OPEN ACCESS
IOP Publishing
Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **54** (2021) 155201 (32pp)
https://doi.org/10.1088/1751-8121/abe58c

Lorentz- and permutation-invariants of particles

Ben Gripaios[1], Ward Haddadin[2,*] and Christopher G Lester[1]

[1] Cavendish Laboratory, University of Cambridge, J.J. Thomson Avenue, Cambridge, CB3 0HE, United Kingdom
[2] DAMTP, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom

E-mail: w.haddadin@damtp.cam.ac.uk

Received 12 August 2020, revised 29 January 2021
Accepted for publication 11 February 2021
Published 22 March 2021

Abstract

Two theorems of Weyl tell us that the algebra of Lorentz- (and parity-) invariant polynomials in the momenta of n particles are generated by the dot products and that the redundancies which arise when n exceeds the spacetime dimension d are generated by the $(d+1)$-minors of the $n \times n$ matrix of dot products. We extend the first theorem to include the action of an arbitrary permutation group $P \subset S_n$ on the particles, to take account of the quantum-field-theoretic fact that particles can be indistinguishable. Doing so provides a convenient set of variables for describing scattering processes involving identical particles, such as $pp \to jjj$, for which we provide an explicit minimal set of Lorentz- and permutation-invariant generators. Additionally, we use the Cohen–Macaulay structure of the Lorentz-invariant algebra to provide a more direct characterisation in terms of a Hironaka decomposition. Among the benefits of this approach is that it can be generalized straightforwardly to when parity is not a symmetry and to cases where a permutation group acts on the particles. In the first non-trivial case, $n = d + 1$, we give a homogeneous system of parameters that is valid for the action of an arbitrary permutation symmetry and make a conjecture for the full Hironaka decomposition in the case without permutation symmetry. An appendix gives formulæ for the computation of the relevant Hilbert series for $d \leqslant 4$.

* Author to whom any correspondence should be addressed.

Keywords: mathematical physics, invariant theory, Hironaka decomposition, minimal algebra generators, invariant polynomial generators

1. Introduction

Given the momentum vectors p_i of n particles in d spacetime dimensions, an old theorem of Weyl [1] tells us that the Lorentz- and parity-invariant polynomials are generated by the dot products $p_i \cdot p_j$. This theorem (or rather its obvious generalization from polynomials to the field of rational functions, the ring of formal power series, and thence to the whole gamut of functions typically considered in physics) has become so ubiquitous that it is, by and large, taken for granted nowadays.

Weyl's important result is actually composed of a pair of theorems, the precise statement of which goes as follows. By allowing the momenta to take values in the complex numbers rather than the reals, we can replace the action of the Lorentz group including parity transformations on the momenta with the action of the orthogonal group $O(d,\mathbb{C})$. Weyl's first fundamental theorem (FFT) [1] states that the algebra of polynomials in the p_i's invariant under $O(d,\mathbb{C})$ is generated by the $n(n+1)/2$ dot products $p_i \cdot p_j$.[3] Weyl's second fundamental theorem (SFT) characterises the relations between the generators: when $n \leqslant d$ there are no relations (so the dot products are algebraically independent and the algebra of invariants is a polynomial algebra), while when $n > d$, the relations are generated by the $(d+1)$-minors[4] of the $n \times n$ matrix whose entries are given by $p_i \cdot p_j$.

As useful as these results are in their current form, they are perhaps in need of a makeover given relatively recent developments in the area of commutative algebra and what we know about quantum field theory, namely that the particles that correspond to excitations of a single quantum field are indistinguishable. We will update these results using two approaches. In the first part of this work, we consider a system of n particles of which some subsets are identical (e.g. in a process in which two protons at the Large Hadron Collider (LHC) collide to produce three jets). In this case, it is apposite to consider not just arbitrary Lorentz-invariant polynomials, but rather to restrict to those that are, in addition, invariant under the group of permutations of the identical particles (e.g. $S_2 \times S_3$ in our $pp \to jjj$ example). We attack the problem in a manner similar to Weyl's and provide a method for constructing minimal algebra generators of Lorentz- and permutation-invariant polynomials in the momenta, i.e. a generalisation of the FFT to include permutation groups. We do not however generalise the SFT (for technical reasons discussed later). In the second part, we make use of commutative algebra methods not available at the time of Weyl and provide an alternative, redundancy-free, description of Lorentz- and permutation-invariant polynomials via what is called a *Hironaka decomposition* where the invariant polynomials are uniquely expressed in terms of the generators of the decomposition. This has the added advantage of being easily generalised to cases when parity is not a symmetry.

To give a first explicit example of why this might be helpful in phenomenological analyses, it is useful to consider the situation in which the analysis is carried out, as is increasingly the case, by a supremely *un*intelligent being, namely via machine learning. There, experience has shown that, rather than let the machine learn about Lorentz invariance for itself, it is far

[3] Without parity transformations, the group becomes $SO(d,\mathbb{C})$, and we have additional generators given by the possible contractions of the d-dimensional epsilon tensor with the momenta.

[4] We define a $(d+1)$-minor of a matrix to be the determinant of a $(d+1) \times (d+1)$ submatrix.

more efficient to feed event data to the machine in a Lorentz-invariant form[5]. There is no reason to expect that permutation invariance should be any different. Symmetrizing in this way has the related benefit of preventing the machine chasing wild geese, in the sense of looking for spurious Lorentz- or permutation-violating signals.[6] Similarly, a description via Hironaka decompositions, where invariant polynomials are uniquely expressed in terms of the decomposition, is also desirable to ensure that the algorithm wastes no time recognising redundancies in the input data. Symmetrizing may even be an astute tactic in situations where the particles in question are known to be *not* identical, but where one wishes to deliberately blind oneself to the difference between them, because the associated physics is not under control. A good (though politically incorrect) example from the LHC might be a Swiss proton and a French proton (or rather beams thereof), where one can be fairly sure that there are observable differences between them, but one can be equally sure that such differences are not due to fundamental new physics, but have a rather more mundane, to wit intermural, origin.

We also hope that these new descriptions using invariant polynomials will also be of use in analyses carried out by rather more intelligent beings. To give just one example, a common method for computing multi-loop amplitudes in quantum field theory is to first relate them using integration-by-parts identities [3, 4]. These are linear equations whose coefficients may be written as Lorentz- and permutation-invariant polynomials in the momenta of external particles. Thus, in setting up and carrying out such calculations, it would presumably be useful to know a (redundancy-free) set of generators of such polynomials[7].

Our goal then in this work will be to generalize Weyl's FFT (namely supplying an explicit set of generators) to the situation where an arbitrary subgroup $P \subset S_n$ of the permutation group acts on the n-particles, which we do in section 3, and to provide an alternative description of the Lorentz- and permutation-invariant algebra in terms of Hironaka decompositions, which we do in section 4. In each approach, we present solved examples for phenomenologically relevant cases. It would be an insult to Weyl's memory not to do so in a rigorous fashion, which requires the mathematical machinery of commutative algebra, the pertinent parts of which we review in appendix A.

2. Hironaka decompositions of Cohen–Macaulay invariant algebras

Before we begin our discussion, it will be useful to present some important concepts that will come up throughout this work. A central idea in both of our approaches is that of Hironaka decompositions of invariant algebras [5]. This is a feature of algebras that possess what is called the Cohen–Macaulay property [6].[8] We need not worry about what this property is precisely, only that it implies the existence of Hironaka decompositions. This powerful tool will allow us to construct systematic methods to characterise the invariant algebras of Lorentz- and permutation-invariant polynomials.

[5] Indeed, as far as we are aware, no computer has yet discovered Lorentz invariance by itself. But, given an arbitrary symmetric metric, a neural network can be trained to converge on the Minkowski metric [2].

[6] Of course, this 'benefit' will be considered a disbenefit by readers who are interested in the possibility that Lorentz invariance is violated, or that, say, two protons are not identical; we tactfully suggest that it would be better for all concerned if they were not to read any further.

[7] Such permutation invariant polynomials may also be of use in analysing correlation functions in cosmology, but we will not consider this possibility further here.

[8] For readers who are not *au courant*, it is perhaps consoling to note that even Macaulay himself professed to being ignorant of this property.

Let V be a finite-dimensional vector space over K carrying a representation of a group G, and $K[V]$ the polynomial algebra on V. Here and elsewhere in this work, K will denote an algebraically-closed field of characteristic zero. The algebra $K[V]$ carries a grading with $(K[V])_0 = K$, which is inherited by the invariant subalgebra $K[V]^G = \{f \in K[V] | f^g = f \ \forall\ g \in G\}$, where f^g denotes the action of g on f. A famous result of invariant theory is that if G is a linearly reductive group[9] and V is a rational representation of G, then $K[V]^G$ is finitely-generated.[10] Another important result, due to Noether, is that any finitely-generated graded algebra R with $R_0 = K$ admits a (not necessarily unique) homogeneous system of parameters (HSOPs). A HSOP (also termed *primary invariants*) is a set of homogeneous polynomials, $\{\theta_i\}$, which satisfy special properties to be discussed later (subsection 4.2). Importantly though, this implies that the algebra can then be expressed as a finitely generated module over the subalgebra generated by the HSOP, $K[\theta_1, \ldots, \theta_l]$. In particular, we may write $K[V]^G = \sum_k \eta_k K[\theta_1, \ldots, \theta_l]$, where we call the η_j *secondary invariants*.

Now comes perhaps the most significant result, namely that if $K[V]^G$ is Cohen–Macaulay, then it is a *free* (and as we have already seen, finitely-generated) module over any HSOP. Thus, we in fact have a *Hironaka decomposition* $K[V]^G = \bigoplus_k \eta_k K[\theta_1, \ldots, \theta_l]$ and we are able to use the full power of linear algebra. In particular, each element in $K[V]^G$ can be written uniquely as $\sum_j \eta_j f^j$, where $f^j \in K[\theta_1, \ldots, \theta_l]$, and the product of any two secondaries is uniquely given by $\eta_k \eta_m = \sum_j \eta_j f^j_{km}$, where $f^j_{km} \in K[\theta_1 \ldots, \theta_l]$. This specifies the multiplication in $K[V]^G$ unambiguously.

Some simple examples will perhaps be illuminating. When G is the trivial group acting on a basis vector $x \in \mathbb{C}$, we may set $\eta_1 = 1$ and $\theta_1 = x$, such that $K[V]^G = 1 \cdot \mathbb{C}[x]$. But we may also set $\eta_1 = 1, \eta_2 = x$, and $\theta_1 = x^2$, such that $K[V]^G = 1 \cdot \mathbb{C}[x^2] + x \cdot \mathbb{C}[x^2]$. This already shows that a Hironaka decomposition is not unique. For a slightly less trivial example, let G be the group $\mathbb{Z}_2$ whose non-trivial element sends basis vectors $x, y \in \mathbb{C}^2$ to minus themselves. Then we may set $\eta_1 = 1, \eta_2 = xy$ and $\theta_1 = x^2, \theta_2 = y^2$.

So the important question now is when is an invariant algebra Cohen–Macaulay? This answer to this is provided by the relatively recent and very powerful *Hochster–Roberts* theorem [7] (1974). The theorem states that an invariant algebra $K[V]^G$ is Cohen–Macaulay if G is a linearly reductive group. Luckily, all groups that we will discuss in the following sections are linearly reductive (all finite groups and the Lorentz group are linearly reductive) which allows us to use the aforementioned tools to obtain descriptions of our invariant algebras.

3. Minimal algebra generators of Lorentz- and permutation-invariants

We begin our expedition by tackling the first proposed problem, namely generalising Weyl's FFT. In layman's terms, the FFT is the statement that every Lorentz-invariant polynomial can be obtained by taking an arbitrary polynomial in variables y_{ij} (where $i, j \in \{1, \ldots, n\}$ and $i \leqslant j$), and replacing $y_{ij} \mapsto p_i \cdot p_j$. The first result of this section is that every Lorentz- and permutation-invariant polynomial can be obtained by taking a permutation-invariant polynomial in y_{ij} (where the permutation group P acts on the indices i, j in the obvious way) and making the same replacement. In a sense, this result *is* the generalization of Weyl's FFT, but not only is it apparently completely trivial (though the proof will show it to be not quite so), but also it is completely useless as it stands, because of the difficulty of describing

[9] A linear algebraic group G is called *linearly reductive* if for every rational representation V and every $v \in V^G \backslash \{0\}$, there exists a linear invariant function $f \in (V^*)^G$ such that $f(v) \neq 0$.

[10] This is the answer to Hilbert's 14th problem. Hilbert himself proved the case when G is a finite group.

the permutation-invariant polynomials in y_{ij}. Indeed, while permutations act in the natural way on the subset $\{y_{ii}\}$ and lead to a simple description of the invariant polynomials (going back, in the case $P = S_n$, to Gauss [8]), the action of permutations on $\{y_{ij}|i < j\}$ is non-standard and a description of the invariants (for the case $P = S_n$) is unknown for $n \geqslant 5$ [5]! Fortunately, such high multiplicities of identical particles are relatively rare in applications. Our second 'result', then, is to describe and carry out a strategy for finding a set of generators of the permutation-invariant polynomials in y_{ij} for specific cases of n and P, with at most four identical particles (such as for the $pp \to jjj$ example) using the tools of invariant theory discussed in section 2.

The list of generators obtained in this way is somewhat lengthy in practice and so we turn to ways of shortening it. Again, there are standard ways in invariant theory of doing so, which we describe.[11] We also describe a more ad hoc method: the observables $p_i \cdot p_i$ for a particle are somewhat redundant, since they return the mass of the particle (for a jet, we assume that all jet masses are negligible, since to do otherwise would invalidate the assumption that jets are identical). As such, we are less interested in invariant polynomials involving $p_i \cdot p_i$. Unfortunately, one cannot simply throw them away, because when $n > d$ there are relations between $p_i \cdot p_j$ which mix pairs with $i = j$ and $i \neq j$ (with $n = 2$ and $d = 1$, for example, we have that $(p_1 \cdot p_1)(p_2 \cdot p_2) = (p_1 \cdot p_2)^2$). Our third 'result' is to replace this by a kosher procedure (which is essentially to form a quotient with respect to the ideal generated by the polynomials $p_i \cdot p_i - m_i^2$, or rather the permutation invariant combinations thereof) and to provide a set of generators thereof.

As we will see, these results eventually lead to a manageable set of generators describing the Lorentz- and permutation-invariant polynomials. In the example of $pp \to jjj$, for example, we end up with a set of 26 generators, given explicitly in table 3. In fact, this set of generators is minimal in number, so one can do no better.

Similar ideas were explored in [9], in the context of classifying higher-dimensional operators in effective scalar field theories. A significant difference there is that one studies the action of permutations on quotient algebras with respect to an ideal which features the relation $\sum_i p_i = 0$ (corresponding to an integration-by-parts identity) in addition to the relations $p_i^2 = 0$ studied here (corresponding there to the leading order equations of motion). These additional relations make it difficult to compare our results directly with those in [9], though we hope that some of the results obtained here could nevertheless be usefully applied to the study of that problem. For a rather different approach, see [10], which studies permutation invariance directly at the level of quantum field theory amplitudes.

3.1. Technical statement of results

Let us now give a more technical statement of the results. Firstly, it is convenient to regard the momenta as taking values in a vector space $V \cong \mathbb{C}^{nd}$ over the algebraically-closed field of complex numbers. Doing so not only leads to simplifications on the commutative algebra side, but also allows us to replace the Lorentz group by its complexification $O(d, \mathbb{C})$. The polynomials in the momenta then form an algebra[12], which we denote $\mathbb{C}[V]$ and the Lorentz-invariant polynomials form a subalgebra $\mathbb{C}[V]^{O(d)} \subset \mathbb{C}[V]$. A 'set of generators' of $\mathbb{C}[V]^{O(d)}$ is equivalent to a surjective algebra map from some polynomial algebra to $\mathbb{C}[V]^{O(d)}$. Phrased in these terms, Weyl's FFT is that there exists such a map $W : \mathbb{C}[y_{ij}] \twoheadrightarrow \mathbb{C}[V]^{O(d)}$, where $\mathbb{C}[y_{ij}]$ is the polynomial algebra in variables $y_{ij}, i, j \in \{1, \ldots, n\}, i \leqslant j$, given explicitly

[11] There is a price to be paid for doing so, which we describe shortly.

[12] In this work, 'algebra' will always be understood to mean 'graded algebra over $\mathbb{C}$', unless stated otherwise.

on the generators by $W : y_{ij} \mapsto p_i \cdot p_j$ and extended to an arbitrary polynomial in the obvious way. The SFT is then the statement that the kernel of this map, $\ker W$, is non-trivial when $n > d$, being the ideal $I \subset \mathbb{C}[y_{ij}]$ generated by the $(d+1)$-minors of the matrix whose ijth entry is y_{ij} for $i \leqslant j$ and y_{ji} for $i > j$.

Our first result, which follows almost immediately from Weyl's, is that W restricts to a surjective map between $\mathbb{C}[y_{ij}]^P \subset \mathbb{C}[y_{ij}]$ and $\mathbb{C}[V]^{O(d)\times P} \subset \mathbb{C}[V]^{O(d)\times P}$, the subalgebras that are invariant under $P \subset S_n$. Thus a set of generators of $\mathbb{C}[y_{ij}]^P$ provides us with a set of generators of the object of interest, $\mathbb{C}[V]^{O(d)\times P}$. Finding a set of generators of $\mathbb{C}[y_{ij}]^P$ is where the real hard work begins. Indeed, while the action of P on the subalgebra $C[y_{ii}]$ is via the natural permutation representation group, whose invariant algebra is well-understood (a result due to Gauss in the 'worst-case scenario' $P = S_n$ tells us, for example, that $\mathbb{C}[y_{ii}]^{S_n}$ is isomorphic to the polynomial algebra in n variables with degrees $1, \ldots, n$), the invariants of the action of P on the subalgebra $\mathbb{C}[y_{ij}|i < j]$ are rather harder to describe, with a known description of $\mathbb{C}[y_{ij}|i < j]^{S_n}$ only known for $n < 5$, even though an algorithm is available [5].

Thus, we content ourselves with finding generators for n particle events in which at most four particles are identical, using the fact that the algebra of invariants is Cohen–Macaulay (as P is a finite group and thus linearly reductive) and therefore possesses a Hironaka decomposition. That is, we may write $\mathbb{C}[y_{ij}]^P = \bigoplus_k \eta_k \mathbb{C}[\theta_l]$, where η_k and θ_l, the secondaries and HSOP respectively, are polynomials in y_{ij}. Evidently, η_k and θ_l collectively generate $\mathbb{C}[y_{ij}]^P$.

There exist algorithms for computing η_k and θ_l, though even modern computers quickly run out of steam (hence the difficulties when $n \geqslant 5$). In this way, we are able to find a set of generators, whose number is typically rather large (for $pp \to jjj$, for example, we have 10 primaries and 360 secondaries for $\mathbb{C}[y_{ij}]^{S_2\times S_3}$). To pare it down to a more manageable number, we employ two further strategies. Firstly, the form of the Hironaka decomposition implies that the algebra multiplication is encoded in the relations $\eta_k\eta_m = \sum_j f^j_{km}\eta_j, f^n_{km} \in \mathbb{C}[\theta_l]$, and these can often be used to remove some generators, which are redundant in the sense that they can be obtained as algebraic combinations of other generators. This reduction results in a minimal set of generators of the algebra. (The price to pay is that the description of the algebra in terms of the remaining generators becomes more complicated.) Secondly, since the dot product $p_i \cdot p_i$ does not vary from event to event, being fixed equal to the invariant mass m_i^2, we repeat our construction starting from the quotient algebra $\mathbb{C}[V]/\langle p_i^2 - m_i^2 | \forall\, i\rangle$, showing that there is a surjection of algebras (which is now no longer graded, since the ideal is not homogeneous) $\mathbb{C}[y_{ij}|i < j]^P \twoheadrightarrow \mathbb{C}[V]^P/(\langle p_k^2 - m_k^2 | \forall\, k\rangle \cap \mathbb{C}[V]^P)$.

We describe the effects of removing parity-invariance (which after complexification amounts to replacing $O(d, \mathbb{C})$ by its subgroup $SO(d, \mathbb{C})$) in section 3.5. This is conceptually straightforward, in that it can be achieved by adding further objects $z_{i_1\ldots i_d}$ to the y_{ij}, which map under W to contractions of the epsilon tensor in d dimensions with d momenta. But in practice, elucidating the structure of the corresponding algebra of permutation invariants quickly becomes complicated.

Even without permutation invariance, the SFT implies that the map to $\mathbb{C}[V]^{O(d)\times P}$ does not inject for $d > n$ (as the example given earlier with $d = 1$ and $n = 2$ illustrates). This means that there are yet further relations between the generators of $\mathbb{C}[V]^{O(d)\times P}$ (beyond those in $\mathbb{C}[y_{ij}]^P$), which may be rather obscure and which may yet further frustrate phenomenological analyses. In section 4, we exploit the fact that the algebras $\mathbb{C}[V]^{O(d)\times P}$ are themselves Cohen–Macaulay, meaning that they too admit a Hironaka decomposition, to describe them directly and give some explicit examples.

3.2. General arguments

3.2.1. Generators for Lorentz and permutation invariants. Let a subgroup $P \subset S_n$ of the permutation group act in the standard way on the indices $i \in \{1, \ldots, n\}$. This action induces, in an obvious way, actions on $\{p_i\}$ and $\{y_{ij}\}$ (with the obvious rule that we replace y_{ij} by y_{ji} if $i > j$) and thence on $\mathbb{C}[y_{ij}]$, $\mathbb{C}[V]$, and (since the action of permutations commutes with that of Lorentz transformations) on $C[V]^{O(d)}$. Moreover, it is easily checked that the Weyl map $W : \mathbb{C}[y_{ij}] \twoheadrightarrow \mathbb{C}[V]^{O(d)}$ is equivariant with respect to P. That is, given $p \in P$, the diagram

$$\begin{array}{ccc} \mathbb{C}[y_{ij}] & \xrightarrow{W} & \mathbb{C}[V]^{O(d)} \\ \downarrow p & & \downarrow p \\ \mathbb{C}[y_{ij}] & \xrightarrow{W} & \mathbb{C}[V]^{O(d)} \end{array}$$

commutes.

From here, we wish to show that W restricts to a surjective map $\mathbb{C}[y_{ij}]^P \twoheadrightarrow \mathbb{C}[V]^{O(d)\times P}$, so that a set of generators of $\mathbb{C}[y_{ij}]^P$ furnishes us with a set of generators of $\mathbb{C}[V]^{O(d)\times P}$ via evaluating $y_{ij} \mapsto p_i \cdot p_j$.

To do so, we first note that W sends a P-invariant polynomial to a P-invariant polynomial; in other words $W(\mathbb{C}[y_{ij}]^P) \subset \mathbb{C}[V]^{O(d)\times P}$ and so there is a well-defined restriction map $W| : \mathbb{C}[y_{ij}]^P \to \mathbb{C}[V]^{O(d)\times P}$. It remains to show that the $W|$ map surjects. To do so, let $q \in \mathbb{C}[V]^{O(d)\times P} \subset \mathbb{C}[V]^{O(d)}$. Since W is onto, there exists $r \in \mathbb{C}[y_{ij}]$ such that $W(r) = q$. But r is not necessarily P-invariant, so consider instead $\overline{r} = \frac{1}{p}\sum_{p\in P} r^p$, where r^p denotes the result of acting on r with $p \in P$. This is P-invariant and moreover, we have that $W|(\overline{r}) = W\left(\frac{1}{p}\sum_{p\in P} r^p\right) = \frac{1}{p}\sum W(r^p) = \frac{1}{p}\sum (W(r))^p = \frac{1}{p}\sum q^p = q$, where we used the facts that W is an algebra map, that W is P-equivariant, that $P \subset S_n$ is a finite group, and that q is P-invariant by assumption. Thus $W|$ is onto.

3.2.2. Generators for permutation invariants. Our next goal is to find a set of generators of the algebra $\mathbb{C}[y_{ij}]^P$, which will in turn provide us with a set of generators for $\mathbb{C}[V]^{O(d)\times P}$. In the case considered by Weyl, where P is the trivial group, this is a triviality, since $\mathbb{C}[y_{ij}]$ is a polynomial algebra and so a set of generators (which is moreover a minimal set of generators) is given by $\{y_{ij}\}$. In cases where P is not the trivial group, finding a set of generators is rather harder than it may first appear. To see why this is the case, consider the 'worst case scenario' $P = S_n$. The group S_n acts reducibly on the subspaces with bases $\{y_{ii}\}$ and $\{y_{ij}\}$, so there is a well-defined action (for any $P \subset S_n$, in fact) on the polynomial subalgebras $\mathbb{C}_= := \mathbb{C}[y_{ii}]$ and $\mathbb{C}_< := \mathbb{C}[y_{ij}|i < j]$; to begin with, it is helpful to consider these separately.

The action of S_n on $\{y_{ii}\}$ is via the natural permutation representation (in terms of irreducible representations in partition notation it is $1 \bigoplus (n-1, 1)$) and a complete description of the invariant algebra $\mathbb{C}_=^{S_n}$ was given by Gauss: it is isomorphic (as a graded $\mathbb{C}$-algebra) to the polynomial algebra in n variables with degrees $1, \ldots, n$. For an explicit isomorphism, one can take e.g. the symmetric polynomials $\sum_i y_{ii}, \sum_{i<j} y_{ii}y_{jj}, \ldots, \prod_i y_{ii}$ or the power sum polynomials $\sum_i y_{ii}^k$ with $k \in \{1, \ldots, n\}$.

The action of S_n on $\{y_{ij}\}$ is non-standard (in terms of irreducible representations it is $1 \bigoplus (n-1, 1) \bigoplus (n-2, 2)$ [11]). A description of the invariant algebra is trivial in $n = 2, 3$, being given by polynomial algebras in 1 and 3 variables, respectively, but was only determined relatively recently for $n = 4$ [12] and is unknown for $n \geqslant 5$. It is important to note that the invariant algebra is not a polynomial algebra for $n \geqslant 4$. Rather, like any algebra of invariants under the action of a finite group, it has the more general structure of a Cohen–Macaulay algebra. Such algebras admit a Hironaka decomposition as a free, finitely-generated module over a polynomial subalgebra.

Since an explicit description of $\mathbb{C}_<^{S_n}$ is, in general, unavailable, it is unrealistic to expect one to be available for the full invariant algebra $\mathbb{C}_\leqslant^P$ (where we use $\mathbb{C}_\leqslant$ as a shorthand to denote the full $\mathbb{C}[y_{ij}|i \leqslant j]$). But, since it too has the Cohen–Macaulay property, we can use the available algorithms to find a Hironaka decomposition in simple cases. As we will see, the number of primaries and secondaries that arise in such cases is rather large, so before describing the algorithms and their outputs explicitly, we first describe a way of reducing the number of generators, by 'removing' the invariant masses $p_i \cdot p_i$. To do so in a rigorous way requires us to form quotients of the algebras with respect to the ideal generated by $p_i^2 - m_i^2$, for all i, (or rather its intersection with the invariant algebra) which we do in the next subsection.

3.2.3. Removing invariant masses. *Without permutations.* Let us warm up by returning to the case considered by Weyl, without permutation symmetry. Consider the algebra formed by taking the quotient of $\mathbb{C}[V]^{O(d)}$ with respect to the ideal I generated by the $O(d, \mathbb{C})$-invariant elements $p_i^2 - m_i^2$, for all i, $\langle p_i^2 - m_i^2 | \forall\, i\rangle$ (where we allow the particle mass-squareds m_i^2 to be arbitrary complex numbers). We wish to show that there is a surjective algebra map[13] $\mathbb{C}_< \twoheadrightarrow \mathbb{C}[V]^{O(d)}/I$, such that we can use the y_{ij} with $i < j$ as a set of generators. Of course, this result will hardly come as a surprise to readers, but making a careful proof in this case will help us to avoid potential pitfalls once we add the requirement permutation invariance.

The proof has two parts. One part is to show that the Weyl map W induces a surjective algebra map $\mathbb{C}_\leqslant/\langle y_{ii} - m_i^2 | \forall\, i\rangle \twoheadrightarrow \mathbb{C}[V]^{O(d)}/I$. The other part is to exhibit an algebra isomorphism $\mathbb{C}_\leqslant/\langle y_{ii} - m_i^2 | \forall\, i\rangle \xrightarrow{\sim} \mathbb{C}_<$.

For the first part, it is enough to note that the map is well-defined on equivalence classes, because any element in $\langle y_{ii} - m_i^2 | \forall\, i\rangle$ lands in I. (Surjectivity follows automatically from the surjectivity of W.)

For the other part, consider the polynomial algebra $R[x]$ in one variable over an arbitrary algebra R. Let $f(x) \in R[x]$, let $r \in R$, and let $\mathrm{ev} : R[x] \twoheadrightarrow R$ be the evaluation map, viz the R-algebra map defined by $x \mapsto r$. Since $(x - r)$ is a monic polynomial, by the division algorithm we have that $f(x) = g(x) \cdot (x - r) + s$, with $g(x) \in R[x]$ and $s \in R$. Thus $\mathrm{ev}(f(x)) = s$ and $\ker \mathrm{ev} = \langle x - r\rangle$. By the first isomorphism theorem, $R[x]/\langle x - r\rangle \xrightarrow{\sim} R$. Now apply this successively to $\mathbb{C}_\leqslant \cong \mathbb{C}[y_{ij}|(i, j) \neq (1, 1)][y_{11}]$, $\mathbb{C}[y_{ij}|(i, j) \neq (1, 1)] \cong \mathbb{C}[y_{ij}|(i, j) \neq (1, 1), (2, 2)][y_{22}]$, &c. to get the desired result. Equivalently, an explicit isomorphism $\mathbb{C}_\leqslant/\langle y_{ii} - m_i^2 | \forall\, i\rangle \xrightarrow{\sim} \mathbb{C}_<$ can be obtained from the evaluation map (which is ungraded, except in the $m_i^2 = 0$ case) from $\mathbb{C}_\leqslant$ to $\mathbb{C}_<$ given by

$$\mathrm{ev} : \mathbb{C}_\leqslant \twoheadrightarrow \mathbb{C}_< : y_{ii}, y_{ij} \mapsto m_i^2, y_{ij} \tag{1}$$

whose kernel is indeed $\ker \mathrm{ev} = \langle y_{ii} - m_i^2 | \forall\, i\rangle$.

With permutations. Now that we have tackled the case without permutations, we turn to address the cases with permutation symmetry. Our goal is to show that there exists a surjective algebra map[14] $\mathbb{C}_<^P \twoheadrightarrow \mathbb{C}[V]^{O(d)\times P}/J$ where $J = \langle p_i^2 - m_i^2 | \forall\, i\rangle \cap \mathbb{C}[V]^{O(d)\times P}$. Again, the proof has two parts. One is to show that the restricted Weyl map $W|$ induces a surjective algebra map

[13] It is important to note that, unless $m_i^2 = 0$ for all i, such that I is homogeneous, $\mathbb{C}[V]^{O(d)}/I$ is not graded, and so nor is the map.

[14] Again, ungraded unless $m_i^2 = 0$.

$\mathbb{C}^P_{\leqslant}/J' \twoheadrightarrow \mathbb{C}[V]^{O(d)\times P}/J$, where $J' = \langle y_{ii} - m_i^2 | \forall\, i\rangle \cap \mathbb{C}^P_{\leqslant}$, and the other is to exhibit an algebra isomorphism $\mathbb{C}^P_{\leqslant}/J' \xrightarrow{\sim} \mathbb{C}^P_{<}$.

For the first part, we begin by showing that the image $W(J') \subset J$. For an element $j' \in J'$, $j' \in \langle y_{ii} - m_i^2 | \forall\, i\rangle$ and $j' \in \mathbb{C}^P_{\leqslant}$ by definition. But since the image $W(\langle y_{ii} - m_i^2 | \forall\, i\rangle) \subset I$, the image $W(j') \in I$. Furthermore, the element j' is P-invariant by assumption and as the map W is P-equivariant, the image $W(j')$ is also P-invariant. So, $W(j') \in J$ and hence $W(J') \subset J$.[15] It is then enough to note that the map is well-defined on the equivalence classes because any element of J' lands in J. (Surjectivity again follows from the surjectivity of W.)

For the second part, it turns out that the required result follows from a more general theorem.

Proposition 3.1. *Suppose that a finite group G acts reducibly on a vector space $V = X \bigoplus Z$ and suppose that the representation carried by X is further reducible, containing the trivial representation. Let $\{x_i\}$ and $\{z_i\}$, respectively, be bases of the dual spaces Hom$(X, \mathbb{C})$ and Hom$(Z, \mathbb{C})$, respectively, and let $a \in X$ denote a G-invariant vector with components $a_i = x_i(a) \in \mathbb{C}$. Further, consider the algebras $R = \mathbb{C}[x_1, \ldots, x_m, z_1, \ldots, z_n]$ and $S = \mathbb{C}[z_1, \ldots, z_n]$ along with the evaluation map* ev $: R \twoheadrightarrow S, x_i \mapsto a_i$*, with kernel $\langle x_i - a_i | \forall\, i\rangle$. Then, there exists an isomorphism of (ungraded if $a_i \neq 0$) algebras $R^G/J \xrightarrow{\sim} S^G$, where R^G, S^G are the G-invariant subalgebras of R, S respectively and $J = \langle x_i - a_i | \forall\, i\rangle \cap R^G$ is an ideal of R^G.*

Proof. To prove this, we start by explicitly defining the action of $g \in G$ on $h \in R^G$ and $f \in S$ via the reducible representation $\rho : G \to GL(V) : g \mapsto \rho_X(g) \bigoplus \rho_Z(g)$ to be as follows

$$h \mapsto h^g = h(\rho_X(g)x_i, \rho_Z(g)z_i) = h(x_i, z_i) = h, \tag{2}$$

$$f \mapsto f^g = f(\rho_Z(g)z_i). \tag{3}$$

Next, we define the inclusion map $i : R^G \hookrightarrow R$ and compose it with the evaluation map to get the restricted algebra map ev$| :=$ ev $\circ\, i : R^G \to S$. It can then be checked that the evaluation map ev$|$ is equivariant with respect to G. That is, given $g \in G$, the diagram

$$\begin{array}{ccc} R^G & \xrightarrow{\text{ev}|} & S \\ \big\downarrow g & & \big\downarrow g \\ R^G & \xrightarrow{\text{ev}|} & S \end{array}$$

commutes. Now as the map ev$|$ is G-equivariant, it sends a G-invariant polynomial to a G-invariant polynomial; in other words ev$|(R^G) \subset S^G$ and so we have a well-defined restriction map ev$| : R^G \to S^G$. It remains to show that ev$|$ is surjective. To do so, let $s \in S^G \subset S$. Since ev is onto, there exists $r \in R$ such that ev$(r) = s$. But r is not necessarily G-invariant, so consider instead $\bar{r} = \frac{1}{|G|}\sum_{g\in G} r^g$, where again r^g denotes the result of acting on r with $g \in G$. This is G-invariant and we have, furthermore, that $\text{ev}|(\bar{r}) = \text{ev}|\left(\frac{1}{|G|}\sum_{g\in G} r^g\right) = \frac{1}{|G|}\sum_{g\in G}\text{ev}|(r^g) = \frac{1}{|G|}\sum_{g\in G}\left(\text{ev}|(r)\right)^g = \frac{1}{|G|}\sum_{g\in G} s^g = s$, where we have used the fact that ev$|$ is an (ungraded for $a_i \neq 0$) algebra map, that ev$|$ is G-equivariant, that G is a finite group, and that s is G-invariant by assumption. Thus, ev$|$ is onto. The last ingredient of the proof is to note that the kernel of the map ker ev$|$ is the restriction of the ideal $\langle x_i - a_i | \forall\, i\rangle$ to the G-invariant subalgebra $J = \langle x_i - a_i | \forall\, i\rangle \cap R^G$. Finally, by the first isomorphism theorem, $R^G/J \xrightarrow{\sim} S^G$. □

[15] Actually, $W(J') = J$, but equality is unnecessary for our purposes.

In our specific case, the variables y_{ii}, y_{ij} transform under reducible representations of the permutation group P with the representation of y_{ii}, $(1 \bigoplus (n-1,1))$, being further reducible containing the trivial representation. Furthermore, the masses m_i^2 clearly form an invariant vector when the particles (and hence the masses) are identical. Hence, the previous theorem applies and we have an isomorphism of (ungraded, except in the massless case) algebras $\mathbb{C}^P_{\leqslant}/J' \xrightarrow{\sim} \mathbb{C}^P_{<}$.

3.3. Generators of permutation invariants

We now describe results from the theory of invariants which together may be used to find sets of generators for the algebras of permutation invariants, such as $\mathbb{C}^P_{<}$. For more details, see e.g. [5, 13].

As discussed previously, given a Hironaka decomposition of an invariant algebra $K[V]^G = \bigoplus_k \eta_k K[\theta_1, \ldots, \theta_l]$, the set containing the primary and secondary invariants, $\{\eta_i, \theta_j\}$, forms a set of generators of $K[V]^G$, which is what we seek. A Hironaka decomposition can be found by a two-step process. The first step is to find a HSOP. It turns out that necessary and sufficient conditions for a set of homogeneous elements in $K[V]^G$ to form such a system are that they be algebraically independent and, in the case where G is a finite group, that the locus of points where all elements of strictly positive degree simultaneously vanish is given by the zero vector in V (this is more generally called the nullcone condition, to be discussed in detail in subsection 4.2).

Finding a HSOP has been reduced to an (unwieldy) algorithm [14, 15], but we will not need it here. Indeed, the group P acts on $\mathbb{C}^P_{<}$, say (an analogous result holds for $\mathbb{C}^P_{\leqslant}$), by permuting the y_{ij} amongst themselves; but it is easily shown (cf [5], example 2.4.9) that for any permutation subgroup of $S_{n(n-1)/2}$, a HSOP is given by the $n(n-1)/2$ elementary symmetric polynomials in y_{ij}.

For our purposes, this HSOP is sometimes less than optimal, because it introduces primary invariants of unnecessarily high degrees, leading to more secondary invariants (as can easily be seen by considering the case where P is the trivial group, such that $\{y_{ij}\}$ is a HSOP, with primary invariants all of degree 1). A HSOP with primary invariants of lower degrees can be found by partitioning the y_{ij} into their orbits under P and forming the respective sets of elementary symmetric polynomials. Again, one may easily show that the union of these forms a HSOP.

Let us make this explicit in our $pp \to jjj$ example. Labelling the protons by 4, 5 and the jets by 1, 2, 3, we have the following orbits: $\{y_{45}\}, \{y_{12}, y_{13}, y_{23}\}, \{y_{14}, y_{15}, y_{24}, y_{25}, y_{34}, y_{35}\}$. Following our prescription, the HSOP will be

$$\begin{aligned}
e_1(y_{45}), \qquad & e_1(y_{12}, y_{13}, y_{23}), \qquad e_1(y_{14}, y_{24}, y_{34}, y_{15}, y_{25}, y_{35}), \\
& e_4(y_{14}, y_{24}, y_{34}, y_{15}, y_{25}, y_{35}), \\
& e_2(y_{12}, y_{13}, y_{23}), \qquad e_2(y_{14}, y_{24}, y_{34}, y_{15}, y_{25}, y_{35}), \\
& e_5(y_{14}, y_{24}, y_{34}, y_{15}, y_{25}, y_{35}), \\
& e_3(y_{12}, y_{13}, y_{23}), \qquad e_3(y_{14}, y_{24}, y_{34}, y_{15}, y_{25}, y_{35}), \\
& e_6(y_{14}, y_{24}, y_{34}, y_{15}, y_{25}, y_{35}).
\end{aligned} \tag{4}$$

Having found a HSOP, we turn to the second step in finding a Hironaka decomposition, which is to find the corresponding secondary invariants. A first observation is that one can read off the degrees of the secondary invariants by comparing the Hilbert series computed using

Table 1. The Hilbert series of some relevant invariant algebras.

	$\mathbb{C}^P_<$
$n=4$, with S_1	$\frac{1}{(1-t)^6}$
$n=4$, with $S_2\times S_2$	$\frac{1+t^3}{(1-t)^3(1-t^2)^3}$
$n=5$, with S_1	$\frac{1}{(1-t)^{10}}$
$n=5$, with $S_2\times S_3$	$\frac{1+t^2+6t^3+8t^4+6t^5+12t^6+14t^7+9t^8+8t^9+5t^{10}+2t^{11}}{(1-t)^3(1-t^2)^4(1-t^3)^2(1-t^6)}$

Molien's formula (for algebras invariant under finite groups)[16]

$$H(K[V]^G,t)=\frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(1-t\cdot\rho_g)} \tag{5}$$

(where ρ_g is the linear operator representing $g\in G$) to the form corresponding to the Hironaka decomposition, viz

$$H\left(\bigoplus \eta K[\theta],t\right)=\frac{1+\sum_k S_k t^k}{\prod_l (1-t^l)^{P_l}}. \tag{6}$$

where there are S_k secondaries at degree k and P_l primaries at degree l. By way of illustration, table 1 lists the Hilbert series for a few of the algebras that we are interested in.

The secondaries may now be found via the following algorithm [5], employing a Groebner basis[17] $\mathcal{G}$ for the ideal $\langle\theta_1,\ldots,\theta_l\rangle\subset K[V]^G$ generated by the primary invariants. An explicit computation using this algorithm is presented in full detail in subsection 4.3. Here we only give a sketch of the main ingredients:

- Read off the degrees of secondaries $d_1,\ldots,d_m$ from the Hilbert series.
- For $i=1,\ldots,m$ perform the following two steps:
  - ∗ Calculate a basis of the homogeneous component $K[V]^G_{d_i}$ (invariant polynomials of degree d_i).
  - ∗ Select an element η_i from this basis such that the normal form $\mathrm{NF}_{\mathcal{G}}(\eta_i)$ (remainder on division by the Groebner basis) is non-zero and is not in the K-vector space generated by the polynomials $\mathrm{NF}_{\mathcal{G}}(\eta_1),\ldots,\mathrm{NF}_{\mathcal{G}}(\eta_{i-1})$.
- The invariants $\eta_1,\ldots,\eta_k$ are the required secondary invariants.

A version of this algorithm is implemented in `Macaulay2` [16] (and other computer packages).

3.4. Redundancies

In the previous subsection, we described a systematic construction of a Hironaka decomposition, and *ergo* a set of generators, for $\mathbb{C}^P_<$ (an analogous construction applies for $\mathbb{C}^P_\leqslant$). Unfortunately, the number of generators is rather large in all but the simplest cases. For the purpose of carrying out phenomenological analyses, one would like to have a set of

[16] The Hilbert series computation for algebras invariant under non-finite group is more involved and is discussed in appendix B.

[17] Readers unfamiliar with these may wish to consult [15] for a gentle introduction.

generators that is as minimal as possible, in the sense of reducing both the number of generators and their degrees. In this subsection, we will see that such a reduction is indeed possible, and leads to a set of generators whose cardinality is minimal (the degrees of the generators in such a set is moreover fixed). Unfortunately, the number of generators in such a set is still rather large. But this is the best one can do.

The reduction may be achieved (at the cost of destroying the neat encoding of the algebraic structure in the Hironaka decomposition, which may in itself be useful for phenomenological analyses) via the following algorithm: for a set of generators S, choose an element $f \in S$ and set up a general element of the same degree as f in the algebra generated by $S \backslash f$ with unknown coefficients. Equate it to f and extract the corresponding system of linear equations by comparison of coefficients. The system is solvable if and only if f can be omitted from S. It turns out [5], though we will not show it here, that this procedure leads to a set of algebra generators whose cardinality is minimal; the degrees of the resulting generators are, moreover, uniquely determined.

It seems that we are home and dry, but there is one remaining issue: although the problem of finding the secondary generators is solved algorithmically, in most non-trivial cases, it is highly inefficient. Even modern computers using state-of-the-art algorithms start struggling with Hironaka decompositions containing more than a few hundred secondaries. Our only hope is if we can somehow get away with finding some, but not all, of the secondaries before using the elimination procedure just described. This hope can be realised by use of arguments going back to Noether, who showed that the maximal degree of an algebra generator in a minimal set is $\leqslant |G|$. When G is non-cyclic (so $P \neq S_1, S_2$ in the case at hand), Noether's bound can be improved to $\frac{3}{4}|G|$ if $|G|$ is even and $\frac{5}{8}|G|$ if $|G|$ is odd [17].[18] Therefore, we only need to find the secondaries up to these bounds before discarding the redundant generators using the process outlined above. Of course, in many cases these bounds are practically useless; the order of S_n is $n!$. But for physically relevant examples such as $S_2 \times S_3$, they reduce the computation time significantly.

3.5. Parity

Finally, we briefly discuss the more general case where parity is not a symmetry. Weyl showed that a generating set of Lorentz invariants in d dimensions is given by the dot products, along with all the possible contractions of momenta with the anti-symmetric d dimensional Levi–Civita epsilon tensor[19]. To include these extra generators in our discussion, one could add some extra variables $z_{i_1,\ldots,i_d}$ which transform in a similar (anti-symmetric) manner to the epsilons under the action of the permutation group and are mapped to the epsilons in the appropriate way under the Weyl map. One then needs to study the algebra $\mathbb{C}[y_{ij}, z_{i_1,\ldots,i_d}]^P$ and find its Hironaka decomposition and consequently a set of minimal algebra generators. The first challenge one runs into in trying to do so is the difficulty in finding a suitable HSOP. Since the elements in P act on $z_{i_1,\ldots,i_d}$ by permutation, a HSOP is given by the elementary symmetric polynomials in $z_{i_1,\ldots,i_d}$, but the degrees of the resulting generators are prohibitively large, with a consequent slew of secondaries. Given the inefficiencies of current algorithms, which already struggle with the simpler case of $\mathbb{C}^P_<$, it seems unlikely that one will be able to find a

[18] In our $pp \to jjj$ example, we have $3|G|/4 = 9$, which comfortably exceeds our highest degree primary, of degree 6; we will see in the next subsection that in fact the highest degree in a minimal set of generators is in fact 6.

[19] There are, of course, relations between the Levi–Civita tensors and the dot products, namely the product of two epsilon tensors contracted with some momenta p_i is equal to the corresponding minor of the $p_i \cdot p_j$ matrix.

Table 2. Table of generators for $n = 4$ with $S_2 \times S_2$.

Degree = 1
$g_{11} = y_{12}$, $g_{12} = y_{34}$, $g_{13} = y_{13} + y_{14} + y_{23} + y_{24}$,
Degree = 2
$g_{21} = y_{13}y_{23} + y_{14}y_{24}$, $g_{22} = y_{13}y_{14} + y_{23}y_{24}$, $g_{23} = y_{13}y_{14} + y_{13}y_{23} + y_{14}y_{23} + y_{13}y_{24} + y_{14}y_{24} + y_{23}y_{24}$,
Degree = 3
$g_{31} = y_{13}y_{14}y_{23} + y_{13}y_{14}y_{24} + y_{13}y_{23}y_{24} + y_{14}y_{23}y_{24}$

minimal set of generators in this way, in all but the simplest cases. We address the issue of generating sets when parity is not a symmetry in another work [18] by approaching the problem in a more direct hands-on method.

3.6. Examples of minimal algebra generators

We will now apply the aforementioned techniques to find sets of generators for common examples of phenomenological interest.

3.6.1. pp → jj. A common scattering problem is the two protons to two jets, $pp \to jj$, though of course jj could be any two objects that we do not want to or cannot distinguish, which corresponds to the $n = 4$ with $S_2 \times S_2$ case.

First, we find the primaries using our prescription. The invariant subspaces are $\{y_{12}\}$, $\{y_{34}\}$, $\{y_{13}, y_{14}, y_{23}, y_{24}\}$, and therefore we take the primaries to be

$$\begin{aligned} &e_1(y_{12}), \qquad e_1(y_{13}, y_{14}, y_{23}, y_{24}), \qquad e_3(y_{13}, y_{14}, y_{23}, y_{24}), \\ &e_1(y_{34}), \qquad e_2(y_{13}, y_{14}, y_{23}, y_{24}), \qquad e_4(y_{13}, y_{14}, y_{23}, y_{24}). \end{aligned}$$

We can already see directly from the improved Noether bound (which is $\frac{3}{4}(2!)(2!) = 3$ in this case) that these generators cannot be part of a minimal set. To read off the degrees of secondaries, we write the Hilbert series in table 1 in the form

$$H(\mathbb{C}[y_{ij}]^{S_2\times S_2}, t) = \frac{1 + 2t^2 + 2t^4 + t^6}{(1-t)^3(1-t^2)(1-t^3)(1-t^4)}.$$

Next, we use the algorithm to compute the secondaries. Using the bound, we only need to find the secondaries up to degree 3. Once found, we can start eliminating redundancies from the union of primaries and secondaries in the fashion described in subsection 3.4. Once this is done, we are left with a set of seven minimal algebra generators given in table 2.

3.6.2. pp → jjj. We now ramp up the level of complexity, by considering $pp \to jjj$, which corresponds to the $n = 5$ with $P = S_2 \times S_3$ case.

The set of primaries was already given in equation (4) of subsection 3.3. Comparing to the Hilbert series in table 1, we see that they are again non-optimal and we need to write the Hilbert series in the modified form

Table 3. Table of generators for $n = 5$ with $S_2 \times S_3$.

Degree = 1

$g_{11} = y_{12},$

$g_{12} = y_{34} + y_{35} + y_{45},$

$g_{13} = y_{13} + y_{14} + y_{15} + y_{23} + y_{24} + y_{25},$

Degree = 2

$g_{21} = y_{13}y_{23} + y_{14}y_{24} + y_{15}y_{25},$

$g_{22} = y_{34}y_{35} + y_{34}y_{45} + y_{35}y_{45},$

$g_{23} = y_{13}y_{14} + y_{13}y_{15} + y_{14}y_{15} + y_{23}y_{24} + y_{23}y_{25} + y_{24}y_{25},$

$g_{24} = y_{13}y_{34} + y_{14}y_{34} + y_{23}y_{34} + y_{24}y_{34} + y_{13}y_{35} + y_{15}y_{35} + y_{23}y_{35} + y_{25}y_{35} + y_{14}y_{45} + y_{15}y_{45} + y_{24}y_{45} + y_{25}y_{45},$

$g_{25} = y_{13}y_{14} + y_{13}y_{15} + y_{14}y_{15} + y_{13}y_{23} + y_{14}y_{23} + y_{15}y_{23} + y_{13}y_{24} + y_{14}y_{24} + y_{15}y_{24} + y_{23}y_{24} + y_{13}y_{25} + y_{14}y_{25} + y_{15}y_{25} + y_{23}y_{25} + y_{24}y_{25},$

Degree = 3

$g_{31} = y_{34}y_{35}y_{45},$

$g_{32} = y_{13}y_{23}y_{34} + y_{14}y_{24}y_{34} + y_{13}y_{23}y_{35} + y_{15}y_{25}y_{35} + y_{14}y_{24}y_{45} + y_{15}y_{25}y_{45},$

$g_{33} = y_{13}y_{14}y_{34} + y_{23}y_{24}y_{34} + y_{13}y_{15}y_{35} + y_{23}y_{25}y_{35} + y_{14}y_{15}y_{45} + y_{24}y_{25}y_{45},$

$g_{34} = y_{13}y_{34}^2 + y_{14}y_{34}^2 + y_{23}y_{34}^2 + y_{24}y_{34}^2 + y_{13}y_{35}^2 + y_{15}y_{35}^2 + y_{23}y_{35}^2 + y_{25}y_{35}^2 + y_{14}y_{45}^2, + y_{15}y_{45}^2 + y_{24}y_{45}^2 + y_{25}y_{45}^2,$

$g_{35} = y_{13}^2y_{34} + y_{14}^2y_{34} + y_{23}^2y_{34} + y_{24}^2y_{34} + y_{13}^2y_{35} + y_{15}^2y_{35} + y_{23}^2y_{35} + y_{25}^2y_{35} + y_{14}^2y_{45} + y_{15}^2y_{45} + y_{24}^2y_{45} + y_{25}^2y_{45}$

$g_{36} = y_{13}^2y_{23} + y_{13}y_{23}^2 + y_{14}^2y_{24} + y_{14}y_{24}^2 + y_{15}^2y_{25} + y_{15}y_{25}^2,$

$g_{37} = y_{13}^2y_{14} + y_{13}y_{14}^2 + y_{13}^2y_{15} + y_{14}^2y_{15} + y_{13}y_{15}^2 + y_{14}y_{15}^2 + y_{23}^2y_{24} + y_{23}y_{24}^2 + y_{23}^2y_{25} + y_{24}^2y_{25} + y_{23}y_{25}^2 + y_{24}y_{25}^2,$

$g_{38} = y_{13}y_{14}y_{15} + y_{13}y_{14}y_{23} + y_{13}y_{15}y_{23} + y_{14}y_{15}y_{23} + y_{13}y_{14}y_{24} + y_{13}y_{15}y_{24} + y_{14}y_{15}y_{24} + y_{13}y_{23}y_{24} + y_{14}y_{23}y_{24} + y_{15}y_{23}y_{24} + y_{13}y_{14}y_{25} + y_{13}y_{15}y_{25} + y_{14}y_{15}y_{25} + y_{13}y_{23}y_{25} + y_{14}y_{23}y_{25} + y_{15}y_{23}y_{25} + y_{13}y_{24}y_{25} + y_{14}y_{24}y_{25} + y_{15}y_{24}y_{25} + y_{23}y_{24}y_{25},$

Degree = 4

$g_{41} = y_{13}^2y_{23}^2 + y_{14}^2y_{24}^2 + y_{15}^2y_{25}^2,$

$g_{42} = y_{13}y_{23}y_{34}^2 + y_{14}y_{24}y_{34}^2 + y_{13}y_{23}y_{35}^2 + y_{15}y_{25}y_{35}^2 + y_{14}y_{24}y_{45}^2 + y_{15}y_{25}y_{45}^2,$

$g_{43} = y_{13}y_{14}y_{34}^2 + y_{23}y_{24}y_{34}^2 + y_{13}y_{15}y_{35}^2 + y_{23}y_{25}y_{35}^2 + y_{14}y_{15}y_{45}^2 + y_{24}y_{25}y_{45}^2,$

$g_{44} = y_{13}^2y_{23}y_{34} + y_{13}y_{23}^2y_{34} + y_{14}^2y_{24}y_{34} + y_{14}y_{24}^2y_{34} + y_{13}^2y_{23}y_{35} + y_{13}y_{23}^2y_{35} + y_{15}^2y_{25}y_{35} + y_{15}y_{25}^2y_{35} + y_{14}^2y_{24}y_{45} + y_{14}y_{24}^2y_{45} + y_{15}^2y_{25}y_{45} + y_{15}y_{25}^2y_{45},$

$g_{45} = y_{13}^2y_{15}y_{34} + y_{14}^2y_{15}y_{34} + y_{23}^2y_{25}y_{34} + y_{24}^2y_{25}y_{34} + y_{13}^2y_{14}y_{35} + y_{14}y_{15}^2y_{35} + y_{23}^2y_{24}y_{35} + y_{24}y_{25}^2y_{35} + y_{13}y_{14}^2y_{45} + y_{13}y_{15}^2y_{45} + y_{23}y_{24}^2y_{45} + y_{23}y_{25}^2y_{45},$

$g_{46} = y_{13}^2y_{14}y_{23} + y_{13}^2y_{15}y_{23} + y_{13}y_{14}^2y_{24} + y_{14}^2y_{15}y_{24} + y_{13}y_{23}^2y_{24} + y_{14}y_{23}y_{24}^2 + y_{13}y_{15}^2y_{25} + y_{14}y_{15}^2y_{25} + y_{13}y_{23}^2y_{25} + y_{14}y_{24}^2y_{25} + y_{15}y_{23}y_{25}^2 + y_{15}y_{24}y_{25}^2,$

$g_{47} = y_{13}y_{14}y_{15}y_{23} + y_{13}y_{14}y_{15}y_{24} + y_{13}y_{14}y_{23}y_{24} + y_{13}y_{15}y_{23}y_{24} + y_{14}y_{15}y_{23}y_{24} + y_{13}y_{14}y_{15}y_{25} + y_{13}y_{14}y_{23}y_{25} + y_{13}y_{15}y_{23}y_{25} + y_{14}y_{15}y_{23}y_{25} + y_{13}y_{14}y_{24}y_{25} + y_{13}y_{15}y_{24}y_{25} + y_{14}y_{15}y_{24}y_{25} + y_{13}y_{23}y_{24}y_{25} + y_{14}y_{23}y_{24}y_{25} + y_{15}y_{23}y_{24}y_{25},$

Degree = 5

$g_{51} = y_{13}^2y_{23}^2y_{34} + y_{14}^2y_{24}^2y_{34} + y_{13}^2y_{23}^2y_{35} + y_{15}^2y_{25}^2y_{35} + y_{14}^2y_{24}^2y_{45} + y_{15}^2y_{25}^2y_{45}$

$g_{52} = y_{13}y_{14}y_{15}y_{23}y_{24} + y_{13}y_{14}y_{15}y_{23}y_{25} + y_{13}y_{14}y_{15}y_{24}y_{25} + y_{13}y_{14}y_{23}y_{24}y_{25} + y_{13}y_{15}y_{23}y_{24}y_{25} + y_{14}y_{15}y_{23}y_{24}y_{25},$

Degree = 6

$g_{61} = y_{13}y_{14}y_{15}y_{23}y_{24}y_{25}.$

$$H(\mathbb{C}[y_{ij}]^{S_2\times S_3}, t) =$$

$$\frac{1+3t^2+6t^3+12t^4+17t^5+32t^6+35t^7+47t^8+48t^9+49t^{10}+38t^{11}+34t^{12}+19t^{13}+12t^{14}+5t^{15}+2t^{16}}{(1-t)^3\left(1-t^2\right)^2\left(1-t^3\right)^2\left(1-t^4\right)\left(1-t^5\right)\left(1-t^6\right)}.$$

Using the algorithm to find the secondaries up to degree $\frac{3}{4}(2!)(3!) = 9$ and eliminating redundancies, we are left with a set of 26 minimal algebra generators. Table 3 contains the explicit list.

4. Hironaka decompositions of Lorentz- and permutation-invariants

In the previous section, we generalized the FFT to include the action of an arbitrary group of permutations of the n particles and provided a systematic method of constructing a set of minimal algebra generators of Lorentz- and permutation-invariant polynomials. (This is relevant, for example, when some of the particles are indistinguishable, which is an inevitable consequence of quantum field theory.) A major difference is that, even when $n \leqslant d$, the algebra of invariants is not a polynomial algebra once we include permutations. This simple observation already suggests that attempts to generalise the SFT to the case where permutations are included will lead to unpleasantness.

In this section, we replace the FFT and SFT by a more direct description of the algebra of Lorentz- and permutation-invariants, using tools of commutative algebra which were not available to Weyl. In particular, we use the fact that (via a theorem of Hochster and Roberts [7]) the algebra of invariants is Cohen–Macaulay, and so admits a Hironaka decomposition as a free, finitely-generated module over a polynomial subalgebra. Thus, a direct description of the invariant algebra can be given in terms of a set of generators of such a polynomial subalgebra, termed either *primaries* or a HSOPs, and a set of basis elements for the module, called *secondaries*. In particular, every element in the algebra can be expressed *uniquely* in terms of primaries and secondaries, and multiplication in the algebra is completely encoded in the finite set of products of secondaries.

Again, the difficulty is in finding these Hironaka decompositions explicitly. In the what follows, we proceed to sketch out the background necessary results of invariant theory and employ them to find Hironaka decompositions in the first non-trivial case, viz $n = d + 1$. We solve the hardest step in the procedure, namely to find HSOPs. We do this both for the case without permutations and for the case with all permutations included; the latter serves as a HSOP for an arbitrary subgroup of permutations. Unfortunately, even though the remaining step of finding the secondaries reduces to a conceptually straightforward exercise in linear algebra, the available algorithm proceeds by brute-force Groebner basis methods [5] and runs out of steam in cases with more than a few particles. But we hope that our results, modest though they are, will inspire others to make more targeted attacks on the problem. In subsection 4.5, we present Hironaka decompositions of the cases with $(n, d) = (5, 4)$ with no permutations included, $(n, d) = (3, 2)$ with all permutations included, and a conjecture for the Hironaka decomposition of the general case of $n = d + 1$ with no permutations. Appendix B gives the details of the relevant Hilbert series computations.

4.1. Technical statement of results

In section 3, we generalised Weyl's FFT to include permutation invariance. That is, we constructed a general method for finding a set of generators of the algebra $\mathbb{C}[y_{ij}]^P$ which surjects onto the Lorentz- and permutation-invariant subalgebra, $W| : \mathbb{C}[y_{ij}]^P \twoheadrightarrow \mathbb{C}[V]^{O(d)\times P}$. What our work did not include is the generalisation of the SFT which amounts to characterising the kernel of the restriction map $\ker W|$. Formally, the kernel is the intersection of the invariant algebra with the ideal $I \subset \mathbb{C}[y_{ij}]$ generated by the $(d+1)$-minors of the matrix whose ijth entry is y_{ij} for $i \leqslant j$ and y_{ji} for $i > j$, i.e. $\ker W| = I \cap \mathbb{C}[y_{ij}]^P$. In practice however, it is difficult to explicitly describe $\ker W|$, for a couple of reasons. For one thing, as stated previously, whereas $\mathbb{C}[y_{ij}]$ is a polynomial algebra, the invariant algebra $\mathbb{C}[y_{ij}]^P$ has a more complicated structure in general: it is Cohen–Macaulay and therefore can be expressed as a free, finitely-generated algebra over a polynomial subalgebra. For another, it turns out that the generators of the ideal I transform in an unpleasant representation of the permutation group, making finding the corresponding permutation-invariant generators difficult.

We therefore follow an alternative approach, seeking a more direct description of the Lorentz- and permutation-invariant algebra $\mathbb{C}[V]^{O(d)\times P}$. The *Hochster–Roberts* theorem [7] states that an invariant algebra $K[V]^G$ is Cohen–Macaulay if G is a linearly reductive group[20]. Since $O(d,\mathbb{C}) \times P$ is linearly reductive, the theorem applies and the algebra $\mathbb{C}[V]^{O(d)\times P}$ can be expressed as a free, finitely-generated module over a polynomial subalgebra. That is, the algebra can be expressed in terms of a Hironaka decomposition as $\mathbb{C}[V]^{O(d)\times P} = \bigoplus_k \eta_k \mathbb{C}[\theta_l]$ where the $\{\eta_k\}$ are the secondaries, the $\{\theta_l\}$ form a HSOP, and multiplication in the algebra is uniquely defined via $\eta_k\eta_m = \sum_j f^j_{km}\eta_j$, with $f^j_{km} \in \mathbb{C}[\theta_l]$. Every element in the algebra is then uniquely expressed as a linear sum of secondaries with coefficients which are polynomials in the HSOP.

The difficult part of finding Hironaka decompositions begins when one tries to find valid HSOPs as, apart from using inefficient algorithms [14], there is no obvious way to obtain them. Furthermore, the properties that a valid HSOP needs to satisfy are non-trivial and difficult to check. Previously, we were able to sidestep this by repurposing Gauss's results on permutation-invariants. Here, we are not so lucky. In subsection 4.2, we propose HSOPs for the algebras $\mathbb{C}[V]^{O(d)\times P}$, with $n = d+1$, in the two cases where $P = 1$ (with 1 denoting the trivial group) and $P = S_n$ and explicitly verify that they satisfy the necessary conditions.

4.2. HSOPs for $\mathbb{C}[V]^{O(d)\times P}$

In this subsection, we find HSOPs for the algebras $\mathbb{C}[V]^{O(d)\times P}$ in the $n = d+1$ case with no permutation symmetry, $P = 1$, and with full permutation symmetry, $P = S_n$. In fact, the HSOP for $P = S_n$ is also a HSOP for any $P \subset S_n$, and so we obtain a complete solution of this part of the problem.

The necessary conditions for a set of polynomials to constitute a HSOP are twofold: firstly, the polynomials must be algebraically independent; secondly, they must satisfy the nullcone condition [5].

A set of polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_k]$ is said to be *algebraically independent* if the only polynomial $h \in K[z_1, \ldots, z_m]$ satisfying $h(f_1, \ldots, f_m) = 0$ is the zero polynomial. Although trivial to define, the algebraic independence of polynomials is less trivial to check. One method proceeds via calculation of a Groebner basis, while another uses the Jacobi criterion. The former quickly becomes inefficient when used with many polynomials of high degree, but more importantly, it is difficult to apply in an abstract way. We therefore resort to using the Jacobi criterion[21] which states that a set of polynomials, $f_1, \ldots, f_m \in K[x_1, \ldots, x_k]$, is algebraically independent if and only if the wedge product of the exterior derivatives[22] of

[20] A linear algebraic group G is called *linearly reductive* if for every rational representation V and every $v \in V^G \backslash \{0\}$, there exists a linear invariant function $f \in (V^*)^G$ such that $f(v) \neq 0$.

[21] For proof of the Jacobi criterion, see for example [19] or [20].

[22] The definition of a derivative requires some care for fields where limits are not defined [20], but here we will only need to consider the case $K = \mathbb{C}$.

the polynomials is non-zero, i.e.

$$\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_m \neq 0.$$

As regards the nullcone condition, the *nullcone*, $N_V \subseteq V$, of an algebra $K[V]^G$ is defined to be the vanishing locus of all homogeneous invariant polynomials of strictly positive degree. That is,

$$N_V = \{v \in V | f(v) = 0,\ \forall f \in K[V]^G_+\}.$$

A set of polynomials, $\{f_1, \ldots, f_m\}$, is said to satisfy the nullcone condition if the vanishing locus of all of its constituent polynomials coincides with N_V. We remark that, in the case of the Lorentz- and permutation-invariant algebra $\mathbb{C}[V]^{O(d)\times P}$, the nullcone is the set $\{p_i \cdot p_j = 0, \forall\ i \leqslant j\}$. The fact that this does not depend on the choice of P will prove to be important when we come to construct a HSOP for arbitrary P.

4.2.1. A HSOP in $n = d + 1$ with $P = 1$. Let us warm up by considering the case without permutations. With $n = d + 1$, the SFT tells us that the relations between the dot products $p_i \cdot p_j$ are generated by the image of a single element under the Weyl map W, namely the determinant of the matrix whose ijth entry is y_{ij} for $i \leqslant j$ and y_{ji} for $i > j$. Thus, $W(\det(y_{ij})) = \det(p_i \cdot p_j) = 0$ where $\det(p_i \cdot p_j) \in \mathbb{C}[V]^{O(d)}$. This will be important for proving that our proposed HSOP satisfies the nullcone condition. We now make the following

Proposition 4.1. *A HSOP for the algebra $\mathbb{C}[V]^{O(d)}$, with $n = d + 1$, is given by the* $d(d + 3)/2$ *polynomials*

$$\begin{aligned} \theta_i &= p_1 \cdot p_1 + p_i \cdot p_i, \quad 2 \leqslant i \leqslant d + 1, \\ \alpha_{ij} &= p_i \cdot p_j, \quad 1 \leqslant i < j \leqslant d + 1. \end{aligned} \tag{7}$$

Proof. We first check that these polynomials satisfy the nullcone condition. Evidently, if all dot products vanish, then both θ_i and α_{ij} vanish. Proceeding in the other direction, suppose that θ_i and α_{ij} vanish. The vanishing of α_{ij} implies not only the vanishing of the dot products with $i < j$, but also implies, together with the vanishing determinant relation, that $\prod_{i=1}^{d+1}(p_i \cdot p_i) = 0$. So either $(p_1 \cdot p_1) = 0$ or $(p_k \cdot p_k) = 0$ for some $2 \leqslant k \leqslant d + 1$. If the former, then the fact that $\theta_i = 0$ implies $p_i \cdot p_i = 0$. If the latter, then $\theta_k = 0$ implies $p_1 \cdot p_1 = 0$, while the vanishing of the other θ_i implies the vanishing of all other $p_i \cdot p_i$ with $i \neq 1, k$. Either way, all dot products vanish and the nullcone condition is satisfied.

To prove algebraic independence, it is sufficient to show that the wedge product of the exterior derivatives of θ_i and α_{ij} is non-zero on at least a single point. We choose to evaluate the wedge product at the point

$$\begin{aligned} p_1 &= (0, 0, 0, \ldots, 0), \\ p_2 &= (1, 0, 0, \ldots, 0), \\ p_3 &= (0, 1, 0, \ldots, 0), \\ &\vdots \\ p_{d+1} &= (0, 0, \ldots, 0, 1), \end{aligned}$$

where the unit entry moves progressively along, as indicated. We claim that the component of the wedge product proportional to

$$\omega = \mathrm{d}p_1^1 \wedge \ldots \wedge \mathrm{d}p_1^d \wedge \mathrm{d}p_2^1 \wedge \ldots \wedge \mathrm{d}p_2^d \wedge \mathrm{d}p_3^2 \wedge \ldots \wedge \mathrm{d}p_3^d \wedge \mathrm{d}p_4^3 \wedge$$
$$\times \ldots \wedge \mathrm{d}p_4^d \wedge \ldots \wedge \mathrm{d}p_d^{d-1} \wedge \mathrm{d}p_d^d \wedge \mathrm{d}p_{d+1}^d,$$

has coefficient at this point given by $2^d \neq 0$ (up to an irrelevant minus sign) and so the Jacobi criterion is satisfied. To establish the claim in detail, one starts by showing that the only non-zero contribution to the wedge product is

$$\mathrm{d}(p_2 \cdot p_2) \wedge \ldots \wedge \mathrm{d}(p_{d+1} \cdot p_{d+1}) \wedge \mathrm{d}(p_1 \cdot p_2) \wedge \ldots \wedge \mathrm{d}(p_1 \cdot p_{d+1})$$
$$\wedge\, \mathrm{d}(p_2 \cdot p_3) \wedge \ldots \wedge \mathrm{d}(p_2 \cdot p_{d+1}) \wedge \mathrm{d}(p_3 \cdot p_4) \wedge \ldots \wedge \mathrm{d}(p_d \cdot p_{d+1}),$$

as contributions with more than one $\mathrm{d}(p_1 \cdot p_1)$ vanish trivially and contributions with a single $\mathrm{d}(p_1 \cdot p_1)$ vanish on the specified point as $\mathrm{d}(p_1 \cdot p_1) = 2\sum_i p_1^i \, \mathrm{d}p_1^i = 0$ there. Now, the coefficient of the component proportional to ω can be thought of as the determinant of an associated matrix[23]. In that form, after some row and column swaps (hence the irrelevant minus sign), one can show that the coefficient is the determinant of a diagonal matrix whose entries are all 1's except for d instances of 2's which come from the $\mathrm{d}(p_i \cdot p_i) = 2\sum_j p_i^j \, \mathrm{d}p_i^j = 2\, \mathrm{d}p_i^{i-1}$, for $2 \leqslant i \leqslant d + 1$. Hence, the coefficient of ω is 2^d as claimed.

Therefore, the proposed set of polynomials θ_i and α_{ij} satisfies the algebraic independence and nullcone conditions and so constitutes a valid HSOP. □

4.2.2. A HSOP in $n = d + 1$ with $P = S_n$. We now move on to the full-permutation case. Previously, in the $n = d + 1$ case with no permutations, the SFT indicated that the relations between the dot products are generated by a single element, $\det(p_i \cdot p_j) \in \mathbb{C}[V]^{O(d)}$, where $\det(p_i \cdot p_j) = 0$. Here, we consider the full permutations case and work in the permutation-invariant subalgebra $\mathbb{C}[V]^{O(d)\times S_n}$. But, since the determinant relation, $\det(p_i \cdot p_j)$, is permutation-invariant, it is also an element of the permutation-invariant subalgebra, $\det(p_i \cdot p_j) \in \mathbb{C}[V]^{O(d)\times S_n}$, and therefore can be safely used in our proof of the validity of the HSOP for $P = S_n$.

As to the HSOP itself, we take inspiration from Gauss who tells us that the m symmetric polynomials in m independent variables satisfy the necessary HSOP conditions. Therefore, the obvious candidates in our case are given by symmetric polynomials in the $d + 1$ variables $p_i \cdot p_i$ and $d(d + 1)/2$ variables $p_i \cdot p_j$ (with $i < j$), giving a total of $d(d + 3)/2 + 1$. But, these cannot satisfy the algebraic independence condition since the dot products are not independent variables. It is therefore reasonable to suppose that in order to fix this, we need to judiciously discard one symmetric polynomial from this set to obtain a valid HSOP. As we will see, taking the power sum polynomials and discarding the highest degree polynomial in $p_i \cdot p_i$ does the job. In fact, taking any set of symmetric polynomials (elementary[24] or complete homogeneous) and discarding the highest degree polynomial in $p_i \cdot p_i$ also does the job. This can be seen by using Newton's identities for the elementary symmetric polynomials or the equivalent relations

[23] Explicitly, it is the matrix with ijth entry being $\frac{\partial f_i}{\partial x_j}$ where $f_i \in \{p_r \cdot p_s | r \leqslant s, s \neq 1\}$ and $x_j \in \{p_1^1, \ldots, p_1^d, p_2^1, \ldots, p_2^d, p_3^2, \ldots, p_3^d, p_4^3, \ldots, p_4^d, \ldots, p_d^{d-1}, p_d^d, p_{d+1}^d\}$.

[24] The kth elementary symmetric polynomial, e_k, on the variables $x_1, \ldots, x_n$ is defined as

$$e_k(x_1, \ldots, x_n) = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n} x_{j_1} \ldots x_{j_k}.$$

for the complete homogeneous symmetric polynomials[25]. Indeed, we have the following

Proposition 4.2. *A HSOP for the algebra $\mathbb{C}[V]^{O(d)\times S_n}$, with $n = d + 1$, is given by the* $\mathrm{d}(d + 3)/2$ *permutation-invariant polynomials*

$$\begin{aligned} \theta_k &= \mathrm{Pow}_k(p_i \cdot p_i) := \sum_{i=1}^{d+1} (p_i \cdot p_i)^k, \quad 1 \leqslant k \leqslant d, \\ \alpha_k &= \mathrm{Pow}_k(p_i \cdot p_j) := \sum_{i<j}^{d+1} (p_i \cdot p_j)^k, \quad 1 \leqslant k \leqslant d(d+1)/2, \end{aligned} \tag{8}$$

where Pow_k *is the kth power symmetric polynomial.*

Proof. We first check that these polynomials satisfy the nullcone condition. Evidently, if all the dot products vanish, then both θ_k and α_k vanish. Proceeding in the other direction, suppose that θ_k and α_k vanish. Using Newton's identities, one can show that the vanishing of the first r power symmetric polynomials implies the vanishing of the first r elementary symmetric polynomials. Therefore, $\{\alpha_k = 0, \forall\, k\}$ implies the vanishing of all the elementary symmetric polynomials on the $p_i \cdot p_j, i < j$. Now, the vanishing of the highest degree elementary symmetric polynomial, $\prod_{i<j}^{n} p_i \cdot p_j = 0$, implies the vanishing of at least one $p_i \cdot p_j, i < j$. This then implies the vanishing of the $\mathrm{d}(d + 1)/2 - 1$ elementary symmetric polynomials on the remaining $\mathrm{d}(d + 1)/2 - 1$ dot products $p_i \cdot p_j, i < j$. Repeating this process recursively, one sees that the vanishing of the α_k implies the vanishing of all $p_i \cdot p_j, i < j$. This result, combined together with $\det(p_i \cdot p_j) = 0$, implies that $\prod_{i=1}^{d+1} (p_i \cdot p_i) = 0$. But $\prod_{i=1}^{d+1} (p_i \cdot p_i)$ is the elementary symmetric polynomial of highest degree, so $\prod_{i=1}^{d+1} (p_i \cdot p_i) = 0$, together with the vanishing of the θ_k, implies the vanishing of all $d + 1$ elementary symmetric polynomials in $p_i \cdot p_i$. From here, one can again recursively show that all $p_i \cdot p_i$ must vanish, so the nullcone condition is satisfied.

To prove algebraic independence, we evaluate (a component of) the wedge product of the exterior derivatives of θ_k and α_k at the point

$$\begin{aligned} p_1 &= (2, 0, \ldots, 0), \\ p_2 &= (3, 0, \ldots, 0), \\ p_3 &= (l_m, 1, 0, \ldots, 0), \\ p_4 &= (l_{m+1}, 0, 1, 0, \ldots, 0), \\ &\vdots \\ p_i &= (l_{m+i-3}, 0, \ldots, 0, 1, 0, \ldots, 0), \\ &\vdots \\ p_{d+1} &= (l_{m+d-2}, 0, \ldots, 0, 1), \end{aligned}$$

[25] Newton's identities relate the mth power sum symmetric polynomial, Pow_m, to the first m elementary symmetric polynomials, e_i, via:

$$\mathrm{Pow}_m = \sum_{\substack{r_1+2r_2+\cdots+mr_m=m,\\ r_1\geqslant 0,\ldots,r_m\geqslant 0}} (-1)^m \frac{m(r_1+\cdots+r_m-1)!}{r_1!r_2!\ldots r_m!} \prod_{i=1}^{m} (-e_i)^{r_i}.$$

Equivalent relations relating the power sum symmetric polynomials to the complete homogeneous symmetric polynomials also exist.

where l_i denotes the ith prime number (with $l_1 = 2$) and $m \geqslant 3$ and where the unit entry moves progressively along, as indicated. The prime numbers will prove useful soon when we require that the dot products $p_i \cdot p_j$ are all distinct. We claim that the component of the wedge product proportional to

$$\omega = \mathrm{d}p_1^1 \wedge \ldots \wedge \mathrm{d}p_1^d \wedge \mathrm{d}p_2^1 \wedge \ldots \wedge \mathrm{d}p_2^d \wedge \mathrm{d}p_3^2 \wedge \ldots \wedge \mathrm{d}p_3^d \wedge \mathrm{d}p_4^3 \wedge \ldots \\ \wedge \mathrm{d}p_4^d \wedge \ldots \wedge \mathrm{d}p_d^{d-1} \wedge \mathrm{d}p_d^d \wedge \mathrm{d}p_{d+1}^d,$$

has a non-zero coefficient. To establish this claim in detail, we first note that the wedge product can be re-expressed as

$$d!\left(\frac{\mathrm{d}(d+1)}{2}\right)!\det(M)\sum_{k=1}^{d+1}\det(L_k)\Omega_k,$$

where M is the Vandermonde matrix[26] on the $p_i \cdot p_j, i < j$, L_k is the Vandermonde matrix on the $p_i \cdot p_i, i \neq k$, and

$$\Omega_k = \mathrm{d}(p_1 \cdot p_1) \wedge \ldots \wedge \mathrm{d}\widehat{(p_k \cdot p_k)} \wedge \ldots \wedge \mathrm{d}(p_{d+1} \cdot p_{d+1}) \wedge \mathrm{d}(p_1 \cdot p_2) \\ \wedge \mathrm{d}(p_1 \cdot p_3) \wedge \ldots \wedge \mathrm{d}(p_d \cdot p_{d+1}),$$

where $\widehat{}$ over a term indicates that term should be omitted. By considering the coefficient of the component proportional to ω of Ω_k as the determinant of an associated matrix[27], one can show that the only contributions to the sum on the specified point come from the instances with $k = 1, 2$. Therefore, the coefficient of the component proportional to ω of the wedge product is

$$2^d d!\left(\frac{\mathrm{d}(d+1)}{2}\right)!\det(M)\left(9\det(L_1) - 4\det(L_2)\right),$$

up to an irrelevant overall minus sign (from row and column swaps). The $\det(M)$ term is non-zero as every dot product $p_i \cdot p_j$, $i < j$, is distinct (our use of prime numbers guarantees that $l_i l_j = l_m l_n$ if and only if either $l_i = l_n$ and $l_j = l_m$ or $l_i = l_m$ and $l_j = l_n$). To show the last term is non zero, we expand it as

$$\left(9\det(L_1) - 4\det(L_2)\right) = 9\prod_{i<j\neq 1}^{d+1}(p_i \cdot p_i - p_j \cdot p_j) - 4\prod_{i<j\neq 2}^{d+1}(p_i \cdot p_i - p_j \cdot p_j) \\ = \prod_{i<j\neq 1,2}^{d+1}(p_i \cdot p_i - p_j \cdot p_j)\left(9\prod_{i=3}^{d+1}(p_2 \cdot p_2 - p_i \cdot p_i) - 4\prod_{i=3}^{d+1}(p_1 \cdot p_1 - p_i \cdot p_i)\right).$$

Since we have the freedom to choose m to be as large as we want (there are infinitely many primes), we can see that this term is non-zero as follows: for large m, where $p_i \cdot p_i \gg p_1 \cdot p_1$, $p_2 \cdot p_2$, it tends to $\sim 5\prod_{i=3}^{d+1}(p_i \cdot p_i)\prod_{i<j\neq 1,2}^{d+1}(p_i \cdot p_i - p_j \cdot p_j)$ which is non-zero as the $p_i \cdot p_i$ are

[26] The Vandermonde matrix V on a set of variables x_i, $i \in \{1, \ldots, n\}$, is the $n \times n$ matrix with entries $V_{ij} = x_i^{j-1}$. The determinant of this matrix can be nicely expressed as $\det(V) = \prod_{1 \leqslant i < j \leqslant n}(x_i - x_j)$ and is non-zero only if all the x_i's are distinct.

[27] Explicitly, it is the matrix with ijth entry being $\frac{\partial f_i}{\partial x_j}$ where $f_i \in \{p_r \cdot p_s | r \leqslant s, s \neq k\}$ and $x_j \in \{p_1^1, \ldots, p_1^d, p_2^1, \ldots, p_2^d, p_3^2, \ldots, p_3^d, p_4^3, \ldots, p_4^d, \ldots, p_d^{d-1}, p_d^d, p_{d+1}^d\}$.

non-zero and distinct (again by our use of the prime numbers). Therefore, the Jacobi criterion is satisfied.

Hence, these polynomials are algebraically independent and satisfy the nullcone condition and so constitute a valid HSOP. □

As we have already remarked, the nullcone of $\mathbb{C}[V]^{O(d)\times P}$, being given by the vanishing locus of the dot products $p_i \cdot p_j$, is independent of the choice of P. Moreover, an algebraically independent set of $O(d,\mathbb{C}) \times S_n$-invariant polynomials is also an algebraically independent set of $O(d,\mathbb{C}) \times P$-invariant polynomials, for any $P \subset S_n$. We thus have the important

Corollary 4.2.1. *A HSOP for the algebra $\mathbb{C}[V]^{O(d)\times P}$, with $n = d+1$, is given by the permutation-invariant polynomials in equation (8), for any $P \subset S_n$.*

As we shall see, this gives us a starting point for finding a Hironaka decomposition for any P in the case $n = d+1$.

4.2.3. A remark on HSOPs for $n \geqslant d+2$. It would obviously be desirable to generalise our methods to cases with $n \geqslant d+2$. The first obstacle in doing so is that the relations between the dot products $p_i \cdot p_j$ given by the higher minors of the matrix whose entries are $p_i \cdot p_j$, are not S_n-invariant. Thus, they do not belong to $\mathbb{C}[V]^{O(d)\times S_n}$ and cannot be used directly in the proofs. To overcome this, one presumably needs to first find a set of invariant polynomials which generate the relations and then work with these. But it is not clear to us what form a HSOP might take.

4.3. Secondaries

Now that we can write down HSOPs of our invariant algebras at will in cases with $n \leqslant d+1$, the corresponding secondaries may be computed via the algorithm sketch-out in subsection 3.3 (which can be found in [5]). Here, we illustrate the algorithm by applying it to a simple example, namely $(n,d) = (3,2)$ with no permutation symmetry, i.e. the algebra $\mathbb{C}[V]^{O(2)}$, with $V \cong \mathbb{C}^6$.

The algorithm is based on the following two observations. Firstly, the number of secondaries required can be read off (along with their degrees) from the Hilbert series, which itself can be computed using standard methods from invariant theory (as we review in appendix B). Indeed, given a Hironaka decomposition of an invariant algebra $K[V]^G = \bigoplus_i \eta_i K[\theta_j]$, its Hilbert series $H(K[V]^G, t)$, takes the form $\frac{1+\sum_{k=1} S_k t^k}{\prod_{l=1}(1-t^l)^{P_l}}$ where S_k is the number of secondary invariants η_i of degree k and P_l is the number of primary invariants (HSOP) θ_j of degree l. Therefore, given a HSOP, which fixes the P_l, one can read off the number and degrees of the secondaries from the numerator of the Hilbert series. Secondly, given a set of polynomial invariants $\{\eta_1, \ldots, \eta_m\}$ of the right cardinality, the set forms the secondaries of the invariant algebra if and only if its constituent polynomials are linearly independent modulo the ideal $I := \langle \theta_1, \ldots, \theta_r \rangle \in K[V]$ generated by the HSOP $\{\theta_i\}$. To show linear independence of a set of polynomials modulo an ideal, one can compute the remainders of the polynomials upon division by a Groebner basis of that ideal and check that the remainders are themselves linearly independent [5].

Turning to our example, the methods described in appendix B show that the Hilbert series is given by

$$H(\mathbb{C}[V]^{O(2)}, t) = \frac{1+t^2+t^4}{(1-t^2)^5}.$$

Here we have written the series in a form such that the denominator reproduces the five primaries of degree 2 corresponding to the HSOP given in equation (7), namely

$$\{(p\cdot p)+(q\cdot q),(p\cdot p)+(r\cdot r),(p\cdot q),(p\cdot r),(q\cdot r)\},$$

where we have labelled the momenta by p, q, and r (we denote the corresponding components of V by $\{p_1, p_2, q_1, q_2, r_1, r_2\}$). We thus read off from the numerator that there is 1 secondary of degree 2 and 1 secondary of degree 4 (and of course the trivial secondary, 1, of degree 0).

The next step in the algorithm is to compute a Groebner basis of the ideal generated by the HSOP, which will later be used to verify the linear independence of the secondaries. To do so, one must first choose a monomial ordering[28]. A common (and often very efficient) choice is graded reverse lexicographic order[29]. In this ordering, a Groebner basis of the ideal generated by our HSOP is given by the set of 20 polynomials

$$\begin{aligned}\{&q_1r_1+q_2r_2, p_1r_1+p_2r_2, q_1^2+q_2^2-r_1^2-r_2^2, p_1q_1+p_2q_2, p_1^2+p_2^2+r_1^2\\&+r_2^2, p_2q_1r_2-p_1q_2r_2, q_2^2r_1-q_1q_2r_2-r_1^3-r_2^2r_1, p_2q_2r_1-p_1q_2r_2, p_2^2r_1\\&-p_1p_2r_2+r_1^3+r_2^2r_1, -p_1q_2^2+p_2q_1q_2+p_1r_2^2-p_2r_1r_2, p_2^2q_1-p_1p_2q_2\\&+q_1r_2^2-q_2r_1r_2, q_2r_1r_2^2-q_1r_2^3, p_2r_1r_2^2-p_1r_2^3, r_2r_1^3+r_2^3r_1, q_2r_2^3\\&+q_2r_1^2r_2, p_2r_2^3+p_2r_1^2r_2, r_1^4-r_2^4, q_2r_1^3+q_1r_2^3, p_2r_1^3+p_1r_2^3, r_2^5+r_1^2r_2^3\}\,.\end{aligned}$$

We then proceed to generate a basis of homogeneous invariant polynomials in the algebra of degree d_i, corresponding to the degrees of secondaries read off of the Hilbert series, using linear algebra methods. If G were a finite group, this would be a simple matter of averaging all possible monomials of degree d_i over G to obtain a basis of invariant polynomials at that degree[30]. But for us G is infinite, so things are not so straightforward. We use the additional information that $\mathbb{C}[V]^{O(d)\times P}\subset\mathbb{C}[V]^{O(d)}$ and that by the FFT, $\mathbb{C}[V]^{O(d)}$ is generated by the set of dot products in the momenta. This allows one to obtain a basis of homogeneous polynomials in $\mathbb{C}[V]^{O(d)\times P}$ of degree d_i by averaging all possible products of $d_i/2$ dot products over the (finite) permutation group P.

From this basis, we consecutively choose elements and compute their remainders upon division by the Groebner basis (also called the normal forms) and keep them only if their remainders are non-zero and lie outside the $\mathbb{C}$-vector space generated by the remainders of previously found secondaries (i.e. the remainders are linearly independent). Once the required number of secondaries is obtained, one proceeds to the next degree and so on until all the secondaries have been found.

In our case (skipping over the trivial case of the secondary 1), we start at degree 2. Here, the basis of polynomials is just the set of dot products. Choosing $p\cdot p$, we compute the remainder upon division to be $-(r_1^2+r_2^2)$, which is non-zero and so we have the required secondary of degree 2. We then move on to degree 4. Here, the basis of polynomials is all possible products of two dot products. We choose $(p\cdot p)^2$ and compute the remainder upon division to be $2r_2^2(r_1^2+r_2^2)$, which is non-zero and is obviously linearly independent from the remainder of

[28] Readers seeking a gentle introduction to Groebner basis methods may wish to consult [15].

[29] Graded reverse lexicographic order, or grevlex for short, is a monomial ordering on some variables $x_1,\ldots,x_n$ where for any two monomials $t=x_1^{a_1}\ldots x_n^{a_n}$ and $t'=x_1^{a'_1}\ldots x_n^{a'_n}$, $t>_{\text{grevlex}}t'$ if $\deg(t)>\deg(t')$ or if $\deg(t)=\deg(t')$ and $a_i<a'_i$ for the largest i with $a_i\neq a'_i$.

[30] The average of a polynomial f over a finite group G is $\frac{1}{|G|}\sum_{g\in G}g\circ f$.

the previous secondary, since it does not have the same degree. We therefore have our required secondary of degree 4[31]. Finally, we obtain the Hironaka decomposition of the algebra as follows

$$\mathbb{C}[V]^{O(2)} = \left(1 \bigoplus (p \cdot p) \bigoplus (p \cdot p)^2\right) \cdot \mathbb{C}[p \cdot p + q \cdot q, p \cdot p + r \cdot r, p \cdot q, p \cdot r, q \cdot r]. \tag{9}$$

Simple though it is, our example already hints at the two bottlenecks that arise when computing the secondaries of $\mathbb{C}[V]^{O(d)\times P}$ in high dimensions with large permutation symmetry. One is the computation of the Groebner basis of the ideal and the other is the computation of a basis of invariant polynomials of a certain degree, which becomes progressively more costly at higher degrees. There are multiple tricks which can be used to mitigate the latter bottleneck [5] (e.g., using products of lower degree secondaries as candidates), but there is still no really effective way of tackling the inefficiency of the Groebner basis computations.

In subsection 4.5, we employ the algorithm to provide Hironaka decompositions for computationally tractable cases. A version of this algorithm is implemented in `Macaulay2` [16], amongst others.

4.4. Parity

A significant advantage of characterising the Lorentz- and permutation-invariant algebra directly via Hironaka decompositions is that the description can be readily extended to the case where parity is not a symmetry. The reason for this is that the algebras with and without parity as a symmetry, i.e. $\mathbb{C}[V]^{O(d)\times P}$ and $\mathbb{C}[V]^{SO(d)\times P}$ respectively, have the same nullcone. This important fact can be traced back to existence of relations between the d-dimensional epsilon tensor contracted with the d momenta and the dot products (the square of a d-dimensional epsilon tensor is equal to the corresponding $d \times d$ subdeterminant of the matrix of dot products). Using these relations, one can readily show that the vanishing set of all the dot products then immediately implies the vanishing set of the epsilon tensors and consequently that the nullcones of the two algebras coincide.

Therefore, the HSOPs found in subsection 4.2, which have been shown to satisfy the nullcone condition for $\mathbb{C}[V]^{O(d)\times P}$, also satisfy the nullcone condition for $\mathbb{C}[V]^{SO(d)\times P}$. Hence, we arrive at the following

Corollary 4.2.2. *A HSOP for the parity-non-invariant algebra* $\mathbb{C}[V]^{SO(d)\times P}$ *is given by the HSOPs for the parity-invariant algebra* $\mathbb{C}[V]^{O(d)\times P}$ *given by equations* (7) *and* (8).

The only extra work needed to find the complete Hironaka decomposition for the algebras $\mathbb{C}[V]^{SO(d)\times P}$ is in computing the secondaries. One now additionally needs to consider the epsilon tensors contracted with the momenta when constructing the basis of monomials and care must be taken to keep track of minus signs that appear as a result of the antisymmetric structure of the epsilon tensors when symmetrising under the permutation group.

4.5. Examples of Hironaka decompositions in $n = d + 1$

We now present two examples with explicit Hironaka decompositions using the above prescriptions.

[31] We could have equally chosen either $(q \cdot q)$ or $(r \cdot r)$ for the degree 2 secondary and either $(q \cdot q)^2$ or $(r \cdot r)^2$ for the degree 4 one. It is also interesting to note that the remainders upon division by the Groebner basis of $(p \cdot p)^3$, $(q \cdot q)^3$, and $(r \cdot r)^3$ are zero and so they lie *in* the ideal generated by the HSOP.

4.5.1. The case of $(n, d) = (5, 4)$ *with* $P = 1$. For the no permutation case with $(n, d) = (5, 4)$, we start by finding the HSOP for the algebra $\mathbb{C}[V]^{O(4)}$ in the way described in subsection 4.2.1. This results in the following set of polynomials

$$\theta_i = p_1 \cdot p_1 + p_i \cdot p_i, \quad 2 \leqslant i \leqslant 5,$$
$$\alpha_{ij} = p_i \cdot p_j, \quad 1 \leqslant i < j \leqslant 5.$$

Using the algorithm described in subsection 4.3, we proceed to find the secondaries in a similar manner. The Hilbert series of the algebra, computed using methods described in appendix B, is

$$H(\mathbb{C}[V]^{O(4)}, t) = \frac{1 + t^2 + t^4 + t^6 + t^8}{(1 - t^2)^{14}}.$$

We are therefore looking for 1 secondary at each of the degrees 0, 2, 4, 6, and 8. We find that the following set of polynomials

$$1, (p_1 \cdot p_1), (p_1 \cdot p_1)^2, (p_1 \cdot p_1)^3, (p_1 \cdot p_1)^4,$$

have remainders upon division by the Groebner basis of the ideal generated by the HSOP which are non-zero and linearly independent. Therefore, we obtain a Hironaka decomposition of the algebra as follows

$$\mathbb{C}[V]^{O(4)} = \left(1 \bigoplus (p_1 \cdot p_1) \bigoplus (p_1 \cdot p_1)^2 \bigoplus (p_1 \cdot p_1)^3 \bigoplus (p_1 \cdot p_1)^4\right) \cdot \mathbb{C}[\{\theta_i, \alpha_{ij}\}].$$

The forms of the Hironaka decompositions for $d = 2$ and 4 given here and in 9 invite an obvious conjecture for their form in arbitrary dimension d. Namely, the secondaries are given by the dot product of any one momenta with itself, raised to the zeroth all the way to the dth powers. An explicit computation shows this to be the case also in $d = 1$ and 3. Therefore, we are led to the following

Conjecture 4.1. The Hironaka decomposition of Lorentz-invariant algebras, $\mathbb{C}[V]^{O(d)}$, in the case of $n = d + 1$, is given by

$$\mathbb{C}[V]^{O(d)} = \bigoplus_{m=0}^{d} (p_1 \cdot p_1)^m \; \mathbb{C}[\{\theta_i, \alpha_{ij}\}],$$

where the HSOP $\{\theta_i, \alpha_{ij}\}$ are as given by equation (7).

4.5.2. The case of $(n, d) = (3, 2)$ *with* $P = S_3$. For the full permutation case with $(n, d) = (3, 2)$, we find the HSOP for the algebra $\mathbb{C}[V]^{O(2) \times S_3}$ in the way described in subsection 4.2.2. This results in the following set

$$\theta_k = \mathrm{Pow}_k(p_i \cdot p_i) = \sum_{i=1}^{3} (p_i \cdot p_i)^k, \quad 1 \leqslant k \leqslant 2,$$

$$\alpha_k = \mathrm{Pow}_k(p_i \cdot p_j) = \sum_{i<j}^{3} (p_i \cdot p_j)^k, \quad 1 \leqslant k \leqslant 3.$$

Using the algorithm described in subsection 4.3, we proceed to find the secondaries in a similar manner. The Hilbert series of the algebra, computed using methods described in appendix B, is

$$H(\mathbb{C}[V]^{O(2)\times S_3}, t) = \frac{1 + t^4 + 2t^6 + t^8 + t^{12}}{\left(1 - t^2\right)^2\left(1 - t^4\right)^2\left(1 - t^6\right)}.$$

We are therefore looking for 1 secondary at degree 0, 1 at degree 4, 2 at degree 6, 1 at degree 8, and 1 at degree 12. We find that the following set of polynomials

$$\begin{aligned}
\eta_1 &= 1,\\
\eta_2 &= (p_1\cdot p_1)(p_2\cdot p_3) + (p_2\cdot p_2)(p_1\cdot p_3) + (p_3\cdot p_3)(p_1\cdot p_2),\\
\eta_3 &= (p_1\cdot p_1)^2(p_2\cdot p_3) + (p_2\cdot p_2)^2(p_1\cdot p_3) + (p_3\cdot p_3)^2(p_1\cdot p_2),\\
\eta_4 &= (p_1\cdot p_1)(p_2\cdot p_3)^2 + (p_2\cdot p_2)(p_1\cdot p_3)^2 + (p_3\cdot p_3)(p_1\cdot p_2)^2,\\
\eta_5 &= (p_1\cdot p_1)^2(p_2\cdot p_3)^2 + (p_2\cdot p_2)^2(p_1\cdot p_3)^2 + (p_3\cdot p_3)^2(p_1\cdot p_2)^2,\\
\eta_6 &= (p_1\cdot p_1)^5(p_2\cdot p_3) + (p_2\cdot p_2)^5(p_1\cdot p_3) + (p_3\cdot p_3)^5(p_1\cdot p_2),
\end{aligned}$$

have remainders upon division by the Groebner basis of the ideal generated by the HSOP which are non-zero and linearly independent. Therefore, we obtain a Hironaka decomposition of the algebra as follows

$$\mathbb{C}[V]^{O(2)\times S_3} = \bigoplus_{i=1}^{6} \eta_i \mathbb{C}[\{\theta_k, \alpha_k\}].$$

5. Discussion

In this work, we have updated the results of Weyl using two different approaches. In the first, we developed a systematic method which produces sets of minimal algebra generators for the Lorentz- and permutation-invariant polynomials using tools of invariant theory. Our method results in manageable sets of generators for phenomenologically-relevant examples, at least when the number of particles is sufficiently small, and we hope that the results will prove to be useful in future phenomenological analyses. This approach has some shortcomings though. One is that it is computationally intractable to apply to the case where parity is not a symmetry. We address this in another work [18]. Another problem is that our generators are not able to fully separate the orbits[32], which is certainly a useful thing to do from a physicist's point of view (for example in searching for parity violating LHC signals, as explored in [21]).

In the second approach, we have addressed the problem of redundancies in the description of the Lorentz- and permutation-invariant algebras via generating sets. Instead of providing a set of generators (FFT) and the relations between them (SFT), we observed that one may provide (via the theorem of Hochster and Roberts) a more direct characterization in terms of a Hironaka decomposition, that is as a free, finitely-generated module over a polynomial subalgebra. This approach has the added advantage that it readily generalises to the case where parity is not a symmetry. In cases where $n \leqslant d + 1$, we gave an explicit solution (for an arbitrary permutation symmetry) to the 'hard' part of finding such a decomposition, namely the identification of a HSOPs. The 'easy' part of finding a decomposition, namely the identification of

[32] To give a somewhat trivial example, the invariant $p \cdot p$ is unable to separate the orbits with $p \cdot p = 0$ and with either $p = 0$ or $p \neq 0$.

suitable secondary generators, reduces to a linear algebra algorithm, but is nonetheless inefficient. We provided Hironaka decompositions in the examples of $(n, d) = (5, 4)$ with $P = 1$ and $(n, d) = (3, 2)$ with $P = S_3$ and a conjecture in the general case of $n = d + 1$ with no permutations. As is the case with minimal algebra generators, Hironaka decompositions also fail to fully separate the orbits[33].

These two approaches attempt to reformulate our description of Lorentz- and permutation-invariant polynomials in particle momenta. The second description, i.e. via Hironaka decompositions, although initially more mathematically involved, reduces to a straightforward algorithm and provides a completely redundancy-free description of these polynomials which is directly generalisable to when parity is not a symmetry. Therefore, it seems like it should be the preferred method of choice. But, as discussed previously, the resulting set of secondary generators can become very large very quickly in cases with a large number of particles and high permutation symmetry. In such cases, a description via a set of minimal algebra generators (which can be significantly smaller) might be preferred either in theoretical work or for efficient computational use. It is difficult to comment on the performance of either method computationally without further investigation. In future work, we hope to implement and test the efficiency of both methods in a neural network setting which would provide us with more concrete empirical knowledge of their performance.

Acknowledgments

We thank Scott Melville and other members of the Cambridge Pheno Working Group for helpful advice and comments. This work has been partially supported by STFC consolidated Grants ST/P000681/1 and ST/S505316/1. WH is supported by the Cambridge Trust. BG thanks King's College, Cambridge, and the University of Canterbury, New Zealand, where part of this work was carried out.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Definitions from commutative algebra

Here we recall some relevant definitions (of terms in italics) and results from commutative algebra (see, e.g. [22, 23], for more details). The most important concepts are those of a *ring* and an *algebra*, and the corresponding structure-preserving *maps* between them.

A *ring* R (which for our purposes will always be a commutative ring with unit) is an Abelian group (with addition $+$, identity 0, and element $r \in R$ having inverse $-r$) that is also a commutative monoid (with multiplication $\cdot$, which we often omit, and identity 1), such that $\cdot$ is distributive over $+$. An example is the ring $\mathbb{Z}$ of integers.

A *ring map* $f : R \to S$ (which we sometimes write less explicitly as $R \to S$) is a map that preserves sums, products, and 1. A *ring isomorphism* $R \xrightarrow{\sim} S$ is a bijective ring map.

An R-*algebra* (or *algebra* for short) is a ring S equipped with a ring map $f : R \to S$. An example is the polynomials in one variable $R[x]$ over a ring R (where the ring map is $r \mapsto rx^0$).

[33] For a slightly non-trivial example, consider the case of $n = 3, d = 2$ with no permutation symmetry given above. Here, the Hironaka decomposition fails to separate the orbits $\{p = (p, p), q = (q, q), r = (r, r)\}$ and $\{p = (0, 0), q = (0, 0), r = (0, 0)\}$, on which all the primaries and secondaries (except the trivial secondary 1) vanish.

Given R-algebras S and T with structure maps f, g (respectively), an *R-algebra map* is a ring map $h : S \to T$ such that $h \circ f = g$.

Given an R-algebra S, the subalgebra $R[\{s_\lambda | \lambda \in \Lambda\}]$ *generated* by $s_\lambda \in S$ is the smallest R-subalgebra that contains them. It consists of all polynomial combinations of the s_λ with coefficients in R. If there exist $s_1, \ldots, s_n \in S$ such that $S = R[s_1, \ldots, s_n]$, we say that S is *finitely-generated* (as an R-algebra).

The *kernel* $\ker f$ of a ring map f is $f^{-1}(0)$. An *ideal* $I \subset R$ is the kernel of a ring map. Equivalently, an ideal contains 0 and is such that given $a, b \in I$ and $r \in R$, $a + b \in I$ and $ar \in I$ (indeed, this is the kernel of the map $R \to R/I$ that sends r to the equivalence class $r + I$, the set of which forms the *quotient ring* R/I). The *first isomorphism theorem* states that $R/\ker f \xrightarrow{\sim} \mathrm{im}\, f$.

The ideal $\langle r_\lambda | \forall\, \lambda \in \Lambda \rangle$ generated by $r_\lambda \in R$ for some set Λ, is the smallest ideal in R that contains the r_λ. A *field* is a ring in which $\langle 0 \rangle$ is a *maximal ideal*, that is, is not contained in any proper ideal. Equivalently, $1 \neq 0$ and every non-zero element is a *unit*, that is has a multiplicative inverse.

An *R-module* (or just *module*) M is an abelian group (written additively) together with a *scalar multiplication* $R \times M \to M : (r, m) \mapsto rm$ that is distributive over the addition in both R and M, is associative, and is such that $1m \mapsto m$. An ideal in R and an R-algebra are both examples of R-modules.

We say that a subset $\{m_\lambda | \lambda \in \Lambda\} \subset M$ *generates* M (as a module) if M is the smallest submodule of M that contains $\{m_\lambda\}$. We say that M is *finitely-generated* if there exists a finite set of generators. We say that the m_λ are *free* if $\sum_\lambda r_\lambda m_\lambda = 0 \implies r_\lambda = 0$, for all λ and that they are a *basis* if they also generate M. A *free module* is one that has a basis.

A ring R is *graded* if we can write it as a direct sum $R = \bigoplus_{n \in \mathbb{N}} R_n$ of subgroups R_n (in fact R_0 is always a subring) such that $R_n R_m \subset R_{n+m}$. A *homogeneous element (of degree n)* is an element belonging to some factor (or specifically to the factor R_n). An algebra is graded if it is graded as a ring.

Given a graded algebra R over a field K with $R_0 = K$, a *HSOPs* is a set of homogeneous elements $\theta_1, \ldots, \theta_m \in R$ which are algebraically independent and are such that R is a finitely-generated module over $K[\theta_1, \ldots, \theta_m]$.

For a finitely-generated graded K-algebra $R = \bigoplus_{i=0}^{\infty} R_i$, we define the Hilbert series $H(R, t)$ as the formal power series

$$H(R, t) = \sum_{i=0}^{\infty} \dim(R_i) t^i$$

where $\dim(R_i)$ is the dimension of the (homogeneous) vector space R_i.

Appendix B. Hilbert series of $\mathbb{C}[V]^{O(d) \times P}$

In this appendix, we describe how to compute Hilbert series of invariant algebras under the combined (complexified) Lorentz and permutation groups in dimension $2 \leqslant d \leqslant 4$.

To do so, we use a generalisation of Molien's formula valid for a reductive group G, whereby the Hilbert series of an invariant algebra $\mathbb{C}[V]^G$ is given by [5]

$$H(\mathbb{C}[V]^G, t) = \int_C \frac{\mathrm{d}\mu}{\det_V(1 - t \cdot \rho_V)},$$

where C is a maximal compact subgroup of G, $\mathrm{d}\mu$ is a Haar measure on C normalised such that $\int_C \mathrm{d}\mu = 1$, and $\rho_V : C \to GL(V)$ denotes the representation of C carried by V. For what follows, it is useful to note that the integrand is constant within a conjugacy class of G.

We now consider in turn the cases of $d = 2, 3$, and 4, with an arbitrary number of momenta n and an arbitrary permutation group, $P \subset S_n$ acting on those momenta. The complexification of the Lorentz group when parity is a symmetry means that the groups we consider are of the form $G = O(d, \mathbb{C}) \times P$. For completeness, we also discuss the case where parity is not a symmetry, i.e. when $G = SO(d, \mathbb{C}) \times P$.

B.1. The case of $O(2, \mathbb{C}) \times P$

We start by considering the invariant algebra $\mathbb{C}[V]^G$ in the case of n momenta in two dimensions with no permutation symmetry which corresponds to $G = O(2, \mathbb{C})$ and $V \cong \mathbb{C}^{2n}$. The group $O(2, \mathbb{C})$ has maximal compact subgroup $O(2, \mathbb{R}) \cong U(1) \rtimes \mathbb{Z}_2$ and its action on $\mathbb{C}^2$ may be written as[34]

$$M^+(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \qquad M^-(z) = \begin{pmatrix} 0 & z^{-1} \\ z & 0 \end{pmatrix},$$

where $z \in \mathbb{C}$ such that $|z| = 1$ and where M^+ corresponds to the component connected to the identity and M^- corresponds to the other connected component. When acting on n copies of $\mathbb{C}^2$ (corresponding to n particles), we have

$$M_V^\pm = \begin{pmatrix} M^\pm & & & \\ & M^\pm & & \\ & & \ddots & \\ & & & M^\pm \end{pmatrix}.$$

The normalised Haar measure is given by $\frac{1}{2}\frac{1}{2\pi \mathrm{i}}\frac{\mathrm{d}z}{z}$ on each component (which is half the Haar measure for the group $U(1)$ and so takes into account the two disconnected components). The Hilbert series is thus given by

$$H(\mathbb{C}[V]^{O(2)}, t) = \frac{1}{2}\frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{\mathrm{d}z}{z} \left(\frac{1}{\det_V(1 - t \cdot M_V^+)} + \frac{1}{\det_V(1 - t \cdot M_V^-)} \right).$$

For our example of $n = 3$ with $P = 1$, the integral becomes

$$\begin{aligned} H(\mathbb{C}[V]^{O(2)}, t) &= \frac{1}{2}\frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{\mathrm{d}z}{z} \left(\frac{1}{(1 - tz)^3(1 - t/z)^3} + \frac{1}{(1 - t^2)^3} \right) \\ &= \frac{1}{2}\left(\frac{1 + 4t^2 + t^4}{(1 - t^2)^5} + \frac{1}{(1 - t^2)^3} \right) = \frac{1 + t^2 + t^4}{(1 - t^2)^5}. \end{aligned}$$

where the integrals have been carried out using the residue theorem of contour integration.

We now include some permutation group $P \subseteq S_n$ acting on the n momenta so that the combined group becomes $G = O(2, \mathbb{C}) \times P$ and its maximal compact subgroup is just $O(2, \mathbb{R}) \times P$. Here, one must additionally average over the permutation group P, where the action of P simply permutes the n particles, ergo the n copies of $\mathbb{C}^2$. Since the integrand is constant within conjugacy classes, it suffices to pick one representative element from each class, and weight accordingly. The Haar measure is rescaled by $1/|P|$ so that it is still properly normalised.

[34] If we consider $O(2, \mathbb{R}) \subset O(2, \mathbb{C})$ as acting on the real components of the momenta, then the isomorphism $O(2, \mathbb{R}) \cong U(1) \rtimes \mathbb{Z}_2$ corresponds to the linear map $(p_0, p_1) \in \mathbb{C}^2 \mapsto (p_0 + \mathrm{i}p_1, p_0 - \mathrm{i}p_1)$.

For our example of $n = 3$ with $P = S_3$, we have three conjugacy classes: the identity with multiplicity 1, $(\cdot\cdot)$ with multiplicity 3, and $(\cdot\cdot\cdot)$ with multiplicity 2. We use the following representative elements from each permutation conjugacy class

$$\begin{pmatrix} M^{\pm} & 0 & 0 \\ 0 & M^{\pm} & 0 \\ 0 & 0 & M^{\pm} \end{pmatrix}, \qquad \begin{pmatrix} 0 & M^{\pm} & 0 \\ M^{\pm} & 0 & 0 \\ 0 & 0 & M^{\pm} \end{pmatrix}, \qquad \begin{pmatrix} 0 & M^{\pm} & 0 \\ 0 & 0 & M^{\pm} \\ M^{\pm} & 0 & 0 \end{pmatrix}.$$

The contribution of the component connected to the identity then becomes

$$\begin{aligned} H^{+}(\mathbb{C}[V]^{O(2)\times S_3}, t) &= \frac{1}{6}\frac{1}{2}\frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{\mathrm{d}z}{z} \left(\frac{1}{(1-tz)^3(1-t/z)^3} \right. \\ &\quad + \frac{3}{(1-tz)(1-t/z)(1-(tz)^2)(1-(t/z)^2)} \\ &\quad \left. + \frac{2}{(1-(tz)^3)(1-(t/z)^3)} \right) \\ &= \frac{1}{2}\frac{1+3t^4+4t^6+3t^8+t^{12}}{(1-t^2)^2(1-t^4)^2(1-t^6)}. \end{aligned}$$

Similarly, the contribution of the other connected component is

$$H^{-}(\mathbb{C}[V]^{O(2)\times S_3}, t) = \frac{1}{2}\frac{1+t^4}{(1-t^2)^2(1-t^6)},$$

and so finally we obtain

$$\begin{aligned} H(\mathbb{C}[V]^{O(2)\times S_3}, t) &= H^{+}(\mathbb{C}[V]^{O(2)\times S_3}, t) + H^{-}(\mathbb{C}[V]^{O(2)\times S_3}, t) \\ &= \frac{1+t^4+2t^6+t^8+t^{12}}{(1-t^2)^2(1-t^4)^2(1-t^6)}. \end{aligned}$$

Notice that we also get the Hilbert series for the case $G = SO(2,\mathbb{C}) \times P$, corresponding to when parity is not a symmetry, for free, by just considering the component connected to the identity

$$\begin{aligned} H(\mathbb{C}[V]^{SO(2)}, t) &= \frac{1+4t^2+t^4}{(1-t^2)^5}, \\ H(\mathbb{C}[V]^{SO(2)\times S_3}, t) &= \frac{1+3t^4+4t^6+3t^8+t^{12}}{(1-t^2)^2(1-t^4)^2(1-t^6)}. \end{aligned}$$

B.2. The case of $O(3,\mathbb{C}) \times P$

In $d = 3$, the group $O(3,\mathbb{C})$ has maximal compact subgroup $O(3,\mathbb{R}) \cong (SU(2)/\mathbb{Z}_2) \times \mathbb{Z}_2$. Since the integrand is constant on the conjugacy classes, we need consider only the maximal torus of $SU(2)$ with elements

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix},$$

where $|z| = 1$ [35] acting on $\mathbb{C}^3$ as[36]

$$M^{+}(z) = \begin{pmatrix} z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-2} \end{pmatrix}, \qquad M^{-}(z) = \begin{pmatrix} -z^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -z^{-2} \end{pmatrix},$$

where $\pm$ again distinguishes the two connected components. The normalised Haar measure on each component is $\frac{1}{2}\frac{1}{2\pi \mathrm{i}}\frac{(1-z^2)\mathrm{d}z}{z}$ (which is just half of the usual normalised Haar measure for $SU(2)$). The Hilbert series with n particles is then given by

$$H(\mathbb{C}[V]^{O(3)}, t) = \frac{1}{2}\frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\frac{(1-z^2)\mathrm{d}z}{z} \times \left(\frac{1}{\det_V(1 - t\cdot M_V^{+})} + \frac{1}{\det_V(1 - t\cdot M_V^{-})}\right).$$

For example, with $n = 4$ the integral becomes

$$H(\mathbb{C}[V]^{O(3)}, t) = \frac{1}{2}\frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\frac{(1-z^2)\mathrm{d}z}{z}\left(\frac{1}{(1-t)^4\left(1-\frac{t}{z^2}\right)^4\left(1-tz^2\right)^4} + \frac{1}{(1+t)^4\left(1+\frac{t}{z^2}\right)^4\left(1+tz^2\right)^4}\right),$$

which we evaluate using the residue theorem, obtaining

$$\begin{aligned} H(\mathbb{C}[V]^{O(3)}, t) &= \frac{1}{2}\left(\frac{1+t^2+4t^3+t^4+t^6}{\left(1-t^2\right)^9} + \frac{1+t^2-4t^3+t^4+t^6}{\left(1-t^2\right)^9}\right) \\ &= \frac{1+t^2+t^4+t^6}{\left(1-t^2\right)^9}. \end{aligned}$$

We also obtain the Hilbert series for when $G = SO(3,\mathbb{C})$ for free by only considering the component connected to the identity

$$H(\mathbb{C}[V]^{SO(3)}, t) = \frac{1+t^2+4t^3+t^4+t^6}{\left(1-t^2\right)^9}.$$

To include an arbitrary permutation group $P \subseteq S_n$ acting on the n momenta, one needs to average over the conjugacy classes of P as discussed previously.

[35] Strictly speaking, one should consider only half of the unit circle, since z and $-z$ yield the same element in $SU(2)/\mathbb{Z}_2$. But since the integral will turn out to be symmetric under $z \to -z$, we can get away with integrating over the whole circle.

[36] Here, the isomorphism $O(3,\mathbb{R}) \cong (SU(2)/\mathbb{Z}_2)\times\mathbb{Z}_2$ corresponds to the linear map $(p_0, p_1, p_2) \in \mathbb{C}^3 \mapsto (p_0 - \mathrm{i}p_1, p_2, p_0 + \mathrm{i}p_1)$.

B.3. The case of $O(4,\mathbb{C})\times P$

In $d=4$, $O(4,\mathbb{C})$ has maximal compact subgroup $O(4,\mathbb{R})\cong\left((SU(2)\times SU(2))/\mathbb{Z}_2\right)\rtimes\mathbb{Z}_2$, where the automorphism in the semi-direct product corresponds to interchanging the $2SU(2)$ factors. Since the integrand is constant on the conjugacy classes, we need consider only the maximal torus with elements

$$\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}, \qquad \begin{pmatrix} w & 0 \\ 0 & w^{-1}\end{pmatrix}\right),$$

where $|z|=|w|=1$.[37] The action on $\mathbb{C}^4$ is given by[38,39]

$$M^{+}(z,w)=\begin{pmatrix} zw & 0 & 0 & 0 \\ 0 & zw^{-1} & 0 & 0 \\ 0 & 0 & wz^{-1} & 0 \\ 0 & 0 & 0 & (zw)^{-1}\end{pmatrix},$$

$$M^{-}(z)=\begin{pmatrix} z & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & z^{-1} & 0 & 0 \\ 0 & 0 & 0 & z^{-1}\end{pmatrix},$$

The normalised Haar measure on the component connected to the identity is $\frac{1}{2}\frac{1}{(2\pi\mathrm{i})^2}\frac{(1-z^2)\mathrm{d}z}{z}\frac{(1-w^2)\mathrm{d}w}{w}$ and the Haar measure on the disconnected component is $\frac{1}{2}\frac{1}{2\pi\mathrm{i}}\frac{(1-z^2)\mathrm{d}z}{z}$. The Hilbert series with n particles is then given by

$$H(\mathbb{C}[V]^{O(4)},t)=\frac{1}{2}\frac{1}{(2\pi\mathrm{i})^2}\oint_{|z|=|w|=1}\frac{(1-z^2)(1-w^2)\mathrm{d}z\,\mathrm{d}w}{zw}\frac{1}{\det_V(1-t\cdot M_V^{+})}$$
$$+\frac{1}{2}\frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}\frac{(1-z^2)\mathrm{d}z}{z}\frac{1}{\det_V(1-t\cdot M_V^{-})}.$$

In our example of $n=5$ with $P=1$, the integral becomes

$$H(\mathbb{C}[V]^{O(4)},t)=\frac{1}{2}\frac{1}{(2\pi\mathrm{i})^2}\oint_{|z|=|w|=1}\frac{\mathrm{d}z\,\mathrm{d}w}{zw}$$
$$\times\frac{(1-z^2)(1-w^2)}{(1-t/(wz))^5(1-(tw)/z)^5(1-(tz)/w)^5(1-twz)^5}$$
$$+\frac{1}{2}\frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}\frac{\mathrm{d}z}{z}\frac{(1-z^2)}{(1-t^2)^5(1-t/z)^5(1-tz)^5},$$

which we evaluate using the residue theorem, obtaining

[37] As in $d=3$, there is no need to take care in projecting to $(SU(2)\times SU(2))/\mathbb{Z}_2$.

[38] The asymmetry in the formulæ arises from the fact that the conjugacy classes in the disconnected component can be parameterized by a single $U(1)$; for details see [24].

[39] Here, the isomorphism $O(4,\mathbb{R})\cong\left((SU(2)\times SU(2))/\mathbb{Z}_2\right)\rtimes\mathbb{Z}_2$ corresponds to the linear map $(p_0,p_1,p_2,p_4)\in\mathbb{C}^4\mapsto(p_0+\mathrm{i}p_3,p_1+\mathrm{i}p_2,p_1-\mathrm{i}p_2,p_0-\mathrm{i}p_3)$.

$$H(\mathbb{C}[V]^{O(4)}, t) = \frac{1}{2}\left(\frac{1+t^2+6t^4+t^6+t^8}{\left(1-t^2\right)^{14}} + \frac{1+3t^2+t^4}{\left(1-t^2\right)^{12}}\right)$$
$$= \frac{1+t^2+t^4+t^6+t^8}{\left(1-t^2\right)^{14}}.$$

We also obtain the Hilbert series for when $G = SO(4, \mathbb{C})$ for free by only considering the component connected to the identity

$$H(\mathbb{C}[V]^{SO(4)}, t) = \frac{1+t^2+6t^4+t^6+t^8}{\left(1-t^2\right)^{14}}.$$

To include an arbitrary permutation group $P \subseteq S_n$ acting on the n momenta, one again needs to average over the conjugacy classes of P as discussed previously.

ORCID iDs

Ward Haddadin https://orcid.org/0000-0002-1217-4775

References

[1] Weyl H 1966 *The Classical Groups: Their Invariants and Representations* (Princeton, NJ: Princeton University Press)
[2] Butter A, Kasieczka G, Plehn T and Russell M 2018 *SciPost Phys.* **5** 28
[3] Tkachov F V 1981 *Phys. Lett.* B **100** 65–8
[4] Kotikov A V 1991 *Phys. Lett.* B **254** 158–64
[5] Derksen H and Kemper G 2002 *Computational Invariant Theory* (Berlin: Springer)
[6] Bruns W and Herzog H J 1998 *Cohen–Macaulay Rings* (Cambridge: Cambridge University Press)
[7] Hochster M and Roberts J L 1974 *Adv. Math.* **13** 115–75
[8] Gauss C F 2010 *Carl Friedrich Gauss Werke V3 (1876)* German edn (Whitefish, MT: Kessinger Publishing, LLC)
[9] Henning B, Lu X, Melia T and Murayama H 2017 *J. High Energy Phys.* JHEP10(2017)199
[10] Komiske P T, Metodiev E M and Thaler J 2020 *Phys. Rev.* D **101** 036019
[11] Fulton W, Harris W and Harris J 1991 *Representation Theory: A First Course* (New York: Springer)
[12] Aslaksen H, Chan S-P and Gulliksen T 1996 *Appl. Algebra Eng. Commun. Comput.* **7** 53–7
[13] Paule P and Sturmfels B 2008 *Algorithms in Invariant Theory* (Berlin: Springer)
[14] Buchberger B 1976 *SIGSAM Bull.* **10** 19–29
[15] Cox D, Little J and O'Shea D 2008 *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra* (New York: Springer)
[16] Grayson D R and Stillman M E 1992 Macaulay2, a software system for research in algebraic geometry available at http://math.uiuc.edu/Macaulay2/
[17] Domokos M and Hegedű̋s P 2000 *Arch. Math.* **74** 161–7
[18] Lester C G, Haddadin W and Gripaios B 2020 arXiv:2008.05206
[19] Ehrenborg R and Rota G-C 1993 *Eur. J. Comb.* **14** 157–81
[20] Beecken M, Mittmann J and Saxena N 2013 *Inf. Comput.* **222** 2–19
[21] Lester C G and Schott M 2019 *J. High Energy Phys.* JHEP12(2019)120
[22] Atiyah M and MacDonald I 1994 *Introduction to Commutative Algebra* (New York: Avalon Publishing)
[23] Kleiman S and Altman A 2013 *A Term of Commutative Algebra* (Cambridge, MA: Worldwide Center of Mathematics, LLC)
[24] Gripaios B, Müller T, Parker M A and Sutherland D 2014 *J. High Energy Phys.* JHEP08(2014)171