}/2$ generated by the reflection $(z,w)\mapsto(z,-w)$, then $\mathbb{C}^2/G\cong\mathbb{C}^2$ is already non-singular but does not remember $G$.*

1.2. **The generalised McKay correspondence.** More generally, for any $n\geq 2$, one considers resolutions of quotient singularities for finite subgroups $G\subset SL(n,\mathbb{C})$,

$$\pi: Y\to X=\mathbb{C}^n/G, \tag{1.1}$$

viewing $\mathbb{C}^n/G=\mathrm{Spec}\,\mathbb{C}[z_1,\ldots,z_n]^G$ as an affine variety. There is no longer a preferential resolution, so one requires $\pi$ to be *crepant*, meaning the canonical bundles satisfy $K_Y=\pi^*K_X$ (which is therefore trivial). Resolutions of such $X$ always exist by Hironaka [28], but crepant resolutions may not exist;[2] when they exist they need not be unique even though they always admit the same collection of divisors [31, Thm.1.4]. For $n\leq 3$, they exist [14, Thm.1.2]. For $n=3$, they are related by flops.

The conjecture $\chi(Y)=|\mathrm{Conj}(G)|$ dates back to work of Dixon-Harvey-Vafa-Witten [20], Atiyah-Segal [3] and Hirzebruch-Höfer [29]. In the early 1990s, the conjecture was refined by Miles Reid [45, 46] by taking into account the grading of $H^*(Y)$. Namely, consider the dual action[3] of $g\in SL(n,\mathbb{C})$ on $\mathbb{C}^n$, $g\cdot x=g^{-1}(x)$, and let $\lambda_1,\ldots,\lambda_n\in U(1)$ denote the unordered eigenvalues (which are $|G|$-th roots of unity). Writing $\lambda_j=e^{ia_j}$ for $a_j\in[0,2\pi)$, define the *age grading* on $\mathrm{Conj}(G)$ by[4]

$$\mathrm{age}(g)=\tfrac{1}{2\pi}\sum a_j\in[0,n). \tag{1.2}$$

The generalised McKay correspondence, as reformulated by Reid [45], is the following.

**Theorem 1.3.** $\dim H^{2k}(Y,\mathbb{C})=|\mathrm{Conj}_k(G)|$ *where* $\mathrm{Conj}_k(G)$ *denotes the conjugacy classes of* age $k$*, and the odd cohomology of $Y$ vanishes.*

This was proved for $n=3$ by Ito-Reid [31]; for general $n$ and abelian $G$ it was proved using toric geometry by Batyrev-Dais [4]; in full generality it was proved using motivic integration machinery by Batyrev [5] and later by Denef-Loeser [19]. We refer to Craw's thesis [17] and the references therein for an extensive history of the generalisations of the McKay correspondence, in particular on the extensions to a statement about the K-theory of $Y$ in terms of $\mathrm{Rep}(G)$, and more generally about relating the derived categories of coherent sheaves on $Y$ and of

[1] The image of $\nu_2:\mathbb{C}^2/\{\pm 1\}\hookrightarrow\mathbb{C}^3$, $(x,y)\mapsto(x^2,xy,y^2)$.

[2] If $Y\to\mathbb{C}^4/\pm 1$ were crepant then by Theorem 1.3, $H^*(Y)$ would have two generators, in degrees $0,4$ (twice the age grading of $\pm 1$), contradicting that $E=\pi^{-1}(0)$ is a projective variety with $H^*(E)\cong H^*(Y)$.

[3] In the notation of Ito-Reid [31], we are taking the age grading of $\varphi_g\in\mathrm{Hom}(\mu_r,G)$, $\varphi_g(e^{2\pi i/r})=g^{-1}$, where $\mu_r\subset\mathbb{C}^*$ is the group of $r$-th roots of unity. The inverse reflects the fact that we do not dualise $H^{2k}(Y,\mathbb{C})$ (compare [31, Theorem 1.6]). This choice agrees with Kaledin [32], where a representation $g:\mu_r\to GL(\mathbb{C}^n)$ labels eigensummands $V_j\subset\mathbb{C}^n$ so that the action is $\lambda\cdot x=\lambda^{-b_j}x$ for $b_j\in[0,r)\cap\mathbb{Z}$; the dual action on the coordinate ring $\mathcal{O}(\mathbb{C}^n)=\mathbb{C}[(\mathbb{C}^n)^*]$ gets rid of that inversion and $\mathrm{age}(g)=\frac{1}{r}\sum b_j\dim V_j$ (so our $a_j=2\pi b_j/r$).

[4] This is an integer, as $\sum a_j$ is divisible by $2\pi$ as $\prod\lambda_j=\det g^{-1}=1$, and it only depends on $[g]\in\mathrm{Conj}(G)$.

$G$-equivariant sheaves on $\mathbb{C}^n$. In particular, on the latter generalisations, we highlight the work of Bridgeland-King-Reid [14] and Bezrukavnikov-Kaledin [6], and we refer the reader to Craw's expository notes [18] and the references therein. Finally, we mention that Reid [45] also strengthened the above correspondence statement with the following conjecture:

**Open Problem.** *There is a natural basis of $H^*(Y)$ labelled by the conjugacy classes of $G$.*

Although the precise meaning of natural is not known, reasonable labellings are known for $n = 2$ by the classical correspondence; for $n = 3$ by Ito-Reid [31]; and by Kaledin [32] for even $n = 2m$ when $G$ preserves the complex symplectic form on $\mathbb{C}^{2m}$.

Our approach to the McKay Correspondence, which we describe below, uses only tools from symplectic topology and thus it differs significantly from the above algebraic geometry literature. Our way of thinking about the McKay Correspondence is fundamentally new, and we expect that it will lead to new insights into the crepant resolution conjecture, which we will address in a subsequent paper [41].

1.3. **Isolated singularities.** In this paper we consider the case when the singularity is isolated. Our approach via Floer theory works in examples of non-isolated singularities, however generalising the proofs is harder as the moduli space of Hamiltonian orbits in $Y$ lying over the singular locus is difficult to pin down. In a subsequent paper [41], we will prove the McKay Correspondence in the non-isolated case in a slightly different way, but based on the foundational work of this paper.

**Lemma 1.4.** *For $G \subset SL(n, \mathbb{C})$ any finite subgroup, $\mathbb{C}^n/G$ is an isolated singularity if and only if $G$ acts freely away from $0 \in \mathbb{C}^n$ (i.e. the eigenvalues of $g \neq 1 \in G$ are not equal to 1).*

*Proof.* Given any finite subgroup $Q \subset GL(n, \mathbb{C})$, the Chevalley-Shephard-Todd theorem [15] states that $\mathbb{C}^n/Q$ is smooth if and only if $Q$ is generated by quasi-reflections[1], in which case $\mathbb{C}^n/Q \cong \mathbb{C}^n$. However, a finite order element of $SL(n, \mathbb{C})$ cannot be a quasi-reflection. So finite $G \subset SL(n, \mathbb{C})$ are *small*, i.e. contain no quasi-reflections. The singular set of $\mathbb{C}^n/G$ is

$$\mathrm{Sing}(\mathbb{C}^n/G) = \{v \in \mathbb{C}^n : g \cdot v = v \text{ for some } 1 \neq g \in G\}/G. \tag{1.3}$$

Indeed, where $G$ acts freely the quotient is easily seen to be smooth. Conversely, at a point $v$ as above, pick a $\mathrm{Stab}_G(v)$-invariant analytic neighbourhood $V \subset \mathbb{C}^n$ of $v$. Then $V/\mathrm{Stab}_G(v)$ is analytically isomorphic to a neighbourhood of $[v] \in \mathbb{C}^n/G$. By the same theorem, this is isomorphic to $\mathbb{C}^n$ if and only if $\mathrm{Stab}_G(v)$ is generated by quasi-reflections. But the latter fails as $G$ is small, so $v$ is indeed singular. We refer to [22, 44] for more precise details. □

**Remark 1.5.** *The Kleinian singularities in Sec.1.1 are always smoothable. This fails in dimension $n \geq 3$ for the above isolated singularities $\mathbb{C}^n/G$ by Schlessinger's rigidity theorem* [51].

**Examples.** *Finite groups $G$ admitting a fixed point free faithful complex representation were classified by Wolf* [56, Theorem 7.2.18] (*cf. also the final comments in* [27, Example 1.43]). *Those representations which yield subgroups in $SL(n, \mathbb{C})$ have been classified by Stepanov* [54]. *For abelian groups, it forces $G$ to be a cyclic group.*[2] *When $n$ is an odd prime, in particular for $n = 3$, the only finite subgroups $G \subset SL(n, \mathbb{C})$ that give rise to an isolated singularity are cyclic groups, by Kurano-Nishi* [36]. *A simple example is $X = \mathbb{C}^3/(\mathbb{Z}/3)$, where $\mathbb{Z}/3$ acts diagonally by third roots of unity, which admits the (unique) crepant resolution $\pi : Y \to X$ given by blowing up 0, with exceptional divisor $E = \pi^{-1}(0) \cong \mathbb{CP}^2$. Lens spaces* [27, Example

[1] A quasi-reflection is a non-identity element $A$ for which $A - I$ has rank one (i.e. $\mathrm{codim}_{\mathbb{C}} \mathrm{Fix}(A) = 1$).

[2] After a conjugation, one may assume all matrices in $G$ are diagonal, then the projection to the $(1, 1)$-entry gives an injective group homomorphism into $S^1$, and finite subgroups of $S^1$ are cyclic.

2.43] *yield a family of examples of cyclic actions, namely* $G = \mathbb{Z}/m$ *acts on* $\mathbb{C}^n$ *by rotation by* $(e^{2\pi i \ell_1/m}, \ldots, e^{2\pi i \ell_n/m})$ *where* $\ell_j \in \mathbb{Z}$ *are coprime to* $m$ *and* $\sum \ell_j \equiv 0 \bmod m$. *For higher dimensional non-abelian examples, we refer to the detailed discussion by Stepanov* [54].

1.4. **An outline of our proof using Floer theory.** Let (1.1) be a crepant resolution of an isolated singularity. By an averaging argument[1] we may assume $G \subset SU(n)$. As $Y$ is quasi-projective, it inherits a Kähler form $\omega$ from an embedding into a projective space. One can modify the Kähler form so that away from a small neighbourhood of

$$E = \pi^{-1}(0) \subset Y$$

it agrees via $\pi$ with the standard Kähler form on $\mathbb{C}^n/G$ (Lemma 2.8). The Floer theory of $(Y, \omega)$ comes into play, as the diagonal $\mathbb{C}^*$-action on $\mathbb{C}^n/G$ lifts to $Y$ (this relies on $Y$ being crepant [5, Prop.8.2], we will give a self-contained proof in Proposition 3.6). The underlying $S^1$-action is Hamiltonian, corresponding to the standard Hamiltonian $\frac{1}{2}|z|^2$ on $\mathbb{C}^n$ away from a neighbourhood of $E$. We use this Hamiltonian, rescaled by large constants, to define Floer cohomology groups of $Y$ and their direct limit, symplectic cohomology $SH^*(Y)$.

Loops in $Y \setminus E$ are naturally labelled by $\mathrm{Conj}(G)$ via their free homotopy class,[2]

$$[S^1, Y \setminus E] = \pi_0(\mathcal{L}(Y \setminus E)) \cong \mathrm{Conj}(G). \tag{1.4}$$

This follows from an analogous statement for based loops: $\pi_1(Y \setminus E) \cong \pi_1((\mathbb{C}^n \setminus 0)/G) \cong \pi_1(S^{2n-1}/G) \cong G$. An analogous isomorphism holds also in the non-isolated case.[3]

**Lemma 1.6.** *Any eigenvector* $v \in \mathbb{C}^n \setminus 0$ *of* $g \in G$ *yields a closed orbit* $x_g$ *of the* $S^1$*-action, corresponding to* $\mathbf{g} = [g] \in \mathrm{Conj}(G)$ *via* (1.4)*. Namely, if* $g(v) = e^{i\ell} v$ *for* $0 < \ell \leq 2\pi$,

$$x_g(t) = e^{i\ell t} v : S^1 \to (\mathbb{C}^n \setminus 0)/G \cong Y \setminus E, \text{ where } x_g(0) = [v] = [gv] = [e^{i\ell} v] = x_g(1). \tag{1.5}$$

*Conversely,* $\mathbf{g} \in \mathrm{Conj}(G)$ *can be uniquely recovered from* $[v]$ *and the eigenvalue* $e^{i\ell}$.

*Proof.* We check that $[v]$ determines $\mathbf{g} \in \mathrm{Conj}(G)$. If $g_1 \neq g_2 \in G$ with $g_1(v) = e^{i\ell} v = g_2(v)$, then $g_2^{-1} g_1 \in \mathrm{Stab}(v)$ implies $v$ is singular by (1.3), yielding the contradiction $v = 0$ (the isolated singularity). Conjugation $g \mapsto hgh^{-1}$ corresponds to changing eigenvectors by $v \mapsto h(v)$. □

Recall that the eigenvalue $e^{i\ell}$ above contributes $\frac{1}{2\pi} a = 1 - \frac{\ell}{2\pi}$ to the age$(g)$ in (1.2). Given $\mathbf{g} \in \mathrm{Conj}(G)$, call $e^{i\ell}$ a **minimal eigenvalue** of $\mathbf{g}$ if $0 < \ell \leq 2\pi$ achieves the minimal possible value amongst eigenvalues of $\mathbf{g}$. If $v \in S^{2n-1}$ satisfies $g(v) = e^{i\ell} v$, and $e^{i\ell}$ is minimal, then we call $x_g(t) = [e^{i\ell t} v] \in S^{2n-1}/G$ a **minimal Reeb orbit**.

To simplify our outline, let us assume that we are using the quadratic radial Hamiltonian $H = \frac{1}{4} R^2$ to define Floer cohomology, where $R = |z|^2$ on $\mathbb{C}^n/G$, and that this agrees via $\pi$ with the Hamiltonian used on $Y \setminus E$ (Section 2.6 discusses these details). This implies that the time-$t$ Hamiltonian flow on $Y \setminus E$ equals multiplication by $e^{iRt}$ on each **slice**[4]

$$\mathcal{S}(R) = \pi^{-1}\{[z] \in \mathbb{C}^n/G : |z|^2 = R\} \cong S^{2n-1}/G. \tag{1.6}$$

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[1]By using an element $h \in SL(n, \mathbb{C})$ we can change the standard basis of $\mathbb{C}^n$ to a basis of eigenvectors for the $G$-invariant inner product $\frac{1}{|G|} \sum_{g \in G} \langle g \cdot, g \cdot \rangle_{\mathbb{C}^n}$. Then the $h$-conjugate of $G$ lies in $SU(n)$.

[2]If the Floer solutions $\mathbb{R} \times S^1 \to Y$ counted by the Floer differential did not intersect $E$, then the Floer differential would preserve these conjugacy classes. But this assumption is most likely false.

[3]Namely, $\pi_1(Y \setminus E) \cong G$ where $E = \pi^{-1}(\mathrm{Sing}(\mathbb{C}^n/G))$. Indeed, let $F_2$ denote the union of all $\mathrm{codim}_{\mathbb{C}} \geq 2$ fixed point loci in $\mathbb{C}^n$ of all subgroups of $G$. Then any non-identity element in $G$ fixing a point in $\mathbb{C}^n \setminus F_2$ must be a quasi-reflection, but there are no quasi-reflections in a finite subgroup $G \subset SL(n, \mathbb{C})$. So $G$ acts freely on $\mathbb{C}^n \setminus F_2$. Thus $\pi_1((\mathbb{C}^n \setminus F_2)/G) \cong G$ (note that $\mathbb{C}^n \setminus F_2$ is simply connected due to the codimension of $F_2$). Finally $Y \setminus E \cong (\mathbb{C}^n \setminus F_2)/G$ via $\pi$, using (1.3). On the other hand, $\pi_1(Y) = 1$ is a general feature of resolutions of quotient singularities [33, Theorem 7.8].

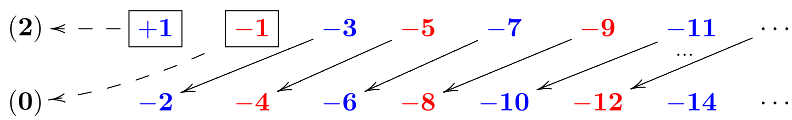
[4]$\mathcal{S}(R) \cong S^{2n-1}/G$ is an $S^1$-equivariant isomorphism, as the region $R > 0$ avoids the isolated singularity.

Under this identification, each 1-periodic Hamiltonian orbit $y : S^1 \to Y \setminus E$ corresponds uniquely to a **Reeb orbit** $x_{\mathbf{g}} \subset S^{2n-1}/G$ satisfying (1.5) (where we also allow $\ell \geq 2\pi$), and $\mathbf{g} \in \mathrm{Conj}(G)$ is the class that $y$ determines via (1.4). Fixing $\mathbf{g}$ and $\ell \in (0, \infty)$ defines

$$\mathcal{O}_{\mathbf{g},\ell} = \{\text{parametrized Hamiltonian 1-orbits in } \mathcal{S}(\ell) \text{ in the class } \mathbf{g}\} \subset S^{2n-1}/G, \tag{1.7}$$

where the "inclusion" is defined by taking the initial point of the corresponding Reeb orbit, so $y \mapsto x(0)$. Although the $\mathcal{O}_{\mathbf{g},\ell+2\pi k}$ yield the same subset of $S^{2n-1}/G$ via (1.6) for each $k \in \mathbb{N}$, they consist of 1-orbits in $Y \setminus E$ arising in different slices. So, loosely, $\mathbb{N}$ copies of $\mathcal{O}_{\mathbf{g},\ell} \subset S^{2n-1}/G$ for $0 < \ell \leq 2\pi$ contribute to the symplectic chain complex $SC^*(Y)$.

**Example 1.7.** *Continuing the Example $G = \mathbb{Z}/2$, $Y = T^*\mathbb{CP}^1$, the chain complex $SC^*(Y)$ is*




*Those numbers are the Conley-Zehnder indices*[1] *of the orbits (Appendix C). The "zero-th column" is a Morse complex for $Y$ and computes $H^*(\mathbb{CP}^1)$ (this arises from constant orbits). The other columns are the local Floer contributions of $\mathcal{O}_{-I,\pi}, \mathcal{O}_{+I,2\pi}, \mathcal{O}_{-I,3\pi}$, etc. Each $\mathcal{O}_{\mathbf{g},\ell}$ equals $\mathcal{S}(\ell) \cong S^3/G \cong \mathbb{RP}^3$ as any point in the slice yields an orbit. For* **even multiples** *of $\pi$ the orbits lift to iterates of great circles in $S^3$, for* **odd multiples** *they lift to non-closed orbits in $S^3$ travelling that odd number of half-great circles. These correspond to* **+1** *and* **−1** *eigenvectors in $\mathbb{C}^2$, for $\mathbf{g} = +\mathbf{I}$ and $\mathbf{g} = -\mathbf{I}$, and (disregarding the zero-th column) they arise in the* **even columns** *and the* **odd columns***. Using a Morse-Bott model (Appendix E) each column is a copy of $H^*(\mathcal{O}_{\mathbf{g},\ell}) \cong H^*(\mathbb{RP}^3, \mathbb{C})$ with grading suitably shifted, so two generators separated by 3 in grading. The jump by $4 = 2n$ in grading every two columns is due to a full rotation of $\varphi_t^* K_Y$ along the $S^1$-action $\varphi_t$ compared to the standard trivialisation of $K_{\mathbb{C}^n}$.*

Loosely, the positive complex $SC^*_+(Y)$ is the quotient of $SC^*(Y)$ by the Morse subcomplex of constant 1-periodic Hamiltonian orbits, which appear in $E = \pi^{-1}(0) \subset Y$. The Morse subcomplex computes $H^*(Y)$ and thus gives rise to the long exact sequence

$$\cdots \to H^*(Y) \xrightarrow{c^*} SH^*(Y) \to SH^*_+(Y) \to H^{*+1}(Y) \to \cdots \tag{1.8}$$

This construction is known for exact convex symplectic manifolds [55, 9], in which case the Floer action functional $A_H : \mathcal{L}Y \to \mathbb{R}$ provides the necessary filtration to make the argument rigorous. As our Kähler form $\omega$ on $Y$ is non-exact (so $A_H$ becomes multi-valued), we construct a novel filtration in Appendix D in order achieve the same result for any convex symplectic manifold. Then $SC^*_+(Y)$ is generated by the union $\cup \mathcal{O}_{\mathbf{g},\ell}$ over $\mathbf{g} \in \mathrm{Conj}(G)$, $\ell \in (0, \infty)$.

**Example 1.8.** *Continuing above, $SH^*(Y) = SH^*(\mathcal{O}_{\mathbb{CP}^1}(-2)) = 0$ by [49], so the chain complex is acyclic and all arrows are isomorphisms.*[2] *In $SC^*_+(Y)$, the zero-th column is quotiented, so the two boxed generators survive to $SH^*_+(Y)$: they are the maxima of the first two Morse-Bott manifolds of 1-orbits; in $S^3$ they become a half-great circle and a great circle. Working over $\mathbb{C}$, and using the notation $A[d]$ to mean $A$ with grading shifted down by $d$ for any $\mathbb{Z}$-graded group $A = \oplus A_m$, so $(A[d])_m = A_{m+d}$, we deduce that*

$$SH^{*-1}_+(Y) = \mathbb{C}[-\mathbf{1}][-1] \oplus \mathbb{C}[+\mathbf{1}][-1] \cong \mathbb{C}[-2] \oplus \mathbb{C} \cong H^2(Y, \mathbb{C}) \oplus H^0(Y, \mathbb{C}) \cong H^*(Y, \mathbb{C}).$$

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[1] Which is a $\mathbb{Z}$-grading on $SC^*(Y)$ as $Y$ is Calabi-Yau by the triviality of $K_Y$.

[2] Those orbits are also the generators when using the canonical exact symplectic form on $T^*S^2$, and the grading is consistent with the Viterbo isomorphism $SH^*(T^*S^2) \cong H_{2-*}(\mathcal{L}S^2)$ provided differentials vanish.

**Theorem 1.9.** *$SH^*(Y) = 0$, so there is a canonical isomorphism*

$$SH_+^{*-1}(Y) \to H^*(Y).$$

That vanishing follows by mimicking the argument in [47] (analogously to $\mathbb{C}^n$ [42, Sec.3]): the Hamiltonians $L_k = kR$ yield the flow $e^{ikt}$, and for generic $k \in \mathbb{R}_{>0}$ the only period 1 orbits are constant orbits in $E$ (Lemma 2.10) whose Conley-Zehnder index becomes unbounded as $k \to \infty$ (Theorem 2.12). As $SH^*(Y)$ is the direct limit of $HF^*(L_k)$ under grading-preserving maps, in any given finite degree no generators appear for large $k$.

**Theorem 1.10.** *$SH_+^*(Y)$ has rank $|\mathrm{Conj}(G)|$. More precisely,*

$$\operatorname{rank} SH_+^{2k-1}(Y) = |\mathrm{Conj}_k(G)|.$$

The remainder of this Section will explain the proof of the above theorem. A Morse-Bott argument (Appendix E) yields a convergent spectral sequence

$$E_1^{*,*} = \bigoplus H^*(\mathcal{O}_{\mathbf{g},\ell})[-\mu_{\mathbf{g},\ell}] \Rightarrow SH_+^*(Y), \tag{1.9}$$

where $\mu_{\mathbf{g},\ell}$ denotes the shift in grading that needs to be applied to $H^*(\mathcal{O}_{\mathbf{g},\ell})$ in the Morse-Bott model for the symplectic chain complex ($\mu_{\mathbf{g},\ell}$ represents the grading of the orbit corresponding to the minimum of $\mathcal{O}_{\mathbf{g},\ell}$). It turns out that $\mu_{\mathbf{g},\ell}$ is always an even integer (Equation (2.6)).

In particular, in the Morse-Bott model, each $\mathbf{g} \in \mathrm{Conj}(G)$ gives rise to a maximum $x_g$ (i.e. top degree generator) of $H^*(\mathcal{O}_{\mathbf{g},\ell})$, which is an orbit associated to the minimal eigenvalue $e^{i\ell}$ of $\mathbf{g}$, and its Conley-Zehnder index $\mu(x_g)$ satisfies

$$\tfrac{1}{2}(\mu(x_g) + 1) = \mathrm{age}(g). \tag{1.10}$$

As the Example illustrated, without knowing $H^*(Y)$ it would be difficult to predict which generators survive in the limit of the spectral sequence (1.9). We can however compute the $S^1$-equivariant analogue of (1.9), because in that case all generators on the $E_1$-page have odd total degree so the spectral sequence degenerates on that page. Our goal is to recover $SH_+^*(Y)$ from this fact.

The $S^1$-equivariant theory $ESH^* = SH^*_{S^1}$ was defined by Seidel [52, Sec.(8b)]. It was constructed in detail by Bourgeois-Oancea [11] and we review it in Appendix B. In the equivariant setup, the generators involve the moduli spaces of unparametrized orbits $\mathcal{O}_{\mathbf{g},\ell}/S^1$ and we prove in Theorem 2.6 that these can be identified with $\mathbb{P}_{\mathbb{C}}(V_{g,\ell})/G_{g,\ell}$, where we projectivise the $e^{i\ell}$-eigenspace $V_{g,\ell}$ of $g$, and $G_{g,\ell} \subset G$ is the largest subgroup which maps $V_{g,\ell}$ to itself (in fact $G_{g,\ell} = C_G(g)$ is the centraliser, by Lemma 2.2). If the characteristic of the underlying field does not divide $|G|$, that quotient by $G_{g,\ell}$ does not affect cohomology (Remark 4.4), so

$$H^*(\mathcal{O}_{\mathbf{g},\ell}/S^1) \cong H^*(\mathbb{P}_{\mathbb{C}}(V_{g,\ell})) \cong H^*(\mathbb{CP}^{\dim_{\mathbb{C}} V_{g,\ell}-1}).$$

Up to an additional grading shift by one, which we will explain later, when working over a field of characteristic zero we deduce that the $S^1$-equivariant spectral sequence has generators in odd degrees as claimed. The case of positive characteristic is discussed in Remark 1.14.

**Example 1.11.** *Continuing the above example, the equivariant complex $ESC_+^*(Y)$ becomes*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| +1 | −1 | −3 | −5 | −7 | −9 | −11 | ⋯ |
| −1 | −3 | −5 | −7 | −9 | −11 | −13 | ⋯ |

*where each column is a shifted copy of $H^*(\mathbb{RP}^3/S^1) \cong H^*(\mathbb{CP}^1)$, for example the second column is shifted up by $1 + \mu_{I,2\pi} = -3$.*

Before we can continue the outline, we need a remark about the coefficients used in the equivariant theory. In the Calabi-Yau setup ($c_1(M) = 0$), symplectic cohomology $SH^*(M)$ is a $\mathbb{Z}$-graded $\mathbb{K}$-module, where $\mathbb{K}$ is the **Novikov field**,

$$\mathbb{K} = \left\{ \sum_{j=0}^{\infty} n_j T^{a_j} : a_j \in \mathbb{R}, a_j \to \infty, n_j \in \mathcal{K} \right\}. \tag{1.11}$$

Here $\mathcal{K}$ is any given field of characteristic zero (we discuss non-zero characteristics later), and $T$ is a formal variable in grading zero. Let $\mathbb{K}(\!(u)\!)$ denote the formal Laurent series in $u$ with coefficients in $\mathbb{K}$, where $u$ has degree 2, and abbreviate by $\mathbb{F}$ the $\mathbb{K}[\![u]\!]$-module

$$\mathbb{F} = \mathbb{K}(\!(u)\!)/u\mathbb{K}[\![u]\!] \cong H_{-*}(\mathbb{CP}^\infty).$$

Here $u^{-j}$ in degree $-2j$ formally represents $[\mathbb{CP}^j] \in H_{-*}(\mathbb{CP}^\infty)$ negatively graded, and the $\mathbb{K}[\![u]\!]$-action is induced by the (nilpotent) cap product action by $H^*(\mathbb{CP}^\infty) = \mathbb{K}[u]$.

In Appendix B, we construct the $S^1$-equivariant symplectic cohomology as a $\mathbb{K}[\![u]\!]$-module $ESH^*(Y)$ together with a canonical $\mathbb{K}[\![u]\!]$-module homomorphism

$$c^* : EH^*(Y) \cong H^*(Y) \otimes_{\mathbb{K}} \mathbb{F} \to ESH^*(Y),$$

where in general $EH^*(Y)$ denotes the locally finite $S^1$-equivariant homology $H^{\mathrm{lf},S^1}_{2n-*}(Y)$, not $H^*_{S^1}(Y)$. It becomes $H^*(Y) \otimes_{\mathbb{K}} \mathbb{F}$ above, as the $S^1$-action is trivial on constant orbits. Similar to Theorem 1.9,[1] we have $ESH^*(Y) = 0$ and there is a canonical $\mathbb{K}[\![u]\!]$-module isomorphism

$$ESH^{*-1}_+(Y) \cong EH^*(Y) \cong H^*(Y) \otimes_{\mathbb{K}} \mathbb{F}. \tag{1.12}$$

This implies that the $\mathbb{K}[\![u]\!]$-module $ESH^*_+(Y)$ is in fact a free $\mathbb{F}$-module, and we will see in the proof of Corollary 2.13 that its rank equals the Euler characteristic of $Y$,

$$|\mathrm{Conj}(G)| = \mathrm{rank}_{\mathbb{F}}\, ESH^*_+(Y) = \dim_{\mathbb{K}} ESH^{-1}_+(Y) = \sum \dim_{\mathbb{K}} H^{2j}(Y) = \chi(Y).$$

As anticipated previously, the equivariant analogue of (1.9), working in characteristic zero, yields the following isomorphism of $\mathbb{K}$-vector spaces (but not as $\mathbb{K}[\![u]\!]$-modules)

$$ESH^*_+(Y) \cong \oplus EH^*(\mathcal{O}_{\mathbf{g},\ell})[-\mu_{\mathbf{g},\ell}] \cong \oplus H^*(\mathcal{O}_{\mathbf{g},\ell}/S^1)[-1-\mu_{\mathbf{g},\ell}] \cong \oplus H^*(\mathbb{CP}^{\dim_{\mathbb{C}} V_{g,\ell}-1})[-1-\mu_{\mathbf{g},\ell}]$$

where we now explain the second isomorphism. For any closed orientable manifold $X$ with an $S^1$-action with finite stabilisers, and working in characteristic zero, $EH^*(X) \cong H^{*-1}(X/S^1)$ as $\mathbb{K}[\![u]\!]$-modules, with $u$ acting by cup product with the negative of the Euler class of $X \to X/S^1$ (Theorem 4.3). In our case, the $u$-action on $H^*(\mathbb{CP}^{\dim_{\mathbb{C}} V_{g,\ell}-1})$ is cup product by $-\mathrm{PD}[H]$, where $H$ is the hyperplane class.

Whilst the usual symplectic chain complex is generated by 1-orbits[2] over a Novikov field $\mathbb{K}$, the equivariant theory is generated by 1-orbits over the $\mathbb{K}[\![u]\!]$-module $\mathbb{F}$. There is a natural inclusion $SC^*_+(Y) \to ESC^*_+(Y)$ as the $u^0$-part, so a $\mathbb{K}[d]$ summand in $SC^*_+(Y)$ belongs to a copy of $\mathbb{F}[d] = \mathbb{K}[d] \oplus \mathbb{K}[d+2] \oplus \mathbb{K}[d+4] \oplus \cdots$ in $ESC^*_+(Y)$ where the $u$-action translates the copies $\mathbb{K}[d+2j] \to \mathbb{K}[d+2j-2]$. These summands may however unexpectedly disappear in cohomology. We proved above that $ESH^*_+(Y)$ is a free $\mathbb{F}$-module, so the $\mathbb{K}$-summands appearing in $ESH^*_+(Y)$ must organise themselves into free $\mathbb{F}$-summands, in particular the

[1] The vanishing follows by the spectral sequence for the $u$-adic filtration (see (4.5)), and (1.12) follows by the equivariant analogue of (1.8) (Corollary 6.5).

[2] Strictly, in Floer theory one must pick a reference loop for each 1-orbit $x : [0,1] \to Y$, as the action functional is multi-valued. In our setup, $Y$ is simply connected [33, Theorem 7.8], so one can just pick a smooth filling disc $\widetilde{x} : \mathbb{D} \to Y$, $\partial\widetilde{x} = x$. Two choices of filling disc $\widetilde{x}_1, \widetilde{x}_2$ differ by a sphere $S = [\widetilde{x}_1 \# -\widetilde{x}_2] \in H_2(Y)$ and one identifies $T^{\omega(S)}\widetilde{x}_2 = \widetilde{x}_1$. As $T$ has grading zero (as $c_1(Y) = 0$), these choices do not matter. A canonical choice of filling disc for $x_g$, for an eigenvalue $e^{i\ell}$ of $g \in G$, is obtained by applying the action of $\{re^{it} : 0 \le r \le 1, 0 \le t \le \ell\} \subset \mathbb{C}^*$ to $x_g(0)$ to define a map $\mathbb{D} \to Y$.

number of $\mathbb{F}$-summands in odd degrees is simply the dimension $\dim_{\mathbb{K}} ESH_+^{-2m-1}(Y)$ for sufficiently large $m$ (these dimensions must stabilize). By considering the gradings of the $\mathcal{O}_{\mathbf{g},\ell}$, we check explicitly in Corollary 2.13 that there are $|\mathrm{Conj}(G)|$ free $\mathbb{F}$-summands in $ESH_+^*(Y)$.

**Example 1.12.** *Continuing the above example, the $ESC_+^*(Y)$ complex must have two $\mathbb{F}$-summands in $ESH_+^*(Y)$, generated by the two orbits labelled by $\pm I \in G = \mathbb{Z}/2$,*

$$ESH^*_{+,\mathbf{-I}}(Y) = \mathbb{K}[\mathbf{-1}] \oplus \mathbb{K}[1] \oplus \mathbb{K}[3] \oplus \cdots \cong \mathbb{K}[\mathbf{-1}] \oplus \mathbb{K}[\mathbf{-1}]u^{-1} \oplus \mathbb{K}[\mathbf{-1}]u^{-2} \oplus \cdots \cong \mathbb{F}[\mathbf{-1}]$$
$$ESH^*_{+,\mathbf{+I}}(Y) = \mathbb{K}[\mathbf{+1}] \oplus \mathbb{K}[3] \oplus \mathbb{K}[5] \oplus \cdots \cong \mathbb{K}[\mathbf{+1}] \oplus \mathbb{K}[\mathbf{+1}]u^{-1} \oplus \mathbb{K}[\mathbf{+1}]u^{-2} \oplus \cdots \cong \mathbb{F}[\mathbf{+1}]$$

*Thus we obtain one free $\mathbb{F}$-summand for each conjugacy class.*

**Example 1.13.** *In the case of $A_n$ surface singularities $\mathbb{C}^2/G$, so $G$ generated by $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ where $\zeta = e^{2\pi i/(n+1)}$, the work of Abbrescia, Huq-Kuruvilla, Nelson and Sultani* [2] *is an independent computation of $ESH_+^*(Y)$, and indeed it has rank $n+1 = |\mathrm{Conj}(G)|$ (their grading is by* CZ *whereas our grading is by $n - \mathrm{CZ}$, see Remark 5.3).*

To recover $SH_+^*(Y)$ from $ESH_+^*(Y)$ we use the Gysin sequence [11] (see Appendix B)

$$\cdots \to SH_+^*(Y) \xrightarrow{\text{in}} ESH_+^*(Y) \xrightarrow{u} ESH_+^{*+2}(Y) \xrightarrow{b} SH_+^{*+1}(Y) \to \cdots \tag{1.13}$$

We work under the assumption that the characteristic of the field is coprime to $2, 3, \ldots, |G|$. As $ESH_+^*(Y)$ lives in odd grading, the sequence splits as

$$0 \to SH_+^{\text{odd}}(Y) \hookrightarrow ESH_+^{\text{odd}}(Y) \stackrel{u}{\to} ESH_+^{\text{odd}+2}(Y) \to SH_+^{\text{odd}+1}(Y) \to 0.$$

So $SH_+^{\text{odd}}(Y)$ is the $u^0$-part of $ESH_+^{\text{odd}}(Y)$, and $SH_+^{\text{even}}(Y) = 0$ since $u$ acts surjectively on $ESH_+^*(Y)$ as it is a free $\mathbb{F}$-module. This yields the $\mathbb{K}$-vector space isomorphism

$$SH_+^*(Y) \cong \ker\left(u : ESH_+^*(Y) \to ESH_+^*(Y)\right).$$

Theorem 1.10 now follows, since we showed that $ESH_+^*(Y)$ has $|\mathrm{Conj}(G)|$ free $\mathbb{F}$-summands. In particular, Corollary 2.13 shows that the $u^0$-parts of those $\mathbb{F}$-summands in $ESH_+^*(Y)$ can be labelled by the maxima mentioned in (1.10) (the labelling is non-canonical, see Sec.1.5).

**Remark 1.14** (Coefficients)**.** *We showed $SH_+^{*-1}(Y)$ recovers the ordinary cohomology $H^*(Y, \mathbb{K})$ over the Novikov field $\mathbb{K}$. But $\mathbb{K}$ is flat over the base field $\mathcal{K}$ (indeed free, a $\mathcal{K}$-vector space) so $H^*(Y, \mathbb{K}) \cong H^*(Y, \mathcal{K}) \otimes_{\mathcal{K}} \mathbb{K}$ determines $H^*(Y, \mathcal{K})$. Thus we recover the McKay correspondence over fields $\mathcal{K}$ of characteristic zero, in particular over $\mathbb{Q}$.*

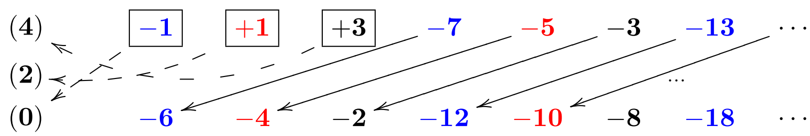
*Our proof also works when the characteristic is coprime to all integers $2, 3, \ldots, |G|$. The key idea is that the claim really only relies on understanding the spectral sequence for $ESH_+^*(Y)$ in odd degrees in $[-1, \dim_{\mathbb{R}} Y - 3]$. In particular the degree $-1$ part suffices to determine the rank of $ESH_+^*(Y)$ over $\mathbb{F}$, which is equal to both $|\mathrm{Conj}(G)|$ and $\chi(Y)$ (see Corollary 2.13). The assumption on the characteristic allows us to relate cohomologies of spaces before and after quotienting by finite groups which involve finite stabilisers, whose size is at most $|G|$ (Theorems 2.6 and 4.3).*

*If $\mathcal{K}$ is any commutative Noetherian ring, the Novikov ring $\mathbb{K}$ is flat over $\mathcal{K}$. The obstruction to running the above proof is the failure of the isomorphism $EH^*(\mathcal{O}_{\mathbf{g},\ell}) \cong H^{*-1}(\mathbb{P}V_{g,\ell})^{G_{g,\ell}}$. If this fails, there may be unexpected contributions in the Floer cohomology.*

Combining Theorems 1.9 and 1.10, we deduce the McKay Correspondence:

**Corollary 1.15** (Generalised McKay Correspondence)**.** *Let $Y$ be any quasi-projective crepant resolution of an isolated singularity $\mathbb{C}^n/G$, where $G \subset SL(n, \mathbb{C})$ is a finite subgroup. Let $\mathcal{K}$ be any field of characteristic zero, or assume $\mathrm{char}\, \mathcal{K}$ is coprime to all integers $\leq |G|$. Then $H^*(Y, \mathcal{K})$ vanishes in odd degrees and has rank $|\mathrm{Conj}_k(G)|$ in even degrees $2k$.*

**Example 1.16.** *For $X = \mathbb{C}^3/G$, with $G = \mathbb{Z}/3$ acting diagonally by powers of $\zeta = e^{2\pi i/3}$. The blow up $Y$ of $X$ at 0 is a crepant resolution with exceptional locus $E \cong \mathbb{CP}^2$. The Morse-Bott submanifolds $\mathcal{O}_{\zeta,2\pi/3}$, $\mathcal{O}_{\zeta^2,4\pi/3}$, $\mathcal{O}_{I,2\pi}$, $\mathcal{O}_{\zeta,8\pi/3}$, etc. are copies of $S^5/G$. Following analogous notation as in the example of $T^*\mathbb{CP}^1$, the acylic chain complex $SC^*(Y)$ over $\mathcal{K} = \mathbb{C}$ is:*




*where the generators in round brackets yield $H^*(Y) \cong H^*(\mathbb{CP}^2) \cong \mathbb{K} \oplus \mathbb{K}[-2] \oplus \mathbb{K}[-4]$, and the boxed generators yield $SH^*_+(Y) \cong \mathbb{K}[+1] \oplus \mathbb{K}[-1] \oplus \mathbb{K}[-3]$. The columns in the Morse-Bott complex for $ESC^*_+(Y)$ are shifted copies of $H^*(\mathbb{CP}^2, \mathbb{K})$ instead of $H^*(S^5, \mathbb{K})$:*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\boxed{-1}$ | $\boxed{+1}$ | $\boxed{+3}$ | $-7$ | $-5$ | $-3$ | $-13$ | $\cdots$ |
| $-3$ | $-1$ | $+1$ | $-9$ | $-7$ | $-5$ | $-15$ | $\cdots$ |
| $-5$ | $-3$ | $-1$ | $-11$ | $-9$ | $-7$ | $-17$ | $\cdots$ |

*Thus $ESH^*_+(Y) \cong \mathbb{F}[+1] \oplus \mathbb{F}[-1] \oplus \mathbb{F}[-3]$. The three summands correspond to the three conjugacy classes of $G$. This also holds for any field $\mathcal{K}$ of characteristic coprime to 2 and 3.*

1.5. **Naturality of the basis.** We return to the Open Problem at the end of Section 1.2. Assume for now that $\mathbb{K}$ has characteristic zero. In Corollary 2.13 we showed that given $\mathbf{g} \in \mathrm{Conj}(G)$, the sum $F_{\mathbf{g}} = \bigoplus_{\ell>0} EH^*(\mathcal{O}_{\mathbf{g},\ell})[-\mu_{\mathbf{g},\ell}]$ stacks together as a $\mathbb{K}$-vector space to yield a copy of the $\mathbb{K}$-vector space $\mathbb{F}$. This does not hold as $\mathbb{K}[\![u]\!]$-modules because the $E_1$-page of the Morse-Bott spectral sequence has forgotten the structure given by multiplication by $u$, yielding only a non-canonical $\mathbb{K}$-linear isomorphism

$$\bigoplus_{\mathbf{g}\in \mathrm{Conj}(\mathbf{G})} F_{\mathbf{g}} \cong ESH^*_+(Y) \cong H^{*+1}(Y) \otimes_{\mathbb{K}} \mathbb{F},$$

using (1.12). We conjecture that each maximum $x_g$ mentioned in (1.10) gives rise to a generator $[x_g + c_g] \in SH^*_+(Y)$, where $c_g$ is a "correction term"[1] with strictly higher $F$-filtration value than $x_g$ (in the sense of Appendix E). Theorem 1.9 would then yield generators of $H^*(Y)$ labelled by $\mathrm{Conj}(G)$, respecting (1.10). Nevertheless, the correction terms are not canonical.

When $\mathbb{K}$ has positive characteristic (coprime to $2, 3, \ldots, |G|$), the stacking mentioned for $F_{\mathbf{g}}$ only holds up to possible torsion summands (by the universal coefficient theorem), but such torsion must eventually cancel out in the spectral sequence as $ESH^*_+(Y)$ is free over $\mathbb{F}$. It is plausible that the Conjecture would persist to hold in positive characteristic.

Our modified approach [41] mentioned in Section 1.3 is expected to yield a clean naturality statement, in addition to discussing product structures and extending the results to the case of non-isolated singularities.

The map from (1.8),

$$SH^{*-1}_+(Y) \to H^*(Y) \cong H^{\mathrm{lf}}_{2\dim_{\mathbb{C}} Y - *}(Y), \tag{1.14}$$

is given by applying the Floer boundary operator of the full $SC^*(Y)$. Once we extend our work to non-isolated singularities in [41], it would be interesting to investigate how our basis compares via (1.14) to the bases built for $H^*(Y)$ by Ito-Reid [31] for $n = 3$, and by Kaledin [32] for $n = 2m$ and $G \subset \mathrm{Sp}(2m; \mathbb{C}) \cap U(2m)$.

---

[1] the correction term is not unique, and arises because the $E_1^{*,*}$ page for the equivariant Morse-Bott spectral sequence is isomorphic to the associated graded algebra of $ESH^*_+(Y)$.

**Remark 1.17.** *In the work of Kollár-Némethi* [35, Corollary 29] *a natural bijection arose between the conjugacy classes of* $G$ *and the irreducible components of the space* $\mathrm{ShArc}(0 \in X)$ *of* short complex analytic arcs $\overline{\mathbb{D}} \to X = \mathbb{C}^n/G$ *which hit the singular point only at* $0 \in \overline{\mathbb{D}}$, *where* $G \subset GL(n, \mathbb{C})$ *is any finite subgroup acting freely on* $\mathbb{C}^n \setminus \{0\}$. *This in turn gives rise to a natural map* [35, Paragraph 27] *from conjugacy classes to subvarieties in any resolution* $Y$ *of* $X$, *by considering the subsets swept out by the lifts of the arcs under evaluation at* $0 \in \overline{\mathbb{D}}$. *The operator in* (1.14) *counts finite energy Floer cylinders* $u : \mathbb{R} \times S^1 \to Y$ *converging to a Hamiltonian* 1*-orbit at the positive end. Such maps have a removable singularity at the negative end, and yield an extension* $u : \mathbb{C} \to Y$ *with* $u(0) \in E$. *Evaluation at* 0 *sweeps the required locally finite pseudo-cycle in* $H_*^{\mathrm{lf}}(Y)$. *As* $u$ *is asymptotically holomorphic near* $0 \in \mathbb{C}$, *the projection to* $\mathbb{C}^n/G$ *should approximate an analytic arc through* 0. *A possible approach to obtain genuine analytic arcs, would be to first perform a neck-stretching argument in the sense of Bourgeois-Oancea* [9] *so that (the main component of the) Floer solution converges to a holomorphic map* $u : \mathbb{C} \to Y$ *that is asymptotic to a Reeb orbit at infinity. It would be interesting to investigate more closely the relationship between these two points of view.*

**Remark 1.18.** *Abreu-Macarini* [1, Theorem 1.12] *proved that the* mean Euler characteristic $\chi(M, \xi)$ *of a Gorenstein toric contact manifold* $(M, \xi)$ *is equal to half of the Euler characteristic of any crepant toric symplectic filling* $Y$. *Recall* $\chi(M, \xi) = \lim_{k\to\infty} \frac{1}{2k} \sum_{j=0}^{k} \dim HC_{2j}(M, \xi)$ *is defined in terms of the linearised contact homology, cf. Ginzburg-Gören* [23]. *By Bourgeois-Oancea* [12], *this homology is isomorphic to the positive* $S^1$*-equivariant symplectic homology. Our Corollary 1.15 implies that* $\chi(M, \xi)$ *is half of the number of* $\mathbb{F}$*-summands in* $ESH_+^{*-1}(Y) \cong H^*(Y) \otimes_{\mathbb{K}} \mathbb{F}$, *thus it yields an alternative perspective of the result of Abreu-Macarini.*

## 2. Proofs

2.1. **Symplectic description of quotient singularities.** Let $X = \mathbb{C}^n/G$ for any finite subgroup $G \subset SU(n)$ acting freely on $\mathbb{C}^n \setminus 0$, and recall Lemma 1.4. Viewed as a convex symplectic manifold (in the sense of Sec.6.1), $\mathbb{C}^n$ has data

$$\omega = \sum dx_j \wedge dy_j, \qquad \theta = \tfrac{1}{2} \sum x_j dy_j - y_j dx_j, \qquad Z = \tfrac{1}{2} \sum x_j \partial_{x_j} + y_j \partial_{y_j} = \tfrac{1}{4} \nabla R, \qquad R = |z|^2$$

in coordinates $z_j = x_j + iy_j$. As $G \subset SU(n)$ preserves $R$ and the metric, it preserves all of the above data, so that descends to corresponding data $\omega_G$, $\theta_G$, $Z_G$, $R_G$ on $(X \setminus 0)/G$. Call $\pi_G : S^{2n-1} \to S_G = S^{2n-1}/G$ the induced quotient map on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ (note $S_G$ is smooth as $G$ acts freely). Using terminology from Appendix C, the **round contact form** $\alpha_0 = \theta|_{S^{2n-1}}$ yields a contact form $\alpha_G = \theta_G|_{S_G}$ on $S_G$, and they define contact structures $\xi = \ker \alpha_0$ on $S^{2n-1}$, $\xi_G = \ker \alpha_G$ on $S_G$. The Reeb flow $\phi_{2t}$ for $\alpha_0$ descends to the Reeb flow on $S_G$ for $\alpha_G$, where

$$\phi_t : S^{2n-1} \to S^{2n-1}, \quad \phi_t(z) = e^{it} z. \tag{2.1}$$

**Remark 2.1.** *We will from now on refer to* $\phi_t$ *as the* **Reeb flow** *(rather than* $\phi_{2t}$*) so the periods/lengths of Reeb orbits we will refer to are, strictly, the double of their actual values.*

2.2. **The closed Reeb orbits in the quotient.** Let $g \in G$ and let $V$ be the $e^{i\ell}$-eigenspace of $g$ for some given $\ell > 0 \in \mathbb{R}$,

$$V = V_{g,\ell} = \{v \in \mathbb{C}^n : g(v) = e^{i\ell} v\}.$$

By Lemma 1.6, $h(V) = V_{hgh^{-1},\ell}$ is the $e^{i\ell}$-eigenspace of $hgh^{-1}$. We sometimes abusively write $\dim_{\mathbb{C}} V_{\mathbf{g},\ell}$ for a class $\mathbf{g} = [g] \in \mathrm{Conj}(G)$, since the dimension does not depend on the choice of representative $g$. Let $\mathbb{P}(V)$ be the complex projectivisation. Define:

$$\begin{aligned} G_v &= \{h \in G : v \text{ is an eigenvector of } h\} \qquad (\text{where } v \in \mathbb{C}^n \setminus 0). \\ G_V &= \{h \in G : h(V) \subset V\}. \\ G_{g,\ell} &= \bigcup_{v \in V \setminus 0} G_v = \{h \in G : V_{h,\ell'} \cap V \neq \{0\} \text{ for some } \ell' \in \mathbb{R}\}. \end{aligned}$$

Observe that $G_{h(v)} = hG_vh^{-1}$, so we sometimes abusively write $|G_p|$ for $p = [v] \in (\mathbb{C}^n \setminus 0)/G$ as the size of the subgroup $G_v$ does not depend on the choice of representative $v$.

**Lemma 2.2.**

*(1) $G_v \subset G$ is a cyclic subgroup of size $|G_v| = |\{\lambda \in S^1 : h(v) = \lambda v$ for some $h \in G\}|$.*
*(2) $\{h \in G : V \cap h(V) \neq \{0\}\} = C_G(g)$ recovers the centraliser of $g$.*
*(3) $G_{g,\ell} = G_V = C_G(g)$.*
*(4) $G_V$ acts on $\mathbb{P}(V)$ with stabilisers $\mathrm{Stab}_{G_V}([v]) = G_v$, and the size $|C_G(g)|/|G_v|$ of the orbit of $[v]$ is the size of the fibre of $\mathbb{P}(V) \to \mathbb{P}(V)/G_V$.*

*Proof.* (1) Consider $\{\lambda \in S^1 : h(v) = \lambda v$ for some $h \in G_v\}$. As this is a finite subgroup of $S^1$, it is cyclic. Pick a generator $\lambda$, associated to $h \in G_v$ say. Then for any $h' \in G_v$, there is a $k \in \mathbb{N}$ satisfying $h'(v) = \lambda^k v = h^k(v)$, which forces $h' = h^k$ since $G$ acts freely on $\mathbb{C}^n \setminus 0$.

(2) $g(hv) = e^{i\ell}hv$ implies $h^{-1}ghv = e^{i\ell}v = gv$, so $h^{-1}gh = g$ ($G$ acts freely). Conversely, if $h \in C_G(g)$, then $h, g$ have a common basis of eigenvectors, so $hv = \lambda v \in V \cap h(V)$.

(3) Let $h$ be the generator of $G_v$ from (1), so $g = h^k$ for some $k$. Thus $h$ and $g$ commute, so $h \in C_G(g)$. Conversely, if $h \in C_G(g)$, then $h, g$ have a common basis of eigenvectors, and a subcollection will be a basis for $V$ consisting of eigenvectors of $h$. Thus $h \in G_v$ for any $v$ from this subcollection, and also $h(V) = V$ so $h \in G_V$. Finally (2) implies $G_V \subset C_G(g)$.

(4) Observe that $h \in G_V = C_G(g)$ fixes $[v] \in \mathbb{P}(V)$ precisely if $v$ is an eigenvector of $h$. □

**Corollary 2.3.** *$B = B_{\mathbf{g},\ell} = \pi_G(V \cap S^{2n-1}) = (V \cap S^{2n-1})/G_{g,\ell} \subset S_G$ is a submanifold of real dimension $\dim B = 2\dim_{\mathbb{C}} V - 1$.*

*Proof.* $\pi_G(V \cap S^{2n-1}) = (GV/G) \cap S_G$ where $GV = \cup h(V)$ over all $h \in G$, and $GV/G \cong V/G_{g,\ell}$ by Lemma 2.2. Finally, $V \cap S^{2n-1}$ is a transverse intersection and $G_{g,\ell}$ acts freely on it (as $G$ acts freely on $\mathbb{C}^n \setminus 0$). □

By Lemma 1.6, $B$ is precisely the moduli space

$$\mathcal{O} = \mathcal{O}_{\mathbf{g},\ell}$$

of parametrized closed Reeb orbits in $S_G$ of length $\ell$ associated to the class $\mathbf{g} \in \mathrm{Conj}(G)$ via (1.4), as $p = [v] \in B$ determines the Reeb orbit $[0, \ell] \to S_G$, $\phi_t(p) = [e^{it}v] \in S_G$ with initial point $\phi_0(p) = p$. We often blur the distinction by identifying $B \equiv \mathcal{O}$. Note however that the subset $B_{\mathbf{g},\ell+2\pi k} \subset S_G$ does not depend on $k \in \mathbb{N}$, whilst $\mathcal{O}_{\mathbf{g},\ell+2\pi k}$ does, due to the length. The **short Reeb orbits** are those of **length** $\ell \in (0, 2\pi]$, and they determine the **age** in (1.2):

$$\mathrm{age}(\mathbf{g}) = \frac{1}{2\pi} \sum_{0<\ell\leq 2\pi} (2\pi - \ell) \dim_{\mathbb{C}} V_{\mathbf{g},\ell}, \tag{2.2}$$

in particular the sum of all these lengths counted with dimension-multiplicity is $2\pi(n - \mathrm{age}\, g)$. Recall from the Introduction, the **minimal Reeb orbits** associated to $g$ are the short Reeb orbits occurring for the smallest value of $\ell$ (amongst eigenvalues of $g$). Also, recall from Appendix C the Definition 5.1 of **Morse-Bott submanifold**.

**Lemma 2.4.** *$B \subset S_G$ is a Morse-Bott submanifold of real dimension $\dim B = 2\dim_{\mathbb{C}} V - 1$.*

*Proof.* Condition (1) of Definition 5.1 is a consequence of Lemma 1.6, as explained above. To check condition (2), let $\gamma : [0,\ell] \to S_G$ be a closed Reeb orbit. Lift $\gamma$ to a path in $V$, $\widetilde{\gamma} : [0,\ell] \to S^{2n-1}$, $\widetilde{\gamma}(t) = \phi_t(v)$ with $g(v) = e^{i\ell}v$. After an $SU(n)$-change of coordinates, we may assume $\widetilde{\gamma}(0) = (0,\dots,0,1)$, so $\widetilde{\gamma}(t) = (0,\dots,0,e^{it})$. Fix the obvious trivialisation $\gamma(0)^*\xi_G \cong \widetilde{\gamma}(0)^*\xi \cong \mathbb{C}^{n-1} \times 0 \subset \mathbb{C}^n$, then this lifts to a trivialisation of $\widetilde{\gamma}(\ell)^*\xi = (g\widetilde{\gamma}(0))^*\xi$ by $g(\mathbb{C}^{n-1} \times 0)$. So the linearised return map in this trivialisation is $z \mapsto (g^{-1} \circ D\phi_\ell)(z) = g^{-1}(e^{i\ell}z)$ for $z \in \mathbb{C}^{n-1} \times 0 \subset \mathbb{C}^n$, whose 1-eigenspace is $E = V \cap (\mathbb{C}^{n-1} \times 0)$. This is a transverse intersection, as $\mathbb{C} \cdot \widetilde{\gamma}(0) = 0 \times \mathbb{C} \subset V$, thus $\dim E = 2\dim_{\mathbb{C}} V - 2 = \dim B - 1$, moreover the intersection equals $TB \cap \xi|_B$. □

Each Reeb orbit $\gamma : \mathbb{R}/\ell\mathbb{Z} \to S_G$ defines iterates $\gamma^k : \mathbb{R}/k\ell\mathbb{Z} \to S_G$, $\gamma^k(t) = \gamma(t)$, for $k \in \mathbb{N}$, and the **multiplicity** of $\gamma$ is the largest $k$ with $\gamma = \eta^k$ for some closed Reeb orbit $\eta$.

**Lemma 2.5.** *The multiplicity of the Reeb orbit of length $\ell$ corresponding to $p \in B$ equals*

$$m_p = \max_{b\in(0,\infty)} \left\{ \tfrac{\ell}{b} : h(v) = e^{ib}v \text{ for some } h \in G \text{ and } v \in \pi_G^{-1}(p) \subset S^{2n-1} \right\} = \ell\,|G_p|/2\pi, \quad (2.3)$$

*so it may depend on the orbit in $B$. For short Reeb orbits ($0<\ell\leq 2\pi$), $m_p \leq |G_p| \leq |G|$ and it is determined by $g = h^{m_p}$ where $h$ is the generator of $G_v$ and $p = [v]$.*

*Proof.* The first equality in (2.3) is immediate, as the achieved maximum yields a minimal $b$ for which $[e^{ib}v] = [v] \in S_G$. The second equality follows by Lemma 2.2(1), since the generator $h$ of $G_v$ will achieve the maximum in (2.3) and satisfies $h(v) = e^{i2\pi/|G_v|}v$, so $b = 2\pi/|G_v|$. □

2.3. **The associated $S^1$-action on the Morse-Bott submanifolds of Reeb orbits.** There are two circle actions on $B = \mathcal{O}$: the $S^1$-action that $B$ inherits as a subset $B \subset \mathbb{C}^n/G$; and the **associated $S^1$-action** of Definition 5.1: the circle $S^1_\ell = \mathbb{R}/\ell\mathbb{Z}$ acts on $\mathcal{O}$ by time-translation $\gamma \mapsto \gamma(\cdot + c)$ for $c \in S^1_\ell$. The orbits of both actions agree geometrically as images in $\mathbb{C}^n/G$, yielding the same circle $C \subset \mathbb{C}^n/G$, but the degrees of the quotient maps $S^1 \to C$, $t \mapsto [e^{it}v]$ and $S^1_\ell \to C$, $t \mapsto \gamma(t)$ can differ. Indeed the degrees are respectively $|G_p|$ and $m_p = \ell\,|G_p|/2\pi$ (by (2.3)), so the two actions coincide only for $\ell = 2\pi$.

**Theorem 2.6.** *One can identify $B/S^1_\ell$ with the quotient $\mathbb{P}V/G_V$. The fiber of $\mathbb{P}V \to \mathbb{P}V/G_V$ over $p = [v]$ has size $|C_G(g)|/|G_v|$ which divides $|G|$. Via $\mathbb{P}V \cong \mathbb{CP}^{\dim_{\mathbb{C}} V-1}$, we get*

$$H^*(B/S^1_\ell) \cong H^*(\mathbb{CP}^{\dim_{\mathbb{C}} V-1})$$

*over any field of characteristic not dividing $|G|$.*

*The equivariant cohomology of $B = \mathcal{O}$ for the $S^1_\ell$-action, in the sense of Appendix B, is*

$$EH^*(\mathcal{O}) \cong H^{*-1}(B/S^1_\ell) \quad (2.4)$$

*in characteristic zero, and for $\ell \leq 2\pi$ also over fields of characteristic coprime to $2, 3, \dots, |G|$.*

*Proof.* $B/S^1_\ell = (GV \cap S^{2n-1})/(S^1 \cdot G) = \mathbb{P}(GV)/G = \mathbb{P}V/G_V$, where the last equality uses Lemma 2.2 (2),(3). The statement about the fiber follows by Lemma 2.2 (4). Remark 4.4 implies $H^*(\mathbb{P}V/G_V) \cong H^*(\mathbb{P}V)^{G_V}$ by pulling back via the projection $p : \mathbb{P}V \to \mathbb{P}V/G_V$. But $H^*(\mathbb{P}V)^{G_V} = H^*(\mathbb{P}V)$, indeed $p^*$ is surjective since $p^*\omega_G = \omega$ is the Fubini-Study form on $\mathbb{P}V \subset \mathbb{CP}^{n-1}$ that generates the ring $H^*(\mathbb{P}V) \cong H^*(\mathbb{CP}^{\dim_{\mathbb{C}} V-1})$. Theorem 4.3 implies (2.4) (using that the sizes of the stabilisers are $m_p \leq |G|$ when $\ell \leq 2\pi$, by Lemma 2.5). □

2.4. **Conley-Zehnder indices of Reeb orbits in the quotient.** Appendix C is a survey of Conley-Zehnder indices and their properties (CZ1)-(CZ4). For $n \geq 2$, let $\kappa$ be the canonical bundle on $S^{2n-1}$ induced by the standard complex structure $J$ inherited from the inclusion $S^{2n-1} \subset \mathbb{C}^n$. Let $E = \sum z_j \partial z_j$ denote the **Euler vector field**, and let $E^* = \sum \overline{z}_j \, dz_j$. Then, as complex vector bundles, $T_{\mathbb{C}}\mathbb{C}^n$ and $T^*_{\mathbb{C}}\mathbb{C}^n$ pulled back to $S^{2n-1}$ split as $\xi \oplus \mathbb{C}E$ and dually $\xi^* \oplus \mathbb{C}E^*$ (in particular, the contraction $\iota_E \xi^* = 0$), where $\xi$ was defined in Sec.2.1. It follows that $\Lambda^{n-1}_{\mathbb{C}}(\mathbb{C}^n)^*$ pulled back to $S^{2n-1}$ has a one-dimensional summand determined by the forms which vanish when contracted with $E$. Thus $\kappa$ can be trivialized by the section

$$K = \iota_E(dz_1 \wedge \cdots \wedge dz_n). \tag{2.5}$$

This ensures that a compatible trivialisation of $\xi$ together with the field[1] $Z + iY$ from Remark 5.3 yields, up to homotopy, the standard trivialisation of the anti-canonical bundle $\mathcal{K}^*_{\mathbb{C}^n}$ of $\mathbb{C}^n$. Conversely, observe that the standard trivialisation of $\mathcal{K}_{\mathbb{C}^n}$ by $dz_1 \wedge \cdots \wedge dz_n$ is induced by any complex frame for $T\mathbb{C}^n$ arising as the image under $SL(n, \mathbb{C})$ of the standard frame $\partial_{z_1}, \ldots, \partial_{z_n}$ (since $\det = 1$). Restricting to $S^{2n-1} \subset \mathbb{C}^n$, if the first vector field of that frame is $E$ then the remaining vector fields of the frame induce the section (2.5) on $\kappa$.

Recall from Sec.2.1, $G \subset SU(n)$ is a finite subgroup acting freely on $\mathbb{C}^n \setminus 0$. Observe that $SU(n)$ preserves the field $E$, the canonical bundle $\mathcal{K}_{\mathbb{C}^n}$, and more generally the above splittings; so it preserves $K$. So via the quotient $\pi_G : S^{2n-1} \to S_G = S^{2n-1}/G$, we obtain induced data $\alpha_G, \xi_G, J_G, \kappa_G, K_G$ from the analogous data $\alpha, \xi, J, \kappa, K$ defined on $S^{2n-1} \subset \mathbb{C}^n$.

**Theorem 2.7.** *For $0 < \ell \leq 2\pi$, the Conley-Zehnder index of $B = B_{\mathbf{g},\ell}$ is*

$$CZ(B) = n - 2\,\mathrm{age}(\mathbf{g}) + 2\sum_{\ell' < \ell} \dim_{\mathbb{C}} V_{\mathbf{g},\ell'} + \tfrac{1}{2}\dim B + \tfrac{1}{2},$$

*and the associated grading (5.4) is*

$$\mu(B) = 2\,\mathrm{age}(\mathbf{g}) - 2\sum_{\ell' < \ell} \dim_{\mathbb{C}} V_{\mathbf{g},\ell'} - \dim B - 1. \tag{2.6}$$

*In general,* $\mathrm{CZ}(B_{\mathbf{g},\ell+2\pi k}) = \mathrm{CZ}(B_{\mathbf{g},\ell}) + 2kn$*, and thus* $\mu(B_{\mathbf{g},\ell+2\pi k}) = \mu(B_{\mathbf{g},\ell}) - 2kn$.

*If $e^{i\ell}$ is the minimal eigenvalue of $\mathbf{g}$ then $\mu(B) = 2\,\mathrm{age}(\mathbf{g}) - \dim B - 1$, so the maximum of $B$ (in the sense of Remark 5.3) is a minimal Reeb orbit in grading $\mu = 2\,\mathrm{age}(\mathbf{g}) - 1$.*

*Proof.* We first show that the middle claim follows from the first. Note that $\phi_t^* K = e^{int} K$. As $\phi_{2k\pi} = \mathrm{Id}_{S_G}$, an orbit $\gamma$ in $B_{\mathbf{g},\ell+2\pi k}$ can be viewed as a concatenation of an orbit $\gamma_1 \in B_{\mathbf{g},\ell}$ together with the orbit $\gamma_2 : \mathbb{R}/2\pi k\mathbb{Z} \to S_G$, $t \mapsto e^{it}\gamma_1(\ell)$. Thus $\mathrm{CZ}(\gamma) = \mathrm{CZ}(\gamma_1) + \mathrm{CZ}(\gamma_2)$ by property (CZ1). By properties (CZ2) and (CZ4) one deduces $\mathrm{CZ}(\gamma_2) = 2kn$.

We now prove the first claim. Abbreviate $d = \dim_{\mathbb{C}} V$. By an $SU(n)$-change of coordinates to a basis of unitary eigenvectors for $g$, we may assume the Reeb orbit is $\gamma(t) = \pi_G(\widetilde{\gamma}(t))$ for $\widetilde{\gamma}(t) = (e^{it}, 0, \ldots, 0)$, with $t \in [0, \ell]$, and that the first $d$ standard basis vectors are eigenvectors of $g$ with eigenvalue $e^{i\ell}$ (i.e. a basis for $V = V_{g,\ell}$). Thus

$$g = \mathrm{diag}(e^{i\ell}, \ldots, e^{i\ell}, e^{i\ell'}, \ldots)$$

with $e^{i\ell}$ in the first $d$ diagonal entries, and the other eigenvalues $e^{i\ell'}, \ldots$ of $g$ in the remaining diagonal entries, where $0 < \ell' \leq 2\pi$. This basis splits $T_{\gamma(t)}\mathbb{C}^n = \mathbb{C}^n = \mathbb{C} \oplus \xi$ where $\mathbb{C} = \mathbb{C} \times 0$, $\xi = 0 \times \mathbb{C}^{n-1}$. Picking a trivialisation of $\gamma^*\xi_G$ is equivalent to picking a **$G$-compatible trivialisation** of $\widetilde{\gamma}^*\xi$, meaning that if $h(\widetilde{\gamma}(t)) = \widetilde{\gamma}(s)$ then the trivialisations of $\widetilde{\gamma}(t)^*\xi$ and $\widetilde{\gamma}(s)^*\xi$ are related by multiplication by $h$. By Lemma 2.2, it suffices to ensure compatibility for $h \in G$, with $h^k = g$ (and $k \in \mathbb{N}$ maximal such). Because we may assume that the above

[1] where $Z, Y$ are the vector fields defined in Sec.6.1.

eigenbasis for $g$ arose from an eigenbasis for $h$, the compatibility with $h$ will in fact also be guaranteed by our construction. We define two auxiliary families in $SL(n,\mathbb{C})$,

$$g_t = \mathrm{diag}(e^{i\ell\frac{t}{\ell}},\dots,e^{i\ell\frac{t}{\ell}},e^{i\ell'\frac{t}{\ell}},\dots) = \mathrm{diag}(e^{it},\dots,e^{it},e^{i\ell'\frac{t}{\ell}},\dots)$$
$$a_t = \mathrm{diag}(1,\dots,1,e^{2\pi i(\mathrm{age}(g)-n)\frac{t}{\ell}},1,\dots,1)$$

for $t\in[0,\ell]$, where for $a_t$ the non-unit entry can be placed in any position except the first (to ensure that the first section $E=(a_t\cdot g_t)(1,0,\dots,0)$ is the Euler vector field along $\widetilde{\gamma}$). The image under $a_t\cdot g_t$ of the eigenbasis defined at $t=0$ gives a trivialisation of the canonical bundle of $\mathbb{C}^n$ along $\widetilde{\gamma}$, which is compatible with the standard trivialisation as $\det(a_t\cdot g_t)=1$, using (2.2). It is also $G$-compatible since at $t=\ell$, we have $a_\ell\cdot g_\ell = g$ (using that $\mathrm{age}(g)\in\mathbb{N}$). Omitting the first section (the Euler vector field) yields a compatible trivialisation of $\widetilde{\gamma}^*\xi$. The (linearised) flow $\phi_t=\mathrm{diag}(e^{it},\dots,e^{it})$ in this trivialisation becomes $(a_t\cdot g_t)^{-1}\cdot\phi_t$, so

$$a_t^{-1}\cdot\mathrm{diag}(1,\dots,1,e^{i(\ell-\ell')\frac{t}{\ell}},\dots).$$

As the lengths $\ell,\ell'\in(0,2\pi]$, all differences satisfy $|\ell-\ell'|<2\pi$, so the function (5.2) satisfies $W(\ell-\ell')=+1$ if $\ell'<\ell$, and $-1$ if $\ell'>\ell$. Thus properties (CZ2) and (CZ4) imply

$$\mathrm{CZ}(B) = 2(n-\mathrm{age}(g)) + \sum_{\ell'<\ell}\dim_{\mathbb{C}} V_{\mathbf{g},\ell'} - \sum_{\ell'>\ell}\dim_{\mathbb{C}} V_{\mathbf{g},\ell'}.$$

By replacing one copy of $n$ by $n=\dim_{\mathbb{C}} V+\sum_{\ell'\neq\ell}\dim_{\mathbb{C}} V_{\mathbf{g},\ell'}$, and using $\dim_{\mathbb{C}} V=\frac{1}{2}\dim B+\frac{1}{2}$ from Lemma 2.4, the claim follows. □

2.5. **Convex symplectic manifold structure for resolutions of isolated singularities.** Continuing with the notation from Sec.2.1, let $\pi: Y\to X=\mathbb{C}^n/G$ be any resolution, and let $B_\epsilon=\{z:|z|\leq\epsilon\}\subset\mathbb{C}^n/G$ for $0<\epsilon\ll 1$.

**Lemma 2.8.** *There is a Kähler form $\omega_Y$ on $Y$ such that $(Y,\omega_Y)$ is convex symplectic and its data $\omega,\theta,Z,R$ on $Y\setminus\pi^{-1}(B_\epsilon)$ agrees via $\pi$ with the data $\omega_G$, $\theta_G$, $Z_G$, $R_G$ on $(\mathbb{C}^n\setminus B_\epsilon)/G$.*

This Lemma is immediate from Lemma 3.2 because $Y$ admits a Kähler form $\omega_Y$ which agrees with $\pi^*\omega_X$ outside of an arbitrarily small neighbourhood of $\pi^{-1}(0)$.

When $Y$ is a crepant resolution, the natural diagonal $\mathbb{C}^*$-action on $\mathbb{C}^n$ (and thus on $\mathbb{C}^n/G$) lifts to $Y$ by Proposition 3.6. So $\pi: Y\to\mathbb{C}^n/G$ is $\mathbb{C}^*$-equivariant. The $S^1\subset\mathbb{C}^*$ defines an $S^1$-action $\widetilde{\phi_t}$ on $Y$ that lifts the action $\phi_t[z]=[e^{it}z]$ on $\mathbb{C}^n/G$. By averaging, namely replacing $\omega_Y$ by $\int_{S^1}\widetilde{\phi}_t^*\omega_Y$, we may assume that the $\omega_Y$ above is $S^1$-invariant.[1]

2.6. **Vanishing of symplectic cohomology of crepant resolutions.** Below we prove directly that $SH^*(Y)=0$ based on [47], but we remark that $SH^*(M)=0$ is a general result [49] for convex symplectic manifolds $M$ with Chern class $c_1(M)=0$ whose Reeb flow at infinity arises from a Hamiltonian $S^1$-action $\phi_t$ with non-zero Maslov index $2I(\phi)\in\mathbb{Z}$ (in our setup, $I(\phi)=n$ is the winding number arising in the proof of Theorem 2.12). Indeed, the assumptions imply that $SH^*(M)$ is $\mathbb{Z}$-graded, $c^*: QH^*(M)\to SH^*(M)$ is a quotient $\mathbb{K}$-algebra homomorphism, and the $S^1$-action defines a $\mathbb{K}$-linear automorphism $\mathcal{R}_g\in\mathrm{Aut}(SH^*(M))$ of degree $2I(\phi)$, but $\dim_{\mathbb{K}} QH^*(M)=\dim_{\mathbb{K}} H^*(M,\mathbb{K})<\infty$ so $SH^*(M)=0$.

Recall crepant resolutions $\pi: Y\to X=\mathbb{C}^n/G$ admit a $\mathbb{C}^*$-action, yielding an action $\phi_t$ by $S^1\subset\mathbb{C}^*$, and by averaging the Kähler form by $\int\phi_t^*\omega_Y$ we may assume $\omega_Y$ is $S^1$-invariant. In

[1] such an averaging does not affect $\pi^*\omega_G$, which is already $\phi_t$-invariant.

general, for a symplectic manifold $(M,\omega)$ with an $S^1$-action, if $\omega$ is $S^1$-invariant the action is symplectic; and if $H^1(M,\mathbb{R})=0$ (e.g. if $M$ is simply connected) the action is Hamiltonian.[1]

**Lemma 2.9.** *Let $\pi: Y\to X=\mathbb{C}^n/G$ be a crepant resolution, where $G\subset SL(n,\mathbb{C})$ is a finite subgroup. If the Kähler form on $Y$ is $S^1$-invariant, then the $S^1$-action is Hamiltonian.*

*Proof.* Resolutions of quotient singularities are always simply connected [33, Thm.7.8]. □

Now assume $G\subset SU(n)$ acts freely on $\mathbb{C}^n\setminus 0$, $\omega_Y$ is $S^1$-invariant, and $\omega_Y=\pi^*\omega_G$ except on a neighbourhood $Y_\epsilon=\pi^{-1}(B_\epsilon)$ of $\pi^{-1}(0)$ (see Sec.2.5). Recall $H_G=\frac{1}{2}|z|^2:\mathbb{C}^n/G\to\mathbb{R}$ generates the $S^1$-action $z\mapsto e^{it}z$ on $\mathbb{C}^n/G$. As $\pi$ is $S^1$-equivariant, the Hamiltonian $H:Y\to\mathbb{R}$ generating the $S^1$-action on $Y$ agrees with $\pi^*(H_G)$ on $Y\setminus Y_\epsilon$ up to an additive constant.

**Lemma 2.10.** *The period of closed orbits of $H:Y\to\mathbb{R}$ outside $\pi^{-1}(0)$ are integer multiples of $2\pi/|G|$. For $k\notin\frac{2\pi}{|G|}\mathbb{Z}$, all 1-orbits of $kH$ are contained in $\pi^{-1}(0)$, and for $k\notin 2\pi\mathbb{Q}$ they are constant.*

*Proof.* As the map $\pi$ is $S^1$-equivariant, the 1-orbits in $Y\setminus\pi^{-1}(0)$ agree with the 1-orbits in $(\mathbb{C}^n\setminus 0)/G$. By Lagrange's theorem, the $|G|$-th iterate of a closed Reeb orbit in $S^{2n-1}/G$ will lift to a closed Reeb orbit in $S^{2n-1}$, whose period must be in $2\pi\mathbb{Z}$. The first two claims follow. For the final claim, consider the initial point $p$ of a 1-orbit in $\pi^{-1}(0)$. The stabilizer in $S^1$ of $p$ contains $e^{ik}$, which has a dense orbit in $S^1$ for $k\notin 2\pi\mathbb{Q}$ and so $p$ is fixed by $S^1$. □

**Remark 2.11.** *The Hamiltonians $L_k=kH$ for $k\notin 2\pi\mathbb{Q}$ determine $SH^*(Y)=\varinjlim HF^*(L_k)$, but they cannot be used in the construction of $SH^*_+(Y)$ (see Appendix D). We briefly clarify the meaning of the notation $HF^*(L_k)$. Recall that the Kähler metric on $Y$ is $S^1$-invariant under the Hamiltonian $S^1$-action, and the fixed locus of that action lies in the compact set $\pi^{-1}(0)$. The proof Lemma 1 in Frankel* [21] *proves that the fixed locus is a compact Riemannian submanifold $C\subset Y$ lying inside $\pi^{-1}(0)$, and that it is a Morse-Bott submanifold for the Hamiltonian. Observe that the (constant) 1-orbits of $L_k$ are the points of $C$. When we write $HF^*(L_k)$, it is understood that one either uses a Morse-Bott Floer complex (see Remark 5.3) or one uses a generic small compactly supported perturbation of $L_k$ (in the sense of Hofer-Salamon* [30, Theorem 3.1]*). Up to isomorphisms, the choice of perturbation does not affect $HF^*(L_k)$ nor the continuation maps that define the direct limit $SH^*(Y)$. By a judicious choice of perturbation, using an auxiliary Morse function $f_S:S\to\mathbb{R}$ on each connected Morse-Bott submanifold $S$ (see* [16, Prop.2.2] *and* [37, Appendix B]*) the generators of the Floer complex after perturbation can be identified with the critical points $x$ of the functions $f_S$, and the grading of those generators is $\mu(S)+\mathrm{ind}_{f_S}(x)$ where $\mu(S)$ is as in* (5.4) *and $\mathrm{ind}_{f_S}(x)$ is the Morse index of $x\in\mathrm{Crit}(f_S)$.*

**Theorem 2.12.** *The generators of $CF^*(L_k)$, for $k\notin 2\pi\mathbb{Q}$, lie in arbitrarily negative degree for large $k$, therefore*

$$SH^*(Y)=0,\quad ESH^*(Y)=0,\quad SH^*_+(Y)\cong H^{*+1}(Y,\mathbb{K}),\quad ESH^*_+(Y)\cong H^{*+1}(Y,\mathbb{K})\otimes_{\mathbb{K}}\mathbb{F}.$$

*Proof.* As $Y$ is crepant and[2] $H^1(\partial Y_\epsilon,\mathbb{R})\cong H^1(S^{2n-1}/G,\mathbb{R})=0$ (using Remark 4.4), there is a nowhere zero smooth section $s$ of the canonical bundle $\mathcal{K}$ of $Y$ agreeing with the pull-back of the standard section on $Y\setminus Y_\epsilon\cong(\mathbb{C}^n\setminus B_\epsilon)/G$ for $\mathcal{K}_{\mathbb{C}^n}$,

$$s|_{Y-Y_\epsilon}=dz_1\wedge\cdots\wedge dz_n. \tag{2.7}$$

[1] Indeed, the vector field $v=v_t$ that generates the $S^1$-action $\phi_t$ on $M$ defines a form $\sigma=\sigma_t=\omega(\cdot,v)$, which is closed due to the Cartan formula $d(\iota_v\omega)+\iota_v d\omega=\mathcal{L}_v\omega=\partial_t|_{t=0}\phi_t^*\omega$. Thus $[\sigma]\in H^1(M,\mathbb{R})=0$, so $\sigma=dH$ and $v=X_H$ is Hamiltonian, where $H=H_t:M\to\mathbb{R}$.

[2] The second condition ensures that sections of $\mathcal{K}$ agree on the boundary, up to homotopy.

The $\mathbb{C}^*$-action $\phi_w$ on $Y$ induces a $\mathbb{C}^*$-action on $\mathcal{K}$, thus it defines a function $f_w : Y \to \mathbb{C}^*$ by

$$\phi_w^*(s|_{\phi_w(y)}) = f_w(y)\, s|_y.$$

By (2.7), $f_w(y) = w^n$ for $y \in Y \setminus Y_\epsilon$. The map $f : \mathbb{C}^* \times Y \to \mathbb{C}^*$, $f(w,y) = f_w(y)$, defines a homotopy class of maps in

$$[\mathbb{C}^* \times Y, \mathbb{C}^*] \cong H^1(\mathbb{C}^* \times Y) = (H^1(\mathbb{C}^*) \otimes H^0(Y)) \;\oplus\; (H^0(\mathbb{C}^*) \otimes H^1(Y)).$$

Only $H^1(\mathbb{C}^*)$ matters, as $Y$ is connected and $H^1(Y) = 0$ (as $Y$ is simply connected). Thus $f$ is homotopic to the map $(w,y) \mapsto w^n$. Given a (constant) 1-orbit $x$ of $L_k = kH$, let $p = x(0) \in \pi^{-1}(0)$ be the initial point. As $p$ is a fixed point, we may linearise the $\mathbb{C}^*$-action on $T_pY$:

$$\mathbb{C}^* \times T_pY \to T_pY, \quad (w, Z_j) \mapsto w^{m_j} Z_j$$

for some $m_1, \cdots, m_n \in \mathbb{Z}$, where $Z_1, \cdots, Z_n$ are a basis of $T_pY$ induced by a choice of $\mathbb{C}$-linear coordinates near $p$ (by a linear change of basis, we diagonalised the action at $p$). Thus, the action on $\mathcal{K}$ is by multiplication by $w^{-\sum m_j}$, so it must equal $w^n$, so $-\sum m_j = n$. As the time $t$ flow of $L_k = kH$ is $\phi_{w^k} = \phi_w^k$ for $w = e^{it}$, using Appendix C we deduce

$$\mathrm{CZ}(x) = \sum W(-km_j) \geq \sum 2\lfloor \tfrac{-km_j}{2\pi} \rfloor \geq \sum (\tfrac{-km_j}{\pi} - 2) = (\tfrac{k}{\pi} - 2)n$$

(using (CZ2), (CZ4), and $W(t) \geq 2\lfloor \frac{t}{2\pi} \rfloor$), so the grading $\mu(x) \leq (3 - \frac{k}{\pi})n$. Thus we conclude that $\mu(x) \to -\infty$ as $k \to \infty$. The final claim then follows, because $SH^*(Y) = \varinjlim HF^*(L_k)$ is a direct limit over grading-preserving maps.[1] The same argument applies to $ESH^*(Y)$, using that $\mathbb{F}$ lies in negative degrees (alternatively, it follows from $SH^*(Y) = 0$ by (4.5)). The final two results follow by Corollary 6.5. ☐

2.7. **Computation of $ESH^*_+$ of crepant resolutions.** We refer to Appendices B and D for the construction of $ESH^*_+$. We choose a specific sequence $H_k$ of **admissible Hamiltonians** (see 6.2) with final slope $k \notin 2\pi\mathbb{Q}$, to compute $ESH^*_+(Y) = \varinjlim EHF^*_+(H_k)$. Recall $H : Y \to \mathbb{R}$ is the Hamiltonian generating the $S^1$-action on $Y$, and by Lemma 2.8 the radial coordinate $R$ on $Y \setminus Y_\epsilon$ (in the sense of Sec.6.1) agrees via $\pi : Y \to \mathbb{C}^n/G$ with the radial coordinate $R_G = |z|^2$ on $(\mathbb{C}^n \setminus B_\epsilon)/G$; in that region $H^2 = \frac{1}{4}R^2$. Define $H_k = H^2$ except on the region where $H^2$ has slope $\geq k$ in $R$, and extend by $H_k = kR$ outside of that region. By projecting via $\pi$ and then projecting to $S_G = S^{2n-1}/G$, there is a 1-to-1 correspondence between the 1-orbits defining $ECF^*_+(H_k)$ and the Reeb orbits in $S_G$ of length $\leq k$ (analogously to Remark 2.11, it is understood that a small perturbation of $H$ is needed near $\pi^{-1}(0)$, but the resulting generators near $\pi^{-1}(0)$ of the Floer complex will be quotiented out by definition of $ECF^*_+(H_k)$, see Section 6). So we may abusively write $B_{\mathbf{g},\ell} \subset Y$ when referring to those orbits in $Y$ (recall $B_{\mathbf{g},\ell} \subset S_G$ from Lemma 2.3).

**Corollary 2.13.** *Assume* $\mathrm{char}\,\mathbb{K} = 0$, *or more generally* $\mathrm{char}\,\mathbb{K}$ *coprime to all integers* $\leq |G|$. *For* $\mathbf{g} = [g] \in \mathrm{Conj}(G)$, *the orbits in* $\cup_{\ell > 0} B_{\mathbf{g},\ell} \subset Y$ *contribute a copy of the* $\mathbb{K}$*-vector space* $\mathbb{F}[-\mu_{\mathbf{g}}]$ *to the* $E_1$*-page of the Morse-Bott spectral sequence for*[2] $ESC^*_+(Y)$ *(see Appendix E), where*

$$\mu_g = \mu_{\mathbf{g}} = 2\,\mathrm{age}(\mathbf{g}) - 1.$$

*Moreover, as a* $\mathbb{K}$*-vector space,* $ESH^*_+(Y)$ *has one summand* $\mathbb{F}[-\mu_{\mathbf{g}}]$ *for each* $\mathbf{g} \in \mathrm{Conj}(G)$.

---

[1] A small perturbation of $L_k$ to a generic Hamiltonian as described in Remark 2.11 will change Conley-Zehnder indices by at most $\dim_{\mathbb{R}}(Y)$, so $\mu(x) \to -\infty$ as $k \to \infty$ still holds even after perturbation. We remark that one can also prove directly (and more generally) that Conley-Zehnder indices change by at most $\dim_{\mathbb{R}}(Y)$ after perturbation, without appealing to the Morse-Bott argument in Remark 2.11, by the same argument as in McLean [40, Lemma 4.10].

[2] more precisely, as we only take the direct limit on cohomology, $ECF^*_+(Y, H_k)$ sees at least the summand $\mathbb{K}[-\mu_g] \oplus \mathbb{K}[-\mu_g + 2] \oplus \mathbb{K}[-\mu_g + 4] \oplus \cdots \oplus \mathbb{K}[-\mu_g + 2mn]$ of $\mathbb{F}[-\mu_g] = \mathbb{K}[-\mu_g] \oplus \mathbb{K}[-\mu_g + 2] \oplus \cdots$ if $k \geq (m+1)\pi$.

*Proof.* Suppose char $\mathbb{K} = 0$. By Theorems 2.6-2.7, each eigenspace $V = V_{g,\ell}$ of $g$ for $0 < \ell \leq 2\pi$ yields a submanifold $B = B_{\mathbf{g},\ell} \subset Y$ which contributes a copy of $EH^*(B) \cong H^{*-1}(\mathbb{CP}^{\dim_{\mathbb{C}} V - 1})$ (by Theorem 2.6), shifted up in grading by the $\mu(B)$ in (2.6). By Theorem 2.7, if $e^{i\ell}$ is the minimal eigenvalue of $\mathbf{g}$ then the maximum of $B/S^1$ contributes a generator in grading $\mu = \mu_g$,

$$\mu_g = \mu(B) + \dim B = 2\,\mathrm{age}(g) - 1.$$

So $B/S^1 \cong \mathbb{P}(V)/G$, which has dimension $2\dim_{\mathbb{C}} V - 2$, contributes one generator in each odd degree in the range $[\mu_g - 2\dim_{\mathbb{C}} V + 2, \mu_g]$. The next smallest eigenvalue $e^{i\ell'}$ of $g$, corresponding to an eigenspace $V' = V_{\mathbf{g},\ell'}$ and a submanifold $B' = B_{\mathbf{g},\ell'}$, will have a maximum in degree $\mu(B') + \dim B' = \mu_g - 2\dim_{\mathbb{C}} V$, so it contributes one generator in each odd degree in the range $[\mu_g - 2\dim_{\mathbb{C}} V - 2\dim_{\mathbb{C}} V' + 2, \mu_g - 2\dim_{\mathbb{C}} V]$. Inductively, the eigenvalues of $g$ will account for one generator in each odd degree in the range $[\mu_g - 2n + 2, \mu_g]$. The iteration formula in Theorem 2.7, i.e. the cases $2k\pi < \ell \leq 2(k+1)\pi$ for $k \in \mathbb{N} \setminus 0$, contribute generators in all odd degrees $[\mu_g - 2n + 2 - 2kn, \mu_g - 2kn] = [\mu_g - 2(k+1)n + 2, \mu_g - 2kn]$.

The second claim follows because the Morse-Bott spectral sequence degenerates: all generators are in odd total degree, so all differentials $d_r^{pq}$ on all pages $E_r^{pq}$ for $r \geq 1$ will vanish.

When $\mathbb{K}$ has non-zero characteristic, Theorem 2.12 implies that $ESH_+^*(Y) \cong H^{*+1}(Y) \otimes_{\mathbb{K}} \mathbb{F}$ is a free $\mathbb{F}$-module with generators in degrees $* = -1, 0, 1, \ldots, \dim_{\mathbb{R}} Y - 2$ (not $\dim_{\mathbb{R}} Y - 1$ as $H^{\dim_{\mathbb{R}} Y}(Y) = 0$ since $Y$ is non-compact, and we can also exclude all even degrees including $\dim_{\mathbb{R}} Y - 2$ since the generators of $ESH_+^*(Y)$ are in odd degree, $\mathbb{F}$ lies in even degrees and $\dim_{\mathbb{R}} Y$ is even). Moreover, the number of $\mathbb{F}$-summands in $ESH_+^*(Y)$ equals

$$\dim_{\mathbb{K}} ESH_+^{-1}(Y) = \sum \dim_{\mathbb{K}} H^{2j}(Y) = \chi(Y),$$

the Euler characteristic of $Y$, because the generators of $ESH_+^*(Y)$ are in odd degree and $\mathbb{F}$ as a $\mathbb{K}$-vector space has exactly one generator in each non-positive even degree.

In the range of degrees $-1, 1, 3, \ldots, \dim_{\mathbb{R}} Y - 3$ mentioned above, only generators corresponding to Reeb orbits of period $\ell \leq 2\pi$ can contribute because those of period $\ell > 2\pi$ have grading $\mu_g - 2kn \leq -3$ for $k \geq 1$, as $\mu_g = 2\,\mathrm{age}(g) - 1 \leq 2n - 3$ using that the age grading lies in $[0, n-1]$ by (1.2) (thus their grading and that of their differentials does not land in the range $[-1, \dim_{\mathbb{R}} Y - 3]$). Finally, under the assumptions on char $\mathbb{K}$, we can apply Theorem 2.6 to the Morse-Bott manifolds of Reeb orbits of period $\ell \leq 2\pi$. We refer the reader back to the closely related discussion below (1.12) for additional clarifications. □

## 3. Appendix A: Weil divisors, Cartier divisors and Resolutions

In the paper, we work with analytic geometry, so the words *regular*, *rational*, *isomorphism* below are replaced respectively by *holomorphic*, *meromorphic*, *biholomorphic*. In this section, codimension always refers to the complex codimension. By a **variety** $X$ we mean an irreducible normal quasi-projective complex variety. Recall **normal** means each point has a normal affine neighbourhood, and an affine variety is normal if its coordinate ring $\mathbb{C}[X]$ of regular functions is integrally closed (i.e. elements of its fraction field satisfying a monic polynomial over the ring must lie in the ring). Equivalently, all local rings of $X$ are integrally closed. Non-singular quasi-projective varieties are normal, since the local rings are UFDs. Normality ensures that for a codimension one subvariety $Z \subset X$, there is some affine open of $X$ on which the ideal for $Z$ is principal in $\mathbb{C}[X]$. It follows [53, Chp.II.5.1 Thm.3] that the subvariety of singular points of $X$ has codimension at least 2. Any quotient $Y = X/G$ of a normal affine variety $X$ by a finite group $G$ of automorphisms is also normal: if $f \in \mathbb{C}(Y)$ is integral over $\mathbb{C}[Y] = \mathbb{C}[X]^G$ then it is integral over $\mathbb{C}[X]$, so $f \in \mathbb{C}[X]$ by normality, but functions in $\mathbb{C}(Y)$ are $G$-invariant, so $f \in \mathbb{C}[Y]$. In particular, $\mathbb{C}^n/G$ is normal for any finite subgroup $G \subset SL(n, \mathbb{C})$.

**Remark 3.1.** *Normality is equivalent to requiring that rational functions bounded in a neighbourhood of a point must be regular at that point. The removable singularities theorem shows* $\mathbb{C}$ *is normal, and Hartogs' extension theorem becomes: for any subvariety* $V \subset X$ *of codimension at least* 2*, any regular function on* $X \setminus V$ *extends to a regular function on* $X$*.*

As we work with singular varieties, we need to distinguish two notions of divisor, which coincide for non-singular varieties. A **Weil divisor** is a finite formal $\mathbb{Z}$-linear combination $\sum a_m V_m$ of irreducible closed subvarieties of codimension one. It is **effective** if all $a_m \geq 0$. A rational section $s$ of a line bundle $L \to X$ defines a Weil divisor $(s) = \sum \mathrm{ord}_s(Z)\, Z$, where we sum over irreducible closed subvarieties $Z \subset X$ of codimension one, and $\mathrm{ord}_s(Z)$ is the associated valuation.[1] Similarly, a global non-zero rational function $f$ on $X$ defines a **principal** Weil divisor $(f)$, and in this notation, $(f/g) = (f) - (g)$ for such functions $f, g$. Two Weil divisors $D_1, D_2$ are *linearly equivalent* if their difference is principal, $D_1 - D_2 = (f)$. The corresponding equivalence classes of Weil divisors define the *Weil divisor class group* $\mathrm{Cl}(X)$. The **support** of a Weil divisor $\sum a_m V_m$ is the subset $\cup V_m$ taking the union over all $a_m \neq 0$.

A **Cartier divisor** $D$ is defined by an equivalence class of data: an open cover $U_i$ of $X$ together with non-zero rational functions $f_i$ on $U_i$, such that $f_i/f_j$ is regular on the overlap $U_i \cap U_j$. One identifies two data sets if one can pass to a common refinement of the cover or rescale the $f_i$ by invertible regular functions. The **support** is the union of zeros and poles of the $f_i$. A **principal** Cartier divisor is given by the data $(X, f)$ for a global meromorphic function $f$. Two Cartier divisors are **linearly equivalent** if they differ by a principal Cartier divisor. Cartier divisors up to linear equivalence correspond to complex line bundles on $X$ up to isomorphism. The associated bundle $\mathcal{O}(D)$ is constructed from the $U_j \times \mathbb{C}$ using $f_i/f_j$ as transition function $U_j \times \mathbb{C} \to U_i \times \mathbb{C}$. The line bundle admits a rational section $s$ given by $s = f_j$ on $U_j$, so in particular the Weil divisor $(s)$ agrees locally with $(f_j)$.

As the variety is normal, Cartier divisors can also be defined as the "locally principal Weil divisors", namely a Weil divisor that locally is equal to $(f)$ for some meromorphic function $f$ on $X$. Explicitly $D = \sum a_m Z_m$ with $a_m = \mathrm{ord}_{f_i}(Z_m)$ for any $f_i$ satisfying $U_i \cap Z_m \neq \emptyset$.

A Weil divisor $D$ is $\mathbb{Q}$**-Cartier** if $mD$ is Cartier for some $m \in \mathbb{N}$. A quasi-projective variety $X$ is $\mathbb{Q}$**-factorial** if all Weil divisors are $\mathbb{Q}$-Cartier. Algebraic or analytic varieties over $\mathbb{C}$ with only quotient singularities are $\mathbb{Q}$-factorial [34, Prop.5.15]. So $\mathbb{C}^n/G$ is $\mathbb{Q}$-factorial for any finite group $G \subset SL(n, \mathbb{C})$. The idea is that, although in general Weil divisors do not pull back to Weil divisors,[2] in this case a Weil divisor $D$ in $\mathbb{C}^n/G$ pulls back to a Weil divisor in $\mathbb{C}^n$, in particular this is Cartier so locally it is cut out as $(f)$, then the averaged function $\sum_{g \in G} f \circ g^{-1} \in \mathbb{C}[\mathbb{C}^n]^G = \mathbb{C}[\mathbb{C}^n/G]$ locally cuts out $|G| \cdot D$, so $|G| \cdot D$ is Cartier.

Let $\pi : Y \to X$ be a morphism of varieties. The **push-forward** of Weil divisors is defined by $\pi_*(\sum a_i V_i) = \sum a_i' \overline{\pi(V_i)}$ with $a_i = a_i'$ if the closure $\overline{\pi(V_i)} \subset X$ is a codimension one subvariety, and $a_i' = 0$ otherwise. This is in general only a Weil divisor, even if $\sum a_i V_i$ is Cartier. If $\pi$ is a dominant map (i.e. with dense image), then the **pull-back** of a Cartier divisor given by data $(U_i, f_i)$ on $X$ is the Cartier divisor $(\pi^{-1}(U_i), \pi^* f_i)$ on $Y$. This corresponds to pull-back for the corresponding line bundles.

[1] In analytic geometry, in a local trivialisation near a generic point $p \in Z$, $s$ is given by a meromorphic function $f = gz^k$, where $g$ is an invertible holomorphic function, and $z$ is a holomorphic coordinate extending a local basis of holomorphic coordinates for $Z$ near $p$. Then one defines $\mathrm{ord}_s(Z) = k$.

[2] Pull-backs of Weil divisors are not usually defined. However, $\pi : \mathbb{C}^n \to \mathbb{C}^n/G$ is a finite flat degree $|G|$ cover over the complement of the singular set which has codimension $\geq 2$ (consisting of points of $\mathbb{C}^n$ with non-trivial stabilizer). So the pre-image $\pi^{-1}(Z)$ of an irreducible codimension 1 subvariety of $\mathbb{C}^n/G$ is a codimension 1 subvariety over that complement and can then be uniquely extended to a Weil divisor on $\mathbb{C}^n$.

By a **resolution** of $X$, we mean a non-singular variety $Y$ with a proper birational morphism $\pi : Y \to X$, such that the restriction $\pi : \pi^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}}$ is an isomorphism over the smooth locus $X_{\mathrm{reg}} = X \setminus \mathrm{Sing}(X)$ of $X$. Resolutions always exist by Hironaka's theorem [28].

**Lemma 3.2.** *Let $X$ be a $\mathbb{Q}$-factorial variety with only one singular point, at $0 \in X$, and let $\omega_X$ be a Kähler form on $X \setminus 0$. Any resolution $\pi : Y \to X$ admits a Kähler form $\omega_Y$ such that $\omega_Y = \pi^*\omega_X$ outside of an arbitrarily small neighbourhood of $\pi^{-1}(0)$.*

*Proof.* We first make an observation. Given any Cartier divisor $D$ on $Y$, the push-forward $\pi_* D$ is a Weil divisor on $X$, so $m\pi_* D$ is Cartier for some $m \in \mathbb{N}$. Let $f$ be a meromorphic function on $X$ such that $m\pi_* D = (f)$ near $0 \in X$. Then $mD - (\pi^* f)$ is a Cartier divisor on $Y$ whose support intersects some open neighbourhood $U \subset Y$ of $\pi^{-1}(0)$ only in codimension one subvarieties contained in $\pi^{-1}(0)$. The Cartier divisor yields a line bundle $\mathcal{O}(mD - (\pi^* f))$ on $Y$ with a meromorphic section[1] $S$ whose only zeros and poles in $U$ lie in $\pi^{-1}(0)$.

As $Y$ is quasi-projective, we may pick a very ample line bundle $L \to Y$. Let $D$ denote a choice of associated Cartier divisor. The above argument yields a meromorphic section $S$ of $L^{\otimes m}$ whose only zeros and poles in a neighbourhood $U \subset Y$ of $\pi^{-1}(0)$ are contained in $\pi^{-1}(0)$.

As $L^{\otimes m}$ is very ample, we can choose a Hermitian metric $|\cdot|$ on $L^{\otimes m}$ such that the curvature $\Omega$ of the Chern connection determines a positive $(1,1)$-form $\frac{i}{2\pi}\Omega = \frac{i}{2\pi}\partial\overline{\partial}\log|S|^2$ on $Y$ (using the fact that the latter is the Chern form for any non-zero meromorphic section $S$ of a holomorphic Hermitian line bundle). Let $c : Y \to [0,1]$ be a smooth function, with $c = 1$ near $\pi^{-1}(0)$ and $c = 0$ outside of $U$. The claim follows by taking $\omega_Y = \pi^*\omega_X + \delta\frac{i}{2\pi}\partial\overline{\partial}(c \cdot \log|S|^2)$, and picking $\delta > 0$ sufficiently small so that this is a positive form where $0 < c < 1$ (observe that our particular choice of section $S$ ensures that $\omega_Y$ is well-defined on $0 < c < 1$). Note that $\omega_Y = \pi^*\omega_X$ outside of $U$, and $\omega_Y = \pi^*\omega_X + \delta\frac{i}{2\pi}\Omega$ where $c = 1$. ∎

**Remark 3.3.** *From the preceding proof, we obtain*[2] *a Weil divisor $D = (s)$ in $Y$ arising from a rational section $s$ of a very ample line bundle $L \to Y$, such that the only zeros and poles of $s$ in some neighbourhood $U \subset Y$ of $\pi^{-1}(0)$ lie in $\pi^{-1}(0)$. Thus $D = A + B$ decomposes into a Weil divisor $A$ supported in $\pi^{-1}(0)$, and a Weil divisor $B$ supported in $Y \setminus U$. We now show that one can construct $L$ and $s$ so that $B = 0$ if one makes the additional assumption that Weil divisors of $X$ supported away from $0$ are torsion in $\mathrm{Cl}(X)$. By the assumption, $m \cdot \pi_* B = (f)$ for some positive integer $m \geq 1$ and some meromorphic function $f$ on $X$. As $\pi$ is a biholomorphism over $X \setminus \{0\}$, the Weil divisor $mD - (\pi^* f)$ is supported in $\pi^{-1}(0)$. Therefore if we replace $L, s$ by $L^{\otimes m}$ and $s^{\otimes m}/\pi^* f$ respectively, we obtain $B = 0$ above.*

As normal varieties $X$ are smooth in codimension one, the canonical bundle $\Lambda^{\mathrm{top}}_{\mathbb{C}} T^* X_{\mathrm{smooth}}$ defined on the smooth locus extends to a Weil divisor class $\mathcal{K}_X$ on $X$, the **canonical divisor**. A variety $X$ is **quasi-Gorenstein** if $\mathcal{K}_X$ is Cartier, i.e. there is a line bundle $\omega_X$ which restricts to the canonical bundle on the smooth part $X_{\mathrm{smooth}} \subset X$. In particular $\mathbb{C}^n/G$, for finite subgroups $G \subset SL(n,\mathbb{C})$, are quasi-Gorenstein, as $g \in G$ acts on $\Lambda^{\mathrm{top}}_{\mathbb{C}} T^*\mathbb{C}^n$ by $\det g = 1$. (They are in fact Gorenstein, although we will not define this notion here).

For a birational morphism $\pi : Y \dashrightarrow X$, a closed codimension one subvariety $V \subset Y$ is an **exceptional divisor** if $\pi(V) \subset X$ has codimension $\geq 2$ (i.e. the Weil divisor $\pi_*(V) = 0$). The **exceptional divisor of** $\pi$ is the Weil divisor $\sum V_i$ summing over the exceptional $V_i$.

If $\pi$ is a regular birational morphism of quasi-Gorenstein varieties, and $V$ is exceptional and irreducible, then the **discrepancy** of $V$ is $a_V = \mathrm{ord}_f(V)$ where $f = s_Y/\pi^* s_X$ is a rational section of $\omega_Y \otimes \pi^*\omega_X$ determined by a choice of non-zero rational sections $s_X, s_Y$ of $\omega_X, \omega_Y$

[1] namely $S = s^{\otimes m}/\pi^* f$ where $s$ is a meromorphic section for the line bundle associated to $D$ with $D = (s)$.

[2] After relabelling $L^{\otimes m}$, $S$ in the proof of Lemma 3.2 by $L$ and $s$ respectively.

such that $\pi^* s_X$ and $s_Y$ agree in the region where $\pi$ is an isomorphism. $V$ is called a **crepant divisor** if $\mathrm{ord}_f(V) = 0$. The **total discrepancy** of $X$ is the infimum of the discrepancies over all such possible $V$ and $\pi : Y \to X$. The total discrepancy of a smooth variety $X$ is one [34, Corollary 2.31]. The **discrepancy divisor** is the Cartier divisor $\sum a_V V$ of $Y$, summing over irreducible exceptional divisors. The discrepancies $a_V$ are in fact independent of the choices of $s_X, s_Y, \pi$, and $\pi$ is a **crepant resolution** if all $a_V = 0$, so $\pi^*\omega_Y = \omega_X$.

**Lemma 3.4** (Negativity Lemma)**.** *Let $\pi : Y \to X$ be a resolution, where $X$ has only one singular point at $0 \in X$. Let $D$ be a homologically trivial $\mathbb{Q}$-Cartier divisor in $Y$ with $\pi_*(D) = 0$. Then $D = 0$.*

*Proof.* We can apply the negativity lemma [34, Lemma 3.39] to the divisors $\pm D$ (they are both $\pi$-nef, since they are homologically trivial, and $\pi_*(\pm D)$ are effective since zero). It follows that $\pm D$ are effective, therefore $D = 0$. □

**Lemma 3.5.** *Let $\pi : Y \to X$ be a resolution, where we assume $X$ has only one singular point at $0 \in X$ and $\mathcal{K}_X$ is $\mathbb{Q}$-Cartier. Then $Y$ is crepant if and only if $c_1(Y)|_U = 0$ on a neighbourhood $U \subset Y$ of $\pi^{-1}(0)$.*

*Proof.* Suppose first $\mathcal{K}_X$ is Cartier. As $\pi$ is an isomorphism away from $\pi^{-1}(0)$, all exceptional divisors lie in $\pi^{-1}(0)$. As $\pi^{-1}(0)$ is of codimension one, the irreducible components $V_i$ of $\pi^{-1}(0)$ are the exceptional divisors, and $\sum V_i$ is the exceptional divisor of $\pi$. In the notation above, we may pick $s_X$ so that near $0 \in X$ it is regular and nowhere-vanishing. If $\pi$ is crepant then $f = s_Y/\pi^* s_X$ has no zeros or poles along the $V_i$ so $s_Y$ is regular and nowhere-vanishing near $\pi^{-1}(0)$, so $\omega_Y$ is trivial near $\pi^{-1}(0)$ as required. Conversely, suppose $c_1(Y)|_U = 0$. Near $0 \in X$ we can pick a local nowhere vanishing section $s_X$ of $\omega_X$. Then $\pi^* s_X$ defines a rational section for $\omega_Y|_U$ (shrinking $U$ if necessary). By construction, the support of the divisor $(\pi^* s_X)$ on $U$ lies entirely in $\pi^{-1}(0)$. Then $c_1(Y)|_U = 0$ implies $(\pi^* s_X)$ is homologically trivial on $U$. Lemma 3.4 implies $(\pi^* s_X) = 0$ on $U$. Thus $\pi^* s_X$ is a regular nowhere vanishing section for $\omega_Y|_U$, as required (taking $s_Y = \pi^* s_X$ in the definition of crepant). When $\mathcal{K}_X$ is $\mathbb{Q}$-Cartier, say $m\mathcal{K}_X$ is Cartier, one considers $f^{\otimes m}$, $\omega_Y^{\otimes m}$, $\omega_X^{\otimes m}$ instead of $f, \omega_Y, \omega_X$. □

**Proposition 3.6.** *Let $X$ be a $\mathbb{Q}$-factorial variety with only one singularity, at $0 \in X$, admitting a regular $\mathbb{C}^*$-action $\mu : \mathbb{C}^* \times X \to X$ which fixes $0$. Assume that the Weil divisors of $X$ supported away from $0$ are torsion in $\mathrm{Cl}(X)$. Suppose $\pi : Y \to X$ is a crepant resolution. Then $\mu$ lifts to a $\mathbb{C}^*$-action on $Y$, so that $\pi$ is a $\mathbb{C}^*$-equivariant morphism.*

**Remark 3.7.** *Let $X = \mathbb{C}^n/G$ for a finite subgroup $G \subset SL(n, \mathbb{C})$ acting freely on $\mathbb{C}^n \setminus \{0\}$. Then $X$ with the standard $\mathbb{C}^*$-action satisfies the assumptions of Proposition 3.6. Indeed, given a Weil divisor $D$ in $\mathbb{C}^n/G$ supported away from $0$, we can define a Weil divisor $\widetilde{D}$ in $\mathbb{C}^n$ by picking a "lift" of $D$ via the quotient $\psi : \mathbb{C}^n \to \mathbb{C}^n/G$, meaning each subvariety $S$ arising in $D$ gets replaced by a choice of lift of $S$ via $\psi$. There are $|G|$ distinct choices of such a lift of $S$, and the lifted subvarieties are freely permuted by $G$. By construction, $\psi_*\widetilde{D} = D$. Weil divisors in $\mathbb{C}^n$ are known to be principal,[1] so $\widetilde{D} = (\widetilde{f})$ for a meromorphic function $\widetilde{f}$ on $\mathbb{C}^n$. Observe that for any $g \in G$ the Weil divisor $(g^*\widetilde{f})$ also has push-forward $\psi_*(g^*\widetilde{f}) = D$, since $G$ permutes the lifted subvarieties. Thus the averaged $G$-invariant meromorphic function $\sum_{g\in G} g^*\widetilde{f}$ on $\mathbb{C}^n$ descends to a well-defined meromorphic function $f$ on $\mathbb{C}^n/G$ with associated Weil divisor $(f) = |G| \cdot D$. Thus $|G| \cdot D$ is a principal divisor, so $D$ is torsion in $\mathrm{Cl}(X)$.*

*The above argument can also be adapted to the situation when $G$ does not act freely on $\mathbb{C}^n \setminus \{0\}$, as follows. Let $S, D$ be as before. Recall that $\mathbb{C}^n/G$ is normal, so the singular set*

[1] More generally this holds whenever the coordinate ring is a unique factorization domain [26, Prop.II.6.2], which in our case is $\mathbb{C}[x_1, \ldots, x_n]$.

*$\mathrm{Sing}(\mathbb{C}^n/G)$ has codimension at least two. Therefore the intersection $S_{\mathrm{sing}} = S \cap \mathrm{Sing}(\mathbb{C}^n/G)$ has codimension at least one in $S$. As $G$ acts freely on $\mathbb{C}^n \setminus \psi^{-1}(\mathrm{Sing}(\mathbb{C}^n/G))$, there are $|G|$ choices of lifts of $S \setminus S_{\mathrm{sing}}$ to $\mathbb{C}^n$. We make one such choice of lift, and then we take the Zariski closure, call it $\widetilde{S} \subset \mathbb{C}^n$. Summing over $S$, the sum of these subvarieties $\widetilde{S}$ defines a Weil divisor $\widetilde{D}$ on $\mathbb{C}^n$ such that $\psi_* \widetilde{D} = D$. The remainder of the previous argument then holds verbatim, showing that $|G| \cdot D$ is principal, so $D$ is torsion in $\mathrm{Cl}(X)$.*

*Proof of Proposition 3.6.* We first lift the action over the smooth locus, to obtain a rational map $\mu : \mathbb{C}^* \times Y \dashrightarrow Y$. Let $\mu_w = \mu(w, \cdot) : Y \dashrightarrow Y$ be the restriction to $\{w\} \times Y$, for $w \in \mathbb{C}^*$. The set of points at which $\mu_w$ is not regular is of codimension at least two (the proof of [53, Chp.II.3 Thm.3] applies to the quasi-projective non-singular variety $Y$). Let $Y' \subset Y$ be the locus where $\mu_w$ is regular. The differential $d\mu_w : T_{\mathbb{C}}Y'|_y \to T_{\mathbb{C}}Y|_{\mu_w(y)}$ induces a rational section $S = (\Lambda^{\mathrm{top}}_{\mathbb{C}} d\mu_w)^\vee$ of $\mathcal{L} = \omega_Y \otimes (\mu_w^* \omega_Y)^\vee$ which is regular over $Y'$. It is locally the determinant of the Jacobian matrix for $\mu_w$. Suppose by contradiction that $\mu_w$ has an exceptional divisor $V$.

Because $\pi : Y \setminus \pi^{-1}(0) \to X \setminus 0$ is a $\mathbb{C}^*$-equivariant isomorphism, it follows that $V \subset \pi^{-1}(0)$. By construction, the section $S$ must vanish along $V$. The effective divisor $(S)$ on $Y'$ yields an effective divisor on $Y$ by taking the closure (recall $\operatorname{codim} Y \setminus Y' \geq 2$). As before, its support lies in $\pi^{-1}(0)$. If $(S)$ were null-homologous on $U$ then, since $\pi_*(\pm(S)) = 0$, Lemma 3.4 would imply $(S) = 0$ on $U$, contradicting that $(S)$ involves a strictly positive multiple of $V$. Therefore $(S)$ is not null-homologous. This implies that $c_1(\mathcal{L})|_U \neq 0$. Finally, we check that this is false. As in the proof of Lemma 3.5, it suffices to consider the case when $\mathcal{K}_X$ is Cartier (if $m\mathcal{K}_X$ is Cartier we consider the bundle $\mathcal{L}^{\otimes m}$ etc.). Let $s_X$ be a rational section of $\omega_X$ that is regular and nowhere zero near $0 \in X$. Then $s_X \cdot (\underline{\mu}_w^* s_X)^\vee$ is a rational section of $\omega_X \otimes (\underline{\mu}_w^* \omega_X)^\vee$ where $\underline{\mu}_w$ is the $\mathbb{C}^*$-action on $X$. As $\pi$ is $\mathbb{C}^*$-equivariant, $\sigma = \pi^* s_X \cdot (\mu_w^* \pi^* s_X)^\vee$ is a rational section of $\mathcal{L}$. As $\pi$ is crepant, $s_Y/\pi^* s_X$ has trivial orders of vanishing near $\pi^{-1}(0)$, so $\sigma$ trivialises $\mathcal{L}$ near $\pi^{-1}(0)$, thus $c_1(\mathcal{L})|_U = 0$ for a neighbourhood $U$ of $\pi^{-1}(0)$, the required contradiction.

Thus $\mu_w$ has no exceptional divisors for all $w \in \mathbb{C}^*$.

By Remark 3.3 (and using the assumption about Weil divisors) we can construct a very ample line bundle $L$ on $Y$ with a rational section $s$ whose only zeros and poles lie in $\pi^{-1}(0)$, in particular $\pi_*(D) = 0$ as $\pi$ collapses the divisors in $\pi^{-1}(0)$. From now on, by divisor we mean the equivalence class of the divisor.

As $\mu_w$ has no exceptional divisors, the $D_w = (\mu_w)_*(D)$ define a smooth family of Weil divisors parameterized by $w \in \mathbb{C}^*$. In particular all $D_w$ are homologous. Also note that $\pi_*(D_w) = 0$, since $\pi_* D = 0$ and $\pi$ is $\mathbb{C}^*$-equivariant.

Abbreviate $D_{w,w'} = D_w - D_{w'}$ for any $w, w' \in \mathbb{C}^*$. As both $\pm D_{w,w'}$ are null homologous and $\pi_*(\pm D_{w,w'}) = 0$, Lemma 3.4 implies $D_{w,w'} = 0$, so $D_w = D_1$ for all $w \in \mathbb{C}^*$.

Recall that a section of $L$ is equivalent to a rational function $f$ such that $(f)+D$ is effective. Since $D_w = D$ for all $w \in \mathbb{C}^*$, we deduce that $\mu_w$ pulls back sections of $L$ to sections of $L$ by pulling back its respective rational functions. Thus we have an action $\mu^* : \mathbb{C}^* \times H^0(L)^* \to H^0(L)^*$ and each map $\mu_w^* = \mu^*|_{\{w\} \times H^0(L)^*} : H^0(L)^* \to H^0(L)^*$ is linear. As $L$ is very ample, it is also relatively very ample (i.e. the restriction to fibers of $\pi$ is very ample), so we have a natural embedding $\iota : Y \to \mathrm{Proj}(\oplus_{k \geq 0} H^0(L^{\otimes k})^*)$ and the induced action of $\mu^*$ on $H^0(L^{\otimes k})^*$ preserves $\mathrm{image}(\iota)$ and its restriction to $Y$ is $\mu$. This extends $\mu$ to an action $\mathbb{C}^* \times Y \to Y$ compatibly with the $\mathbb{C}^*$-action on $X$ via projection. $\square$

## 4. Appendix B: Equivariant symplectic cohomology

4.1. **Classical $S^1$-equivariant cohomology.** Recall that $\mathbb{K}$ is the Novikov field from (1.11), and (co)homology is computed with coefficients in $\mathbb{K}$ unless indicated otherwise. For a topological space $X$ with an $S^1$-action, $H^*_{S^1}(X) = H^*(ES^1 \times_{S^1} X)$ is a module over $H^*_{S^1}(\text{point}) = H^*(BS^1)$ by applying the functor $H^*_{S^1}(\cdot)$ to $X \to \text{point}$. We take $ES^1 = S^\infty$ to be the direct limit of $S^1 \subset S^3 \subset S^5 \subset \cdots$ where $S^{2n-1} \subset \mathbb{C}^n$, then $BS^1 = ES^1/S^1 = \mathbb{CP}^\infty$ and we identify $H^*(\mathbb{CP}^\infty) = \mathbb{K}[u]$, with $u$ in degree 2. So $H^*_{S^1}(X)$ is naturally a $\mathbb{K}[u]$-module.

Similarly, $H^{S^1}_*(X) = H_*(ES^1 \times_{S^1} X)$ admits a cap product action $u : H^{S^1}_*(X) \to H^{S^1}_{*-2}(X)$ making $H^{S^1}_{-*}(X)$ a $\mathbb{K}[u]$-module (notice the negative grading). We can identify $H^{S^1}_{-*}(\text{point}) = H_{-*}(\mathbb{CP}^\infty) \cong \mathbb{K}[u^{-1}, u]/u\mathbb{K}[u]$ as $\mathbb{K}[u]$-modules, where $u^{-j}$ represents the class $[\mathbb{CP}^j]$ graded negatively. Equivalently, completing in $u$, we can view them as $\mathbb{K}[\![u]\!]$-modules

$$\mathbb{F} = \mathbb{K}(\!(u)\!)/u\mathbb{K}[\![u]\!] \cong H_{-*}(\mathbb{CP}^\infty),$$

where $\mathbb{K}[\![u]\!]$ and $\mathbb{K}(\!(u)\!) = \mathbb{K}[\![u]\!][u^{-1}]$ are respectively formal power series and Laurent series.

**Motivation.** *One wants an equivariant Viterbo theorem* [55]*: for closed oriented spin $N$, we want a $\mathbb{K}[\![u]\!]$-module isomorphism $ESH^*(T^*N) \cong H^{S^1}_{n-*}(\mathcal{L}N)$ (using the natural $S^1$-action on the free loop space $\mathcal{L}N = C^\infty(S^1, N)$), compatibly with the inclusion of constant loops $EH^*(T^*N) \cong H^{S^1}_{n-*}(N) \to H^{S^1}_{n-*}(\mathcal{L}N)$ via $c^* : EH^*(T^*N) \to ESH^*(T^*N)$ (the equivariant analogue of the canonical map $c^* : H^*(T^*N) \to SH^*(T^*N)$). As the $S^1$-action on constant loops is trivial, $H^{S^1}_{-*}(N) \cong H_{-*}(N) \otimes H_{-*}(\mathbb{CP}^\infty) \cong H_{-*}(N) \otimes_{\mathbb{K}} \mathbb{F}$.*

4.2. **$S^1$-complexes and equivariant symplectic cohomology.** Let $C^* = CF^*(H)$ be a Floer chain complex used in the construction of symplectic cohomology $SH^*(M)$ (for $M$ as in Sec.6.1). Following Seidel [52, Sec.(8b)], $C^*$ admits degree $1-2k$ maps

$$\delta_k : CF^*(H) \to CF^*(H)[1-2k]$$

for $k \in \mathbb{N}$, where $\delta_0$ is the usual Floer differential and $\sum_{i+j=k} \delta_i \circ \delta_j = 0$. In general such data $(C^*, \delta_k)$ is called an **$S^1$-complex**, and we recall the specific Floer construction of $\delta_k$ later.

Given an $S^1$-complex $C^*$, we define the **equivariant complex** by

$$EC^* = C^* \otimes_{\mathbb{K}} \mathbb{F}, \qquad d = \delta_0 + u\delta_1 + u^2\delta_2 + \cdots \tag{4.1}$$

so $\mathbb{K}$-linearly extending

$$d(yu^{-j}) = \sum u^{k-j}\delta_k(y) = \delta_0(y)u^{-j} + \delta_1(y)u^{-j+1} + \cdots + \delta_j(y)u^0.$$

Notice $d$ is naturally a $\mathbb{K}[\![u]\!]$-module homomorphism (but not for $\mathbb{K}[u^{-1}]$), in particular $u$ acts by zero on $u^0C^*$. The resulting cohomology $EH^*$ is a $\mathbb{K}[\![u]\!]$-module.

For the Floer complexes, the direct limit $ESH^*(M)$ of the equivariant Floer cohomologies $EHF^*(H)$ over the class of Hamiltonians admits a canonical $\mathbb{K}[\![u]\!]$-module homomorphism

$$c^* : EH^*(M) \cong H^*(M) \otimes_{\mathbb{K}} \mathbb{F} \to ESH^*(M), \tag{4.2}$$

where $EH^*(M)$ arises from the Morse-theoretic analogue of the construction (4.1) for the 1-orbits of a $C^2$-small Hamiltonian (these are constant orbits, so involve a trivial $S^1$-action).

4.3. **Construction of the $\delta_k$ in Floer theory.** We follow work of Viterbo [55, Sec.5] and Seidel [52, Sec.(8b)], and for details we refer to Bourgeois-Oancea [12, Sec.2.3]. The function

$$f : S^\infty \to \mathbb{R}, \qquad f(z) = \textstyle\sum_{j=1}^\infty j|z_j|^2 \tag{4.3}$$

induces a Morse function on $\mathbb{CP}^\infty = S^\infty/S^1$ with critical points $c_0, c_1, c_2, \ldots$ in degrees $0, 2, 4, \ldots$. One picks a connection on the $S^1$-bundle $S^\infty \to \mathbb{CP}^\infty$ that is trivial near all $c_i$ (in a chosen trivialisation of the bundle near each $c_i$). This induces a connection on

$$E = S^\infty \times_{S^1} \mathcal{L}M \to \mathbb{CP}^\infty, \tag{4.4}$$

using the natural $S^1$-action on the free loop space $\mathcal{L}M = C^\infty(S^1, M)$. One picks a family of Hamiltonians $H_z : M \to \mathbb{R}$ parametrized by $z \in \mathbb{CP}^\infty$ such that locally near $c_i$ the $H_z = h_i$ are some fixed Hamiltonians $h_i : M \to \mathbb{R}$. Similarly, one picks generic almost complex structures $J_z$ on $E_z$ that locally near $c_i$ are some fixed $J_i$ on $M$. It is understood that all Hamiltonians $h_i$ and almost complex structures $J_i$ must be of the type allowed by the construction of $SH^*(M,\omega)$ (so generic time dependent perturbations are tacitly understood), and in a neighbourhood $V$ of infinity we require $H_z$ to be radial of the same slope as the given $H$ (so Floer solutions will stay in the compact region $M \setminus V$ by a maximum principle).

If we do not work with a Morse-Bott model, then the given Hamiltonian $H$ has to have been time-dependently perturbed, say $H = H(t,\cdot)$, so as to ensure that Hamiltonian 1-orbits are non-degenerate. The $H_z$ must then be $S^1$-equivariant: $H_{e^{i\tau}z}(t,\cdot) = H_z(t-\tau,\cdot)$, and similarly for the $J_z$.

We now count pairs $(w,v)$,

$$w : \mathbb{R} \to \mathbb{CP}^\infty \qquad\qquad v : \mathbb{R} \to E$$

where $w$ is a $-\nabla f$ flowline for the Fubini-Study metric on $\mathbb{CP}^\infty$, and $v$ is a lift of $w$ which satisfies the Floer equation $\frac{Dv}{ds} + J_z(\frac{Dv}{dt} - X_{H_{w(s)}}) = 0$, where the derivatives are induced by the connection on $E$. More precisely, one fixes asymptotics $c_{i_-}, c_{i_+}$ for $w$ and asymptotic 1-orbits $x_-, x_+$ for $v$ for the Hamiltonians $h_{i_-}, h_{i_+}$, and the moduli space $\mathcal{M}(c_{i_-}, x_-; c_{i_+}, x_+)$ consists of the rigid solutions $[(w,v)]$ modulo the natural $\mathbb{R}$-action in $s$.

The shift $\sigma : \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, $(z_0, z_1, z_2, \ldots) \mapsto (0, z_0, z_1, \ldots)$ is compatible with the Fubini-Study form and $\nabla f$ (as $\sigma^* f = f + 1$), so we may pick all data compatibly with the natural lifted action $\sigma : E \to E$. So for each $i$, $h_i = H$ and $J_i = J$. The moduli spaces can be naturally identified if we add the same positive constant to both $i_-, i_+$. Define $\delta_k$ as the $\mathbb{K}$-linear extension of

$$\delta_k(y) = \sum \#\mathcal{M}(c_k, x; c_0, y) \cdot x$$

summing over the 1-orbits $x$ of $H$ for which the moduli spaces are rigid, and # denotes the algebraic count with orientation signs (and Novikov weights, if present).

Then $d = \delta_0 + u\delta_1 + u^2\delta_2 + \cdots$ operates on $CF^*(H) \otimes_{\mathbb{K}} C_{-*}(\mathbb{CP}^\infty) \cong CF^*(H) \otimes_{\mathbb{K}} \mathbb{F}$ using the above Morse model for $\mathbb{CP}^\infty$, so viewing the formal variable $u^{-j}$ as playing the role of $c_j$ (equivalently $[\mathbb{CP}^j] \in H_{-*}(\mathbb{CP}^\infty)$ negatively graded with $u$ acting by cap product). For a detailed description of the Morse-Bott construction of the differentials, we refer to Seidel [52, Sec.(8b)], Bourgeois - Oancea [9, 11], and Kwon - van Koert [37, Appendix B].

4.4. **The $u$-adic spectral sequence.** Following Seidel [52, Sec.(8b)], the $u$-adic filtration is bounded below and exhausting, so it gives rise to a spectral sequence converging to $ESH^*(M)$ with $E_1^{**} = H^*(C^*, \delta_0) \cong SH^*(M) \otimes_{\mathbb{K}} \mathbb{F}$. Dropping $u^{-j}$ to avoid confusion, as this is only a spectral sequence of $\mathbb{K}$-vector spaces, and adjusting gradings,[1]

$$E_1^{pq} \Rightarrow ESH^*(M), \text{ where } E_1^{pq} = SH^{q-p}(M) \text{ for } p \le 0, \text{ and } E_1^{pq} = 0 \text{ for } p > 0. \qquad (4.5)$$

4.5. **Gysin sequence.** Following Bourgeois-Oancea [11], any $S^1$-complex $(C^*, \delta_k)$ admits a short exact sequence $0 \to C^* \xrightarrow{\text{in}} EC^* \xrightarrow{u} EC^{*+2} \to 0$ using the natural inclusion of $C^*$ as $u^0 C^*$ (recall the differential $\delta_0$ on $C^*$ agrees with $d$ on $u^0 C^*$), and using the $\mathbb{K}[\![u]\!]$-module action by $u$ on $EC^*$. The induced long exact sequence is called **Gysin sequence**,

$$\cdots \longrightarrow H^* \xrightarrow{\text{in}} EH^* \xrightarrow{u} EH^{*+2} \xrightarrow{b} H^{*+1} \longrightarrow \cdots \qquad (4.6)$$

[1] Abbreviating the total degree by $k = p+q$, the filtration is $F^p(EC^k) = C^{k-2p}u^p + C^{k-2p-2}u^{p+1} + \cdots$, which vanishes for $p > 0$, and $E_0^{pq} = F^p(EC^k)/F^{p+1}(EC^k) \cong C^{k-2p}u^p$ with $d_0^{pq} = \delta_0$, so $E_1^{pq} \cong SH^{k-2p}(M)$.

The boundary map $b$ is induced by the maps $b(yu^{-j}) = \delta_{j+1}(y)$ for $j \geq 0$, since that is the $u^0$-term of $d$ applied to the preimage $yu^{-j-1}$ of $yu^{-j}$ under multiplication by $u$.

In our setup above, this exact sequence becomes

$$\cdots \longrightarrow HF^*(H) \xrightarrow{\text{in}} EHF^*(H) \xrightarrow{u} EHF^{*+2}(H) \xrightarrow{b} HF^{*+1}(H) \longrightarrow \cdots$$

then taking the direct limit over continuation maps yields

$$\cdots \longrightarrow SH^*(M) \xrightarrow{\text{in}} ESH^*(M) \xrightarrow{u} ESH^{*+2}(M) \xrightarrow{b} SH^{*+1}(M) \longrightarrow \cdots$$

The positive symplectic cohomology version is analogous, yielding (1.13).

**Remark 4.1.** *For context, the classical Gysin sequence for an $S^1$-bundle $\pi : E \to M$ is*

$$\cdots \to H_*(E) \xrightarrow{\pi_*} H_*(M) \xrightarrow{\cap e} H_{*-2}(M) \longrightarrow H_{*-1}(E) \to \cdots \tag{4.7}$$

*where the middle arrow is cap product with the Euler class of the bundle. Now consider the free loop space $\mathcal{L}N$. If $M = \mathcal{L}N \times_{S^1} S^\infty$ (and $E = \mathcal{L}N \times S^\infty$), then* (4.7) *becomes*

$$\cdots \to H_*(\mathcal{L}N) \to H_*^{S^1}(\mathcal{L}N) \to H_{*-2}^{S^1}(\mathcal{L}N) \to H_{*-1}(\mathcal{L}N) \to \cdots$$

*so resembles the Gysin sequence* (4.6) *via the Viterbo isomorphism $H_*(\mathcal{L}N) \cong SH^{n-*}(T^*N)$.*

4.6. **Construction of the $\delta_k$ in Morse theory.** For a $C^2$-small Morse Hamiltonian $H$, taking $J_z = J$ and $H_z = H$, the Floer theory in Sec.4.3 reduces to Morse theory: the Hamiltonian orbits and the Floer solutions become time-independent, so $v : \mathbb{R} \to E$ will solve $\frac{Dv}{ds} = -\nabla H_z$ using the Riemannian metric $g_z = \omega(\cdot, J_z \cdot)$ on $M$ over $z$. As the two equations for $(w, v)$ have decoupled, this gives rise to the isomorphism $EH^*(M) \cong H^*(M) \otimes_{\mathbb{K}} H_{-*}(\mathbb{CP}^\infty)$.

Let $X$ be an oriented closed manifold with an $S^1$-action and a Morse function $H : X \to \mathbb{R}$, or a convex symplectic manifold (Sec.6.1) with an $S^1$-action with a $C^2$-small Morse Hamiltonian $H$ radial at infinity with positive slope (so $-\nabla H$ is inward pointing at infinity).

Then replacing $E$ in (4.4) by $E = S^\infty \times_{S^1} X$ gives a Morse theory analogue of Sec.4.3, and via Sec.4.2 yields a $\mathbb{K}[\![u]\!]$-module $EH^*(X)$. One can identify the Morse complex with the associated complex of pseudo-manifolds obtained by taking the stable manifold $W^s(p)$ of each critical point $p$ with grading $|W^s(p)| = 2n - |p|$ (recall this is how one can classically prove that Morse cohomology recovers locally finite homology, and thus the ordinary cohomology by Poincaré duality). It follows that there is an isomorphism to $S^1$-equivariant lf-homology,

$$EH^*(X) \cong H_{2n-*}^{\mathrm{lf}, S^1}(X). \tag{4.8}$$

When $X$ is a closed manifold, $H_{2n-*}^{\mathrm{lf}, S^1}(X) = H_{2n-*}^{S^1}(X)$.

When the $S^1$-action is trivial, $EH^*(X) \cong H_{2n-*}^{\mathrm{lf}}(X) \otimes_{\mathbb{K}} \mathbb{F} \cong H^*(X) \otimes_{\mathbb{K}} \mathbb{F}$ (in Sec.4.1 we had $EH^*(T^*N) \cong H^*(T^*N) \otimes_{\mathbb{K}} \mathbb{F}$ and by Poincaré duality $H^*(T^*N) \cong H^*(N) \cong H_{n-*}(N)$).

**Remark 4.2.** (4.8) *is not $H_*^{S^1}(X)$ because Poincaré duality in the equivariant setup is only well-behaved using lf-homology, and the $\mathbb{K}[u]$-actions by cup product on $H^*(\mathbb{CP}^\infty)$ and by cap product on $H_*(\mathbb{CP}^\infty)$ are substantially different.*

## 4.7. Equivariant cohomology for $S^1$-actions with finite stabilisers.

**Theorem 4.3.** *Let $X$ be a closed oriented smooth manifold with a free $S^1$-action. Let $e$ denote the Euler class of the $S^1$-bundle $X \to X/S^1$. There are $\mathbb{K}[\![u]\!]$-module isomorphisms*

$$EH^*(X) \cong H_{2n-*}(X/S^1) \cong H^{*-1}(X/S^1),$$

*where the second isomorphism is Poincaré duality for $X/S^1$, where $u$ acts on $H_*(X/S^1)$ by cap product by $-e$, and $u$ acts on $H^*(X/S^1)$ by cup product by $-e$.*

*This result holds more generally if the $S^1$-action has finite stabilisers, provided the characteristic of the underlying field of coefficients is coprime to the sizes of the stabilisers.*

*Proof.* Let $E = (S^\infty \times X)/S^1$ with $S^1$ acting by $(z, x) \mapsto (ze^{it}, e^{-it}x)$. The natural projection map $p : E \to X/S^1$ is a homotopy equivalence since the fibres $S^\infty$ are contractible. By naturality, the following $S^1$-bundles have the Euler classes labelled on the vertical maps

$$\begin{array}{ccccc} S^\infty & \longrightarrow & S^\infty \times X & \xrightarrow{\simeq} & X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u-e} & & \downarrow{\scriptstyle -e} \\ \mathbb{CP}^\infty & \longrightarrow & E = S^\infty \times_{S^1} X & \xrightarrow{\simeq} & X/S^1 \end{array}$$

where on $X$ we have reversed the $S^1$-action. By naturality of the classical Gysin sequence (4.7), this implies that the cap product action of $u$ on $H_*^{S^1}(X) = H_*(E)$ is identified with cap product by $-e$ on $H_*(X/S^1)$ via $p_* : H_*(E) \cong H_*(X/S^1)$. The claim follows upon identifying the classical Gysin sequence (4.7) with (4.6), by first using (4.8) to identify

$$EH^*(X) \cong H_{2n-*}^{\mathrm{lf},S^1}(X) = H_{2n-*}^{S^1}(X) = H_{2n-*}(E) \cong H_{2n-*}(X/S^1),$$

and then applying Poincaré duality: $H_{2n-*}(X/S^1) \cong H^{2n-1-(2n-*)}(X/S^1) = H^{*-1}(X/S^1)$. The final claim, about the case of finite stabilisers, is proved analogously but requires three technical lemmas to justify why the above techniques also work when $X/S^1$ has finite quotient singularities. We prove those lemmas separately, below. □

Before proving the lemmas required to justify the final part of Theorem 4.3, we recall the following motivation behind the assumption on the coefficients.

**Remark 4.4.** *For any quotient $M/G$ of a Hausdorff topological space $M$ by a finite group action $G$ (not necessarily acting freely), the projection $p : M \to M/G$ induces an isomorphism*

$$p^* : H^*(M/G) \cong H^*(M)^G \subset H^*(M)$$

*over any field of characteristic coprime to $|G|$, where $H^*(M)^G$ is the group of $G$-invariant elements. When the action is free, this follows by considering the classical transfer homomorphism $C_*(M/G) \to C_*(M)$ which sends a singular simplex $\sigma$ in $M/G$ to the sum $\sum_{g\in G} g(\widetilde{\sigma})$ of the possible lifts of $\sigma$ to $M$ (where $\widetilde{\sigma}$ is any choice of such a lift). In the non-free case one needs to consider an analogous map acting on the sheaf of coefficients, this is proved for example in* [24, Corollary to Prop.5.2.3] *or* [13, Thm.II.19.2]. *The result also holds for cohomology with compact supports.*[1] *If we assume in addition that $M$ is locally compact, then the homology version holds:*[2] *there is an isomorphism $H_*^{\mathrm{lf}}(X/G) \cong H_*^{\mathrm{lf}}(X)^G \subset H_*^{\mathrm{lf}}(X)$ induced by a transfer map (and when $M$ is compact, recall that lf-homology is just ordinary homology).*

In the following lemmas, we assume that $X$ is a closed oriented smooth manifold with an $S^1$-action whose stabilisers are finite, and that (co)homology is taken with coefficients in a field whose characteristic is coprime to the sizes of the stabilisers. We again consider the natural projection $p : E_X \to X/S^1$, where

$$E_X = (S^\infty \times X)/S^1$$

with $S^1$ acting by $(z, x) \mapsto (ze^{it}, e^{-it}x)$ on $S^\infty \times X$. The fibres are $p^{-1}(x) = S^\infty/\mathrm{Stab}(x)$, so under the assumption on the field $\mathbb{K}$ of coefficients the fibres have trivial cohomology:

$$H^*(p^{-1}(x)) = H^*(S^\infty/\mathrm{Stab}(x)) \cong H^*(S^\infty) = \mathbb{K} \cdot 1 \cong \mathbb{K} \tag{4.9}$$

[1] In [13, Thm.II.19.2], we use the family of supports on $M/G$ given by compact subsets, whose preimage in the sense of [13, Definition I.6.3] is the family of compact supports in $M$ since $M \to M/G$ is a proper continuous map (which in turn follows from the fact that it is a closed continuous map with compact fibres).

[2] By [13, Discussion below Proposition V.19.2] (and [13, Paragraph above Sec.V.2]) one can construct a transfer map on locally finite homology, $\mu_* : H_*^{\mathrm{lf}}(X/G) \to H_*^{\mathrm{lf}}(X)$ such that $\mu_* \circ p_* = \sum g_*$ is the averaging operator by the $G$-action, and $p_* \circ \mu_*$ is multiplication by $|G|$. Under our assumptions on the coefficients, $p_* \circ \mu_*$ is an isomorphism, and it follows that $\mu_*$ is an injection onto the $G$-invariant classes.

living in degree zero, by Remark 4.4.

**Lemma 4.5.** *The quotient map induces an isomorphism* $p_* : H_*(E_X) \cong H_*(X/S^1)$.

*Proof.* We will prove the statement by an inductive Mayer-Vietoris argument [7, Sec.5], by inducting on a cover of $X/S^1$. The inductive step is the following. Assume that $U, V$ are open subsets of $X/S^1$ such that the claim holds for $X$ replaced by any of $U$, $V$ or $U \cap V$. Then we can prove the claim for $X' = U \cup V$, as follows. By naturality, the two Mayer-Vietoris sequences for the open covers $X' = U \cup V$ and $X'/S^1 = U/S^1 \cup V/S^1$ fit into a commutative diagram via the map $p_*$. By the assumption and the five-lemma [7, Exercise 5.5], $p_* : H_*(X') \to H_*(X'/S^1)$ is also an isomorphism, as required. We now build the cover.

Observe that $X = \cup_{n \geq 1} X_n$ can be stratified by considering the sizes of the stabilisers (which are cyclic subgroups of $S^1$), by defining

$$X_n = \{x \in X : |\mathrm{Stab}(x)| = n\}.$$

Abbreviate $X_{\leq n} = \cup_{m \leq n} X_m$. Fixing $n$, a sequence of points in $X_n$ cannot converge to a point in $X_m$ for $m < n$, by a continuity argument. So $X_n \subset X_{\leq n}$ is a closed subset, and $X_{\leq n} \subset X$ is an open subset.

For any subset $Y \subset X$ on which $S^1$ acts freely, the natural projection map $p : E_Y \to Y/S^1$ is a homotopy equivalence since the fibres $S^\infty$ are contractible, so the claim holds for $Y$. For example, this applies to the case $Y = X_1$.

We claim that $X_n \subset X$ is a smooth submanifold. Pick any Riemannian metric on $X$. By an averaging argument, we may assume that the Riemannian metric is $S^1$-invariant. The exponential map and the $S^1$-action $\psi_t$ for time $t \in S^1 = \mathbb{R}/\mathbb{Z}$ therefore satisfy

$$\psi_t \circ \exp_p = \exp_{\psi_t(p)} \circ d_p\psi_t. \tag{4.10}$$

Suppose a point $p \in X$ is fixed by $\psi_t$. Consider the chart near $p \in X$ given via $\exp_p : T_pX \to X$ in a neighbourhood $N$ of $0 \in T_pX$. The induced $S^1$-action on $N$ becomes[1] the linear action by $d_p\psi_t$. It follows that $X_n$ correponds locally via $\exp_p$ to the *linear* subspace of $T_pX$ of vectors that have stabiliser of size $n$, therefore $X_n \subset X$ is a smooth submanifold. In particular, $X_n$ is a submanifold of the open submanifold $X_{\leq n} \subset X$.

We now consider the inductive Mayer-Vietoris argument for $X_{\leq 2} = U \cup V$, where $U = X_1$ and $V$ is an open $S^1$-invariant tubular neighbourhood of $X_2 \subset X_{\leq 2}$ (we apply the exponential map to a neighbourhood of the zero section of the normal bundle of $X_2 \subset X_{\leq 2}$). The claim holds for $U$ and $U \cap V$ since the $S^1$ action is free there, so once we prove the claim for $V$ we deduce it also for $X_{\leq 2}$. By construction, $V$ deformation retracts $S^1$-equivariantly onto $X_2$, so it remains to prove the claim for $X_2$. For $X_2$, we can quotient the $S^1$ action by $\mathbb{Z}/2$ (without affecting the claim), which reduces us again to the known case of a free $S^1$ action.

By induction, we can assume that the claim is known for open manifolds for which the stabilisers are at most of size $n-1$, and we now prove it for $X_{\leq n} = U \cup V$ taking $U = X_{\leq n-1}$ and $V$ an $S^1$-invariant tubular neighbourhood of $X_n \subset X_{\leq n}$. By induction the claim holds for $U$ and $U \cap V$, so we reduce to proving it for $V$. It suffices to prove it for $X_n$ since we can $S^1$-equivariantly deformation retract $V$ onto $X_n$. For $X_n$ we may use the quotiented action by $S^1/(\mathbb{Z}/n)$, which is a free action, therefore the claim holds as required. ☐

**Lemma 4.6.** *The quotient map* $X \to X/S^1$ *admits a Gysin sequence* (4.7) *which corresponds to the Gysin sequence for the circle bundle* $S^\infty \times X \to E_X$ *via the projection isomorphism* $p_*$. *In particular, there is a well-defined Euler class in* $H^2(X/S^1)$ *which agrees with the usual Euler class in* $H^2(X_1/S^1)$ *over the locus* $X_1 \subset X$ *where* $S^1$ *acts freely.*

---

[1] $\exp_p^{-1} \circ (\psi_t \circ \exp_p) = \exp_p^{-1} \circ (\exp_p \circ d_p\psi_t) = d_p\psi_t$, using (4.10) and $\psi_t(p) = p$.

*Proof.* One approach is to define the Gysin sequence for $X \to X/S^1$ as the Gysin sequence for the circle bundle $S^\infty \times X \to E_X$ after applying the projection isomorphisms $H_*(S^\infty \times X) \cong H_*(X)$ and $H_*(E_X) \cong H_*(X/S^1)$ (using Lemma 4.5). One can alternatively construct a Gysin sequence directly by applying the inductive argument as in the previous proof as follows, by using the fact that Gysin sequences are natural with respect to maps of spaces. The classical Gysin sequence applies to any subset $Y \subset X$ on which $S^1$ acts freely, since in that case $Y \to Y/S^1$ is an $S^1$-fibre bundle. Similarly to the previous proof, we assume that the claim is known for open manifolds for which the stabilisers are at most of size $n-1$, and we then prove it for $X_{\leq n} = U \cup V$ taking $U = X_{\leq n-1}$ and $V$ an $S^1$-invariant tubular neighbourhood of $X_n \subset X_{\leq n}$. By assumption, the claim holds for $U$ and $U \cap V$. For $V$, since $V$ deformation retracts $S^1$-equivariantly onto $X_n$, we can replace $V$ by $X_n$. For $X_n$ we first consider the quotiented action by $S^1_n = S^1/(\mathbb{Z}/n)$, which is a free action and thus yields a Gysin sequence.

We claim that the Euler class constructed for the $S^1_n$-action on $X_n$ (so for the circle bundle $X_n \to X_n/S^1_n$), after viewing it as a class in $H^2(V/S^1)$ via the deformation retraction, will pull back to $n$ times the Euler class constructed for $U \cap V \to (U \cap V)/S^1$. One way to see this, is to construct the Euler class by considering a certain edge homomorphism[1] in a spectral sequence construction of the Gysin sequence, then along a fibre of the circle bundle the Euler class corresponds to a generator of $H^1(S^1)$. The factor $n$ we mentioned above is then caused by the fact that the quotient map $S^1 \to S^1_n = S^1/(\mathbb{Z}/n)$ has degree $n$. A more concrete way to prove this, is to define the Euler class in terms of the Thom class of a 2-disc bundle, namely the mapping cylinder of the projection map of the circle bundle [27, Below Theorem 4D.10]. In this case, along a fibre the Euler class corresponds to a generator of $H^2(D^2, S^1)$, and again the factor of $n$ above arises because of the degree of the map $S^1 \to S^1_n$.

We work with (co)homology with coefficients in a field of characteristic coprime to $n$ (since $n$ arises as the size of a stabiliser, otherwise $X_n = \emptyset$ and there is nothing to prove). So we may rescale by $n$ the Euler class in the Gysin sequence for $X_n$ without affecting the exactness of the sequence. After this rescaling, and via the deformation retraction, we therefore obtain a Gysin sequence for $V$ which is functorial with respect to the inclusion $U \cap V \subset V$. Therefore the Mayer-Vietoris argument from the previous proof yields a Gysin sequence for $X_{\leq n}$ that satisfies the claim, as required. □

**Lemma 4.7.** *There is a Poincaré duality isomorphism* $H_*(X/S^1) \cong H^{\dim X - 1 - *}(X/S^1)$, *under which cap product on homology corresponds to cup product on cohomology.*

*Proof.* Recall that there is a proof of Poincaré duality by using an inductive Mayer-Vietoris sequence argument (see [27, Lemma 3.36] or [7, Lemma 5.6]), where the inductive Poincaré duality statement is reformulated for open orientable manifolds $U$ by using compactly supported cohomology: $H_*(U) \cong H_c^{\dim U - *}(U)$. In our setup, we consider the inductive Mayer-Vietoris argument from the proof of Lemma 4.5, and we prove inductively on $n$ the Poincaré duality statement for open orientable manifolds whose stabilisers have size at most $n-1$.

The initial step of the induction uses the known Poincaré duality statement $H_*(U/S^1) \cong H_c^{\dim U - 1 - *}(U/S^1)$ for the open subsets $U \subset X$ on which the $S^1$-action is free (so $U/S^1$ is an open orientable manifold). In the inductive step, we consider an $S^1$-invariant tubular open neighbourhood $T$ of $X_n \subset X_{\leq n}$, and we need to justify the Poincaré duality statement for $T/S^1$. Via the exponential map, we may view $T$ as an open neighbourhood of the zero section of the normal bundle $V \to X_n$ of the submanifold $X_n \subset X_{\leq n}$. Observe that $X_n/S^1$ can be identified geometrically with the smooth oriented (but possibly non-compact)

[1] Over real coefficients this is carried out in Bott-Tu [7, Sec.14 above Proposition 14.33], where the Euler class can also be constructed as a de Rham form, as an angular form which equals 1 under integration along fibres.

manifold $Y = X_n/S^1_n$, where $S^1_n = S^1/(\mathbb{Z}/n)$ has quotiented out the subgroup generated by $1/n \in \mathbb{R}/\mathbb{Z} = S^1$. A neighbourhood of $Y \subset X_{\leq n}/S^1$ can be identified via the exponential map with an open neighbourhood of the "zero section" of the fibre bundle

$$\pi : V/S^1 \to X_n/S^1_n = Y, \tag{4.11}$$

where $S^1$ acts on $V$ by $d\psi_t$, where $\psi_t$ is the $S^1$-flow for time $t \in S^1 = \mathbb{R}/\mathbb{Z}$. Notice that (4.11) is not quite a vector bundle: the fiber over a point $[x]$ is[1] the quotient $V_x/\Gamma_x$ of the vector space $V_x$ by the finite cyclic $n$-group $\Gamma_x$ generated by $d_x\psi_{1/n}$. We also remark that $X_n$ and $Y$ need not be orientable (although they are in our applications by Remark 4.8).

Let $\mathcal{O}_Y$ denote the orientation sheaf for $Y$, and $\mathcal{O}_F$ the orientation sheaf for the fiber bundle (4.11) (the latter being the orientation sheaf associated[2] to the vertical tangent bundle $\ker(d\pi)$ of $V$). By construction their tensor product,

$$\pi^*(\mathcal{O}_Y) \otimes \mathcal{O}_F \cong \mathcal{O}_{V/S^1},$$

recovers the orientation sheaf for $V/S^1$, which is a *constant* sheaf, since a chosen orientation on $X$ determines an orientation for the total space of $V$ and thus for $V/S^1$. Let $r = \dim X - \dim X_n$ denote the rank of $V \to X_n$ (in this proof, dim will denote the real dimension). Let $V' = V \setminus (\text{zero section})$. We now consider the following diagram:

$$\begin{array}{ccc}
H^{p+r}_c(V/S^1) \cong H^{p+r}_c(V/S^1, V'/S^1; \mathcal{O}_{V/S^1}) & & H_{\dim Y - p}(V/S^1) \\
\downarrow \cong & & \uparrow \cong \\
H^p_c(Y; \mathcal{O}_Y) & \xrightarrow{\cong} & H_{\dim Y - p}(Y)
\end{array}$$

The top-left cohomology groups in the above diagram are isomorphic because $\mathcal{O}_{V/S^1}$ is a constant sheaf. The bottom horizontal map is the Poincaré duality isomorphism for the smooth manifold $Y$ (the orientation sheaf corrects the possibility that $Y$ may not be orientable). The right vertical map is an isomorphism since the spaces are homotopy equivalent ($V$ is $S^1$-equivariantly contractible onto its zero section $X_n$). The left vertical map is (a mild generalisation of) a version of the non-orientable Thom isomorphism [13, Sec.IV.7.9] (compare also the simpler [7, Theorem 7.10]), where the orientation sheaf $\mathcal{O}_F$ corrects for the possibility that the bundle $V$ is non-orientable, and where we twisted the Thom isomorphism by the sheaf $\mathcal{O}_Y$ (so taking $\mathcal{B} = \mathcal{O}_Y$ in [13, Sec.IV.7.9 Equation (22)]). The version we are using is slightly more general than [13, Sec.IV.7.9] since (4.11) is not quite a vector bundle. The proof of [13, Sec.IV.7.9] for a rank $r$ vector bundle $U \to Y$ is a Leray-Serre spectral sequence argument, using as sheaf over $Y$ the cohomology of the fibre pair $(U, U \setminus (\text{zero section}))$. The key observation is that, on a local trivialisation, the fiber directions are modelled on the pair $(D^r, D^r \setminus 0)$ where $D^r$ is the $r$-disc in $\mathbb{R}^r$, and the relative cohomology of that pair is a copy of the base field in degree $r$ (the identification with the base field is canonical up to sign, and it is precisely the orientation sheaf of the bundle that keeps track of signs). In our setup, the local model is the pair $(D^r/\Gamma, (D^r \setminus 0)/\Gamma)$ where $\Gamma$ is a cyclic group of order $n$ acting by orientation preserving maps. The assumption on the characteristic of our base field ensures by Remark 4.4 that the relative cohomology of that pair is canonically isomorphic, via pull-back by the quotient map, to the relative cohomology of the pair $(D^r, D^r \setminus 0)$. So the Thom isomorphism also holds in our setting. The above diagram thus yields the Poincaré duality statement for $T/S^1$, namely $H^q_c(V/S^1) \cong H_{\dim X - 1 - q}(V/S^1)$.

[1] Observe that the geodesic $\gamma(s) = \exp_x(s \cdot v)$ maps via $\psi_{1/n}$ to the geodesic $\exp_x(s \cdot d\psi_{1/n} v)$.

[2] There are various equivalent ways to define orientation sheaves (e.g. [13, Sec.IV.7.9]). In the approach of Bott-Tu [7, End of Sec.7] the orientation sheaf for the bundle $V \to X_n$ is a line bundle on $X_n$ whose transition maps are multiplication by the sign of the determinant of the Jacobian of the transition functions used for the bundle $V$. This descends to a real line bundle $\mathcal{O}_F$ on $Y$, as the $S^1$-action is orientation-preserving.

The Thom isomorphism is given by cup product by the Thom class $\tau$ (so $\alpha \mapsto \tau \cup \pi^*\alpha$ in the left vertical map in the diagram), and the analogous statement for locally finite homology is the isomorphism $H^{\mathrm{lf}}_{p+r}(V/S^1) \to H_p(Y;\mathcal{O}_Y)$ given by cap product by $\tau$ followed by $\pi_*$ (so $\beta \mapsto \pi_*(\tau \cap \beta)$). In particular, a fundamental class $[V/S^1] \in H^{\mathrm{lf}}_{\dim X-1}(V/S^1)$ is determined by requiring that $\pi_*(\tau \cap [V/S^1]) \in H^{\mathrm{lf}}_{\dim X_n -1}(Y;\mathcal{O}_Y)$ is Poincaré dual to $1 \in H^0(Y)$. The Poincaré duality statement $H^q_c(V/S^1) \cong H_{\dim X-1-q}(V/S^1)$ above is then given by cap product by $[V/S^1]$. In particular, by construction $[V/S^1]$ is a fundamental class which restricts to the local orientation generators $\mu_{[x]} \in H_{\dim X-1}(X/S^1, X/S^1 - \{[x]\})$ at points $[x] \in V/S^1 \subset X/S^1$, and these generators are determined by the orientation of $X/S^1$ that is canonically induced by a chosen orientation of $X$.

Recall the inductive step in the previous proof involved $X_{\leq n} = U \cup V$ where $U = X_{\leq n-1}$. Now, our inductive hypothesis is that Poincaré duality holds for $U/S^1$, in the sense that the Poincaré duality isomorphism can be described by cap product by a locally finite fundamental class $[U/S^1]$ which is consistent with the local orientation generators induced by the orientation of $X/S^1$. Above, we proved that the same statement holds for $V/S^1$. The consistency of the two Poincaré duality isomorphisms on the overlap $(U \cap V)/S^1$ is guaranteed by the fact that the fundamental classes $[U/S^1]$ and $[V/S^1]$ can be compared (and agree) with the local orientation generators $\mu_{[x]}$ at points $[x]$ in the overlap. The Mayer-Vietoris proof of Poincaré duality therefore applies, and yields the Poincaré duality statement for $X_{\leq n}/S^1$, which completes the proof of the inductive step. □

**Remark 4.8.** *We will use the above results for complex manifolds with $S^1$ actions arising from $\mathbb{C}^*$-actions. In that case, the submanifold $X_n$ of points with stabiliser of size n is automatically a complex submanifold (via the exponential map argument above, $T_pX_n$ corresponds to the* complex *linear subspace of $T_pX$ of vectors with stabiliser of size n for the* complex-linear *linearised action). Once an orientation is chosen for $X_n$, an orientation for the normal bundle of $X_n \subset X$ can also be canonically determined from the chosen orientations for $X_n$ and $X$ (similarly, orientations for quotients by $S^1$ can be determined canonically using the canonical orientation for $S^1$).*

## 5. Appendix C: Conley-Zehnder indices

Let $(C^{2n-1}, \xi, \alpha)$ be a **contact manifold** admitting a global contact form: so $\alpha$ is a 1-form on $C$ such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form, and $\xi = \ker(\alpha)$ is the contact structure. The **Reeb vector field** $Y$ on $C$ is determined by $\alpha(Y) = 1$, $d\alpha(Y, \cdot) = 0$, and it defines the **Reeb flow**. Let $J$ be a complex structure on $\xi$ compatible with $d\alpha|_\xi$. The **anti-canonical** bundle $\kappa^* = \Lambda^{\mathrm{top}}_{\mathbb{C}}\xi$ is the highest exterior power of this complex bundle, and its dual $\kappa$ is called the **canonical bundle**. Now assume $\kappa$ is trivial and fix[1] a nowhere zero section $K$ of $\kappa$.

By **Reeb orbit $\gamma$ of length $\ell$** we mean a periodic orbit of period $\ell$ of the Reeb vector field, so $\gamma : \mathbb{R}/\ell\mathbb{Z} \to C$. Up to homotopy, there is a unique trivialisation $\tau : \gamma^*\xi \to (\mathbb{R}/\ell\mathbb{Z}) \times \mathbb{C}^{n-1}$ so that the corresponding trivialisation $\Lambda^{n-1}\tau : \gamma^*\kappa \to (\mathbb{R}/\ell\mathbb{Z}) \times (\Lambda^{\mathrm{top}}_{\mathbb{C}}\mathbb{C}^{n-1})^*$ satisfies $(\Lambda^{n-1}\tau)(K) = dz_1 \wedge \cdots \wedge dz_{n-1}$. Expressing the derivative of the Reeb flow $\phi_t : C \to C$ in the trivialisation at $\gamma(0)$ yields a family of symplectic matrices

$$(\mathrm{pr}_2 \circ \tau|_{\gamma(t)}) \circ D\phi_t \circ (\mathrm{pr}_2 \circ \tau|_{\gamma(0)})^{-1} : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1} \tag{5.1}$$

[1]The argument below would similarly apply if we were given a nowhere zero section $K^*$ of the anti-canonical bundle $\kappa^*$, in which case we use a trivialisation $\tau$ with the property that the induced map $\wedge^{n-1}(\tau)^*$ : $(\mathbb{R}/\ell\mathbb{Z}) \times (\Lambda^{\mathrm{top}}_{\mathbb{C}}\mathbb{C}^{n-1}) \to \kappa_J$ satisfies $\Lambda^{n-1}(\tau)^*(z_1 \wedge \cdots \wedge z_{n-1}) = K^*$.

where $\mathrm{pr}_2 : \mathbb{R}/\ell\mathbb{Z} \times \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ is the projection. The **Conley-Zehnder index** of $\gamma$ is the Conley-Zehnder index of this family, which is a half-integer satisfying the following properties (e.g. see [50] and [25]):

(CZ1) If $A_t, B_t$ are two paths of symplectic matrices then the Conley-Zehnder index of their catenation is the sum $CZ(A_t) + CZ(B_t)$.

(CZ2) If $A_t, B_t$ are two paths of symplectic matrices then $CZ(A_t \oplus B_t) = CZ(A_t) + CZ(B_t)$.

(CZ3) The Conley-Zehnder index is invariant under homotopies with fixed end points.

(CZ4) The Conley-Zehnder index of $(e^{is})_{s\in[0,t]}$ is $W(t)$, where

$$W\colon \mathbb{R} \to \mathbb{N}, \quad W(t) = \begin{cases} 2\lfloor t/2\pi \rfloor + 1 & \text{if } t \notin 2\pi\mathbb{Z} \\ t/\pi & \text{if } t \in 2\pi\mathbb{Z} \end{cases}. \tag{5.2}$$

Let $\phi_t : C \to C$ be the Reeb flow. The **linearized return map** associated to the Reeb orbit $\gamma$ of length $\ell$ is the restriction $D\phi_\ell : \xi|_{\gamma(0)} \to \xi|_{\gamma(0)}$.

**Definition 5.1.** *A* **Morse-Bott submanifold** $B \subset C$ *of length* $\ell$ *is a submanifold such that:*

*(1) Through each point of* $B$*, there is a Reeb orbit of length* $\ell$ *contained in* $B$.

*(2) The linearized return map for each such Reeb orbit has* 1*-eigenspace equal to* $TB \cap \xi|_B$.

*So the Reeb flow satisfies* $\phi_t(B) \subset B$ *and* $\phi_\ell|_B = \mathrm{id}$*, and the real dimension of the* 1*-eigenspace of each return map is* $\dim B - 1$*. We call* $\phi_{\ell t}|_B$ *the* **associated $S^1$-action** *on* $B$.

*For a connected Morse-Bott submanifold* $B \subset C$*, we define* $\mathrm{CZ}(B) = \mathrm{CZ}(p) \in \mathbb{Z}$*, for any point* $p \in B$ *(independence of the choice of* $p$ *is a consequence of property (CZ3) above).*

**Remark 5.2.** *Convex symplectic manifolds* $M$ *(with* $J$ *as in Sec.6.1) contain a contact hypersurface* $\Sigma$ *with a complex splitting* $TM = \xi \oplus \mathbb{C}$ *where* $\mathbb{C} \cong \mathbb{R}Z \oplus \mathbb{R}Y$ *for the vector fields* $Z, Y$ *defined in Sec.6.1. If the canonical bundle* $\mathcal{K}_M = \Lambda^{\mathrm{top}}_{\mathbb{C}} T^*M$ *is trivial, then so is the canonical bundle* $\kappa$ *for* $\Sigma$*. For a convex symplectic manifold* $M^{2n}$ *with trivial canonical bundle* $\mathcal{K}$ *and a choice of trivialisation, our* **Conley-Zehnder grading** *on the Floer complex* $CF^*(H)$ *is*

$$\mu(x) = n - \mathrm{CZ}_H(x), \tag{5.3}$$

*where* $\mathrm{CZ}_H(x)$ *is computed analogously to the above, except now* $x$ *is a Hamiltonian* 1*-orbit for* $H$ *so we consider the linearisation* $D\phi_H^t$ *of the Hamiltonian flow in a trivialisation of* $x^*TM$ *that is compatible with the given trivialisation of* $x^*\mathcal{K}$*. If we chose a different trivialisation of* $\kappa$*, involving a section of* $\kappa$ *that is obtained from the restriction of the section for* $\mathcal{K}_M$ *multiplied by a function* $f : \Sigma \to \mathbb{C}^*$*, then the CZ-index of a Reeb orbit in a free homotopy class* $c$*, so a conjugacy class of* $\pi_1(\Sigma)$*, changes by* $-2\langle [f], c\rangle$ *where* $[f] \in H^1(\Sigma, \mathbb{Z}) \cong [\Sigma, \mathbb{C}^*]$*, and the grading on* $SH^*$ *changes by* $+2\langle [f], c\rangle$*. A similar argument applies to the choice of trivialisation of* $\mathcal{K}_M$*; that choice will not matter if* $M$ *is simply connected.*

**Remark 5.3.** *The convention in* (5.3) *ensures*[1] *that for a* $C^2$*-small Morse Hamiltonian* $H$*, a critical point* $x$ *of* $H$ *will have Morse index* $\mu(x)$*. In the notation of Sec.6.1 on the end* $\Sigma \times [1, \infty)$*, using a radial Hamiltonian* $H = h(R)$*, a* 1*-orbit* $x$ *in the slice of slope* $h'(R) = \ell$ *corresponds to a Reeb orbit* $\gamma(t) = x(t/\ell)$ *of length* $\ell$ *in* $\Sigma$*. We pick a basis of sections to trivialize* $x^*\xi$ *so that together with* $Z, Y$ *we obtain a trivialisation of* $x^*TM \cong x^*\xi \oplus (\mathbb{R}Z \oplus \mathbb{R}Y)$ *that is compatible with the given trivialisation of* $\mathcal{K}$*. If* $h''(R) > 0$*, then the family of symplectic matrices obtained for* $\varphi_H^t$ *can be identified with the family obtained by* (5.1)*, together with a shear of type* $\left(\begin{smallmatrix} 1 & 0 \\ \text{positive} & 1 \end{smallmatrix}\right)$ *in the* $(Z, Y)$*-plane contributing* $\frac{1}{2}$ *to* $\mathrm{CZ}_H$ *(e.g. see* [25, Prop.4.9]*). Thus* $\mathrm{CZ}_H(x) = \mathrm{CZ}(\gamma) + \frac{1}{2}$*. The correction* $+\frac{1}{2}$ *however will cancel out, once one takes into account that there is an* $S^1$*-family of* 1*-orbits* $x(\cdot + \text{constant})$*. Indeed if* $x$ *is transversally non-degenerate then in* $CF^*(H)$ *it would give rise, after perturbation, to a copy of* $H^*(S^1)$

[1] This agrees with [50, Exercise 2.8] despite the sign, because we use the convention $\omega(\cdot, X_H) = dH$.

*shifted up by* $n - \mathrm{CZ}(\gamma) - 1$ [16, Prop.2.2]*. In the case of a connected Morse-Bott submanifold* $S \subset \Sigma$ *of orbits, using a Morse-Bott Floer complex for* $H$ *as in* [9, 10]*, one analogously obtains a copy of* $H^*(S)$ *shifted up in degree by*

$$\mu(S) = n - \mathrm{CZ}(S) - \tfrac{1}{2} - \tfrac{1}{2}\dim S \tag{5.4}$$

*because half of the signature of the Hessian of an auxiliary Morse function* $f_S : S \to \mathbb{R}$ *used to perturb the moduli space of* 1*-orbits* $S$ *would contribute to* $\mathrm{CZ}_H$ [42, Section 3.3]*. In our conventions, a radial Hamiltonian with* $h''(R) > 0$ *on* $\mathbb{C}^n$ *gives rise to a Morse-Bott submanifold* $S = S^{2n-1}$ *when the flow undergoes one full rotation, and* $\mathrm{CZ}(S) = 2n$*, so* (5.4) *equals* $\mu(S) = -2n$ *(the grading of* $\min f_S$*), and* $\max f_S \in H^{\mathrm{top}}(S)$ *contributes in grading* $-1$ *and has non-trivial Floer differential exhibiting the unit* $1 \in SH^0(\mathbb{C}^n)$ *as a boundary.*

## 6. Appendix D: the $F$-filtration and positive symplectic cohomology

6.1. **Convex symplectic manifolds.** We consider non-compact symplectic manifolds $(M, \omega)$ where $\omega$ is allowed to be non-exact, but outside of a bounded domain $M_0 \subset M$ there is a symplectomorphism $(M \setminus M_0, \omega|_{M\setminus M_0}) \cong (\Sigma \times [1,\infty), d(R\alpha))$, where $(\Sigma, \alpha)$ is a contact manifold, and $R$ is the coordinate on $[1,\infty)$ (**radial coordinate**). The **Liouville vector field** $Z = R\partial_R$ is defined at infinity via $\omega(Z, \cdot) = \theta$. The **Reeb vector field** $Y$ on the $\Sigma$ factor is defined by $\alpha(Y) = 1,\ d\alpha(Y, \cdot) = 0$. By "the" contact hypersurface $\Sigma \subset M$ we mean the level set $R = 1$. By **Reeb periods** we mean the periods of Reeb orbits on this $\Sigma$. The almost complex structure $J$ is always assumed to be $\omega$-compatible and of **contact type at infinity** (meaning $JZ = Y$ or equivalently $\theta = -dR \circ J$). Let $H : M \to \mathbb{R}$ be smooth. By **1-orbits** we mean 1-periodic Hamiltonian orbits (i.e. using the Hamiltonian vector field $X_H$ where $\omega(\cdot, X_H) = dH$). The data $(H, J, \omega)$ determines the Floer solutions which define the Floer chain complex $CF^*(H)$. We recall (e.g. see [48]) that when $H$ is a linear function of $R$ at infinity of slope different than all Reeb periods, the $R$-coordinate of Floer solutions satisfies a maximum principle. This ensures that its cohomology $HF^*(H)$ is defined (to avoid technicalities in defining Floer cohomology, one assumes $M$ satisfies a weak monotonicity condition [47], for example this holds if $c_1(M) = 0$).

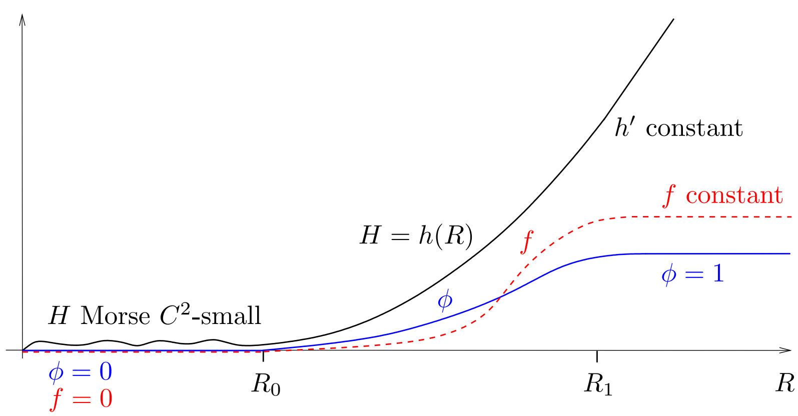



Figure 1. An illustration of the graphs of $H$, $\phi$, and $f$.

6.2. **Admissible Hamiltonians.** Observe Figure 1. We assume that for some $R_1 > R_0 > 0$,

(1) $J$ is of contact type for $R \geq R_0$,
(2) $H = h(R)$ only depends on the radial coordinate for $R \geq R_0$,
(3) $h'(R_0) > 0$ is smaller than the minimal Reeb period,
(4) $h''(R_0) > 0$, and $h''(R) \geq 0$ for $R \geq R_0$ (so $h'$ is increasing),
(5) if $h''(R) = 0$ for some $R \geq R_0$ then we require that $h'(R)$ is not a Reeb period,
(6) for $R \geq R_1$, $h'(R)$ is a constant (and not equal to a Reeb period),
(7) For $R \leq R_0$, $H$ is Morse and $C^2$-small so that all 1-orbits in $R \leq R_0$ are constant (i.e. critical points of $H$) and the Floer complex generated by these 1-orbits is quasi-isomorphic to the Morse complex, and on cohomology recovers $QH^*(M, \omega)$.

The last condition is not strictly necessary, but one can apply a Floer continuation isomorphism to homotope $H$ on $R \leq R_0$ to ensure that condition. That the complex in (7) is well-defined follows from a maximum principle and it is known that one recovers quantum cohomology [48]. Moreover, the filtration argument in Sec.6.4 will show that this complex is a subcomplex $CF_0^*(H)$ of the Floer complex $CF^*(H)$ of $H$.

Let $H_s : M \to \mathbb{R}$ depend on $s \in \mathbb{R}$ on a compact subset of $\mathbb{R}$, with $H_s = H_-$ for $s \ll 0$ and $H_s = H_+$ for $s \gg 0$, where $H_\pm : M \to \mathbb{R}$ are admissible (i.e. satisfy the above conditions). Then $H_s$ is an **admissible homotopy** of Hamiltonians if $H_s = h_s(R)$ on $R \geq R_0$ such that

(8) $\partial_s h_s' \leq 0$ (to ensure that the maximum principle for Floer solutions applies [48]),
(9) each $h_s$ satisfies the above conditions (1)-(4),
(10) and $h_s'$ is constant for $R \geq R_1$ (but may depend on $s$, or be equal to a Reeb period).

Also $J = J_s$ may vary with $s$, subject to the above compatibility and contact type conditions.

6.3. **Cut-off function.** Observe Figure 1. Let $\phi : \mathbb{R} \to [0, 1]$ be a smooth and increasing function such that

(1) $\phi = 0$ for $R \leq R_0$, $\phi > 0$ for $R > R_0$,
(2) $\phi' > 0$ for $R_0 < R < R_1$ (recall $h'(R)$ is constant for $R \geq R_1$),
(3) and $\phi = 1$ for large $R$.

One could omit (2) at the cost of losing the strictness of the filtration in Theorem 6.2.

The cut-off function determines an exact two-form on $M$,

$$\eta = d(\phi(R)\alpha) = \phi(R)\, d\alpha + \phi'(R)\, dR \wedge \alpha,$$

and an associated 1-form $\Omega_\eta$ on the free loop space $\mathcal{L}M = C^\infty(S^1, M)$ given by

$$\Omega_\eta \colon\ T_x\mathcal{L}M = C^\infty(S^1, x^*TM) \to \mathbb{R},\ \ \xi \mapsto -\textstyle\int \eta(\xi, \partial_t x - X_H)\, dt. \tag{6.1}$$

**Lemma 6.1.** *The* 1*-form* $\Omega_\eta$ *is negative (or zero) on Floer trajectories* $u : \mathbb{R} \times S^1 \to M$*.*

*Proof.* Substituting the Floer equation $\partial_t u - X_H = J\partial_s u$, and abbreviating $\rho = R \circ u$,

$$\begin{aligned}\eta(\partial_s u, \partial_t u - X_H) &= \eta(\partial_s u, J\partial_s u)\\ &= \phi(\rho)\cdot d\alpha(\partial_s u, J\partial_s u) + \phi'(\rho)\cdot (dR\wedge\alpha)(\partial_s u, J\partial_s u)\\ &= \text{positive}\cdot\text{positive} + \text{positive}\cdot (dR\wedge\alpha)(\partial_s u, J\partial_s u)\end{aligned}$$

To estimate the last term, we may assume that $R \geq R_0$ since $\phi' = 0$ otherwise. Since $J$ is of contact type for $R \geq R_0$, we can decompose

$$\partial_s u = C \oplus yY \oplus zZ \in \ker\alpha \oplus \mathbb{R}Y \oplus \mathbb{R}Z$$

where $Z = R\partial_R$. Thus: $dR(\partial_s u) = \rho z$ and $\alpha(J\partial_s u) = \alpha(JzZ) = \alpha(zY) = z$. Using $\theta = R\alpha$,

$$(dR\wedge\alpha)(\partial_s u, J\partial_s u) = dR(\partial_s u)\alpha(J\partial_s u) + \alpha(\partial_s u)\theta(\partial_s u) = \rho z^2 + \rho y^2 \geq 0.$$

The claim then follows from

$$\eta(\partial_s u, \partial_t u - X_H) = \phi(\rho) \cdot |C|^2 + \rho\, \phi'(\rho) \cdot (z^2 + y^2) \geq 0. \qquad \square \tag{6.2}$$

6.4. **Filtration functional.** Observe Figure 1. Let $f : \mathbb{R} \to [0, \infty)$ be the smooth function defined by

$$f(R) = \int_0^R \phi'(\tau)\, h'(\tau)\, d\tau.$$

Notice it is a primitive for $\phi'(R)\, h'(R)\, dR$, and satisfies[1]

(1) $f = 0$ for $R \leq R_0$, $f > 0$ for $R > R_0$,
(2) and $f$ is bounded.

Define the **filtration functional** $F : \mathcal{L}M \to \mathbb{R}$ on the free loop space by

$$F(x) = -\int_{S^1} x^*(\phi\alpha) + \int_{S^1} f(R \circ x)\, dt.$$

where $R \circ x$ is the $R$-coordinate of $x(t)$.

**Theorem 6.2.** *The filtration functional $F$ satisfies:*

*(1)* Exactness*: $F$ is a primitive of $\Omega_\eta$, so thus $dF \cdot \xi = -\int_{S^1} \eta(\xi, \partial_t x - X_H)\, dt$.*
*(2)* Negativity*: $dF \cdot \partial_s u \leq 0$ for any Floer trajectory $u$ for $(H, J, \omega)$, thus $F(x_-) \geq F(x_+)$ if $u$ travels from $x_-$ to $x_+$.*
*(3)* Separation*: $F = 0$ on all loops in $R \leq R_0$, and $F < 0$ on the 1-orbits in $R \geq R_0$.*
*(4)* Compatibility*: $F$ decreases along any Floer continuation solution $u$ for any admissible homotopy of $(H, J)$.*
*(5)* Strictness*: $F(x_-) > F(x_+)$ for any Floer trajectory joining distinct orbits $x_-$, $x_+$ with $R(x_+) \geq R_0$.*

*Thus $F$ determines a filtration on the Floer chain complex for a given admissible pair $(H, J)$, and this filtration is respected by Floer continuation maps for admissible homotopies of $(H, J)$. We use cohomological conventions,[2] so the Floer differential increases the filtration.*

We prove this in Section 6.5. Recall[3] that non-constant 1-orbits in $R = \rho \geq R_0$ are in 1-to-1 correspondence with closed Reeb orbits in $\Sigma$ of period $\tau = h'(\rho)$; the filtration value is

$$F(x) = T(\rho) \equiv -\phi(\rho)\, h'(\rho) + f(\rho). \tag{6.3}$$

Note $T : [0, \infty) \to \mathbb{R}$ satisfies the following properties, arising from the conditions on $\phi, h$,

(1) $T(\rho) = 0$ for $\rho \leq R_0$,
(2) $T$ is decreasing,[4]
(3) $T(\rho) < 0$ for $R > R_0$,[5]
(4) $T$ strictly decreases near $\rho \geq R_0$ if $h'(\rho)$ is a Reeb period.[6]

From this, we also deduce that $T$ is a strict filtration for $R \geq R_0$, as follows.

[1] using that $\phi' \geq 0, h' \geq 0$, and those are strict just above $R = R_0$, and that $\phi' = 0$ for large $R$.
[2] $x_-$ contributes to $\partial x_+$.
[3] since $X_H = h'(R)Y$ for $R \geq R_0$, a 1-orbit $x(t)$ corresponds to the Reeb orbit $x(t/\ell)$ of period $\ell = h'(\rho)$.
[4] Indeed $T'(\rho) = -\phi(\rho)\, h''(\rho) \leq 0$.
[5] by integrating $T'$, using that $\phi, h'' > 0$ just above $R_0$.
[6] since then $h''(\rho) > 0$.

**Corollary 6.3.** *If $u$ is a Floer trajectory joining distinct 1-orbits $x_-, x_+$, with $R(x_+) \geq R_0$,*[1]

$$R(x_-) < R(x_+).$$

*In particular, given a 1-orbit $y$ in $R = \rho \geq R_0$, the Floer differential $\partial y$ is determined by the 1-orbits in $R < \rho$ and the Floer trajectories that lie entirely in $R \leq \rho$.*

*Proof.* Note $T(R(x_-)) = F(x_-) > F(x_+) = T(R(x_+))$, and now use that $T$ is decreasing (and use the Strictness in Theorem 6.2). The final claim follows from the maximum principle. □

Define $CF^*_0(H) \subset CF^*(H)$ to be the subcomplex generated by 1-orbits with $F \geq 0$ (which by Sec.6.2.(7) is quasi-isomorphic to $QH^*(M)$), and let $CF^*_+(H)$ be the corresponding quotient complex. Define **positive symplectic cohomology** as the direct limit $SH^*_+(M) = \varinjlim HF^*_+(H)$ of the cohomologies of $CF^*_+(H)$.

**Corollary 6.4.** *Positive symplectic cohomology does not depend on the choice of $\phi$.*

*Proof.* This follows from the fact that Corollary 6.3 does not depend on the choice of $\phi$, and the Floer theory for $(H, J, \omega)$ does not use $\phi$. □

**Corollary 6.5.** *There is a long exact sequence of $\mathbb{K}$-algebra homomorphisms*

$$\cdots \to QH^*(M) \xrightarrow{c^*} SH^*(M) \to SH^*_+(M) \to QH^{*+1}(M) \to \cdots$$

*In particular, if $SH^*(M) = 0$ then $SH^*_+(M) \cong QH^{*+1}(M)$ canonically as vector spaces.*

*In the equivariant case, there is a long exact sequence of $\mathbb{K}[\![u]\!]$-module homomorphisms*

$$\cdots \to H^*(M) \otimes_{\mathbb{K}} \mathbb{F} \xrightarrow{c^*} ESH^*(M) \to ESH^*_+(M) \to H^{*+1}(M) \otimes_{\mathbb{K}} \mathbb{F} \to \cdots$$

*Proof.* The subcomplex yields the long exact sequence $QH^*(M) \to HF^*(H) \to HF^*_+(H) \to QH^{*+1}(M)$ which, using the Compatibility in Theorem 6.2, yields the claim by taking the direct limit over continuation maps, as we make the final slope of $h$ increase.

The equivariant setup follows analogously from the filtration, provided the $H_z$ in Sec.4.3 are chosen to belong to the class of admissible Hamiltonians (we fix the cut-off function $\phi$). For Theorem 6.2 (4) to apply to the Floer solutions in the equivariant construction, we need $\partial_s h'_{w(s)} \leq 0$ for $R \geq R_0$, where $w : \mathbb{R} \to \mathbb{CP}^\infty$ is any $-\nabla f$ trajectory (this $f$ refers to the function (4.3)). Here $h_z = h(\cdot, z)$, for $h : [R_0, \infty) \times \mathbb{CP}^\infty \to \mathbb{R}$, is $H_z$ in the region $R \geq R_0$. We achieve this by requiring[2] that $h_z$ is independent of $z$ for $R \geq R_0$. □

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[1] Since $H = h(R)$ is time-independent for $R \geq R_0$, each 1-orbit of $H$ arises as an $S^1$-family of orbits, due to the choice of the starting point of the orbit. With a time-dependent perturbation localized near the orbit, one can split the $S^1$-family into two non-degenerate 1-orbits (lying in the same $R$-coordinate slice), such that locally there are precisely two rigid Floer trajectories connecting these orbits, and they lie in the same $R$-slice and give cancelling contributions to the Floer differential (so this local Floer complex computes $H^*(S^1)$ up to a degree shift). For the purposes of Floer cohomology we can ignore these two Floer trajectories, and with this proviso the above Corollary continues to hold and implies that there cannot be other Floer trajectories connecting the two perturbed 1-orbits. The same argument holds more generally when there is a Morse-Bott manifold $\mathcal{O}_a$ of orbits, in which case the local Floer complex computes $H^*(\mathcal{O}_a)$ up to a degree shift, and the Corollary refers to Floer trajectories that are not already accounted for in this local Floer cohomology.

[2] As we require $J$ to be of contact type not just for $R \geq R_1$ but also on the region $R_0 \leq R \leq R_1$ (due to Lemma 6.1), on this region we cannot perturb $J$ in the span$(Z, Y)$ directions, so the standard transversality argument [38, Prop.3.4.1] may fail there (this issue does not arise for $R \geq R_1$ as Floer solutions do not reach $R \geq R_1$ due to the maximum principle). If the Floer solution $u$ enters $R < R_0$, it suffices to perturb $J$ there. So transversality is only problematic if $u$ is contained in $R_0 \leq R \leq R_1$ and the image of $du$ lies in span$(Z, Y)$. This implies that $u$ lands inside a cylinder in $\Sigma \times [1, \infty)$ and the ends of $u$ wrap different amounts of time around the two boundary circles of that cylinder (as $h'$ increased), which is not allowed for homotopical reasons. Alternatively (without using that $h'$ is monotone) one could allow small enough perturbations of $J$ at injective points of Floer solutions in $R_0 \leq R \leq R_1$, so as to ensure that the inequalities in Lemma 6.1 remain strictly negative at those points. This way the filtration construction will still hold.

6.5. **Proof of Theorem 6.2.** Define the $\phi$-action by

$$\mathcal{A}_\phi : \mathcal{L}M \to \mathbb{R},\ \mathcal{A}_\phi(x) = -\int_{S^1} x^*(\phi(R)\alpha).$$

This vanishes on loops $x$ which lie entirely in $R \leq R_0$. Suppose now $x$ is a 1-orbit that intersects the region $R \geq R_0$. Then $x$ is forced to lie entirely in the region $R \geq R_0$, indeed it lies in some fixed level set $R = \rho$ since $X_H = h'(R)Y$. In this case, $\mathcal{A}_\phi(x) = -\phi(\rho)\, h'(\rho)$.

A simple calculation shows that

$$d\mathcal{A}_\phi \cdot \xi = -\int_{S^1} \eta(\xi, \partial_t x)\, dt. \tag{6.4}$$

Finally, we need to ensure the exactness of the second term in (6.1),

$$\int \eta(\xi, X_H)\, dt = \int \phi'(\rho) h'(\rho)\, d\rho(\xi)\, dt,$$

where we used the equality $X_H = h'(R)Y$ (and the fact that $\eta = 0$ and $\phi' = 0$ where this equality fails). By definition, $F(x) = \mathcal{A}_\phi(x) + \int_x f \circ R$ so (6.4) and the choice of $f$ imply claim (1). Lemma 6.1 and claim (1) imply claim (2).

In claim (3), that $F$ vanishes on loops inside $R \leq R_0$ follows from $\phi(R) = f(R) = 0$. On a 1-orbit $x$ lying in $R = \rho$ the value of $F$ is (6.3). The rest of claim (3) follows from the properties of $T(\rho)$ mentioned under (6.3). To show claim (4), let

$$f_s(R) = \int_0^R \phi'(\tau)\, h_s'(\tau)\, d\tau, \qquad F_s(x) = \mathcal{A}_\phi(x) + \int_x f_s \circ R.$$

Then

$$d_x F_s \cdot \xi = -\int \eta(\xi, \partial_t x - X_{H_s})\, dt.$$

As in Lemma 6.1, one checks $d_u F_s \cdot \partial_s u \leq 0$ on Floer continuation solutions $u$. Now

$$\partial_s(F_s \circ u) = d_u F_s \cdot \partial_s u + (\partial_s F_s) \circ u$$

where $(\partial_s F_s)(x) = \int_x (\partial_s f_s) \circ R$. But

$$\partial_s f_s(R) = \int_0^R \phi'(\tau)\, \partial_s h_s'(\tau)\, d\tau \leq 0,$$

using that $h_s$ is admissible ($\partial_s h_s' \leq 0$). So $\partial_s(F_s \circ u) \leq 0$. To prove claim (5), note that in (6.2), if $\eta(\partial_s u, \partial_t u - X_H) = 0$ for some $R \geq R_0$, then $C, z, y$ vanish as $\phi(R), \phi'(R) > 0$. Thus $\partial_s u = 0$ and so $\partial_t u = X_H$ (by the maximum principle, $u$ does not enter the region $R \geq R_1$). But $x_-, x_+$ are distinct, so $\partial_s u$ cannot be everywhere zero, so strict negativity holds in Lemma 6.1 for some $s \in \mathbb{R}$. ☐

## 7. Appendix E: Morse-Bott spectral sequence

The Morse-Bott spectral sequences that we use in the paper are analogous to those due to Seidel [52, Eqns.(3.2),(8.9)] that arose from $S^1$-actions on Liouville manifolds. Morse-Bott techniques in Floer theory go back to Poźniak [43] and Bourgeois [8]. For Liouville manifolds (i.e. exact convex symplectic manifolds), Bourgeois-Oancea [9, 10] showed that the Morse-Bott Floer complex for time-independent Hamiltonians $H$ (assuming transversal non-degeneracy of 1-orbits) computes the same Floer cohomology as when using a time-dependent perturbation of $H$. The Morse-Bott Floer complex introduces auxiliary Morse functions on the copies of $S^1$ arising as the initial points of 1-orbits, and uses the critical points of the auxiliary Morse functions as generators, with an appropriate degree shift. The differential now counts **cascades** i.e. alternatingly following the flows of the negative gradients of the auxiliary functions or following Floer solutions that join two 1-orbits. This is the natural complex that would arise from a limit, as one undoes small time-dependent perturbations of $H$ localised near those copies of $S^1$ in $M$. The Morse-Bott complex admits a natural filtration by the action functional. As the functional decreases along Floer solutions, the filtration is exhausting and bounded below, so it induces a spectral sequence converging to $HF^*(H)$ whose $E_1^{pq}$-page consists of the cohomologies of the $S^1$ copies shifted appropriately in degree. It was shown by Cieliebak-Floer-Hofer-Wysocki [16, Prop.2.2] that a suitable time-dependent perturbation of $H$ localised near such an $S^1$-copy creates a local Floer complex in two generators whose cohomology agrees with the (Morse-Bott) cohomology of $S^1$.

Kwon and van Koert [37, Appendix B] carried out a detailed construction of Morse-Bott spectral sequences for symplectic homology of Liouville domains with periodic Reeb flows. So we restrict ourselves to explaining how these ideas generalise for convex symplectic manifolds $M$ (Sec.6.1), using admissible Hamiltonians $H$ and our new filtration $F$ from Appendix D (our filtration replaces the role of the action functional, which is multi-valued in our setup).

**Assumption.** *The subsets of Reeb orbits in* $\Sigma$ *are Morse-Bott submanifolds* (see Def.5.1).

Recall the non-constant 1-orbits $x$ of $H$ arising at slope $h' = \tau$ correspond to Reeb orbits $y(t) = x(t/\tau)$ in $\Sigma$ of period $\tau$. Consider the **slices**

$$\mathcal{S}(c) = \{m : R(m) = c\} \subset M,$$

i.e. the subset of points where the radial coordinate $R$ of Sec.6.1 equals a given value $c$. Let $R_{-1} < R_{-2} < \cdots$ be the values of $R$ for which 1-orbits of $H$ appear in $\mathcal{S}(R)$, equivalently the slopes $\tau_p = h'(R_p)$ for $p < 0$ are the Reeb periods less than the final slope of $h'$.

Let $\mathcal{O}_p = \mathcal{O}_{p,H}$ be the **moduli space of parametrized 1-orbits** of $H$ in $\mathcal{S}(R_p)$. These have $F$-filtration value $F_p = -\phi(R_p)h'(R_p) + f(R_p)$ by (6.3). By construction,

$$0 > F_{-1} > F_{-2} > F_{-3} > \cdots$$

By considering the initial point of the orbits, we view $\mathcal{O}_p \subset \mathcal{S}(R)$ as a subset, which can be identified with the Morse-Bott submanifold $B_p \subset \Sigma$ of initial points of the Reeb orbits of period $\tau_p$. Denote by $\mathcal{O}_0$ the Morse-Bott manifold of constant orbits of $H$, i.e. the critical locus of $H$ (which by admissibility are the 1-orbits of $H$ in $R \leq R_0$, and determine a Morse-Bott complex for $M$). We define $F_0 = 0$, which is the filtration value for $\mathcal{O}_0$, and by convention we define $F_p = p$ for positive integers $p \geq 1$ (there are no 1-orbits with filtration value $F > 0$).

Abbreviate by $\mathbf{k} = \mathbf{p} + \mathbf{q}$ the **total degree**. Let $C^*$ denote the Floer complex $CF_+^*(H)$ or $CF^*(H)$. The filtration is defined by letting $F^p(C^k)$ be the subcomplex generated by 1-orbits with filtration function value $F \geq F_p$ (in particular, $F^p(C^k) = 0$ for $p > 0$ since $F \leq 0$ on all 1-orbits). Recall the spectral sequence for this filtration has $E_0^{pq} = F^p(C^k)/F^{p+1}(C^k)$. As the filtration is exhaustive and bounded below, it yields convergent spectral sequences

$$\begin{array}{lll} E_1^{pq} \Rightarrow HF_+^*(H) & \text{where} & E_1^{pq} = HF^k_{\text{loc}}(\mathcal{O}_p, H) \text{ for } p < 0, \text{and } 0 \text{ otherwise} \\ E_1^{pq} \Rightarrow HF^*(H) & \text{as above, except} & E_1^{0q} = H^q(M) \end{array}$$

where it is understood, that $E_1^{pq} = 0$ for $p \ll 0$, as there are only finitely many $\mathcal{O}_p$ for $H$, and we remark that the same spectral sequences exist in the equivariant setup after replacing $HF$ by $EHF$. Above, $HF^*_{\text{loc}}(\mathcal{O}_p, H)$ refers to the cohomology of the local Morse-Bott Floer complex generated by $\mathcal{O}_p$. By construction, its differential only counts cascades which do not change the filtration value, so the Floer solutions stay trapped in the slice $\mathcal{S}(R_p)$. If one were to make a very small time-dependent perturbation of $H$ supported near $\mathcal{S}(R_p)$, the argument in [16, Prop.2.2] and [42, Sec.3.3] would show that this is quasi-isomorphic to the local Floer complex for that slice, where one only considers Floer solutions whose filtration value stays bounded within a small neighbourhood of the value $F = F_p$.

Let $B_{p,c}$ label the connected components of $B_p$ (and the labelling by $c$ depends on $p$). These have a Conley-Zehnder index $\mathrm{CZ}(B_{p,c})$ and a grading $\mu(B_{p,c}) = n - \mathrm{CZ}(B_{p,c})$ (Appendix C).

**Lemma 7.1.** *Assume that the linearised Reeb flow is complex linear with respect to a unitary trivialisation of the contact structure along every periodic Reeb orbit in* $\Sigma$. *Then*

$$HF^*_{\mathrm{loc}}(\mathcal{O}_p, H) \cong \bigoplus_c H^{*-\mu(B_{p,c})}(B_{p,c}). \tag{7.1}$$

Kwon and van Koert give a detailed discussion of this in [37, Prop.B.4.] and explain in [37, Sec.B.0.2] that there is an obstruction in $H^1(\Sigma, \mathbb{Z}/2)$ to (7.1) caused by orientation signs. Indeed (7.1) always holds if one uses the local system of coefficients on $B_{p,c}$ determined by that $H^1$-class. This obstruction vanishes under the assumptions of Lemma 7.1 (see [37, Lemma B.7]).

By letting the slope of $H$ increase at infinity, and taking the direct limit over continuation maps, one obtains the following spectral sequences.

**Corollary 7.2.** *Under the assumption of Lemma 7.1, there are convergent spectral sequences*

| | | |
|---|---|---|
| $E_1^{pq} \Rightarrow SH^*_+(H)$ | *where* | $E_1^{pq} = \bigoplus_c H^{k-\mu(B_{p,c})}(B_{p,c})$ *for* $p<0$, *and 0 otherwise* |
| $E_1^{pq} \Rightarrow SH^*(H)$ | *as above, except* | $E_1^{0q} = H^q(M)$ |
| $E_1^{pq} \Rightarrow ESH^*_+(H)$ | *where* | $E_1^{pq} = \bigoplus_c EH^{k-\mu(B_{p,c})}(B_{p,c})$ *for* $p<0$, *and 0 otherwise* |
| $E_1^{pq} \Rightarrow ESH^*(H)$ | *as above, except* | $E_1^{0q} = EH^q(M) \cong H^*(M) \otimes_{\mathbb{K}} \mathbb{F}$. |

*(where ordinary cohomology is always computed using* $\mathbb{K}$ *coefficients.)*

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