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Bayesian Inverse Problems

by

Jenovah Rodrigues

PhD Thesis for Doctor of Philosophy

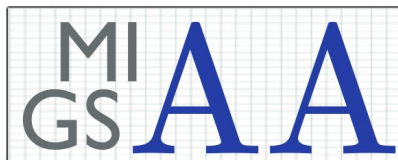
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DECLARATION

I, *Jenovah Rodrigues*, hereafter referred to as *the student* declare

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Sincerely,

Jenovah Rodrigues

Abstract

We consider linear, mildly ill-posed inverse problems in separable Hilbert spaces under Gaussian noise, whose covariance operator is not identity (i.e. it is not a white noise problem), and use the Bayesian approach to find their regularised solution. Specifically, our goal is to regularise the prior in such a way that the posterior distribution achieves the optimal rate of contraction. The object of interest (an unknown function) is assumed to lie in a Sobolev space. Firstly, we consider the so-called conjugate setting where the covariance operator of the noise and the covariance operator of the prior are simultaneously diagonalisable, and the noise has heterogeneous variance. Note this similar to the work done in [Knapik et al., 2011], albeit for the homogeneous variance case. Hence, we derive the minimax rate of convergence, the contraction rate of the posterior distribution and subsequently, discuss the conditions under which these rates coincide. The results are numerically illustrated by the problem of recovering a function from noisy observations. Secondly, motivated by Poisson inverse problems, we consider Gaussian, signal-dependent noise (i.e. non-conjugate setting). Using [Panov and Spokoiny, 2015] we obtain Bernstein von-Mises results for the posterior distribution, and consequently derive the contraction rates and conditions for its optimality as well.

Lay Summary

This thesis studies ill-posed Inverse problems, using the Bayesian approach. Informally, an inverse problem is derived from a cause-effect relationship. Specifically, we wish to estimate the cause given that we have (indirectly) observed the effect. Most readers in fact are already familiar with such a problem, since Medical Imaging using X-rays is ubiquitous in real life.

The complexity of an inverse problem is directly linked to how ill-posed it is. An inverse problem can be ill-posed for several reasons. In this thesis, we shall assume the ill-posedness is due to the indirect nature of our observations. Compounding matters, is the fact these observations will contain noise.

These two properties, (ill-posedness and noise) make it nigh impossible to solve Inverse problems in the classical manner. Specifically, our aim should not be to recover an exact solution, which explains the observed effect perfectly, but an approximate solution, which takes into account the presence of noise. Approaches that do so are called Regularisation methods, and are comprised of Deterministic and Statistical methods.

Deterministic methods account for the noise, but do not exploit the structure of it. Statistical methods, on the other hand, use all the information regarding the noise structure and therefore are deemed as superior. However, note statistical methods will generally be more computationally expensive.

In this thesis, Bayesian statistical methods are used. Specifically, all the information we currently know about the cause (even if we have none) forms part of the prior. Thus, given an observation, we can update the prior, (and consequently the information we have regarding the cause); this updated information is referred to as the posterior.

Consequently, the efficacy of the Bayesian method can be gauged by the quality of the information contained in the posterior. Specifically, as the noise in our observations decreases, the posterior should determine the true solution with increasing accuracy. Informally, the rate at which it does so, is called the contraction rate.

These contraction rates will depend on the properties of the inverse problem, as well as the smoothness of the prior. Thus, by controlling the latter, we can improve our contraction rates.

Typically, the contraction rates are compared to a benchmark of sorts. This benchmark takes into account all possible methods, and every possible true solution. In our setting, this benchmark is referred to as the minimax rate, and will depend on the smoothness of the true solution as well. Consequently, a contraction rate is deemed optimal if it achieves the minimax rate.

Thus, the aim of this thesis is find priors that lead to the optimal contraction rate, for different noise structures. In our setting, optimality will depend on how well the smoothness of the prior matches the smoothness of the true solution.

Subsequently, we consider Gaussian noise with two different covariance structures. In the first case, we assume the Gaussian noise has non-constant variance. Hence, we derive the minimax rates, the contraction rates, and conditions under which the optimal rate coincides with the minimax rate.

In the second case, we assume the Gaussian noise has signal-dependent variance. Informally, this means that the variance of the noise is also dependent on the unknown true solution. Subsequently, we derive the contraction rates, and its corresponding optimality conditions.

Finally, we conclude this thesis by discussing possible extensions and applications of the highlighted results.

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1 Introduction

1.1 Layout of this Thesis

This thesis comprises of six sections. In Section 1, we state the layout of this thesis, provide an overview of inverse problems, along with a relevant literature review, and discuss inverse problems in both the finite and infinite dimensional setting.

In Section 2, we review [Knapik et al., 2011], and highlight the relevant results. Specifically, under the prior $\mu \sim N(0, \Lambda)$, with Gaussian white noise $Z \sim N(0, I)$, the posterior $\mu|Y$ is derived in Proposition 2.1, ([Knapik et al., 2011]’s Proposition 3.1). Subsequently, by explicitly categorising the behaviour of the singular values of K , Λ and μ_0 in Assumption 2.1, the contraction rates are derived in Theorem 2.1, ([Knapik et al., 2011]’s Theorem 4.1).

In Section 3, under the prior $\mu \sim N(0, \Lambda)$, with Gaussian Heterogeneous noise $\tilde{Z} \sim N(0, WW^T)$, the posterior $\mu|Y$ is derived in Proposition 3.1. Subsequently, by explicitly categorising the behaviour of the singular values of K , Λ , W and μ_0 in Assumption 3.1, the minimax and contraction rates are derived in Proposition 3.2 and Theorem 3.2, respectively. Furthermore, contraction rates are also derived in Theorem 3.1 under a general setting, (Assumption 3.2), where the behaviour of the singular values of K , Λ , W aren’t restricted. In addition, we derive contraction rates for the case where there is noise in the covariance operator, and conclude the section by numerically illustrating the problem of recovering a function from noisy observations.

In Section 4, we review [Panov and Spokoiny, 2015] and highlight the relevant results. Specifically, we discuss the assumptions stated in [Panov and Spokoiny, 2015]’s Section 2.1, and subsequently review Theorem 9 (Local Approximation), and Theorem 10 (Concentration). Consequently, we highlight how the previously mentioned theorems are used to derive the BvM results under a Uniform prior (Theorem 1), and a Gaussian prior (Theorem 2).

In Section 5, we consider direct and indirect inverse problems with signal-dependent noise. Specifically, Theorems 5.1 and 5.2 provide Bernstein-von Mises results for the posterior distribution for the direct problem, which is defined in Section 5.1. Subsequently, Theorem 5.3 states Bernstein-von Mises results for the posterior distribution for the indirect problem, which is defined in Section 5.3. Note, these aforementioned theorems rely on a concentration result, which is proved in Theorem 5.4. Consequently, the contraction rates for the direct and indirect problems are derived in Theorems 5.6 and 5.7, respectively.

In Section 6, we identify gaps in the literature and state how the results from Sections 3 and 5 can be used to address them.

1.2 An Overview of Inverse Problems

Remark 1.1. *Most of this report’s introductory material regarding inverse problems and the methods used to tackle them are based on [Alquier et al., 2011], [Kaipio and Somersalo, 2006], and [Engl et al., 1996]. Keywords and statements will be **highlighted** to aid the reader.*

Inverse Problems can be found in many disciplines, such as Astronomy (restoring blurred images from the Hubble Space telescope, [Adorf, 1995]), Financial Mathematics (calibrating derivative pricing models, [Lagnado and Osher, 1997]), Image Processing (radiography, [Hunt, 1970]) and Physics (solving the inverse heat conduction problem, [Beck et al., 1985]). Even differentiation can lead to an inverse problem, see [Engl et al., 1996]’s Section 1.1 and [Anderssen and Bloomfield, 1974].

Generally, an inverse problem is derived from a cause-effect relationship. We define the forward problem as estimating the effect given the cause. Conversely, we define the inverse problem as recovering the cause given that we have (perhaps indirectly) observed the effect.

This notion of indirect observations of some cause (i.e. function) is usually modeled via an operator K . Thus, from a mathematical point of view, inverse problems usually correspond to the *inversion* of K .

Since the base requirement for estimation is the notion of distance (norm) and convergence (completeness), we begin by formulating the ***inverse problem*** in a Banach Space setting:

$$Y = K\mu, \quad (1)$$

where $Y \in \mathcal{Y}$ denotes the effect we observe, $\mu \in \mathcal{X}$ denotes the cause we wish to recover, and $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ are our Banach spaces. The operator $K : \mathcal{X} \rightarrow \mathcal{Y}$ is typically called the *Forward* or *Observation* operator. Hence, in an inverse problem we observe Y and aim to recover μ .

Additionally, if K is the identity operator, the problem is referred to as ***direct***, otherwise it is called ***indirect***. Furthermore, if K is injective and continuous (bounded) the solution is unique and approximable, respectively.

Observe, recovering μ in Model (1) is rather straight-forward assuming K is invertible. Naturally, in most inverse problems K is not invertible and hence we require an alternative method for addressing its *inversion*, so as to obtain a precise reconstruction. One way is to use Least Squares, i.e.

$$\hat{\mu}_{ls} = \inf_{\tilde{\mu} \in \mathcal{X}} \|Y - K\tilde{\mu}\|.$$

Furthermore, one could also encounter the following scenarios, which would make the inverse problem harder to solve:

- There may exist an object, $\tilde{\mu} \neq 0 \in \mathcal{X}$ s.t $K\tilde{\mu} = 0$, thereby resulting in the mapping no longer being unique, (since even though $\mu \neq (\tilde{\mu} + \mu)$, $K\mu = K(\tilde{\mu} + \mu)$). This corresponds to the operator having a non-trivial null space, (*Uniqueness*).
- There may exist $Y \in \mathcal{Y}$ such that $Y \neq K\mu$, $\forall \mu \in \mathcal{X}$, i.e. the solution of the inverse problem may not exist. This may well be the case when dealing with very noisy images, (*Existence*).
- There may exist objects $\mu_1, \mu_2 \in \mathcal{X}$ such that while μ_1 and μ_2 are far apart in \mathcal{X} , $K\mu_1$ and $K\mu_2$ are close in \mathcal{Y} . Thus when solving the inverse problem, one could easily arrive at the wrong μ , (*Stability*).

This motivates the definition of *ill-posedness*. Hadamard introduced the concept of a *well-posed* problem (see [Hadamard, 1932]), using the principles mentioned above and defined below.

Definition 1.1. *An inverse problem is well-posed if there exists a unique solution that depends continuously on Y (i.e. the Uniqueness, Existence and Stability conditions described above hold).*

Furthermore, any problem that is not well-posed is defined as ill-posed.

Note, continuity w.r.t Y , (i.e. the *Stability* condition), is typically the crux of the matter in inverse problems.

Well-posedness is particularly hard to guarantee in the presence of noise. Specifically, consider Model (1) but now with additive noise Z , (**noisy inverse problem**):

$$Y = K\mu + \epsilon Z, \quad (2)$$

where Y, K and μ satisfy the assumptions from Model (1), $Z \in \mathcal{Y}$ is unknown and ϵ is a non-negative constant.

We assume that

$$\|Z\|_{\mathcal{Y}} \leq 1,$$

and refer to ϵ as the **noise level**. Hence, as $\epsilon \rightarrow 0$, the noise disappears and we recover Model (1).

Note, the existence and uniqueness of solutions is akin to identifiability in statistics. Furthermore, while it will not be discussed in this thesis, one can show the existence and uniqueness of a least squares solution via the use of a Moore-Penrose (generalised) inverse of K .

Observe, recovering μ in Model (2) is no longer straight-forward, even if there exists a unique solution and we assume K is invertible, due to the aforementioned stability condition. To elucidate, let

$$\hat{\mu} = K^{-1}Y = K^{-1}(K\mu + Z).$$

There is no guarantee that $\hat{\mu}$ is close to the true μ , even if $\|Z\|_{\mathcal{Y}}$ is small, since K^{-1} may be sensitive to even the smallest perturbations in Y . For instance, this can happen when K^{-1} is unbounded, (see Section 1.3.3).

Thus, the issue is that we are looking for an exact solution rather than an *approximate* solution; approximate in the sense, that our solution must not reproduce the data, Y , exactly, but rather within experimental errors. However this space of approximate solutions is too vast, due to the very nature of our ill-posed problems.

Consequently, methods to enforce stability and choose an approximate solution in a meaningful way led to the birth of *Regularisation Methods*. In this thesis we shall assume ill-posedness is due to the stability condition. As discussed later, said stability condition will be dictated by the behaviour of K 's eigenvalues, (see Definition 1.8).

Remark 1.2. Note, in the finite dimensional setting, non-continuity w.r.t. Y is equivalent to assuming K is ill-conditioned (i.e. a high condition number; for more details and examples see Sections 1.3.1 and 1.4.4). However, a problem can be well-conditioned (i.e. have a low condition number) and still be ill-posed (if for instance, the non-uniqueness or the non-existence scenarios hold).

Classical Regularisation Methods are deterministic in nature, and rely on *Least Squares*. Most importantly, they assume the existence of a **true solution** to Model (2), i.e. μ_0 .

The most famous classical method is the *Tikhonov Regularisation Method* [Engl et al., 1996]:

$$\hat{\mu}_\lambda := \inf_{\tilde{\mu} \in \mathcal{X}} \|Y - K\tilde{\mu}\|_{\mathcal{Y}} + \lambda \|\tilde{\mu}\|_{\mathcal{X}},$$

where $\lambda > 0$ is some **tuning parameter**. Comparing $\hat{\mu}_\lambda$ to $\hat{\mu}_{ls}$, the additional term $\|\tilde{\mu}\|_{\mathcal{X}}$ (referred to as energy) enforces stability by penalising *rough* $\tilde{\mu}$.

Consequently, one can study the **convergence properties** of $\hat{\mu}_\lambda$ to μ_0 , as $\epsilon \rightarrow 0$. As one can expect, the rate of convergence will depend directly on λ , and indirectly on K and μ_0 . Furthermore, we would expect $\lambda \rightarrow 0$ as $\epsilon \rightarrow 0$, hence λ must be a function of ϵ .

Thus, a **choice rule** for λ can be constructed which incorporates information about ϵ , K and μ_0 . One can even use Y in the construction of said rule, (this is described as *data driven* or *Empirical*). [Hämarik et al., 2012] contains choice rules for λ when ϵ is under and over-estimated.

Subsequently, one can establish a **benchmark** by bounding every regularisation method's best convergence rate from below. Hence, the *performance* of a given regularisation method can be gauged by comparing its convergence rate (under a given choice rule for λ) against the aforementioned benchmark.

For the deterministic noise model, (2), we can define the benchmark using the **worst noise risk**:

$$\sup_{\mu \in \mathcal{X}} \sup_{\|Z\|_{\mathcal{Y}} \leq 1} \|\hat{\mu} - \mu\|_{\mathcal{X}}^2.$$

This risk, as well as the rates of convergence, are stated in [Engl et al., 1996]. Furthermore, Table 1.3 in [Alquier et al., 2011] lists the worst noise risk for when μ belongs to different classes of functions as well.

Note, for the Statistical approach, (discussed further below), this benchmark will be embellished under the guise of *minimax error*.

While the regularised least squares approach is well-developed, as stated in [Hofinger and Pikkarainen, 2012], “a main criticism is that ... convergence rate result depends on ϵ which can be seen as a worst-case scenario.” Hence, the use of statistics to improve these convergence rate results.

Statistical Regularisation Methods are used when at least one of the variables in Model (2) is stochastic; typically, and in this thesis, **the noise (Z) will be random, with known distribution**.

In a Frequentist setting, convergence results are usually given using the mean square error [Wahba, 1977]. However, there are results using other metrics, such as [Hofinger and Kindermann, 2000] for the Ky Fan metric and [Engl et al., 2005] for the Prokhorov metric. Note, the latter metric involves convergence in distribution while the former involves convergence in probability.

In this thesis, the *Bayesian approach* will be employed; specifically Z , and therefore Y , will be random, and we will assume a prior on μ . Hence, using Bayes' rule, the posterior distribution $\mu|Y$ will be derived. Thus, under the Bayesian approach, $\mu|Y$ is the regularised solution of Model (2).

Note, [Stuart, 2010] and [Kaipio and Somersalo, 2006] provide an excellent introduction to the Bayesian formulation of inverse problems, as well as containing a myriad of examples. [Stuart, 2010] also gives conditions for the resulting posterior distribution to be well-defined i.e. normalisable and Lipschitz wrt Y (c.f. Theorem 4.1 and 4.2); the latter implying that the posterior mean and covariance operator are continuous wrt Y . This is relevant when considering inverse problems in an infinite dimensional setting with a non-Gaussian prior. This result was later derived for Besov priors as well in [Dashti et al., 2012].

Consequently, having obtained the regularised solution i.e. $\mu|Y$, we can use it to construct estimators (such as the posterior mean) or credible sets. However, the biggest obstacle in most Bayesian applications is obtaining a closed-form expression for $\mu|Y$. Thus, classical Bayesian analysis relied on *conjugate priors*, see [Box and Tiao, 1973].

Due to recent advancements in technology, there do exist several computational methods for sampling the posterior, such as the Monte Carlo method, which is used in [Beskos et al., 2015]. Still, such methods require extensive computer time, therefore it is of importance to have good analytic approximations which are much simpler to compute.

For example, in classical statistics, the Central Limit Theorem (CLT) can be used to obtain normal approximations, the latter being computationally inexpensive. Similarly, in the Bayesian setting, Laplace (1774) (and later Bernstein (1917) and von Mises (1931)) discovered that the posterior distribution could be well-approximated by a normal distribution with mean equal to the Maximum Likelihood Estimate (MLE) and covariance matrix equal to the inverse of the Fisher Information matrix. This result came to be known as a *Bernstein-von Mises (BvM)* type result, and is also called the Bayesian CLT due to its similarity to the classical CLT.

Le Cam [Le Cam, 1953] was the first to formalise this result, in which he assumed a parametric model, with i.i.d observations and large sample sizes. His work has undergone several modifications and extensions, such as being shown to hold for non-independent observations in [Heyde and Johnstone, 1979]. Even the assumption of a parametric model was improved upon, i.e. [Shen, 2002] and [Castillo and Rousseau, 2015] show BvM results for the semi-parametric framework. Furthermore, sharper approximations than that achieved via normal approximation can be obtained by expanding the posterior, as shown in [Johnson, 1970]. More recently, Panov and Spokoiny [Panov and Spokoiny, 2015] obtained BvM results in a non-classical setup allowing for finite samples and model mis-

specification.

Note, BvM results for the infinite dimensional setting are still in their infancy. There is still hope however; Castillo and Nickl (in [Castillo and Nickl, 2013] and [Castillo and Nickl, 2014]) obtained BvM results for the full non-parametric model, under a weaker notion of convergence (based on the Lipschitz metric). In Sections 2 and 3 we will exploit conjugacy to obtain the posterior distribution, while in Sections 4 and 5 we will use BvM type results to approximate the posterior distribution.

Subsequently, all that remains is to analyse the convergence properties of $\mu|Y$ for Model (2). This is easier to compute in the *separable Hilbert Space setting*:

$$Y = K\mu + \epsilon Z, \quad (3)$$

where ϵ is the noise level, $\mu \in H_1$ a separable Hilbert Space, and a known, injective, continuous, linear operator K maps μ into another separable Hilbert space, H_2 . If we assume Z is Gaussian white noise process, (c.f. Definition 1.5), then (3) is referred to as the *Gaussian white noise model*.

As stated previously, one's computations are easier since not only can we define a norm using the Hilbert Space's inner product and therefore use all the theory from the Classical Approach, but, more importantly, we also obtain the notion of orthogonality. The latter allows us to decompose Model (3) into its sequence space counterpart:

$$Y_i = k_i \mu_i + \epsilon Z_i, \text{ for all } i \geq 1, \quad (4)$$

where, for orthonormal bases $\{\phi_i\}_{i=1}^\infty$ and $\{\varphi_i\}_{i=1}^\infty$ of H_1 and H_2 respectively,

$$Z_i := \langle Z, \varphi_i \rangle_{H_2}, \quad \mu_i := \langle \mu, \phi_i \rangle_{H_1}, \quad \text{and } Y_i := \langle Y, \varphi_i \rangle_{H_2},$$

and k_i^2 correspond to the eigenvalues of KK^T , (see Section 1.4.3 for more details).

Note, since the Hilbert space is *separable* it guarantees the existence of a countable orthonormal basis, which is important in our analyses. A thorough treatment of Models (3) and (4) is provided in Section 1.4.

Note, this decomposition also allows us to take an unobservable Gaussian random variable in a Banach (or Hilbert) space and instead address it as a sequence of Gaussian random variables in \mathbb{R} . Furthermore, the norm of any element can be expressed using this decomposition as well, via Parseval's Identity.

Consequently all that remains is to discuss what it means for $\mu|Y$ to converge, its corresponding convergence rates and the construction of a so-called *benchmark*.

In the Bayesian approach, convergence is described in the Frequentist setting and relies on two notions: *Posterior consistency* and *Posterior Contraction Rates*; the latter being the cornerstone of this thesis. Note, both these notions rely on the existence of some *underlying true parameter* μ_0 , (hence the Frequentist setting).

Recall in the Frequentist setting, estimators $\hat{\mu}$ are deemed *consistent* if they converge (in probability) to the truth μ_0 , assuming the data, Y , was generated using the true probability distribution \mathbb{P}_{μ_0} . Likewise in the Bayesian setting, we define consistency on our posterior.

Specifically, we start with a (sometimes imprecise) prior on the parameter space, H_1 and update it with the given data, Y , to obtain our posterior. It is therefore of utmost importance to know whether the posterior becomes increasingly accurate and precise when data is collected indefinitely. This property is known as posterior consistency.

Definition 1.2. *We say the posterior distribution of $\mu|Y$, $\Pi(\cdot|Y)$ is consistent at μ_0 , if for every neighbourhood of μ_0 , $B(\mu_0)$,*

$$\Pi(B(\mu_0)^c|Y) \xrightarrow{\mathbb{P}_{\mu_0}} 0, \text{ as } \epsilon \rightarrow 0,$$

where \mathbb{P}_{μ_0} ensures that Y was generated using μ_0 .

Thus, posterior consistency guarantees the concentration of the posterior distribution around the truth μ_0 . Note, however it does not quantify the performance of the posterior, and hence we cannot compare two different posteriors using the notion of consistency.

It is well known that for virtually all finite dimensional problems, the posterior distribution is consistent under mild conditions, see [Ibragimov and Has'minskii, 1981], [Le Cam, 1986] and [Ghosal et al., 1995]. Informally, this is because the likelihood is prominently peaked near the true parameter for large sample sizes, hence one is able to show that the posterior distribution can be approximated by a normal distribution (which is referred to as a Bernstein-von Mises (BvM) type result).

Verifying consistency in the infinite dimensional however is no easy task. As shown in [Freedman, 1963] consistency can fail to hold in even the most simplest cases; in the paper the constructed prior puts positive mass on every neighbourhood of the truth, yet the posterior converges to the wrong distribution.

There are several results that guarantee posterior consistency in the infinite dimensional setting. For instance, Doob [Doob, 1949] showed that for every prior distribution on a given parameter space, posterior consistency held for any underlying truth, assuming said truth had non-zero prior mass. However, there is no reason to assume that this is the case, i.e. the true parameter may belong to the prior distribution's null set.

Fortunately, Schwartz [Schwartz, 1965] was able to prove consistency for *any parameter* assuming the existence of some tests and positive prior mass on every Kullback–Leibler neighborhood of said parameter. These results were later extended to weak and L_1 neighbourhoods by [Barron et al., 1999], [Ghosal et al., 1999] and [Walker, 2004]. Please see [Diaconis and Freedman, 1986], and [Choi and Ramamoorthi, 2008] for further details regarding posterior consistency.

Subsequently, to judge asymptotic performance, we consider the (related) notion of contraction instead. Specifically, we allow the radius ε of the neighbourhoods (of μ_0), $B(\mu_0)$, to depend on ϵ , and hence aim to find the smallest ε such that consistency holds.

Definition 1.3. *The posterior distribution of $\mu|Y$, $\Pi(\cdot|Y)$, contracts around μ_0 with contraction rate $\varepsilon \downarrow 0$ if*

$$\Pi(\{\mu : \|\mu - \mu_0\|_{H_1} \geq M\varepsilon|Y\}) \xrightarrow{\mathbb{P}_{\mu_0}} 0,$$

for every sequence $M \rightarrow \infty$.

Note again that \mathbb{P}_{μ_0} guarantees that Y was generated using μ_0 . Furthermore, by definition, posterior contraction will imply posterior consistency.

For a finite n -dimensional model, Le Cam [LeCam, 1973] showed that the posterior distribution achieved the optimal rate of contraction under the Hellinger metric. In particular, one can use the Bernstein-von Mises Theorem to show $\varepsilon = n^{-1/2}$, assuming the model is suitably differentiable.

For a infinite dimensional model, with i.i.d observations [Ghosal et al., 2000] obtained general results under the Total Variation, Hellinger and L_2 -metrics. This result was then extended to a non-i.i.d framework in [Ghosal and van der Vaart, 2007]. Note, [Shen and Wasserman, 2001] obtained similar results to [Ghosal et al., 2000], but under stronger conditions.

The seminal paper on posterior contraction rates for linear ill posed problems with conjugate Gaussian priors, in the infinite dimensional setting is [Knapik et al., 2011]. In [Knapik et al., 2013], under the same setup as [Knapik et al., 2011], a severely ill-posed linear inverse problem is studied, and the posterior contraction rate derived. In both of these papers, the noise is assumed to be i.i.d standard normal.

Conversely, [Agapiou et al., 2013] and [Florens and Simoni, 2016] studied the case when the covariance operator of the noise is not a constant (i.e. not identically distributed). Furthermore, the work done in [Agapiou et al., 2013] doesn't require the simultaneous diagonalisable assumption, as required in [Knapik et al., 2011] and [Knapik et al., 2013], however the rates are sub-optimal.

All that is left to discuss, w.r.t asymptotic performance, is our *benchmark*.

Definition 1.4. *In the Frequentist setting, the Maximal Risk of an estimator $\hat{\mu}$ is*

$$\sup_{\mu \in H_1} R(\hat{\mu}, \mu),$$

and the Minimax Rate is

$$r_\epsilon(H_1) = \inf_{\hat{\mu}} \sup_{\mu \in H_1} R(\hat{\mu}, \mu), \quad (5)$$

where $R : H_1 \times H_1 \rightarrow [0, \infty]$ is some risk function and the infimum is taken over all estimators of μ (i.e. all measurable functions $\hat{\mu} : H_1 \rightarrow H_2$).

Note, $\hat{\mu}$ will typically be a function of Y , hence the presence of ϵ in $r_\epsilon(H_1)$.

Typically, and in our thesis, $R(\hat{\mu}, \mu_0)$ will correspond to the L^2 norm (also known as *Mean Integrated Squared Error*):

$$R(\hat{\mu}, \mu_0) = \mathbb{E}_{\mu_0} \|\hat{\mu} - \mu_0\|^2.$$

The Minimax Rate will be the *benchmark* our contraction rates are compared against. Observe how the posterior contraction rate provides an upper bound for the rate of estimation, since any $\varepsilon' \leq \varepsilon$ will also be a contraction rate. Conversely, the minimax rate via Markov's Inequality will provide a lower bound for the posterior contraction rate. Note, it is well-known that these rates of convergence depend on the ill-posedness of the inverse problem and smoothness conditions on μ , (see [Alquier et al., 2011]).

In [Bissantz et al., 2007] minimax rates are derived for several Classical Regularisation methods, including SVD and Tikhonov estimators. Minimax rates for Statistical Approaches generally use the results from [Pinsker, 1980], which provides conditions for a linear estimator to be minimax. For instance, it is used in [Cavalier, 2008] to derive the minimax rates for Model (3) with Gaussian white noise; specifically, the minimax rate for the *Sobolev class of functions* is $n^{-\beta/(1+2\beta+2q)}$. Similarly, minimax rates for severely ill-posed problems with *analytic functions* are given in [Golubev and Khasminskii, 1999] and [Golubev and Khasminskii, 2001].

As seen in both the Classical and Bayesian approach, optimal convergence/contraction requires some optimal choice of the tuning parameter, λ , that often relies on the unknown smoothness of the function μ_0 . This has led to more recent concepts like *adaptive estimation*, which circumvent this condition.

A prior is called *rate-adaptive* if it achieves the optimal rate without knowledge of the truth’s smoothness. Generally the smoothness class is indexed, and one may or may not know the value of said index for the truth μ_0 .

Results are known for Model (3), where Z has Homogeneous (i.e. constant) variance, (aka the Gaussian white noise model). For instance, [Ray, 2013] considered both the mildly and severely ill posed setting using non-conjugate priors, notably sieve and wavelet series priors, obtaining suboptimal rates. More recently, [Ray, 2017] investigated adaptive Bernstein–von Mises (BvM) theorems for the Gaussian white noise model as well.

Additionally, [Knapik et al., 2016] discussed two different ways of obtaining adaptive rates; both involve a data-driven choice of a prior’s tuning parameter in order to automatically achieve an optimal bias-variance trade-off. The first involves putting a prior distribution on the tuning parameter itself, and the second is likelihood-based.

[Szabó et al., 2013] used a plug-in estimator of the prior scale parameter for direct models under white noise, while [Florens and Simoni, 2016] did the same for inverse problems with heterogeneous noise. In a different setting, [Knapik and Salomond, 2018] showed that adaptive hierarchical mixture models lead to optimal posterior contraction rates. Furthermore, [Rousseau and Szabo, 2017] considered the likelihood based approach, i.e. they studied the concentration properties of the Empirical Bayes posterior.

For inverse problems, only the approach of [Johannes et al., 2020] (i.e. using the truncation number) leads to the posterior distribution contracting at the optimal rate (under white noise).

Finally note, inverse problems can be identified as linear or non-linear based on their forward operator, K . Non-linear inverse problems are far more complex, but can generally be linearised [Stefanov and Uhlmann, 2009]. A survey of non-linear inverse problems can be found in [Engl and Kügler, 2005]. Similar to their linear counterparts, non-linear inverse problems were generally tackled using a least squares formulation [Tarantola and Valette, 1982], however recently neural networks have also been used to find solutions [Obornev et al., 2020].

Of particular import is Nickl’s work on non-linear inverse problems using the Bayesian

approach. Note, we touched on some of his work when discussing BvM results ([Castillo and Nickl, 2013] and [Castillo and Nickl, 2014]). Additionally, in [Nickl and Söhl, 2017] posterior contraction rates were obtained for a reflected diffusion model, and most recently, in [Giordano and Nickl, 2020] consistency results were obtained using Gaussian process priors.

In this thesis we shall assume the forward operator K is linear. However, later on in Sections 4 and 5 we will allow the variance of the noise Z to depend on μ , which introduces non-linearity into our probability model for Y . Regardless, even if an inverse problem is non-linear, they can be linearised. Consequently, this thesis's results are still applicable.

In the next section, we shall review how one can construct estimators using the Classical (Deterministic) and Statistical Approaches. Special attention will be paid to *SVD* (Singular Value Decomposition) methods and how one can use them to obtain *Sequence Space models*.

1.3 Finite Dimensional Setting

We begin by assuming Model (3), but in the finite dimensional setting, \mathbb{R}^m :

$$Y = K\mu + \epsilon Z, \quad (6)$$

where Z is a random m -dimensional vector, $\epsilon \in \mathbb{R}^+$, $\mu \in \mathbb{R}^p$, $Y \in \mathbb{R}^m$ and $K \in \mathbb{R}^{m \times p}$. Note, in this section, given a vector x , x_i will refer its i^{th} component.

1.3.1 Ill-Posedness

Recall, we assume an inverse problem is ill-posed if the solution doesn't depend continuously on Y . This in turn can be categorised via the behaviour of K (or more specifically K^{-1} , if it exists.)

Note if $m = p$, then K is a square matrix, and hence the problem reduces to solving a system of linear equations. If $m > p$, which is often the case in real life, i.e. if one has more observations than variables, the problem changes into a Least Squares problem (c.f. Section 1.3.2).

Recall that $K : \mathbb{R}^p \mapsto \mathbb{R}^m$ is continuous if and only if there exists c such that

$$\|K\mu\| \leq c\|\mu\|, \quad \forall \mu \in \mathbb{R}^p,$$

where the norm $\|\cdot\|$ is the standard Euclidean norm. Thus we see that K never lengthens a vector by more than a factor of c . Hence, we can naturally define the *length* of the operator K by setting it to be the smallest c , i.e.

$$\|K\|_{op} := \min\{c \geq 0 : \|K\mu\| \leq c\|\mu\|, \quad \forall \mu \in \mathbb{R}^p\}.$$

Furthermore, it can be shown that

$$\|K\|_{op} = \max\left\{\frac{\|K\mu\|}{\|\mu\|} : \mu \in \mathbb{R}^p \text{ where } \mu \neq 0\right\} = \max_{\|\mu\|=1} \{\|K\mu\|\}.$$

In addition, when Y is observed with error, one defines the *condition number* of K ($\text{cond}(K)$) as the maximum ratio between the relative error in μ (i.e. $\|K^{-1}\epsilon Z\|/\|K^{-1}Y\|$) and the relative error in Y (i.e. $\|\epsilon Z\|/\|Y\|$). Hence since

$$\frac{\|K^{-1}\epsilon Z\|/\|K^{-1}Y\|}{\|\epsilon Z\|/\|Y\|} = \left(\frac{\|K^{-1}\epsilon Z\|}{\|\epsilon Z\|}\right)\left(\frac{\|Y\|}{\|K^{-1}Y\|}\right) \implies \text{cond}(K) := \|K^{-1}\|_{op}\|K\|_{op}.$$

Subsequently, having defined a matrix's condition number, we can investigate its role in the ill-posedness of an inverse problem, (specifically, its effect on the non-continuity w.r.t Y). Therefore, let us assume we solve the noisy problem (Model 6) and obtain the solution $\hat{\mu}$. We then wish to ascertain how close $\hat{\mu}$ is to the exact solution of the non-noisy problem.

Denoting the true solution as μ_0 one can show using *classical perturbation theory*, (c.f. [Ralston and Rabinowitz, 2001]) , that

$$\frac{\|\mu_0 - \hat{\mu}\|}{\|\mu_0\|} \leq \text{cond}(K) \frac{\|\epsilon Z\|}{\|Y\|},$$

where $\|\cdot\|$ is the standard Euclidean norm.

Thus, a large condition number could imply that $\hat{\mu}$ was very far from μ_0 regardless of how small the perturbation, ϵZ , was. Consequently, even the presence of rounding errors could produce a useless approximation of μ_0 . Hence the need for *Regularisation Theory*.

Note that we state a matrix, K , is *well-conditioned* if $\text{cond}(K)$ is small. Subsequently, the larger this number, the more ill-conditioned the matrix becomes. A far more rigorous definition of ill-posedness, via ill-conditionedness, is given for the infinite dimensional case in Section 1.4.4.

1.3.2 Least Squares Method

We begin by recalling some definitions and terminology, with regards to the rank of matrices. A matrix has *full row rank* when each of its rows are linearly independent and *full column rank* when each of its columns are linearly independent. In the case of a square matrix, we say that it has *full rank* if all columns and rows are linearly independent. Note, a square matrix has full rank if and only if its determinant is non-zero.

Furthermore, for a non-square matrix, it will always be the case that either its rows or its columns (whichever is larger in number) are linearly *dependent*. Thus when we say that a non-square matrix is *full rank*, we mean that either the row or the column rank is as high as possible, given the shape of the matrix. So for example, if there are more rows than columns ($m > p$), then the matrix is full rank if the matrix is full column rank.

Observe, if K is square and has full rank then it is invertible and one can find a single solution, whilst if K has full row rank then there can exist an infinite number of solutions or no solution. If K has full column rank and Y is in the range of K , then there will exist a unique solution, otherwise no solution will exist.

Hence, for the cases where no solution exists, the best one can do is find an approximate solution via *Least Squares* i.e.

$$\min_{\mu \in \mathbb{R}^n} \|Y - K\mu\|_2,$$

where $\|\cdot\|_2$ is the standard Euclidean norm. The solution to this equation, i.e. the Least Squares estimator is

$$\hat{\mu}_{ls} = (K^T K)^{-1} K^T Y.$$

Recall that if K has full column rank then $(K^T K)^{-1}$ does indeed exist.

Now, assume K is ill-conditioned. Hence, even if one were to invert the operator the solution obtained would be erroneous (assuming K^{-1} exists).

Numerically, if we knew ϵZ was small, we could obtain a solution via Least Squares by ignoring the term ϵZ in our calculations. However, K is ill-conditioned, hence our Least Squares solution, $\hat{\mu}_{ls}$, would be incorrect, which we demonstrate in the next section.

1.3.3 Ill-Posed Example using Matrices

The problem (taken from Chapter 1 of [Hansen, 2010]) is as follows:

$$\min_{\mu} \|Y - K\mu\|_2, \text{ where } K = \begin{pmatrix} 0.16 & 0.10 \\ 0.17 & 0.11 \\ 2.02 & 1.29 \end{pmatrix} \text{ and } Y = K \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0.01 \\ -0.03 \\ 0.02 \end{pmatrix}.$$

Note $\|\cdot\|_2$ is the standard Euclidean norm. One would expect the solution to be close to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ however, as we shall see, this is not the case. The *least squares* solution, along with some alternate solutions, are

$$\hat{\mu}_{ls} = \begin{pmatrix} 7.01 \\ -8.40 \end{pmatrix}, \hat{\mu}_1 = \begin{pmatrix} 1.65 \\ 0 \end{pmatrix} \text{ and } \hat{\mu}_2 = \begin{pmatrix} 0 \\ 2.58 \end{pmatrix},$$

with residual errors 0.022, 0.031 and 0.036, respectively. The reason we obtain such erratic behaviour is due to K being *ill-conditioned* i.e. it has a very large condition number (1097.5 in fact).

Consequently, let us now consider an alternative method for recovering μ , one which regularises K in some sense (i.e. a Regularisation Method).

1.3.4 Classical (Deterministic) Approach: Tikhonov Regularisation

While there are numerous deterministic regularisation approaches, we shall restrict our attention to the most famous one: the *Tikhonov Regularisation Method*

This method tries to ensure one obtains a smooth solution whilst also fitting the data, Y , adequately, i.e.

$$\hat{\mu}_{\lambda} = \arg(\min_{\mu} \|Y - K\mu\|^2 + \lambda^2 \|\mu\|^2),$$

where λ is our regularisation parameter and $\|\cdot\|$ is the standard Euclidean norm or the norm generated by a Hilbert space's inner product, whichever is relevant. As we can see, the term $\|Y - K\mu\|^2$ controls the goodness of fit whilst $\|\mu\|^2$ ensures that the solution obtained is regular. Furthermore, by alternating λ one could ensure that we either fit the data, Y , more accurately or make our solution smoother.

The Tikhonov approach is akin to a Least Squares approach, hence the latter can be used to easily derive a solution for the former. Recall that for any vectors u and v ,

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2 = \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} u \\ v \end{pmatrix} = \|u\|^2 + \|v\|^2.$$

Hence, one can rewrite the Tikhonov problem as follows:

$$\min_{\mu} \|Y - K\mu\|^2 + \lambda^2 \|\mu\|^2 = \min_{\mu} \left\| \begin{pmatrix} Y \\ 0 \end{pmatrix} - \begin{pmatrix} K \\ \lambda I \end{pmatrix} \mu \right\|^2 \implies \hat{\mu}_{\lambda} = (K^T K + \lambda^2 I)^{-1} K^T Y,$$

where letting $\lambda \rightarrow 0$ recovers the Least Squares problem.

As stated in our Introduction, Model (3) is easier to analyse in the Sequence Space setting. The same can be said for Model (6). Both models can be decomposed using Singular Value Decomposition (SVD) methods, as discussed in the next section.

1.3.5 Singular Value Decomposition Method and Ill-Posedness Revisited

This section requires familiarity with the SVD (Singular Value Decomposition) of a matrix, hence for those unfamiliar with the topic we would recommend [Trefethen and Bau III, 1997]. Subsequently, for $m \geq p$, we can express $K \in \mathbb{R}^{m \times p}$ using its *Singular Value Decomposition*, i.e.

$$K = F \Sigma E^T,$$

where $F \in \mathbb{R}^{m \times m}$, $\Sigma \in \mathbb{R}^{m \times p}$ and $E \in \mathbb{R}^{p \times p}$. Note, $E = (\phi_1, \dots, \phi_p)$ and $F = (\varphi_1, \dots, \varphi_m)$ are orthogonal, i.e. $E^T E = I_{p \times p}$ and $F^T F = I_{m \times m}$. Furthermore,

$$\Sigma = \text{diag}(k_1, \dots, k_p), \text{ where } k_1 \geq k_2 \geq \dots k_p \geq 0,$$

where the $\{k_i\}_{i=1}^p$ are known as the *singular values* of K .

Consequently, if $m = p$, one can define the inverse of K , if it exists, as $K^{-1} = E \Sigma^{-1} F^T$. Note that if K is a symmetric positive-definite matrix, then the singular values equal K 's eigenvalues. Additionally, the matrix K has full rank, and therefore an inverse, only if all of its singular values are non-zero. If K is singular then $k_p = 0$. Thus, one can show that

$$\|K\|_{op} = k_1, \text{ and } \|K^{-1}\|_{op} = k_p^{-1}.$$

Whilst the above is not proved here, one can find the proof in [Trefethen and Bau III, 1997]. Consequently, the condition number of K can then be simplified to

$$\text{cond}(K) = \frac{k_1}{k_p}.$$

First, let us try to recover μ in the absence of noise, whilst continuing to assume that K is ill-conditioned. Note F 's orthogonality implies

$$Y = K\mu \iff Y = F \Sigma E^T \mu \iff F^T Y = \Sigma E^T \mu.$$

Therefore, using Least Squares

$$\mu_{svd} := \min_{\mu} \|Y - K\mu\| = \min_{\mu} \|F^T Y - \Sigma E^T \mu\|.$$

Thus, since Σ is a diagonal matrix with (possibly) some zero diagonal elements,

$$\mu_{svd} = \sum_{i=1}^p \frac{Y^T \varphi_i}{k_i} \phi_i \mathbb{I}_{\{k_i \neq 0\}},$$

where $\mathbb{I}_{\{\cdot\}}$ corresponds to the Indicator function.

Whilst this solution is valid when there is no noise, if the converse is true, this method amplifies any noise present due to the decreasing k_i (a consequence of the large condition number). Thus, one again, the need for Regularisation Methods.

Note that the $\{\varphi_i\}_{i=1}^m$ form an orthonormal basis for the range of K , whilst the $\{\phi_i\}_{i=1}^p$ do the same for \mathbb{R}^p , with $K\phi_i = k_i\varphi_i$. Therefore,

$$\mu = \sum_{i=1}^p (\mu^T \phi_i) \phi_i \implies K\mu = \sum_{i=1}^p (\mu^T \phi_i) k_i \varphi_i.$$

Furthermore, if Y is in the range of K then

$$Y = \sum_{i=1}^m (Y^T \varphi_i) \varphi_i.$$

Thus, the *SVD* method can also be used to decompose the terms μ , $K\mu$ and Y . Additionally, define $k_i = 0$ for $i \geq p$, $Y_i = Y^T \varphi_i$, $\mu_i = \mu^T \phi_i$ and assume

$$Z = \sum_{i=1}^m z_i \varphi_i.$$

Then

$$\begin{aligned} Y = K\mu + \epsilon Z &\iff \sum_{i=1}^m (Y^T \varphi_i) \varphi_i = \sum_{i=1}^m [(\mu^T \phi_i) k_i + \epsilon z_i] \varphi_i \\ &\iff Y_i = \mu_i k_i + \epsilon z_i, \text{ for } i \leq m. \end{aligned}$$

The latter is referred to as the *Sequence Space Model*.

Note, we can use an analogous method to recover μ and obtain the sequence space formulation for the infinite dimensional setting as well. Next, we construct SVD estimators.

1.3.6 SVD Estimators (TSVD, SSVD, and Tikhonov)

Recall, for the noisy model (6), we can construct the naive SVD estimator (by ignoring Z)

$$\hat{\mu}_{svd} = \sum_{i=1}^p \frac{Y_i}{k_i} \phi_i \mathbb{I}_{\{k_i \neq 0\}},$$

where $Y_i := Y^T \varphi_i$.

Observe, there will be some components that are dominated by noise. For instance, assume

$$Z = \sum_{i=1}^m z_i \varphi_i,$$

then

$$\frac{Y_i}{k_i} \approx \frac{\epsilon z_i}{k_i},$$

where the k_i form a decreasing sequence. Hence, the best way to proceed would be to discard these noisy components by retaining the first N components. Doing so leads to the TSVD method and the estimator,

$$\hat{\mu}_{tsvd} = \sum_{i=1}^N \frac{Y_i}{k_i} \phi_i \mathbb{I}_{\{k_i \neq 0\}}.$$

Note that the truncation parameter, N , should be chosen based on the noisy components rather than the size of the k_i .

There is also a variant of the TSVD, known as the Selective SVD (SSVD), where one discards components if their magnitude is below a certain threshold, δ , i.e.

$$\hat{\mu}_{ssvd} = \sum_{|Y_i| > \delta} \frac{Y_i}{k_i} \phi_i \mathbb{I}_{\{k_i \neq 0\}}.$$

This method is advantageous when there exist several Y_i that are small. Obviously, for problems where Y_i decay one can expect little difference between the two methods.

The advantages of both these methods is that they are easy to compute for different N and δ . However, they both require the computation of the SVD which can be computationally overwhelming when tackling large scale problems.

In addition, whilst not obvious at first, the Tikhonov problem is similar to the SSVD as well. Using the SVD of K , i.e. $K = F \Sigma E^T$ and recalling that $EE^T = I$, we can see that

$$\begin{aligned} \hat{\mu}_\lambda &= (E \Sigma^2 E^T + \lambda^2 E E^T)^{-1} E \Sigma F^T Y = E (\Sigma^2 + \lambda^2 I)^{-1} E^T E \Sigma F^T Y \\ &= E (\Sigma^2 + \lambda^2 I)^{-1} \Sigma F^T Y \\ &= \sum_{i=1}^p \tau_i \frac{Y_i}{k_i} \phi_i \mathbb{I}_{\{k_i \neq 0\}}, \end{aligned}$$

where the filtering factor,

$$\tau_i = \frac{k_i^2}{k_i^2 + \lambda^2} \approx \begin{cases} 1, & \text{if } k_i^2 \geq \lambda^2, \\ k_i^2 / \lambda^2, & \text{if } k_i^2 \leq \lambda^2. \end{cases}$$

Consequently, we see that τ_i either retains or shrinks the component $\frac{Y_i}{k_i}$, and unlike the SSVD (and TSVD) does not require the computation of the SVD. Furthermore, the

filtering occurs in a smooth fashion unlike in the SVD methods. Much more could be said about these methods and their effectiveness and we encourage the reader to consult [Hansen, 2010] for further details.

As we can see, most estimators in the Classical Approach are constructed naively, i.e. the noise is ignored. Subsequently, the best way to proceed would be to use our knowledge of Z 's randomness to find a solution, i.e. employ Statistical methods. However, note under certain conditions, these Statistical Methods will in fact coincide with the Deterministic Methods described above.

1.3.7 Statistical Approaches: Maximum Likelihood Method

Whilst regularisation methods ignore the stochastic attributes of the noise, statistical methods view Y as a realisation of a random process. In this section we assume μ deterministic and our goal is to estimate the parameters that characterize Y 's probability distribution (i.e. a Frequentist setting). This section is largely based on [Bertero and Boccacci, 2020].

Note that in Section 1.3.10, we shall discuss methods for when μ is instead a random process (i.e. a Bayesian setting). Additionally, we shall also show, under certain conditions, that the Maximum Likelihood Method is equivalent to the Least Squares Method and the Bayesian Method to Tikhonov Regularisation.

Consider the finite dimensional noisy model (6). To proceed with our analysis we must make several assumptions.

- First, Z and consequently Y , will correspond to m -dimensional random vectors.
- Second, we shall assume the expectation of Y is $K\mu$, i.e. given a probability density, $p_Y(\cdot)$, the

$$\mathbb{E}(Y) = \int_{\mathbb{R}^m} y p_Y(y) dy = K\mu.$$

However μ is unknown, thus we make a third assumption.

- We shall assume $Y \in \{Y(\tilde{\mu})\} := \{K\tilde{\mu} + \epsilon Z | \tilde{\mu} \in \mathbb{R}^p\}$, and the probability density of $Y(\tilde{\mu})$, i.e. $p_Y(\cdot | \tilde{\mu})$, is known.

Consequently, our problem is as follows: Given y , i.e. a noisy image, which is a realization of the random vector Y , which member of the family $\{Y(\tilde{\mu})\}$ is most likely to represent μ ? This can be answered using the *Maximum Likelihood (ML) Method*.

Given $Y = y$, we can evaluate Y 's probability density assuming an arbitrary $\tilde{\mu}$, and denote it as

$$L(\tilde{\mu}) := p_Y(y | \tilde{\mu}),$$

where $L(\tilde{\mu})$ is called the *Likelihood Function*. Subsequently, the *Maximum Likelihood Estimate* of μ is the $\tilde{\mu}$ that maximises $L(\tilde{\mu})$:

$$\hat{\mu}_{ML} = \max_{\tilde{\mu}} L(\tilde{\mu}).$$

Informally, it is the object $\tilde{\mu}$ which maximises the probability of observing the given y . Obviously we can find the maxima of this function by evaluating its derivatives.

However, if there are a large number of variables involved, not only could the computations be quite complex, our Likelihood function may also contain several local maxima; though in that case we would choose the global maximum. Additionally, if the components of our random vector Y were independent, our likelihood function would be the product of functions. Consequently, since evaluating the derivative of the product of functions can be quite cumbersome, we can simplify the forementioned problem by considering

$$l(\tilde{\mu}) := \ln L(\tilde{\mu}) = \ln p_Y(y|\tilde{\mu}).$$

Note the log-likelihood function's ($l(\tilde{\mu})$'s) maxima are the same as its counterpart, $L(\tilde{\mu})$. We conclude by demonstrating the ML method for when the noise is Gaussian, as well as Poisson. Additionally, under these conditions, the Maximum Likelihood method will coincide with the Least Squares estimator.

1.3.8 ML Method: Gaussian Noise

In most cases, we tend to assume the noise, Z , is a zero mean Gaussian random vector, i.e. the components of Z , Z_i , are normal random variables with mean zero. Thus, for the noisy finite dimensional model (6),

$$\mathbb{E}(Z) = 0 \implies \mathbb{E}(Y) = K\mu.$$

Thus assuming the Z_i are correlated, with correlation matrix Σ , the joint density of $Z = z$ is

$$p_Z(z) = [(2\pi)^m |\Sigma|]^{-1/2} \exp\left(-\frac{\Sigma^{-1} z \cdot z}{2}\right),$$

where $|\Sigma|$ is the determinant of Σ , and (\cdot) is the scalar dot product.

If the Z_i have zero expectation then $(\Sigma)_{i,j} = \mathbb{E}(Z_i Z_j)$. This in turn implies that Σ is positive semi-definite, since

$$\Sigma z \cdot z = \sum_j \left(\sum_i \mathbb{E}(Z_j Z_i z_i) \right) z_j = \mathbb{E} \left(\sum_j \sum_i Z_j Z_i z_i z_j \right) = \mathbb{E}((z \cdot Z)^2) \geq 0.$$

In addition, Σ is positive definite and therefore invertible, i.f.f. $\mathbb{P}(Z_i = 0) \neq 1$ for all i . Thus, from now on we shall assume the latter so as to ensure Σ^{-1} , and consequently $p_Z(\cdot)$, exist.

Furthermore, note that the Z_i would be uncorrelated if Σ was a diagonal matrix, i.e.

$$(\Sigma)_{i,j} = \sigma_i^2 \delta_{i,j},$$

where σ_i^2 is the variance of Z_i and $\delta_{i,j}$ is the Kronecker delta function. Consequently, since the Z_i are uncorrelated and normally distributed, this would imply they were independent as well. In addition, if $\sigma_i^2 = \sigma^2$ for all i , then $\Sigma = \sigma^2 I$, and the corresponding model is referred to as the *white noise* model.

Regardless, if the additive noise, Z , has zero expectation we know that $p_Y(y|\mu) = p_{\epsilon Z}(y - K\mu)$, subsequently

$$p_Y(y|\mu) = [(2\pi)^m \epsilon^2 |\Sigma|]^{-1/2} \exp\left(-\frac{\epsilon^{-2} \Sigma^{-1} (y - K\mu) \cdot (y - K\mu)}{2}\right),$$

where $|\Sigma|$ refers to the determinant of Σ .

Therefore,

$$l(\mu) = -\frac{1}{2} [m \ln(2\pi) + \ln(\epsilon^2 |\Sigma|)] - \frac{\epsilon^{-2} \Sigma^{-1} (y - K\mu) \cdot (y - K\mu)}{2}.$$

Hence, we see that the

$$\hat{\mu}_{ML} = \max_{\mu} l(\mu) = \min_{\mu} \epsilon^{-2} \Sigma^{-1} (y - K\mu) \cdot (y - K\mu).$$

Note that in the case of white noise, i.e. when $\Sigma^{-1} = \sigma^{-2} I$,

$$\hat{\mu}_{ML} = \max_{\mu} l(\mu) = \min_{\mu} (y - K\mu) \cdot (y - K\mu) = \min_{\mu} \|y - K\mu\|^2.$$

which is equivalent to the *Least Squares* problem. Hence, the ML Method suffers from the same weaknesses as the Least Squares Method, i.e. not being continuous w.r.t Y . Please see Section 1.3.2 for more details.

1.3.9 ML Method: Poisson Noise

As discussed in Sections 3.5 and 6, we would like to study Poisson inverse problems in the future. Therefore, consider (6) where Z is multivariate Poisson, i.e. $Z_i \sim \text{Pois}(\lambda_i)$.

Then for λ_i sufficiently large,

$$\frac{Z_i - \lambda_i}{\sqrt{\lambda_i}} \xrightarrow{D} N(0, 1),$$

via the Central Limit Theorem (CLT).

Consequently,

$$Y_i \approx N((K\mu)_i + \epsilon \lambda_i, \epsilon^2 \lambda_i) \implies p_{Y_i}(y_i|\mu) \approx [2\pi \epsilon^2 \lambda_i]^{-1/2} \exp\left(-\frac{(y_i - (K\mu)_i - \epsilon \lambda_i)^2}{2\epsilon^2 \lambda_i}\right).$$

Therefore, using our results from Section 1.3.8 with $(\Sigma)_{i,j} = \lambda_i \delta_{i,j}$ implies

$$\hat{\mu}_{ML} = \max_{\mu} l(\mu) = \min_{\mu} \epsilon^{-2} \Sigma^{-1} (\tilde{y} - K\mu) \cdot (\tilde{y} - K\mu), \quad (7)$$

where $\tilde{y}_i = y_i - (K\mu)_i - \epsilon \lambda_i$.

Note that $(\Sigma^{-1})_{i,j} = \lambda_i^{-1} \delta_{i,j}$, and whilst we knew the variance in Section 1.3.8, in this section we shall assume λ_i is a function of μ and is therefore unknown. However, these unknown parameters can be replaced by their likelihood estimates.

Thus, recalling that $Y_i \sim \text{Pois}(\lambda_i)$, we can state its log-likelihood,

$$l(\lambda_i) = y_i \ln \lambda_i - \ln(y_i!) - \lambda_i \implies l'(\lambda_i) = \frac{y_i}{\lambda_i} - 1, \text{ and } l''(\lambda_i) = -\frac{y_i}{\lambda_i^2}.$$

Thus we can use the above to infer the maximum of $l(\lambda_i)$ is $\hat{\lambda}_i = y_i$. Subsequently, using these substitutions in (7) our ML Method would provide us with the following *Weighted Least Squares problem* for $Y = y$:

$$\hat{\mu}_{ML} = \max_{\mu} l(\mu) = \min_{\mu} \sum_{i=1}^m y_i^{-1} ([1 - \epsilon]y - K\mu)_i^2.$$

Thus we have seen that statistical methods are equivalent to classical deterministic methods (i.e. Least Squares) under certain conditions. Next, we shall discuss Bayesian methods and how they can be equivalent to Tikhonov regularisation methods.

1.3.10 Bayesian Methods

In this section, we assume the noisy model (6), where Y and μ are also random processes, with known probability distributions $p_Y(\cdot)$ and $p_{\mu}(\cdot)$, and realisations y and $\tilde{\mu}$, respectively. The distribution of μ is referred to as the *prior*.

Thus, our problem now consists of obtaining information about μ given that we have an observation of Y , i.e. y . This information can be obtained by computing $p_{\mu|Y}(\cdot)$, which is known as the *a posteriori* density function of μ .

Due to the relationship above, the probabilistic nature of the problem above can be encapsulated by the joint distribution of only 2 of the 3 variables (Y , μ , and Z). Typically, we choose μ and Y , and denote their joint probability as $p_{\mu,Y}(\tilde{\mu}, y)$. Consequently, due to the relationship between joint and marginal distributions, we obtain

$$p_{\mu}(\tilde{\mu}) = \int_{\mathbb{R}^m} p_{\mu,Y}(\tilde{\mu}, y) dy.$$

Furthermore, since we know $p_Y(y|\mu)$ and $p_{\mu}(\tilde{\mu})$, one can ascertain that

$$p_{\mu,Y}(\tilde{\mu}, y) = p_Y(y|\tilde{\mu})p_{\mu}(\tilde{\mu}).$$

Subsequently, we can now deduce

$$p_{\mu|Y}(\tilde{\mu}|y) = \frac{p_{\mu,Y}(\tilde{\mu}, y)}{p_Y(y)} = \frac{p_Y(y|\tilde{\mu})p_{\mu}(\tilde{\mu})}{\int_{\mathbb{R}^n} p_Y(y|\tilde{\mu})p_{\mu}(\tilde{\mu}) d\tilde{\mu}},$$

which is known as *the Bayes Formula*.

Now note that whilst $p_{\mu|Y}(\tilde{\mu}|y)$ does not give us a unique estimate but a set of possible estimates, (with corresponding probabilities), we can nonetheless use it to compute various approximations of μ . In practice, we tend to use the Expectation or the Maximum A Posteriori Estimate (MAP) of μ , i.e.

- the Expected value of μ given Y ,

$$\bar{\mu} = \mathbb{E}(\mu|Y) = \int_{\mathbb{R}^n} \tilde{\mu} p_{\mu|Y}(\tilde{\mu}|Y) d\tilde{\mu}.$$

- the MAP of μ given Y ,

$$\hat{\mu}_{MAP} = \max_{\mu} p_{\mu|Y}(\tilde{\mu}|Y).$$

Note that if $p_{\mu}(\tilde{\mu}|Y)$ has a unique global maximum, around which it is concentrated, then $\bar{\mu} \cong \hat{\mu}_{MAP}$. We generally assume μ to be normally distributed, since the Normal distribution is flexible and robust. Furthermore, in the presence of white noise, the posterior distribution can be computed easily as it too will be Gaussian (as we see in the subsequent section). However, whilst Bayesian methods are flexible and allow one to implement sophisticated constraints, they are also computationally expensive.

1.3.11 Bayesian Method: Gaussian Case

For the Gaussian case we shall assume the following:

- $Z \sim N(0, \Sigma)$, which implies

$$p_Z(z) = [(2\pi)^m |\Sigma|]^{-1/2} \exp\left(-\frac{\Sigma^{-1} z \cdot z}{2}\right).$$

- $\mu \sim N(0, \Lambda)$, which implies

$$p_{\mu}(\tilde{\mu}) = [(2\pi)^p |\Lambda|]^{-1/2} \exp\left(-\frac{\Lambda^{-1} \tilde{\mu} \cdot \tilde{\mu}}{2}\right).$$

- μ and Z are independent random vectors, which in turn implies

$$\mathbb{E}(\mu_i Z_j) = 0.$$

Consequently, since $p_Y(y) = p_{\epsilon Z}(y - K\mu)$, using Bayes formula we know the joint probability of (μ, Y) is

$$\begin{aligned} p_{\mu, Y}(\tilde{\mu}, y) &= p_Y(y|\tilde{\mu}) p_{\mu}(\tilde{\mu}) = p_{\epsilon Z}(y - K\tilde{\mu}) p_{\mu}(\tilde{\mu}) \\ &= [(2\pi)^m \epsilon^2 |\Sigma|]^{-1/2} \exp\left(-\frac{\epsilon^{-2} \Sigma^{-1} (y - K\tilde{\mu}) \cdot (y - K\tilde{\mu})}{2}\right) [(2\pi)^p |\Lambda|]^{-1/2} \exp\left(-\frac{\Lambda^{-1} \tilde{\mu} \cdot \tilde{\mu}}{2}\right) \\ &= [(2\pi)^{m+p} \epsilon^2 |\Sigma| |\Lambda|]^{-1/2} \exp\left(-\frac{1}{2} \Phi(\tilde{\mu}, y)\right), \end{aligned}$$

where $\Phi(\tilde{\mu}, y) = \epsilon^{-2} \Sigma^{-1} (y - K\tilde{\mu}) \cdot (y - K\tilde{\mu}) + \Lambda^{-1} \tilde{\mu} \cdot \tilde{\mu}$.

Subsequently, from the above equations, we know the marginal distribution of Y will be normally distributed as well, with $\mathbb{E}(Y) = 0$ and

$$\begin{aligned} \mathbb{V}ar(Y) &= K \mathbb{V}ar(\mu) K^T + \mathbb{V}ar(\epsilon Z) + 2 \mathbb{C}ov(K\mu, \epsilon Z) = K \Lambda K^T + \epsilon^2 \Sigma + 2\epsilon K \mathbb{C}ov(\mu, Z) \\ &= K \Lambda K^T + \epsilon^2 \Sigma. \end{aligned}$$

Let $V := K\Lambda K^T + \epsilon^2\Sigma$. Then $Y \sim N(0, V)$, with

$$p_Y(y) = [(2\pi)^m |V|]^{-1/2} \exp\left(-\frac{V^{-1}y \cdot y}{2}\right),$$

and hence

$$p_{\mu|Y}(\tilde{\mu}|y) = \frac{p_{\mu,Y}(\tilde{\mu}, y)}{p_Y(y)} = [(2\pi)^p \frac{|\Sigma||\Lambda|}{|V|}]^{-1/2} \exp\left(-\frac{1}{2}[\Phi(\tilde{\mu}, y) - V^{-1}y \cdot y]\right).$$

This not only implies that $\mu|Y$ is normally distributed, but that $\bar{\mu} = \hat{\mu}_{MAP}$. Whilst the latter statement is not proved in this project, it is in [Bertero and Boccacci, 2020].

Furthermore,

$$\hat{\mu}_{MAP} = \max_{\tilde{\mu}} p_{\mu|Y}(\tilde{\mu}|Y) = \min_{\tilde{\mu}} \Phi(\tilde{\mu}, g) = \min_{\tilde{\mu}} \epsilon^{-2}\Sigma^{-1}(y - K\tilde{\mu}) \cdot (y - K\tilde{\mu}) + \Lambda^{-1}\tilde{\mu} \cdot \tilde{\mu}.$$

Hence if $\Sigma = \sigma^2 I$ and $\Lambda = 0$, then $\hat{\mu}_{MAP} = \hat{\mu}_{ML}$ from Section 1.3.8, which is a Least Squares problem. Additionally, if $\Sigma = \sigma^2 I$ and $\Lambda = \lambda^2 I$, then we obtain a Tikhonov Regularisation problem, i.e.

$$\hat{\mu}_{MAP} = \min_{\tilde{\mu}} \|y - K\tilde{\mu}\|^2 + \tilde{\lambda}^2 \|\tilde{\mu}\|^2,$$

where $\tilde{\lambda} = \lambda/\sigma$ and $\|\cdot\|$ is the standard Euclidean norm.

Thus we see that statistical methods can be equivalent to the classical deterministic methods under certain conditions. Subsequently, the rest of this project will consist of reviewing and adapting the results presented in [Knapik et al., 2011]. However before doing so it would be wise to review Gaussian processes in Hilbert spaces, Appendix A.1, along with SVD in infinite dimensions, Section 1.4.3.

1.4 Infinite Dimensional Setting

This section will require that the reader be familiar with several aspects of Operator theory and Hilbert Spaces. For a brief review of the following concepts please peruse Appendix A.1. Subsequently, consider the problem of estimating μ given the infinite dimensional model (3).

Note, in this section, given an element x in some Hilbert space, x_i will refer its eigenvalue, and no longer correspond to any vector notation.

1.4.1 Classical (Deterministic) Approaches

Since inner products, and their induced norms, are well defined on Hilbert Spaces, the arguments used in Sections 1.3.2 and 1.3.4 still hold in the infinite dimensional setting.

Consequently, the Least Squares problem is

$$\min_{\mu \in H_2} \|Y - K\mu\|_{H_2},$$

where $\|\cdot\|_{H_i}$ is the norm induced by H_i 's inner product, $\langle \cdot, \cdot \rangle_{H_i}$, for $i = 1, 2$.

The existence of a solution is guaranteed if $Y \in R(K) \oplus R(K)^\perp$. Furthermore, if $(K^T K)^{-1}$ exists, then the solution can be explicitly written as

$$\hat{\mu}_{ls} = (K^T K)^{-1} K^T Y,$$

where K^T is the adjoint of K .

Similarly the Tikhnov estimator is

$$\hat{\mu}_\lambda = \arg \min_{\mu \in H_1} L_\lambda(\mu),$$

where

$$L_\lambda(\mu) = \|Y - K\mu\|_{H_2}^2 + \lambda^2 \|\mu\|_{H_1}^2,$$

and $\lambda > 0$ is some tuning parameter.

By considering the differential of $L_\lambda(\mu)$ in $h \in H_1$, and noting its strict convexity for any $\lambda > 0$,

$$\hat{\mu}_\lambda = (K^T K + \lambda^2 I)^{-1} K^T Y.$$

Consequently, just as in the finite setting, we can decompose model (3) using SVD.

1.4.2 Generalisation of Stochastic Noise in Infinite Space

Before discussing Singular Value Decomposition, it would be wise to consider how one defines stochastic noise in a Hilbert Space. Observe, that Z cannot be realised as a random element in H_2 . It is interpreted as a process instead, i.e. $Z = \{Z_h := \langle Z, h \rangle_{H_2} : h \in H_2\}$. Subsequently, Z_h can now be viewed as a random element, as required.

Formally, $Z : H_2 \rightarrow L^2(\Omega, \mathcal{S}, P)$ is a bounded linear operator, where (Ω, \mathcal{S}, P) is some probability space and $L^2(\cdot)$ is the space of all square integrable measurable functions.

Thus, for all elements $g_1, g_2 \in H_2$, we can define the mean

$$\mathbb{E}Z_{g_i} = \mathbb{E}\langle Z, g_i \rangle_{H_2} = \langle m, g_i \rangle_{H_2}, \text{ for } i = 1, 2,$$

for some $m \in H_2$, hereafter referred to as the mean function. Likewise, the covariance

$$\text{Cov}(Z_{g_i}, Z_{g_j}) = \langle \Lambda g_i, g_j \rangle_{H_2}, \text{ for } i, j = 1, 2,$$

for some bounded linear operator $\Lambda : H_2 \rightarrow H_2$, hereafter referred to as the covariance operator. Consequently, we can now define a white noise Gaussian process in H_2 .

Definition 1.5. *We define Z to be a Gaussian white noise process in H_2 , denoted as $Z \sim N(0, I)$, if the mean function $m = 0$, the covariance operator $\Lambda = I$, and Z_h is normally distributed.*

Note, if Z is Gaussian white noise (i.e. $Z \sim N(0, I)$) then

$$\mathbb{E}Z_{g_i} = \langle 0, g_i \rangle_{H_2} = 0, \text{ and } \text{Cov}(Z_{g_i}, Z_{g_j}) = \langle g_i, g_j \rangle_{H_2} \implies Z_{g_i} \sim N(0, \|g_i\|^2).$$

Additionally, the following definition will be important when considering Heterogeneous Stochastic Noise (Section 3).

Definition 1.6. For any bounded linear operator $A : H_2 \rightarrow H_1$, we define AZ by

$$\langle AZ, h \rangle_{H_2} = \langle Z, A^T h \rangle_{H_2},$$

for all h in H_2 .

1.4.3 Generalisation of Singular Value Decomposition in Infinite Space

Consider the following theorem from [Alquier et al., 2011] (Theorem 1.1):

Theorem. Let $A : H \rightarrow H$ be a self-adjoint, compact linear bounded operator. Then there exists a complete orthonormal set $F := \{\phi_j : j \in I\}$ of H consisting of eigenfunctions of A , with corresponding eigenvalues a_j . Here I is some index set and $A\phi_j = a_j\phi_j$ for $j \in I$. The set $J := \{j \in I : a_j \neq 0\}$ is countable and

$$Ah = \sum_{j \in I} a_j \langle h, \phi_j \rangle \phi_j,$$

for all $h \in H$. Moreover, for any $\delta > 0$ the set $J_\delta := \{j \in I : |a_j| \geq \delta\}$ is finite.

Assume the operator K is compact, then $K^T K : H_1 \mapsto H_1$ is strictly positive and self-adjoint. Thus, applying the above theorem implies $K^T K$ possesses countably many positive eigenvalues (also called *singular values*) $\{k_i^2\}_{i=1}^\infty$ that are non-increasing and converge to 0. Additionally, $K^T K \phi_i = k_i^2 \phi_i$ for $i \in \mathbb{N}$, with $\{\phi_i\}_{i=1}^\infty$ being a complete orthonormal basis of H_1 .

Subsequently, for $\{\phi_j\}_{j=1}^\infty \in H_1$, we can define its (normalised) image under K by $\{\varphi_j\}_{j=1}^\infty \in H_2$, where

$$\varphi_j = k_j^{-1} K \phi_j.$$

Note, that $\{\varphi_j\}_{j=1}^\infty$ will be orthonormal as well, since

$$\langle \varphi_i, \varphi_j \rangle = k_i^{-1} k_j^{-1} \langle K \phi_i, K \phi_j \rangle = k_i^{-1} k_j^{-1} \langle K^T K \phi_i, \phi_j \rangle = k_i k_j^{-1} \langle \phi_i, \phi_j \rangle = k_i^1 k_j^{-1} \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta function.

Therefore, since

$$K^T \varphi_j = k_j^{-1} K^T K \phi_j = k_j^{-1} k_j^2 \phi_j = k_j \phi_j,$$

we have that

$$K \phi_j = k_j \phi_j, \text{ and } K^T \varphi_j = k_j \varphi_j.$$

Consequently, we obtain the following definition:

Definition 1.7. We state A admits a Singular Value Decomposition (SVD) if, for all $h \in H$,

$$A^T A h = \sum_{i=1}^{\infty} a_i^2 \langle h, \phi_i \rangle \phi_i,$$

where a_j are the eigenvalues of A , and $\{\phi_j\}_{j=1}^\infty$ is an orthonormal basis of H .

Thus, assuming K is compact, the SVD of K is

$$K^T K h = \sum_{j=1}^{\infty} k_j^2 \langle h, \phi_j \rangle \phi_j,$$

for any $h \in H_1$.

Furthermore, using $\{\phi_j\}_{j=1}^{\infty}$ and $\{\varphi_j\}_{j=1}^{\infty}$, any $\mu \in H_1$ can be expressed in the form

$$\mu = \sum_{i=1}^{\infty} \mu_i \phi_i,$$

where $\mu_i := \langle \mu, \phi_i \rangle_{H_1}$, and its image in the form

$$K\mu = \sum_{i=1}^{\infty} \mu_i K\phi_i = \sum_{i=1}^{\infty} k_i \mu_i \varphi_i.$$

Additionally, assuming Z is a stochastic process $Z_h \in H_2$,

$$Y_i := \langle Y, \varphi_i \rangle_{H_2} = \langle K\mu, \varphi_i \rangle_{H_2} + \langle \epsilon Z, \varphi_i \rangle_{H_2} = k_i \mu_i + \epsilon Z_i,$$

where $Z_i := \langle Z, \varphi_i \rangle_{H_2}$.

Consequently,

$$Y = \sum_{i=1}^{\infty} Y_i \varphi_i.$$

Thus, analogous to the finite dimensional setting, we obtain the following *Sequence space model*

$$Y = K\mu + \epsilon Z \iff Y_i = k_i \mu_i + \epsilon Z_i, \text{ for all } i \geq 1.$$

1.4.4 Ill-Posedness and SVD Estimators

Subsequently, just as in the finite dimensional setting, we can use the SVD of K to describe the ill-posedness of the model (4).

Recall, we can construct the naive SVD estimator

$$\hat{\mu}_{svd} = \sum_{i=1}^{\infty} \frac{Y_i}{k_i} \phi_i.$$

However, since $k_i \downarrow 0$, the errors (Z_i) in Y_i will eventually explode. Consequently the problem is ill-posed, and this ill-posedness is defined by the rate at which $k_i \downarrow 0$. Informally, this gives us a notion akin to ill-conditionedness even in an infinite dimensional setting.

Definition 1.8. *If there exists $q > 0$ such that the $k_i = \mathcal{O}(i^{-q})$ then q is the degree of ill-posedness.*

- If $0 < q \leq 1$, then the problem is mildly ill-posed.
- If $q > 1$, then the problem is moderately ill-posed.
- If $k_i = \mathcal{O}(e^{-iq})$, then the problem is severely ill-posed.

Remark 1.3. [Wahba, 1977] was one of the first to discuss this method for measuring ill-posedness. As stated in [Cavalier, 2008], there exist other ways of defining the degree of ill-posedness, such as using the noise structure, the smoothness assumptions on μ or the smoothing properties of K . For instance, [Mathé and Pereverzev, 2001] discusses ill-posedness in a Hilbert Scales setting.

Thus, once again just as in the finite dimensional setting, we see that the SVD of K is susceptible to noise, hence the need to employ regularisation methods. Fortunately, as we saw with $\hat{\mu}_{svd}$, the estimators constructed in Section 1.3.6 will hold in the infinite dimensional setting as well.

Specifically, the TSVD estimator is

$$\hat{\mu}_{tsvd} = \sum_{i=1}^N \frac{Y_i}{k_i} \phi_i,$$

with truncation parameter N .

Additionally, the Selective SVD (SSVD) is

$$\hat{\mu}_{ssvd} = \sum_{|Y_i| > \delta} \frac{Y_i}{k_i} \phi_i.$$

Furthermore, just as in the finite dimensional setting, the Tikhonov problem is similar to the SSVD as well. Using the SVD of K ,

$$\langle \hat{\mu}_\lambda, \phi_i \rangle_{H_1} = \langle (K^T K + \lambda^2 I)^{-1} K^T Y, \phi_i \rangle_{H_1} = \tau_i \frac{Y_i}{k_i},$$

where the filtering factor,

$$\tau_i = \frac{k_i^2}{k_i^2 + \lambda^2} \approx \begin{cases} 1, & \text{if } k_i^2 \geq \lambda^2, \\ k_i^2 / \lambda^2, & \text{if } k_i^2 \leq \lambda^2. \end{cases}$$

1.4.5 Statistical Approaches

Whilst the Bayesian Approach for the infinite dimensional setting will be discussed in Sections 2 and 3, in this section we will restrict our attention to the Maximum Likelihood (ML) method for Gaussian white noise. However, Section 5 considers a setting similar to that in Section 1.3.9, (Poisson noise); specifically, the noise is *signal dependent* i.e. it depends on μ .

Note, once again, we will follow the outline depicted in the finite dimensional setting, c.f. Sections 1.3.7 and 1.3.8. Hence, recall for some stochastic process Z with known probability measure,

$$\hat{\mu}_{ML} = \max_{\tilde{\mu}} L(\tilde{\mu}),$$

where $L(\tilde{\mu})$ is the Likelihood Function.

Let Z be Gaussian white noise, as defined in Section 1.4.2. Then,

$$Y|\mu \sim N(K\mu, \epsilon^2 I).$$

Consequently,

$$L(\tilde{\mu}) = \prod_{i=1}^{\infty} [(2\pi)\epsilon^2]^{-1/2} \exp(-\frac{1}{2}\epsilon^{-2}(y_i - k_i\tilde{\mu}_i)^2).$$

Thus, using Parseval's Identity,

$$L(\tilde{\mu}) = \frac{1}{c(y)} \exp(-\frac{1}{2}\epsilon^{-2}||Y - K\tilde{\mu}||_{H_2}^2),$$

where $c(y) \in (0, \infty)$ is the normalising constant. Therefore,

$$\hat{\mu}_{ML} = \max_{\tilde{\mu}} L(\tilde{\mu}) = \min_{\tilde{\mu}} ||Y - K\tilde{\mu}||_{H_2}^2.$$

Note, its equivalence to the Least Squares problem, just as in the finite dimensional setting, c.f. Section 1.3.8.

2 Bayesian Inverse Problems with Homogeneous Variance

Remark 2.1. *In this section, we review the paper: *Bayesian Inverse Problems with Gaussian Priors (2011)* by Knapik et al. Therefore, all of the results stated are from said paper, i.e. [Knapik et al., 2011].*

The objective of the paper, [Knapik et al., 2011], was to estimate a parameter μ from an observation Y given Model (3), i.e.

$$Y = K\mu + \epsilon Z,$$

where ϵ is the noise level, $\mu \in H_1$ a separable Hilbert Space, and a known, injective, continuous, linear operator K maps μ into another separable Hilbert space, H_2 . Note that $K\mu$ is perturbed by scaled Gaussian white noise Z , (c.f. Definition 1.5), and we shall set

$$\epsilon = \frac{1}{\sqrt{n}}.$$

[Brown and Low, 1996] states that, by letting $n \rightarrow \infty$, the model above (i.e. trying to recover μ completely) is asymptotically equivalent to a non-parametric inverse regression model (i.e. estimating μ pointwise via some experiment).

Now the Bayesian approach consists of computing the posterior distribution for μ by putting a prior on the parameter. In this paper, Gaussian priors are used, since they are *conjugate* to the model, and hence result in the posterior being Gaussian as well, and therefore easily derived.

Note that this paper is reviewed partially as we are only interested in whether and at what rate the posterior distributions contract to the true parameter μ_0 as $\epsilon \rightarrow 0$, or equivalently as $n \rightarrow \infty$.

In summary, under the prior $\mu \sim N(0, \Lambda)$, with Gaussian white noise $Z \sim N(0, I)$, the posterior $\mu|Y$ is derived in Proposition 2.1, ([Knapik et al., 2011]’s Proposition 3.1). Subsequently, by explicitly categorising the behaviour of the singular values of K , Λ and μ_0 in Assumption 2.1, the contraction rates are derived in Theorem 2.1, ([Knapik et al., 2011]’s Theorem 4.1).

2.1 Formal description of the problem with Homogeneous Variance

For this subsection, we will require the reader to be familiar with Gaussian processes and Functional Analysis for Hilbert Spaces. Feel free to peruse Sections 1.4.2 and 1.4.3, and Appendix A.2 for more details. Subsequently, let $\langle \cdot, \cdot \rangle_{H_i}$ and $\| \cdot \|_{H_i}$ refer to the inner product and induced norm of H_i , respectively.

Observe, the stochastic noise Z from Model (3) cannot be realised as a random element in H_2 . Therefore, it is interpreted as a process instead, i.e. $Z = \{Z_h := \langle Z, h \rangle_{H_2} : h \in H_2\}$. Consequently, Z_h can now be viewed as a random element.

Thus, for all elements $g_1, g_2 \in H_2$, we can define the mean

$$\mathbb{E}Z_{g_i} = \mathbb{E}\langle Z, g_i \rangle_{H_2} = \langle m, g_i \rangle_{H_2}, \text{ for } i = 1, 2,$$

for some $m \in H_2$, hereafter referred to as the mean function. Likewise, the covariance

$$\mathbb{C}ov(Z_{g_i}, Z_{g_j}) = \langle \Lambda g_i, g_j \rangle_{H_2}, \text{ for } i, j = 1, 2,$$

for some bounded linear operator $\Lambda : H_2 \rightarrow H_2$, hereafter referred to as the covariance operator.

Consequently, we define Z to be a Gaussian white noise process in H_2 , if the mean function $m = 0$, the covariance operator $\Lambda = I$, and Z_h is normally distributed. We denote said process as

$$Z \sim N(0, I).$$

Note, Z has *Homogeneous* variance, (also referred to as constant variance).

Observe,

$$\mathbb{E}Z_{g_i} = \langle 0, g_i \rangle_{H_2} = 0, \text{ and } \mathbb{C}ov(Z_{g_i}, Z_{g_j}) = \langle g_i, g_j \rangle_{H_2} \implies Z_{g_i} \sim N(0, \|g_i\|^2).$$

Additionally, if given an orthonormal basis for H_2 , $\{\varphi_i\}_{i=1}^\infty$, the $\mathbb{E}Z_{\varphi_i} = 0$, and the $\mathbb{C}ov(Z_{\varphi_i}, Z_{\varphi_j}) = \langle \varphi_i, \varphi_j \rangle_{H_2} = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta function. Therefore,

$$Z_{\varphi_i} \sim N(0, 1).$$

Subsequently, Model (3) can be interpreted as a Gaussian process as well, where $Y = \{Y_h := \langle Y, h \rangle_{H_2} : h \in H_2\}$, with

$$\mathbb{E}Y_{g_i} = \mathbb{E}(\langle K\mu + \epsilon Z, g_i \rangle_{H_2}) = \mathbb{E}(\langle K\mu, g_i \rangle_{H_2} + \frac{1}{\sqrt{n}}Z_{g_i}) = \langle K\mu, g_i \rangle_{H_2},$$

and covariance

$$\begin{aligned} \mathbb{C}ov(Y_{g_i}, Y_{g_j}) &= \mathbb{C}ov(\langle K\mu + \frac{1}{\sqrt{n}}Z, g_i \rangle_{H_2}, \langle K\mu + \frac{1}{\sqrt{n}}Z, g_j \rangle_{H_2}) \\ &= \frac{1}{n} \mathbb{C}ov(Z_{g_i}, Z_{g_j}) = \frac{1}{n} \langle g_i, g_j \rangle_{H_2}. \end{aligned}$$

Thus,

$$Y|\mu \sim N(K\mu, \frac{1}{n}I).$$

Having described the statistical properties of the model, let us look at how one can use the SVD decomposition method to recover μ , (c.f. Section 1.4.4).

Recall that if the operator K is compact, then the self-adjoint operator $K^TK : H_1 \mapsto H_1$ possesses countably many positive eigenvalues k_i^2 and $K^TK\phi_i = k_i^2\phi_i$ for $i \in \mathbb{N}$, where the $\{\phi_i\}_{i=1}^\infty$ are the orthonormal basis of H_1 . Additionally, by constructing a sequence $\{\varphi_i\}_{i=1}^\infty$ such that $K\phi_i = k_i\varphi_i$ we form an orthonormal basis for the range of K in H_2 . This also implies that $KK^T\varphi_i = k_i^2\varphi_i$.

Note, from here on out we shall assume there exists an orthonormal basis of eigenfunctions for K^TK , rather than K being compact.

Consequently, any $\mu \in H_1$ and its image (under K) can be expressed in the forms

$$\mu = \sum_i \mu_i \phi_i,$$

and

$$K\mu = \sum_i \mu_i K\phi_i = \sum_i (\mu_i k_i) \varphi_i.$$

Subsequently, since $Z(\varphi_i) \sim N(0, 1)$, the

$$\mathbb{E}(Y(\varphi_i)) = \langle K\mu, \varphi_i \rangle_{H_2} = \langle \sum_i (\mu_i k_i) \varphi_i, \varphi_i \rangle_{H_2} = \mu_i k_i,$$

and the covariance function is

$$\text{Cov}(Y(\varphi_i), Y(\varphi_j)) = \langle \varphi_i, \frac{1}{n} \varphi_j \rangle_{H_2} = \frac{1}{n} \delta_{i,j}.$$

Hence, we see that $Y(\varphi_i)$ are independent with $N(\mu_i k_i, \frac{1}{n})$ distributions, and our original problem of recovering μ is now equivalent to recovering $\{\mu_i\}_{i=1}^\infty$.

Thus, we consider the *Sequence space model* instead,

$$Y_i = k_i \mu_i + \epsilon Z_i, \text{ for all } i \geq 1,$$

where $Y_i := Y_{\varphi_i}$, and $Z_i := Z_{\varphi_i}$.

If $k_i \rightarrow 0$, (this will definitely be the case if K is compact), the problem is ill-posed, and this ill-posedness is defined by the rate at which $k_i \downarrow 0$, (see Definition 1.8). In this paper, it was assumed that the problem was *mildly ill-posed*, i.e. for some $q \geq 0$,

$$k_i \asymp i^{-q}.$$

Note that the larger the value of q , the harder the estimation of μ , since the decay is faster. Please see Section 1.4.4 and Definition 1.8 for further details regarding ill-posedness.

The Minimax rates (c.f. Definition 1.4) are defined over the Sobolev space of order β , i.e.

$$S^\beta = \{\mu \in H_1 : \|u\|_{S^\beta}^2 < \infty\}, \text{ where } \|u\|_{S^\beta}^2 = \sum_{i=1}^\infty \mu_i^2 i^{2\beta}.$$

Remark 2.2. For those unfamiliar with such a Sobolev space the above is akin to the following Hilbert space, which is based on another form of decomposition i.e. the Fourier Decomposition,

$$W^{k,2}(\mathbb{T}) = \{\mu \in L^2(\mathbb{T}) : \sum_{n=-\infty}^\infty (1 + n^2 + n^4 + \dots n^{2k}) |\hat{\mu}(n)|^2 < \infty\},$$

where $\hat{\mu}$ is the Fourier series of μ . Further details regarding such spaces can be found in [Evans, 1998].

Observe, the purpose of both these spaces is to ensure that the singular values of μ decay sufficiently rapidly. Subsequently, the Bayesian method (c.f. Section 1.3.10) is used to find the posterior distribution of μ given Y .

2.2 Prior and Posterior Distributions

Assume the prior for μ is $N(0, \Lambda)$, and that the noise Z is independent of μ . Subsequently, via *conjugacy*, the joint distribution of (Y, μ) is Gaussian, as is the posterior distribution of $\mu|Y$ i.e. the conditional distribution of μ given Y .

Remark 2.3. *The proof for the equivalence of posterior to the prior w.r.t. absolute continuity has been omitted due to it being highly technical, but can be found in [Knapik et al., 2011].*

Note, from here on out we shall assume $K^T K$ and Λ have the same eigenfunctions, i.e. $\{\phi_i\}_{i=1}^\infty$.

Proposition 2.1 (Homogeneous: Posterior). *If μ is $N(0, \Lambda)$ distributed and Y given μ is $N(K\mu, n^{-1}I)$ distributed, then $\Pi_n(\cdot|Y)$, the conditional distribution of μ given Y , is $N(AY, S_n)$ on H_1 where,*

$$S_n = \Lambda - A(n^{-1}I + K\Lambda K^T)A^T,$$

and $A : H_2 \mapsto H_1$ is the continuous linear operator

$$A = \Lambda^{1/2}(n^{-1}I + \Lambda^{1/2}K^T K \Lambda^{1/2})^{-1} \Lambda^{1/2}K^T = \Lambda K^T(n^{-1}I + K\Lambda K^T)^{-1}.$$

The posterior distribution is proper (i.e. S_n has finite trace) and is equivalent to the prior (in the sense of absolute continuity).

Note this proposition was proved by Knapik et al, [Knapik et al., 2011].

The proof for this proposition has been split into several parts and can be found in Appendix B.1. Therefore, we see that even though the distribution of $Y|\mu$ was improper, the posterior distribution is proper. Furthermore, note that the mean of $\mu|Y$ is random whilst its covariance operator S_n is deterministic. Subsequently, having found the posterior distribution we can construct estimators (c.f. Section 1.3.10) and derive contraction rates, which we compare to the minimax rates.

2.3 Minimax Rates

Assume there exists an *underlying true parameter* μ_0 . Consequently via regularisation methods, such as generalised Tikhonov, one can show that the minimax rate of estimation, (see Definition 1.4), over the unit ball of this space, w.r.t $\|\mu - \mu_0\|_{H_1}$ is

$$\varepsilon_n^* := n^{-\beta/(1+2\beta+2q)},$$

where μ is an estimator of μ_0 . The proof for said optimal rate can be found in [Cavalier et al., 2002] and [Cavalier, 2008]. Consequently, our goal is to determine the rate at which μ contracts to μ_0 with probability 1.

2.4 Contraction Rates for the Posterior

Before we can discuss the contraction rates for μ , (c.f. Definition 1.3), we need to make some assumptions.

Assumption 2.1. *The operators $K^T K$ and Λ have the same eigenfunctions $\{\phi_i\}_{i=1}^\infty$, with eigenvalues $\{k_i^2\}_{i=1}^\infty$ and $\{\lambda_i\}_{i=1}^\infty$, satisfying*

$$\lambda_i = \tau_n^2 i^{-1-2\alpha}, \text{ and } C^{-1}i^{-q} \leq k_i \leq Ci^{-q},$$

for some $\alpha > 0, q \geq 0, C \geq 1$ and $\tau_n > 0$ such that $n\tau_n^2 \rightarrow \infty$. Additionally, the true parameter $\mu_0 \in S^\beta$ for some $\beta > 0$.

Remark 2.4. *Observe, that λ_i , or more specifically τ_n , will be a function of the noise level $\epsilon = n^{-1/2}$. Furthermore, there do exist methods for estimating ϵ in the event that it is unknown, see for instance [Coeurjolly et al., 2014].*

We briefly highlight the purpose of α in the above assumption. Recall, we assume $\mu \sim N(0, \Lambda)$, and we require μ to belong to some Sobolev space (S^r) , with probability 1, a priori. Therefore, via Markov's Inequality, this is equivalent to us showing

$$\mathbb{E}\left(\sum_{i=1}^{\infty} \mu_i^2 i^{2r}\right) < \infty \iff \sum_{i=1}^{\infty} \lambda_i i^{2r} = \sum_{i=1}^{\infty} \tau_n^2 i^{-1-2\alpha} i^{2r} < \infty \iff \alpha > r,$$

where the Expectation can be brought inside the sum due to Λ being a trace class operator. Hence α would characterise, a priori, the *regularity* of our estimator, μ . Similarly, β would characterise the regularity of μ_0 , i.e. the true parameter. In addition, assuming $n\tau_n^2 \rightarrow \infty$ ensures the rates described in Theorem 2.1 converge to 0. Note that τ_n allows us to *fine-tune* the rate obtained when we choose α , thereby ensuring that we achieve the optimal rate.

Note that convergence to μ_0 w.r.t a random variable μ means that the probability of our μ generating outcomes that are *close* to μ_0 approach 1 as $n \rightarrow \infty$. Informally, our goal is to show that our estimator μ converges to the true function μ_0 and that it does so at a particular rate ε_n , with probability 1.

This is equivalent to us showing that for any rate greater than ε_n , i.e. $M_n \varepsilon_n$ where $M_n \rightarrow \infty$, our estimator diverges, or more specifically, μ converges to μ_0 via this new rate with probability 0. Note that Markov's Inequality tells us,

$$\Pi_n(\{\mu \in H_1 : \|\mu - \mu_0\|_{H_1}^2 \geq M_n^2 \varepsilon_n^2 | Y\}) \leq \frac{\mathbb{E}(\|\mu - \mu_0\|_{H_1}^2)}{M_n^2 \varepsilon_n^2}, \quad (8)$$

where $\Pi_n(\cdot | Y)$ is the posterior probability distribution of $\mu | Y$.

Consequently, we can find ε_n and show divergence for any M_n by using the above inequality, or more specifically, calculating the expected distance between μ and μ_0 (i.e. $\mathbb{E}(\|\mu - \mu_0\|_{H_1}^2)$) and then choosing ε_n^2 to be *bigger* than said expectation. Therefore, regardless of the M_n chosen, the $\frac{\mathbb{E}(\|\mu - \mu_0\|_{H_1}^2)}{M_n^2 \varepsilon_n^2} \rightarrow 0$. Lastly, note that we are only interested in proving (8) for data, Y , that was generated using μ_0 hence we take the \mathbb{E}_{μ_0} of (8).

Subsequently, given Assumption 2.1, we arrive at the theorem below, (as stated in [Knapik et al., 2011]), which shows that $\Pi_n(\cdot|Y)$ contracts at a rate of ε_n as $n \rightarrow \infty$; a rate that depends on all four parameters of the inverse problem, i.e. α, β, τ_n and q . Note however, that q and β are fixed, and thus one only has control over α and τ_n .

Theorem 2.1 (Homogeneous: Contraction). *Given Assumption 2.1, the $\mathbb{E}_{\mu_0} \Pi_n(\{\mu : \|\mu - \mu_0\|_{H_1} \geq M_n \varepsilon_n | Y\}) \rightarrow 0$, as $n \rightarrow \infty$, for every $M_n \rightarrow \infty$ where*

$$\varepsilon_n = (n\tau_n^2)^{-\beta/(1+2\alpha+2q) \wedge 1} + \tau_n (n\tau_n^2)^{-\alpha/(1+2\alpha+2q)}.$$

The rate is uniform over μ_0 in balls in S^β . In particular,

1. *If $\tau_n \equiv 1$, then $\varepsilon_n = n^{-(\beta \wedge \alpha)/(1+2\alpha+2q)}$.*
2. *If $\beta \leq 1 + 2\alpha + 2q$, then $\varepsilon_n = n^{-\beta/(1+2\beta+2q)}$ for $\tau_n \asymp n^{(\alpha-\beta)/(1+2\beta+2q)}$.*
3. *If $\beta > 1 + 2\alpha + 2q$, then $\varepsilon_n \gg n^{-\beta/(1+2\beta+2q)}$, for every scaling τ_n .*

Note this theorem was proved by Knapik et al, [Knapik et al., 2011].

The proof for this theorem can be found in Appendix B.2. Recall that the minimax rate over a Sobolev ball, S^β , is $\varepsilon_n^* = n^{-\beta/(1+2\beta+2q)}$. Consequently, if the scaling τ_n is fixed, then by the theorem above, the optimal rate can only be achieved if $\alpha = \beta$. Additionally, the forementioned theorem implies that, if $\beta \leq 1 + 2\alpha + 2q$, the optimal rate can still be achieved as long as τ is scaled appropriately. However if $\beta > 1 + 2\alpha + 2q$ then regardless of the scaling used the optimal rate can never be achieved. This is also the case if the scaling is fixed and $\alpha \neq \beta$.

In conclusion, note that the discrepancies between the regularity of μ and μ_0 (i.e. α and β) affect the contraction rates. Thus, we should ensure that μ is at least as *smooth* as μ_0 if we wish to achieve the optimal rate of contraction. Subsequently, let us now obtain the minimax and contraction rates for the *Heterogeneous* case.

3 Bayesian Inverse Problems with Heterogeneous Variance

In this section, our objective will be to estimate a parameter μ from an observation Y given the Model:

$$Y = K\mu + \epsilon\tilde{Z}, \quad (9)$$

where $\mu \in H_1$, a separable Hilbert Space and a known, injective, continuous, linear operator K maps μ into another separable Hilbert space, H_2 . In addition, we shall set the noise level

$$\epsilon = \frac{1}{\sqrt{n}}.$$

Most importantly, we define

$$\tilde{Z} = WZ \sim N(0, WW^T),$$

where Z is the Gaussian white noise process in H_2 , (see Definition 1.5), i.e.

$$Z \sim N(0, I),$$

and $W : H_2 \mapsto H_2$ is a continuous linear operator. Note \tilde{Z} now has *Heterogeneous* variance, unlike in Section 2.

In summary, under the prior $\mu \sim N(0, \Lambda)$, with Gaussian Heterogeneous noise $\tilde{Z} \sim N(0, WW^T)$, the posterior $\mu|Y$ is derived in Proposition 3.1. Subsequently, by explicitly categorising the behaviour of the singular values of K , Λ , W and μ_0 in Assumption 3.1, the minimax and contraction rates are derived in Proposition 3.2 and Theorem 3.2, respectively. Furthermore, contraction rates are also derived in Theorem 3.1 under a far more general setting, (Assumption 3.2), where the behaviour of the singular values of K , Λ , W aren't restricted. In addition, we derive contraction rates for the case where there is noise in the covariance operator $V := WW^T$, and conclude the section by numerically illustrating the problem of recovering a function from noisy observations.

3.1 Formal description of the problem

The format of this subsection will follow that of Section 2.1. Thus, we will require the reader to be familiar with Gaussian processes and Functional Analysis for Hilbert Spaces. Feel free to peruse Sections 1.4.2 and 1.4.3, and Appendix A.2 for more details. Regardless, let $\langle \cdot, \cdot \rangle_{H_i}$ and $\|\cdot\|_{H_i}$ refer to the inner product and induced norm of H_i , respectively.

Observe, the stochastic noise \tilde{Z} cannot be realised as a random element in H_2 . Therefore, it is interpreted as a process instead, i.e. $\tilde{Z} = \{\tilde{Z}_h := \langle \tilde{Z}, h \rangle_{H_2} : h \in H_2\}$. Consequently, \tilde{Z}_h can now be viewed as a random element.

Therefore, Model (3) can be interpreted as a Gaussian process as well, where $Y = \{Y_h := \langle Y, h \rangle_{H_2} : h \in H_2\}$, with

$$\mathbb{E}Y_{g_i} = \mathbb{E}(\langle K\mu + \epsilon\tilde{Z}, g_i \rangle_{H_2}) = \mathbb{E}(\langle K\mu, g_i \rangle_{H_2}) + \frac{1}{\sqrt{n}}\mathbb{E}\tilde{Z}_{g_i} = \langle K\mu, g_i \rangle_{H_2},$$

and covariance

$$\begin{aligned}\mathbb{C}ov(Y_{g_i}, Y_{g_j}) &= \mathbb{C}ov(\langle K\mu + \frac{1}{\sqrt{n}}\tilde{Z}, g_i \rangle_{H_2}, \langle K\mu + \frac{1}{\sqrt{n}}\tilde{Z}, g_j \rangle_{H_2}) \\ &= \frac{1}{n}\mathbb{C}ov(\tilde{Z}_{g_i}, \tilde{Z}_{g_j}) = \frac{1}{n}\langle WW^T g_i, g_j \rangle_{H_2}.\end{aligned}$$

Thus,

$$Y|\mu \sim N(K\mu, \frac{1}{n}V),$$

where $V := WW^T$. Note V , just like the identity element of H_2 in the Homogeneous variance setting, does not have to be of trace class.

Having described the statistical properties of the model, let us look at how one can use the SVD decomposition method to recover μ , (c.f. Section 1.4.4).

Recall that if the operator K is compact, then the self-adjoint operator $K^T K : H_1 \mapsto H_1$ possesses countably many positive eigenvalues k_i^2 and $K^T K \phi_i = k_i^2 \phi_i$ for $i \in \mathbb{N}$, where the $\{\phi_i\}_{i=1}^\infty$ are the orthonormal basis of H_1 . Additionally, by constructing a sequence $\{\varphi_i\}_{i=1}^\infty$ such that $K\phi_i = k_i \varphi_i$ we form an orthonormal basis for the range of K in H_2 . This also implies that $KK^T \varphi_i = k_i^2 \varphi_i$.

Before continuing, we shall assume there exists an orthonormal basis of eigenfunctions, (denoted by $\{\phi_i\}_{i=1}^\infty$), for $K^T K$, rather than K being compact. Furthermore, we shall also assume that there exists an orthonormal set of eigenfunctions for WW^T and that they are the same as KK^T 's eigenfunctions, i.e. $\{\varphi_i\}_{i=1}^\infty$.

Therefore,

$$V\varphi_i = WW^T \varphi_i = \sigma_i^2 \varphi_i.$$

Hence, denote

$$\tilde{Z}_i := \langle \tilde{Z}, \varphi_i \rangle_{H_2}, \quad \mu_i := \langle \mu, \phi_i \rangle_{H_1}, \quad \text{and} \quad Y_i := \langle Y, \varphi_i \rangle_{H_2}.$$

Subsequently, any $\mu \in H_1$ and its image (under K) can be expressed in the forms

$$\mu = \sum_i \mu_i \phi_i,$$

and

$$K\mu = \sum_i \mu_i K\phi_i = \sum_i (k_i \mu_i) \varphi_i.$$

Additionally, we can show

$$\mathbb{E}\tilde{Z}_i = \mathbb{E}\langle \tilde{Z}, \varphi_i \rangle_{H_2} = \langle 0, \varphi_i \rangle_{H_2} = 0,$$

and

$$\mathbb{C}ov(\tilde{Z}_i, \tilde{Z}_j) = \langle V\varphi_i, \varphi_j \rangle_{H_2} = \langle WW^T \varphi_i, \varphi_j \rangle_{H_2} = \sigma_i^2 \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta function. Thus,

$$\tilde{Z}_i \sim N(0, \sigma_i^2).$$

Consequently, the

$$\mathbb{E}(Y(\varphi_i)) = \langle K\mu, \varphi_i \rangle_{H_2} = \left\langle \sum_i (k_i \mu_i) \varphi_i, \varphi_i \right\rangle_{H_2} = k_i \mu_i,$$

and the covariance function is

$$\text{Cov}(Y(\varphi_i), Y(\varphi_j)) = \left\langle \frac{1}{n} V \varphi_i, \varphi_j \right\rangle_{H_2} = \left\langle \frac{1}{n} W W^T \varphi_i, \varphi_j \right\rangle_{H_2} = \frac{1}{n} \sigma_i^2 \delta_{i,j}.$$

Hence, we see that

$$Y_i \sim N(\mu_i k_i, \frac{1}{n} \sigma_i^2),$$

and our original problem of recovering μ is now equivalent to recovering its singular values $\{\mu_i\}_{i=1}^\infty$.

Thus, we consider the *Sequence space model* instead,

$$Y_i = k_i \mu_i + \epsilon \tilde{Z}_i, \text{ for all } i \geq 1.$$

If $k_i \rightarrow 0$, (this will definitely be the case if K is compact), the problem is ill-posed, and this ill-posedness is defined by the rate at which $k_i \downarrow 0$, (see Definition 1.8).

Just as in Section 2, we assume that our problem is *mildly ill-posed*, i.e. for some $q \geq 0$,

$$k_i \asymp i^{-q}.$$

Note that the larger the value of q , the harder the estimation of μ , since the decay is faster. Please see Section 1.4.4 and Definition 1.8 for further details regarding ill-posedness.

The Minimax rates (c.f. Definition 1.4) are defined over the Sobolev space of order β , i.e.

$$S^\beta = \{\mu \in H_1 : \|u\|_{S^\beta}^2 < \infty\}, \text{ where } \|u\|_{S^\beta}^2 = \sum_{i=1}^\infty \mu_i^2 i^{2\beta}.$$

Consequently, we obtain the following posterior distribution.

3.2 Prior and Posterior Distributions

Proposition 3.1 (Heterogeneous: Posterior). *If μ is $N(0, \Lambda)$ distributed and Y given μ is $N(K\mu, n^{-1}V)$ distributed, then $\Pi_n(\cdot|Y)$, the conditional distribution of μ given Y , is $N(\tilde{A}Y, \tilde{S}_n)$ on H_1 where,*

$$\tilde{S}_n = \Lambda - \tilde{A}(n^{-1}V + K\Lambda K^T)\tilde{A}^T,$$

and $\tilde{A} : H_2 \mapsto H_1$ is the continuous linear operator

$$\tilde{A} = \Lambda^{1/2}(n^{-1}V + \Lambda^{1/2}K^TK\Lambda^{1/2})^{-1}\Lambda^{1/2}K^T = \Lambda K^T(n^{-1}V + K\Lambda K^T)^{-1},$$

assuming $\sum_i \frac{\lambda_i k_i^2}{\sigma_i^2} < \infty$. The posterior distribution is proper (i.e. \tilde{S}_n has finite trace).

We shall proceed just as we did in the homogeneous case i.e. by splitting the proof into several segments. We will begin by deriving the joint probability distribution, and then the posterior distribution. Subsequently, for the latter, we will simplify \tilde{A} , and show \tilde{S}_n is proper. Note that we haven't proved the equivalence between the prior and posterior distribution (w.r.t. their absolute continuity) as it is rather technical and is not required in our inferences.

Thus, let us begin by deriving the joint probability distribution. Whilst computing the joint distribution was not necessary for my proof, those wishing to apply the proof described in [Knapik et al., 2011] will find the following useful.

Joint Distribution: Assuming $Y|\mu \sim N(K\mu, n^{-1}V)$ and $\mu \sim N(0, \Lambda)$ we wish to show their joint probability is

$$Y, \mu \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} n^{-1}V + K\Lambda K^T & K\Lambda \\ \Lambda K^T & \Lambda \end{pmatrix}\right) \iff Y_i, \mu_i \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} n^{-1}\sigma_i^2 + k_i^2\lambda_i & k_i\lambda_i \\ \lambda_i k_i & \lambda_i \end{pmatrix}\right).$$

Recall that if the normal random variables X_1, X_2 have the joint density

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]\right) \\ &\implies X_1, X_2 \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_2\sigma_1 & \sigma_2^2 \end{pmatrix}\right). \end{aligned}$$

Hence, since $Y_i|\mu_i \sim N(k_i\mu_i, n^{-1}\sigma_i^2)$ and $\mu_i \sim N(0, \lambda_i)$, we know

$$\begin{aligned} f(y_i, \mu_i) &= f(y_i|\mu_i)f(\mu_i) \propto \exp\left(\frac{-n}{2\sigma_i^2}[y_i - k_i\mu_i]^2\right) \exp\left(\frac{-1}{2\lambda_i}\mu_i^2\right) = \exp\left(\frac{-n}{2\sigma_i^2}[y_i^2 + k_i^2\mu_i^2 - 2y_ik_i\mu_i + \frac{\sigma_i^2\mu_i^2}{n\lambda_i}]\right) \\ &= \exp\left(\frac{-(n\lambda_ik_i^2 + \sigma_i^2)}{2\sigma_i^2}\left[\frac{y_i^2}{k_i^2\lambda_i + n^{-1}\sigma_i^2} + \frac{\mu_i^2}{\lambda_i} - 2\frac{k_i\lambda_i^{1/2}}{(\lambda_ik_i^2 + n^{-1}\sigma_i^2)^{1/2}}\frac{y_i\mu_i}{(\lambda_ik_i^2 + n^{-1}\sigma_i^2)^{1/2}\lambda_i^{1/2}}\right]\right), \end{aligned}$$

which gives us our desired result. Note that $\rho := \frac{k_i\lambda_i^{1/2}}{(\lambda_ik_i^2 + n^{-1}\sigma_i^2)^{1/2}}$ and $1 - \rho^2 = (n\lambda_ik_i^2 + \sigma_i^2)^{-1}\sigma_i^2$ as required. \square

Consequently, we derive the posterior distribution in the sequence space setting.

Posterior distribution: Singular Value form: Since $Y_i \sim N(k_i\mu_i, \frac{\sigma_i^2}{n})$ and $\mu \sim N(0, \Lambda)$, (which implies $\mu_i := \langle \mu, \phi_i \rangle_{H_1} \sim N(0, \lambda_i)$), we can find the posterior distribution of $\mu_i|Y_i$

i.e.

$$\begin{aligned}
f(\mu_i|y_i) &\propto f(y_i|\mu_i)f(\mu_i) \propto \exp\left(\frac{-n}{2\sigma_i^2}[y_i - k_i\mu_i]^2\right) \exp\left(\frac{-1}{2\lambda_i}\mu_i^2\right) \\
&= \exp\left(\frac{-n}{2\sigma_i^2}\left[y_i^2 + k_i^2\mu_i^2 - 2y_ik_i\mu_i + \frac{\sigma_i^2\mu_i^2}{n\lambda_i}\right]\right) \\
&= \exp\left(\frac{-(n\lambda_ik_i^2 + \sigma_i^2)}{2\sigma_i^2\lambda_i}[\mu_i^2 - 2\frac{y_ik_in\lambda_i}{n\lambda_ik_i^2 + \sigma_i^2}\mu_i]\right) \\
&= \exp\left(\frac{-(n\lambda_ik_i^2 + \sigma_i^2)}{2\sigma_i^2\lambda_i}\left[\mu_i - \frac{ny_ik_i\lambda_i}{n\lambda_ik_i^2 + \sigma_i^2}\right]^2\right),
\end{aligned}$$

which implies

$$\mu_i|Y_i \sim N\left(\frac{nY_ik_i\lambda_i}{n\lambda_ik_i^2 + \sigma_i^2}, \frac{\sigma_i^2\lambda_i}{n\lambda_ik_i^2 + \sigma_i^2}\right). \quad (10)$$

Consequently, noting that (10) are exactly the singular values of $\tilde{A}Y$ and \tilde{S}_n concludes the proof. \square

Next, we seek to simplify the form of \tilde{A} .

Simplifying \tilde{A} : Note that for any compact linear operator $B : H_1 \mapsto H_2$ the following identity holds:

$$(I + BB^T)^{-1}B = B(I + B^TB)^{-1}. \quad (11)$$

Thus defining $D = n^{-1}V$, we see that

$$D + BB^T = D(I + D^{-1}BB^T) = D^{1/2}(I + D^{-1/2}BB^TD^{-1/2})D^{1/2}.$$

The last equality following from D 's symmetry. Consequently, setting $B := \Lambda^{1/2}K^T$,

$$\begin{aligned}
\tilde{A} &= \Lambda^{1/2}(D + BB^T)^{-1}B = \Lambda^{1/2}D^{-1/2}(I + D^{-1/2}BB^TD^{-1/2})^{-1}D^{-1/2}B \\
&= \Lambda^{1/2}D^{-1/2}(I + \tilde{B}\tilde{B}^T)^{-1}\tilde{B},
\end{aligned}$$

where $\tilde{B} = D^{-1/2}B = n^{1/2}V^{-1/2}\Lambda^{1/2}K^T$. Hence, we must prove \tilde{B} is a compact linear operator, in order to use the identity (11) and simplify \tilde{A} .

Note that \tilde{B} is a *Hilbert-Schmidt* operator if the $\text{tr}(\tilde{B}\tilde{B}^T) < \infty$, i.e. if

$$\text{tr}(\tilde{B}\tilde{B}^T) = \text{tr}(nV^{-1/2}\Lambda^{1/2}K^TK\Lambda^{1/2}V^{-1/2}) = n \sum_i \frac{\lambda_ik_i^2}{\sigma_i^2} < \infty.$$

Consequently, noting that Hilbert-Schmidt operators are compact operators concludes the proof. Please see Appendix A.1 for more information regarding Hilbert-Schmidt operators. \square

Finally, we show that \tilde{S}_n is proper.

\tilde{S}_n is proper: First note that $\Lambda - \tilde{S}_n$ is non-negative definite i.e. $\langle (\Lambda - \tilde{S}_n)h, h \rangle_{H_1} \geq 0$ since

$$(\Lambda - \tilde{S}_n)h = \sum_j h_j^2 \frac{n\lambda_j^2 k_j^2}{\sigma_j^2 + nk_j^2 \lambda_j} \phi_j \implies \langle (\Lambda - \tilde{S}_n)h, h \rangle_{H_1} = \sum_j h_j^2 \frac{n\lambda_j^2 k_j^2}{\sigma_j^2 + nk_j^2 \lambda_j} \geq 0,$$

where $h_j := \langle h, \phi_j \rangle_{H_1}$. Hence \tilde{S}_n is bounded above by Λ and since Λ is of trace class so is \tilde{S}_n . \square

Subsequently, we now derive the minimax rates for the Heterogeneous case.

3.3 Minimax Rates

Just as we did in the homogeneous case, we must make the following assumption in order to find our contraction rates.

Assumption 3.1. *The operators $K^T K$ and Λ have the same eigenfunctions $\{\phi_i\}_{i=1}^\infty$. Similarly, KK^T and V have the same eigenfunctions $\{\varphi_i\}_{i=1}^\infty$ as well. Furthermore their eigenvalues $\{k_i^2\}_{i=1}^\infty$, $\{\lambda_i\}_{i=1}^\infty$, and $\{\sigma_i^2\}_{i=1}^\infty$ satisfy*

$$\lambda_i = \tau_n^2 i^{-1-2\alpha}, \quad C_1^{-1} i^{-q} \leq k_i \leq C_1 i^{-q}, \quad \text{and} \quad C_2^{-1} i^\gamma \leq \sigma_i \leq C_2 i^\gamma,$$

for some $\alpha > 0, q \geq 0, \gamma \in \mathbb{R}, C_1, C_2 \geq 1$ and $\tau_n > 0$ such that $n\tau_n^2 \rightarrow \infty$. Additionally, the true parameter $\mu_0 \in S^\beta$ for some $\beta > 0$.

Note that there are now additional assumptions involving the decay of σ_i and the eigenfunctions of V . Hence, one can show that the degree of ill-posedness for Model (9) is now $\tilde{q} = (q + \gamma)$, (c.f. proof of Proposition 3.2). In addition, since q is fixed, alternating γ (the rate at which σ_i decays) can lead to our problem becoming well-posed ($\tilde{q} < 0$).

Regardless, before discussing the contraction rates, we shall state the Minimax rates, (see Definition 1.4), for Model (9) in the following proposition.

Proposition 3.2. *The Minimax rates, ε_n^* , for the heterogeneous case are*

$$\varepsilon_n^* := \begin{cases} n^{-\frac{\beta}{1+2\beta+2(q+\gamma)}}, & \text{if } \gamma > -\frac{1+2q}{2}. \\ n^{-1/2}(\ln n)^{1/2}, & \text{if } \gamma = -\frac{1+2q}{2}. \\ n^{-1/2}, & \text{if } \gamma < -\frac{1+2q}{2}. \end{cases}$$

Note, $n^{-\frac{\beta}{1+2\beta+2(q+\gamma)}}$ is generally referred to as the non-parametric rate, and $n^{-1/2}$ as the parametric rate.

Note two proofs have been listed. The first is based on [Cavalier et al., 2002], and is rather elegant and intuitive. In addition, it naturally describes the degree of ill-posedness of Model (9), denoted by \tilde{q} . However, it requires $\tilde{q} > 0$, hence the inclusion of a more rigorous second proof, which is based on [Belitser and Levit, 1994] and [Tsybakov, 2009].

Proof. Recall, the sequence space model

$$Y_i = k_i \mu_i + \epsilon \tilde{Z}_i,$$

can be rescaled by $\sigma_i > 0$ to obtain

$$\tilde{Y}_i = \tilde{k}_i \mu_i + \epsilon Z_i,$$

where

$$\tilde{Y}_i = \frac{Y_i}{\sigma_i}, \quad \tilde{k}_i = \frac{k_i}{\sigma_i} \quad \text{and} \quad Z_i \sim N(0, 1)$$

and $Z_i \sim N(0, 1)$.

The optimal rate of this scaled model is given in ([Cavalier et al., 2002], however in order to derive it we must first evaluate its degree of ill-posedness, \tilde{q} .

We do so by investigating the singular values of \tilde{K} , i.e. $\tilde{k}_i = k_i/\sigma_i \asymp i^{-(q+\gamma)}$, the latter equality following from Assumption 3.1. Consequently $\tilde{q} = (q+\gamma)$ and the optimal contraction rate is

$$n^{-\frac{\beta}{1+2\alpha+2\tilde{q}}} \text{ when } \tilde{q} > 0.$$

□

Proof. In order to prove Proposition 3.2 we shall mainly use Theorem 3 from [Belitser and Levit, 1994], and for γ particularly small we shall use Theorems 2.1 and 2.2 from [Tsybakov, 2009].

Note the model studied in [Belitser and Levit, 1994] is defined using spectral values, i.e.

$$\tilde{Y}_i = \theta_i + \epsilon \tilde{\sigma}_i \xi, \quad \text{where } \xi \sim N(0, 1), \quad \tilde{\sigma}_i \geq 0 \text{ and the small parameter } \epsilon > 0. \quad (12)$$

However, Model (12) is equivalent to the Heterogeneous Model (9) if

$$\tilde{Y}_i := \frac{Y_i}{k_i}, \quad \theta_i := \mu_i, \quad \epsilon := \frac{1}{\sqrt{n}} \quad \text{and} \quad \tilde{\sigma}_i := \frac{\sigma_i}{k_i} \asymp i^{\tilde{q}},$$

where $\tilde{q} = (q + \gamma)$.

Furthermore, in [Belitser and Levit, 1994], it is assumed the true parameter $\theta_0 \in \Theta$, where

$$\Theta = \Theta(Q) = \{\theta : \sum_{i=1}^{\infty} a_i^2 \theta_i^2 \leq Q\},$$

and $\{a_i\}_{i=1}^{\infty}$ is a non-negative sequence converging to infinity. However, we assume $\mu_0 \in S^\beta$, where $\beta > 0$ (see Assumption 3.1). Subsequently, defining

$$a_i := i^\beta \implies \Theta \subset S^\beta.$$

Thus, having described the setting, our goal is to find the minimax risk,

$$\varepsilon_n^{*2} = r_\epsilon = r_\epsilon(\Theta) = \inf_{\theta} \sup_{\theta_0 \in \Theta} \mathbb{E}_{\theta_0} \|\theta - \theta_0\|_{H_1}^2.$$

We can find this by using [Belitser and Levit, 1994]'s Theorem 3, which is as follows

Theorem. Define c_ϵ to be the solution of the equation

$$\epsilon^2 \sum_{i=1}^{\infty} \tilde{\sigma}_i^2 a_i (1 - c_\epsilon a_i)_+ = c_\epsilon Q.$$

and $N := N_\epsilon(\Theta) = \max\{i : a_i \leq c_\epsilon^{-1}\}$. If condition

$$\log \epsilon^{-1} \frac{\sum_{i=1}^{\infty} a_i^2 \tilde{\sigma}_i^4 (1 - c_\epsilon a_i)_+^2}{(\sum_{i=1}^{\infty} a_i \tilde{\sigma}_i^2 (1 - c_\epsilon a_i)_+)^2} = o(1), \quad \epsilon \rightarrow 0, \quad (13)$$

holds, then

$$r_\epsilon = \epsilon^2 \sum_{i=1}^N \tilde{\sigma}_i^2 - \epsilon^2 (c_\epsilon \sum_{i=1}^N \tilde{\sigma}_i^2 a_i) (1 + o(1)), \quad \epsilon \rightarrow 0.$$

In order to use this theorem, we must first find c_ϵ and N . Note

$$(1 - c_\epsilon a_i)_+ = (1 - c_\epsilon i^\beta)_+ = 0 \iff i \geq c_\epsilon^{-\frac{1}{\beta}} \implies N = \lfloor c_\epsilon^{-\frac{1}{\beta}} \rfloor. \quad (14)$$

Consequently,

$$\epsilon^2 \sum_{i=1}^{\infty} \tilde{\sigma}_i^2 a_i (1 - c_\epsilon a_i)_+ = \epsilon^2 \sum_{i=1}^N i^{2\tilde{p}+\beta} (1 - c_\epsilon i^\beta) = \epsilon^2 \sum_{i=1}^N i^{2\tilde{p}+\beta} - c_\epsilon \epsilon^2 \sum_{i=1}^N i^{2\tilde{p}+2\beta}.$$

Note, we could use the following equation (derived by bounding a sum by its integral) to bound the above sums,

$$\sum_{i=1}^N i^\kappa = \frac{N^{\kappa+1}}{\kappa+1} (1 + o(1)) \quad \text{as } N \rightarrow \infty, \quad \text{if } \kappa > -1. \quad (15)$$

However, this requires

$$2\tilde{q} + \beta > -1 \implies q + \gamma > -\frac{1+\beta}{2}, \quad (16)$$

which we will now assume unless stated otherwise, (we consider the case $q + \gamma \leq -\frac{1+\beta}{2}$ at the very end of the proof).

Consequently, one can show via (15) that

$$\epsilon^2 \sum_{i=1}^{\infty} \tilde{\sigma}_i^2 a_i (1 - c_\epsilon a_i)_+ = [\epsilon^2 \frac{N^{2\tilde{q}+\beta+1}}{2\tilde{q} + \beta + 1} - c_\epsilon \epsilon^2 \frac{N^{2\tilde{q}+2\beta+1}}{2\tilde{q} + 2\beta + 1}] (1 + o(1)).$$

Omitting the factor $(1 + o(1))$ for now and using (14) implies

$$\epsilon^2 \sum_{i=1}^{\infty} \tilde{\sigma}_i^2 a_i (1 - c_\epsilon a_i)_+ = \epsilon^2 \frac{c_\epsilon^{-\frac{2\tilde{q}+\beta+1}{\beta}}}{2\tilde{q} + \beta + 1} - c_\epsilon \epsilon^2 \frac{c_\epsilon^{-\frac{2\tilde{q}+2\beta+1}{\beta}}}{2\tilde{q} + 2\beta + 1} = \epsilon^2 c_\epsilon^{-\frac{c_1}{\beta}} \frac{\beta}{c_1 c_2},$$

where $c_1 := 2\tilde{q} + \beta + 1$ and $c_2 := 2\tilde{q} + 2\beta + 1$.
Consequently,

$$\begin{aligned} \epsilon^2 \sum_{i=1}^{\infty} \tilde{\sigma}_i^2 a_i (1 - c_\epsilon a_i)_+ &= c_\epsilon Q \iff \epsilon^2 c_\epsilon^{-\frac{c_1}{\beta}} \frac{\beta}{c_1 c_2} = c_\epsilon Q \implies \\ c_\epsilon &= \left(\frac{\epsilon^2 \beta}{Q c_1 c_2} \right)^{\frac{\beta}{c_2}} (1 + o(1)) \quad \text{and} \quad N = c_\epsilon^{-\frac{1}{\beta}} = \left(\frac{\epsilon^2 \beta}{Q c_1 c_2} \right)^{-\frac{1}{c_2}} (1 + o(1)). \end{aligned} \quad (17)$$

Furthermore, observe that $N \rightarrow \infty$ as $\epsilon \rightarrow 0$. We require the former expression to hold in order to use (15). Thus, having found c_ϵ and N , we can use the above theorem if we verify condition (13).

Observe however that

$$\frac{\sum_{i=1}^{\infty} a_i^2 \tilde{\sigma}_i^4 (1 - c_\epsilon a_i)_+^2}{\left(\sum_{i=1}^{\infty} a_i \tilde{\sigma}_i^2 (1 - c_\epsilon a_i)_+ \right)^2} \leq \frac{\sum_{i=1}^N a_i^2 \tilde{\sigma}_i^4}{\left(\sum_{i=1}^N a_i \tilde{\sigma}_i^2 \right)^2} = \mathcal{O}\left(\frac{N^{4\tilde{q}+2\beta+1}}{N^{4\tilde{q}+2\beta+2}}\right) = \mathcal{O}((\epsilon^2)^{\frac{1}{c_2}}).$$

Furthermore, note that $\epsilon^{\frac{1}{c_2}} \rightarrow 0$ as $\epsilon \rightarrow 0$, since (16) $\implies c_2 > 0$. Hence, we see that condition (13) is indeed satisfied.

Therefore, we can now use the theorem to find the rates for the three different cases in Proposition 3.2. Thus,

$$r_\epsilon \asymp \epsilon^2 \sum_{i=1}^N i^{2\tilde{q}} - \epsilon^2 (c_\epsilon \sum_{i=1}^N i^{2\tilde{q}+\beta}).$$

Subsequently, in order to bound the above terms, we consider 3 cases.

Case 1: $2\tilde{q} > -1 \implies 2\tilde{q} + \beta > -1$. Hence, we can use (15) and (17) to show

$$r_\epsilon \asymp \epsilon^2 \left[\frac{N^{2\tilde{q}+1}}{2\tilde{q}+1} - c_\epsilon \frac{N^{2\tilde{q}+\beta+1}}{2\tilde{q}+\beta+1} \right] = \epsilon^2 c_\epsilon^{-\frac{2\tilde{q}+1}{\beta}} \frac{\beta}{(2\tilde{q}+1)c_1} = \mathcal{O}((\epsilon^2)^{\frac{2\beta}{c_2}}) = \mathcal{O}(n^{-2\frac{\beta}{c_2}}),$$

since we defined $\epsilon := \frac{1}{\sqrt{n}}$ at the start of this proof. Note Proposition 3.2's

$$\varepsilon_n^* = \sqrt{r_\epsilon},$$

since we are interested in $\mathbb{E}_{\theta_0} \|\theta - \theta_0\|_{H_1}$, rather than $\mathbb{E}_{\theta_0} \|\theta - \theta_0\|_{H_1}^2$, (c.f. Theorem 3.2).

Case 2: $2\tilde{q} = -1 \implies 2\tilde{q} + \beta > -1$. Before, we proceed please note that, by bounding a sum via its integral, we can obtain the following

$$\sum_{i=1}^N i^{-1} = \ln N + C_e + o(1), \quad \text{as } N \rightarrow \infty, \quad \text{where } C_e \text{ is the Euler constant.}$$

Consequently, since $N \rightarrow \infty$ as $\epsilon \rightarrow 0$, we can use the above and (17) to derive

$$r_\epsilon \asymp \epsilon^2 \left[\ln N - c_\epsilon \frac{N^\beta}{\beta} \right] = \mathcal{O}(\epsilon^2 \ln N) = \mathcal{O}(n^{-1} \ln n).$$

Finally, setting $\varepsilon_n^* = \sqrt{r_\epsilon}$ concludes this case's proof.

Case 3: $2\tilde{q} < -1$. Using the following asymptotic expansion,

$$\sum_{i=1}^N i^{-\kappa-1} = \frac{1}{\kappa}(1 + o(1)) \quad \text{as } N \rightarrow \infty, \quad \text{if } \kappa > 0,$$

and (15), we can show

$$r_\epsilon \asymp \epsilon^2 \left[\frac{1}{-2\tilde{q}-1} - \frac{c_\epsilon N^{2\tilde{q}+\beta+1}}{2\tilde{q}+\beta+1} \right] = \mathcal{O}(\epsilon^2 [1 - (\epsilon^2)^{-\frac{2\tilde{q}+1}{c_2}}]) = \mathcal{O}(\epsilon^2 - (\epsilon^2)^{\frac{2\beta}{c_2}}) = \mathcal{O}(\epsilon^2),$$

as $\epsilon \rightarrow 0$, since $2\tilde{q} < -1 \implies \frac{2\beta}{c_2} > 1$. Thus, $\varepsilon_n^* = \mathcal{O}(\epsilon) = \mathcal{O}(n^{-\frac{1}{2}})$.

Finally, we consider the case: $q + \gamma \leq -\frac{1+\beta}{2}$. Note, the latter implies $q + \gamma + 1/2 < 0$, since $\beta > 0$.

We can find a lower bound for

$$\mathbb{E}_{\mu_0} \|\hat{\mu} - \mu_0\|_{H_1}^2,$$

using Theorems 2.1 and 2.2 from [Tsybakov, 2009] with $d(\mu_2, \mu_1) = \|\mu_2 - \mu_1\|_{H_1}$ and $\Theta = S^\beta(A)$. For that we need to find two elements μ_2, μ_1 in $S^\beta(A)$ such that $d(\mu_2, \mu_1) \geq B\psi_n$ for some $B > 0$ and $KL(P_{\mu_1}, P_{\mu_2}) \leq \alpha < \infty$ where ψ_n is the rate defined in the proposition. Here P_μ is the probability distribution of Y generated by Model (9). Then, for any estimator $\hat{\mu}$ and $\mu_0 \in S^\beta(A)$,

$$\mathbb{E}_{\mu_0} [\psi_n^{-2} \|\hat{\mu} - \mu_0\|_{H_1}^2] \geq (B/2)^2 \max \left(\frac{1}{4} \exp(-\alpha), \frac{1 - \sqrt{\alpha/2}}{2} \right).$$

We take $\mu_1 = 0$ and $\mu_2 = \sum_i \mu_{2,i} e_i$ such that $\mu_{2,i} = B\psi_n$ for $i = i_0 \geq 1$ and $\mu_{2,i} = 0$ otherwise. Such μ_2 belongs to $S^\beta(A)$ if $\sum_i i^{2\beta} \mu_{2,i}^2 = i_0^{2\beta} (B\psi_n)^2 \leq A^2$, i.e. if $i_0 \leq (A/B)^{1/\beta} \psi_n^{-1/\beta}$. Also, $d(\mu_2, \mu_1) = \sqrt{\sum_i \mu_{2,i}^2} = B\psi_n$. Using spectral decomposition, it is easy to compute the Kullback-Leibler distance:

$$KL(P_{\mu_1}, P_{\mu_2}) = \frac{1}{2} \left[\sum_i \frac{k_i^2 \mu_{2,i}^2}{\epsilon^2 \sigma_i^2} - 1 \right] \leq \frac{1}{2} \left[c_k^2 B^{-2} i_0^{-2(q+\gamma)} \psi_n^2 \epsilon^{-2} - 1 \right].$$

As $q + \gamma + 1/2 < 0$, take $\psi_n = \epsilon$ and i_0 such that $i_0 < c_k^{2/(q+\gamma)} B^{-2/(q+\gamma)}$, then $\alpha = 0.5 \left[c_k^2 B^{-2} i_0^{-2(q+\gamma)} - 1 \right]$, so that the rate is $\psi_n = \epsilon$.

For the upper bound, consider the estimator $\hat{\mu} = y_i/k_i I(i \leq i_1)$ with i_1 to be specified later. Then,

$$\begin{aligned} \mathbb{E}_{\mu_0} \|\hat{\mu} - \mu_0\|_{H_1}^2 &= \mathbb{E}_{\mu_0} \sum_{i \leq i_1} (y_i/k_i - \mu_{0,1})^2 + \sum_{i > i_1} \mu_{0,1}^2 \leq \sum_{i \leq i_1} \epsilon^2 \sigma_i^2 / k_i^2 + i_1^{-2\beta} \sum_{i > i_1} i^{2\beta} \mu_{0,1}^2 \\ &\leq C \epsilon^2 \sum_{i \leq i_1} i^{2(q+\gamma)} + i_1^{-2\beta} A^2. \end{aligned}$$

If $q + \gamma + 1/2 < 0$, then $\sum_{i \leq i_1} i^{2(q+\gamma)} \leq C$, and

$$\mathbb{E}_{\mu_0} \|\hat{\mu} - \mu_0\|_{H_1}^2 \leq C\epsilon^2 + i_1^{-2\beta} A^2 \leq C\epsilon^2,$$

with $i_1 = \epsilon^{-1/\beta}$. □

Having found the optimal rates, we can now discuss the contraction rates achieved by our posterior distribution.

3.4 Contraction Rates for the Posterior

3.4.1 Contraction Rates for General Setting

In order to derive the contraction rates, (c.f. Definition 1.3), in a general setting we require only the following assumption.

Assumption 3.2. *Operators $K^T K$ and Λ have the same eigenfunctions $\{e_i\}$, with eigenvalues $\{k_i^2\}$ and λ_i , respectively. Similarly, Operators $K K^T$ and V have the same eigenfunctions $\{\phi_i\}$, with eigenvalues $\{k_i^2\}$ and σ_i^2 , respectively. Additionally, the true parameter $\mu_0 \in S^\beta$ for some $\beta > 0$.*

Note, that we make no assumptions regarding the behaviour of the eigenvalues, in contrast with [Knapik et al., 2011] who assume the mildly ill-posed setting. Consequently, we present a general result that can be applied for any λ_i , κ_i and σ_i satisfying stated conditions.

Theorem 3.1. *Given Assumption 3.2, and a monotonically increasing sequence $\sigma_i^2/[\lambda_i k_i^2]$, the $\mathbb{E}_{\mu_0} \Pi_n(\{\mu : \|\mu - \mu_0\|_{H_1} \geq M_n \varepsilon_n | Y\}) \rightarrow 0$, as $n \rightarrow \infty$, for every $M_n \rightarrow \infty$ where*

$$\varepsilon_n = \left[\epsilon^2 \sum_{i \leq i_\epsilon} \sigma_i^2 k_i^{-2} + i_\epsilon^{-2\beta} + \sum_{i > i_\epsilon} \lambda_i + \epsilon^4 \max_{i \leq i_\epsilon} \left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2 \right]^{1/2},$$

with $i_\epsilon = \max\{i : \epsilon^2 \sigma_i^2 \leq k_i^2 \lambda_i\}$. The rate is uniform over μ_0 in balls in S^β .

Proof. The main tool we are going to use is Markov inequality:

$$\mathbb{P}(\|\mu - \mu_0\|_{H_1} \geq M \varepsilon_n | Y) \leq M^{-2} \varepsilon_n^{-2} \mathbb{E}(\|\mu - \mu_0\|_{H_1}^2 | Y).$$

Note, if we show that $\mathbb{E}_{\mu_0} \mathbb{E}(\|\mu - \mu_0\|_{H_1}^2 | Y) \leq C \varepsilon_n^2$ for some $C > 0$ independent of ε_n then ε_n is the rate of contraction of the posterior distribution.

Under Assumption 3.2, using Parseval's identity, we have that

$$\mathbb{E}(\|\mu - \mu_0\|^2 | Y) = \sum_i \mathbb{E}[(\mu_i - \mu_{0,i})^2 | Y] = \sum_i [\text{Var}(\mu_i | Y) + (\mathbb{E}[\mu_i | Y] - \mu_{0,i})^2].$$

Taking the expected value with respect to the true distribution of Y and using the explicit form of the posterior distribution (10), we have

$$\begin{aligned}\mathbb{E}_{\mu_0}[\text{Var}(\mu_i | Y)] + \mathbb{E}_{\mu_0}[(\mathbb{E}[\mu_i | Y] - \mu_{0,i})^2] &= \mathbb{E}_{\mu_0} \left[\frac{Y_i k_i \lambda_i}{\lambda_i k_i^2 + \epsilon^2 \sigma_i^2} - \mu_{0,i} \right]^2 + \frac{\sigma_i^2 \lambda_i}{\epsilon^{-2} \lambda_i k_i^2 + \sigma_i^2} \\ &= \frac{\epsilon^2 \sigma_i^2 k_i^2 \lambda_i^2}{[\lambda_i k_i^2 + \epsilon^2 \sigma_i^2]^2} + \mu_{0,i}^2 \left[\frac{k_i^2 \lambda_i}{\lambda_i k_i^2 + \epsilon^2 \sigma_i^2} - 1 \right]^2 + \frac{\sigma_i^2 \lambda_i}{\epsilon^{-2} \lambda_i k_i^2 + \sigma_i^2} \\ &\asymp \frac{\sigma_i^2 \lambda_i}{\epsilon^{-2} \lambda_i k_i^2 + \sigma_i^2} + \mu_{0,i}^2 \left[\frac{\epsilon^2 \sigma_i^2}{\lambda_i k_i^2 + \epsilon^2 \sigma_i^2} \right]^2,\end{aligned}$$

as the first term is less than the third one.

Recall that $\sigma_i^2/[\lambda_i k_i^2]$ is an increasing sequence. Denote $i_\epsilon = \max\{i : \sigma_i^2/[\lambda_i k_i^2] \leq \epsilon^{-2}\}$. Then,

$$S_1 = \sum_i \frac{\sigma_i^2 \lambda_i}{\epsilon^{-2} \lambda_i k_i^2 + \sigma_i^2} \asymp \epsilon^2 \sum_{i \leq i_\epsilon} \sigma_i^2 k_i^{-2} + \sum_{i > i_\epsilon} \lambda_i,$$

and

$$\begin{aligned}S_2 &= \sum_i \mu_{0,i}^2 \left[\frac{\epsilon^2 \sigma_i^2}{k_i^2 \lambda_i + \epsilon^2 \sigma_i^2} \right]^2 = \|\mu_0\|_{S^\beta}^2 \sum_i \bar{\mu}_{0,i}^2 \left[\frac{\epsilon^2 \sigma_i^2 i^{-\beta}}{k_i^2 \lambda_i + \epsilon^2 \sigma_i^2} \right]^2 \\ &\asymp \|\mu_0\|_{S^\beta}^2 \epsilon^4 \sum_{i \leq i_\epsilon} \bar{\mu}_{0,i}^2 \left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2 + \|\mu_0\|_{S^\beta}^2 \sum_{i > i_\epsilon} \bar{\mu}_{0,i}^2 i^{-2\beta} \\ &\leq \|\mu_0\|_{S^\beta}^2 \left[\epsilon^4 \max_{i \leq i_\epsilon} \left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2 + C i_\epsilon^{-2\beta} \right],\end{aligned}$$

where $\bar{\mu}_{0,i}^2 := \mu_{0,i}^2 i^{2\beta} / \|\mu_0\|_{S^\beta}^2$ and $\|\mu_0\|_{S^\beta}^2 = \sum_i \mu_{0,i}^2 i^{2\beta}$. The lower bound can be proved by taking μ_0 such that $\bar{\mu}_{0,i} = 1$ for one of the $i \leq i_\epsilon$ and $\bar{\mu}_{0,i} = 0$ otherwise to get the first term, and $\bar{\mu}_{0,i} = 0$ for $i \neq i_\epsilon + 1$ and $\bar{\mu}_{0,i} = 1$ for $i = i_\epsilon + 1$ to get the second term (up to a constant).

Hence,

$$S_2 \asymp \|\mu_0\|_{S^\beta}^2 \left[\epsilon^4 \max_{i \leq i_\epsilon} \left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2 + i_\epsilon^{-2\beta} \right].$$

Combining these results together, we obtain

$$\mathbb{E}_{\mu_0} \mathbb{E}(\|\mu - \mu_0\|^2 | Y) \asymp \epsilon^2 \sum_{i \leq i_\epsilon} \sigma_i^2 k_i^{-2} + \sum_{i > i_\epsilon} \lambda_i + \epsilon^4 \max_{i \leq i_\epsilon} \left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2 + i_\epsilon^{-2\beta}.$$

Hence, setting ε_n such that

$$\varepsilon_n^{-2} \mathbb{E}_{\mu_0} \mathbb{E}(\|\mu - \mu_0\|^2 | Y) \asymp \varepsilon_n^{-2} \left[\epsilon^2 \sum_{i \leq i_\epsilon} \sigma_i^2 k_i^{-2} + \sum_{i > i_\epsilon} \lambda_i + \epsilon^4 \max_{i \leq i_\epsilon} \left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2 + i_\epsilon^{-2\beta} \right] = O(1),$$

ensures that

$$\mathbb{E}_{\mu_0} \mathbb{P}(\{\mu : \|\mu - \mu_0\| \geq M\varepsilon_n | Y\}) \leq M^{-2} \varepsilon_n^{-2} \mathbb{E}_{\mu_0} \mathbb{E}(\|\mu - \mu_0\|^2 | Y) \rightarrow 0, \text{ as } \varepsilon_n \rightarrow 0,$$

for every $M \rightarrow \infty$. □

The first two terms of ε_n (in Theorem 3.1) represent variance and squared bias terms, respectively, and the remaining terms involve prior parameters λ_i that can be chosen.

Note that this theorem easily generalises to the case of non-monotonic sequence $\lambda_i k_i^2 / \sigma_i^2$, with the range $i > i_\epsilon$ becoming $I_\epsilon = \{i : \sigma_i^2 / [\lambda_i k_i^2] > \epsilon^{-2}\}$.

Remark 3.1. *The posterior distribution can contract at parametric rate ϵ if $\{\sigma_i^2\}_{i=1}^\infty$ is such that $\sum_{i=1}^\infty \sigma_i^2 k_i^{-2} \leq C < \infty$, under the appropriate choice of prior parameters $\{\lambda_i\}_{i=1}^\infty$. See Section 3.4.2 for details in the case of polynomially decaying σ_i .*

3.4.2 Contraction Rates for Mildly Ill-posed Setting

Consequently, let us derive contractions rates, (c.f. Definition 1.3), for the mildly ill-posed setting where $k_i \asymp i^{-q}$, (as done in [Knapik et al., 2011]). As observed in Section 3.3, the sequence space model for Model (9)

$$Y_i = k_i \mu_i + \epsilon \tilde{Z}_i,$$

can be rescaled by $\sigma_i > 0$ to obtain

$$\tilde{Y}_i = \tilde{k}_i \mu_i + \epsilon Z_i,$$

where

$$\tilde{Y}_i = \frac{Y_i}{\sigma_i}, \quad \tilde{k}_i = \frac{k_i}{\sigma_i} \quad \text{and} \quad Z_i \sim N(0, 1)$$

and $Z_i \sim N(0, 1)$.

Thus the proof used to derive the contraction rates in [Knapik et al., 2011] will also hold for Model (9) and therefore Theorem 3.2, albeit with a different degree of ill-posedness $\tilde{q} = q + \gamma$.

Note that the contraction rate depends on all five parameters of the inverse problem, i.e. $\alpha, \beta, \gamma, \tau_n$ and q . However q, β and γ are fixed, and one only has control over α and τ_n .

Theorem 3.2 (Heterogeneous: Contraction). *Given Assumption 3.1, the $\mathbb{E}_{\mu_0} \Pi_n(\{\mu : \|\mu - \mu_0\|_{H_1} \geq M_n \varepsilon_n | Y\}) \rightarrow 0$, as $n \rightarrow \infty$, for every $M_n \rightarrow \infty$ where*

$$\varepsilon_n := \begin{cases} (n\tau_n^2)^{-\left(\frac{\beta}{1+2\alpha+2(q+\gamma)}\wedge 1\right)} + \tau_n(n\tau_n^2)^{-\frac{\alpha}{1+2\alpha+2(q+\gamma)}}, & \text{if } \gamma > -\frac{1+2q}{2}. \\ (n\tau_n^2)^{-\left(\frac{\beta}{1+2\alpha+2(q+\gamma)}\wedge 1\right)} + n^{-1/2}[\ln(n\tau_n^2)]^{1/2}, & \text{if } \gamma = -\frac{1+2q}{2}. \\ (n\tau_n^2)^{-\left(\frac{\beta}{1+2\alpha+2(q+\gamma)}\wedge 1\right)} + n^{-1/2}, & \text{if } -\frac{1+2q}{2} - \alpha < \gamma < -\frac{1+2q}{2}. \end{cases}$$

The rate is uniform over μ_0 in balls in S^β . In particular,

1. If $\tau_n = 1$,

$$\varepsilon_n = \begin{cases} n^{-(\frac{\beta}{1+2\alpha+2(q+\gamma)} \wedge 1)} + n^{-\frac{\alpha}{1+2\alpha+2(q+\gamma)}} = \mathcal{O}(n^{-\frac{\beta \wedge \alpha}{1+2\alpha+2(q+\gamma)}}), & \text{if } \gamma > -\frac{1+2q}{2}. \\ n^{-(\frac{\beta}{1+2\alpha+2(q+\gamma)} \wedge 1)} + n^{-1/2}(\ln n)^{1/2}, & \text{if } \gamma = -\frac{1+2q}{2}. \\ n^{-(\frac{\beta}{1+2\alpha+2(q+\gamma)} \wedge 1)} + n^{-1/2} = \mathcal{O}(n^{-(\frac{\beta}{1+2\alpha+2(q+\gamma)} \wedge \frac{1}{2})}), & \text{if } -\frac{1+2q}{2} - \alpha < \gamma < -\frac{1+2q}{2}. \end{cases}$$

2. If $\beta \leq 1 + 2\alpha + 2(q + \gamma)$ and

$$\tau_n \asymp \begin{cases} n^{\frac{\alpha-\beta}{1+2\beta+2(q+\gamma)}} \implies \varepsilon_n = \mathcal{O}(n^{-\frac{\beta}{1+2\beta+2(q+\gamma)}}), & \text{if } \gamma > -\frac{1+2q}{2}. \\ n^{\frac{1+2\alpha+2(q+\gamma)}{4\beta} - \frac{1}{2}} \implies \varepsilon_n = \mathcal{O}(n^{-\frac{1}{2}}[\ln n]^{\frac{1}{2}}), & \text{if } \gamma = -\frac{1+2q}{2}. \\ n^{\frac{1+2\alpha+2(q+\gamma)-2\beta}{4\beta}} \implies \varepsilon_n = \mathcal{O}(n^{-\frac{1}{2}}), & \text{if } -\frac{1+2q}{2} - \alpha < \gamma < -\frac{1+2q}{2}. \end{cases}$$

3. If $\beta > 1 + 2\alpha + 2(q + \gamma)$ then

$$\begin{cases} \varepsilon_n \gg n^{-\frac{\beta}{1+2\beta+2(q+\gamma)}} \text{ for all } \tau_n, & \text{if } \gamma > -\frac{1+2q}{2}. \\ \varepsilon_n = \mathcal{O}(n^{-\frac{1}{2}}[\ln n]^{\frac{1}{2}}) \text{ for } \tau_n \asymp \mathcal{O}(n^{-\frac{1}{4}}), & \text{if } \gamma = -\frac{1+2q}{2}. \\ \varepsilon_n = \mathcal{O}(n^{-1/2}) \text{ for each } \tau_n \geq Cn^{-\frac{1}{4}}, & \text{if } -\frac{1+2q}{2} - \alpha < \gamma < -\frac{1+2q}{2}. \end{cases}$$

Proof. Just as done in [Knapik et al., 2011], (c.f. Appendix B.2), we must evaluate and asymptotically bound the terms $\|\tilde{A}K\mu_0 - \mu_0\|_{H_1}^2, n^{-1}\text{tr}(\tilde{A}V\tilde{A}^T)$ and $\text{tr}(\tilde{S}_n)$. This is because, given the posterior distribution $\Pi_n(\cdot|Y)$, Markov's Inequality applied on the first moment of $\mu - \mu_0$ implies

$$\Pi_n(\{\mu \in H_1 : \|\mu - \mu_0\|_{H_1}^2 \geq M_n^2 \varepsilon_n^2 | Y\}) \leq \frac{1}{M_n^2 \varepsilon_n^2} \int_{H_1} \|\mu - \mu_0\|_{H_1}^2 d\Pi_n(\mu|Y).$$

However, since $\mu|Y \sim N(\tilde{A}Y, \tilde{S}_n)$ and $\tilde{A}Y|\mu \sim N(\tilde{A}K\mu, n^{-1}\tilde{A}V\tilde{A}^T)$, we know (using Proposition A.1)

$$\begin{aligned} \int_{H_1} \|\mu - \mu_0\|_{H_1}^2 d\Pi_n(\mu|Y) &= \mathbb{E}\|\mu - \mu_0\|_{H_1}^2 = \|\tilde{A}Y - \mu_0\|_{H_1}^2 + \text{tr}(\tilde{S}_n) \implies \\ \mathbb{E}_{\mu_0} \mathbb{E}\|\mu - \mu_0\|_{H_1}^2 &= \|\tilde{A}K\mu_0 - \mu_0\|_{H_1}^2 + n^{-1}\text{tr}(\tilde{A}V\tilde{A}^T) + \text{tr}(\tilde{S}_n). \end{aligned}$$

Consequently, defining $\tilde{k} := k_i/\sigma_i$,

$$\begin{aligned} \bullet \quad \|\tilde{A}K\mu_0 - \mu_0\|_{H_1}^2 &= \|\sum_i k_i \mu_{0,i} \tilde{A}\varphi_i - \sum_i \mu_{0,i} \phi_i\|_{H_1}^2 \\ &= \|\sum_i k_i \mu_{0,i} \frac{n\lambda_i k_i}{\sigma_i^2 + nk_i^2 \lambda_i} \phi_i - \sum_i \mu_{0,i} \phi_i\|_{H_1}^2 = \sum_i \frac{\mu_{0,i}^2 \sigma_i^4}{(\sigma_i^2 + nk_i^2 \lambda_i)^2} = \sum_i \frac{\mu_{0,i}^2}{(1 + n\tilde{k}_i^2 \lambda_i)^2}. \end{aligned}$$

The penultimate equality following from *Parseval's Identity*.

- $\text{tr}(\tilde{A}V\tilde{A}^T) = \text{tr}(\Lambda K^T(n^{-1}V + K\Lambda K^T)^{-2}VK\Lambda^T)$

$$= \sum_i \lambda_i k_i \left(\frac{1}{n^{-1}\sigma_i^2 + k_i \lambda_i k_i} \right)^2 \sigma_i^2 k_i \lambda_i = \sum_i \frac{n^2 \lambda_i^2 k_i^2 \sigma_i^2}{(\sigma_i^2 + n \lambda_i k_i^2)^2} = \sum_i \frac{n^2 \lambda_i^2 \tilde{k}_i^2}{(1 + n \lambda_i \tilde{k}_i^2)^2},$$

noting that V is symmetric.

- $\text{tr}(\tilde{S}_n) = \text{tr}(\Lambda - \tilde{A}(n^{-1}V + K\Lambda K^T)\tilde{A}^T)$

$$= \sum_j \frac{\lambda_j \sigma_j^2}{\sigma_j^2 + n k_j^2 \lambda_j} = \sum_j \frac{\lambda_j}{1 + n \tilde{k}_j^2 \lambda_j}.$$

Hence, given Assumption 3.1, we know the bounds on k_i and σ_i^2 , thus

$$\begin{aligned} 1 + n \lambda_i C_1^{-2} C_2^{-2} i^{-2(q+\gamma)} &\leq 1 + n \lambda_i \tilde{k}_i^2 \leq 1 + n \lambda_i C_1^2 C_2^2 i^{-2(q+\gamma)} \implies \\ C_l(1 + n \lambda_i i^{-2\tilde{q}}) &\leq 1 + n \lambda_i \tilde{k}_i^2 \leq C_u(1 + n \lambda_i i^{-2\tilde{q}}) \implies \\ 1 + n \lambda_i \tilde{k}_i^2 &\asymp 1 + n \lambda_i i^{-2\tilde{q}}, \end{aligned}$$

where $\tilde{q} = q + \gamma$, $C_l = \min(1, C_1^{-2} C_2^{-2})$ and $C_u = \max(1, C_1^2 C_2^2)$. Consequently, via Assumption 3.1 again,

- $\|\tilde{A}K\mu_0 - \mu_0\|_{H_1}^2 =$

$$\sum_i \frac{\mu_{0,i}^2}{(1 + n \tilde{k}_i^2 \lambda_i)^2} \asymp \sum_i \frac{\mu_{0,i}^2}{(1 + n \tau_n^2 i^{-1-2\alpha-2\tilde{q}})^2} = \|\mu_0\|_{S^\beta}^2 \sum_i \frac{\mu_{0,i}^2 / \|\mu_0\|_{S^\beta}^2}{(1 + n \tau_n^2 i^{-1-2\alpha-2\tilde{q}})^2}$$

$$= \|\mu_0\|_{S^\beta}^2 \sum_i \frac{\tilde{\mu}_{0,i}^2}{(1 + n \tau_n^2 i^{-1-2\alpha-2\tilde{q}})^2} \leq \|\mu_0\|_{S^\beta}^2 \sup_{\|\tilde{\mu}\|_{S^\beta} \leq 1} \sum_i \frac{\tilde{\mu}_i^2}{(1 + n \tau_n^2 i^{-1-2\alpha-2\tilde{q}})^2},$$

where $\tilde{\mu}_{0,i}^2 := \mu_{0,i}^2 / \|\mu_0\|_{S^\beta}^2$.

- $\text{tr}(\tilde{A}V\tilde{A}^T) =$

$$\sum_i \frac{n^2 \lambda_i^2 \tilde{k}_i^2}{(1 + n \lambda_i \tilde{k}_i^2)^2} \asymp \sum_i \frac{n^2 \tau_n^4 i^{-2-4\alpha-2\tilde{q}}}{(1 + n \tau_n^2 i^{-1-2\alpha-2\tilde{q}})^2}.$$

- $\text{tr}(\tilde{S}_n) =$

$$\sum_i \frac{\lambda_j}{1 + n \tilde{k}_j^2 \lambda_j} \asymp \sum_i \frac{\tau_n^2 i^{-1-2\alpha}}{1 + n \tau_n^2 i^{-1-2\alpha-2\tilde{q}}}.$$

Finally using Lemma 8.1 from [Knapik et al., 2011], i.e. Lemma B.1, one obtains

$$\sup_{\|\tilde{\mu}\|_{S^\beta} \leq 1} \sum_i \frac{\tilde{\mu}_i^2}{(1 + n \tau_n^2 i^{-1-2\alpha-2\tilde{q}})^2} \asymp (n \tau_n^2)^{-(\frac{2\beta}{1+2\alpha+2\tilde{q}} \wedge 2)},$$

by setting $r = \beta, t = 0, u = 1 + 2\alpha + 2\tilde{q}, v = 2$ and $N = n\tau_n^2$. Consequently $\|\tilde{A}K\mu_0 - \mu_0\|_{H_1}^2 \asymp \|\mu_0\|_{S^\beta}^2 (n\tau_n^2)^{-\left(\frac{2\beta}{1+2\alpha+2\tilde{q}}\wedge 2\right)}$.

Similarly using Lemma 8.2 from [Knapik et al., 2011], i.e. Lemma B.2, along with setting $S(i) = 1, r = -1/2, t = 2 + 4\alpha + 2\tilde{q}, u = 1 + 2\alpha + 2\tilde{q}, v = 2$ and $N = n\tau_n^2$, we obtain

$$\sum_i \frac{i^{-2-4\alpha-2\tilde{q}}}{(1 + n\tau_n^2 i^{-1-2\alpha-2\tilde{q}})^2} \asymp \begin{cases} (n\tau_n^2)^{-\frac{1+4\alpha+2\tilde{q}}{1+2\alpha+2\tilde{q}}}, & \text{if } \gamma > -\frac{1+2q}{2}. \\ (n\tau_n^2)^{-2} \sum_{i \leq (n\tau_n^2)^{(1/u)}} i^{-1}, & \text{if } \gamma = -\frac{1+2q}{2}. \\ (n\tau_n^2)^{-2}, & \text{if } \gamma < -\frac{1+2q}{2}. \end{cases}$$

Thus

$$n^{-1} \text{tr}(\tilde{A}V\tilde{A}^T) \asymp \begin{cases} \tau_n^2 (n\tau_n^2)^{-\frac{2\alpha}{1+2\alpha+2\tilde{q}}}, & \text{if } \gamma > -\frac{1+2q}{2}. \\ \tau_n^2 (n\tau_n^2)^{-1} \sum_{i \leq (n\tau_n^2)^{(1/u)}} i^{-1}, & \text{if } \gamma = -\frac{1+2q}{2}. \\ \tau_n^2 (n\tau_n^2)^{-1}, & \text{if } \gamma < -\frac{1+2q}{2}. \end{cases}$$

Furthermore, setting $S(i) = 1, r = -1/2, t = 1 + 2\alpha, u = 1 + 2\alpha + 2\tilde{q}, v = 1$ and $N = n\tau_n^2$ in Lemma B.2, we obtain

$$\sum_i \frac{i^{-1-2\alpha}}{1 + n\tau_n^2 i^{-1-2\alpha-2\tilde{q}}} \asymp \begin{cases} (n\tau_n^2)^{-\frac{2\alpha}{1+2\alpha+2\tilde{q}}}, & \text{if } \gamma > -\frac{1+2q}{2}. \\ (n\tau_n^2)^{-1} \sum_{i \leq (n\tau_n^2)^{(1/u)}} i^{-1}, & \text{if } \gamma = -\frac{1+2q}{2}. \\ (n\tau_n^2)^{-1}, & \text{if } \gamma < -\frac{1+2q}{2}. \end{cases}$$

Note, in Lemmas B.1 and B.2 $u > 0$, hence from here on out we shall assume

$$\gamma > -\frac{(1 + 2\alpha + 2q)}{2} = -\frac{1 + 2q}{2} - \alpha.$$

This constraint can also be found in Theorem 3.1, where

$$\frac{\sigma_i^2}{\lambda_i k_i^2} = \tau_n^{-2} i^{1+2\alpha+2(q+\gamma)},$$

is a monotonically increasing sequence if

$$1 + 2\alpha + 2(q + \gamma) > 0 \iff \gamma > -\frac{1 + 2q}{2} - \alpha.$$

Subsequently, we can see that the

$$\begin{aligned} \mathbb{E}_{\mu_0} \Pi_n(\{\mu : \|\mu - \mu_0\|_{H_1} \geq M_n \varepsilon_n | Y\}) &\leq \frac{1}{M_n^2 \varepsilon_n^2} (\|\tilde{A}K\mu_0 - \mu_0\|_{H_1}^2 + n^{-1} \text{tr}(\tilde{A}V\tilde{A}^T) + \text{tr}(\tilde{S}_n)) \\ &\leq \begin{cases} \frac{1}{M_n^2 \varepsilon_n^2} [\|\mu_0\|_{S^\beta}^2 (n\tau_n^2)^{-\left(\frac{2\beta}{1+2\alpha+2\tilde{q}}\wedge 2\right)} + 2\tau_n^2 (n\tau_n^2)^{-\frac{2\alpha}{1+2\alpha+2\tilde{q}}}], & \text{if } \gamma > -\frac{1+2q}{2}. \\ \frac{1}{M_n^2 \varepsilon_n^2} [\|\mu_0\|_{S^\beta}^2 (n\tau_n^2)^{-\left(\frac{2\beta}{1+2\alpha+2\tilde{q}}\wedge 2\right)} + 2\tau_n^2 (n\tau_n^2)^{-1} \sum_{i \leq (n\tau_n^2)^{(1/u)}} i^{-1}], & \text{if } \gamma = -\frac{1+2q}{2}. \\ \frac{1}{M_n^2 \varepsilon_n^2} [\|\mu_0\|_{S^\beta}^2 (n\tau_n^2)^{-\left(\frac{2\beta}{1+2\alpha+2\tilde{q}}\wedge 2\right)} + 2\tau_n^2 (n\tau_n^2)^{-1}], & \text{if } \gamma < -\frac{1+2q}{2}. \end{cases} \end{aligned}$$

Hence, setting

$$\varepsilon_n := \begin{cases} (n\tau_n^2)^{-\left(\frac{\beta}{1+2\alpha+2(q+\gamma)}\wedge 1\right)} + \tau_n(n\tau_n^2)^{-\frac{\alpha}{1+2\alpha+2(q+\gamma)}}, & \text{if } \gamma > -\frac{1+2q}{2}. \\ (n\tau_n^2)^{-\left(\frac{\beta}{1+2\alpha+2(q+\gamma)}\wedge 1\right)} + n^{-1/2} \left(\sum_{i \leq (n\tau_n^2)^{(1/u)}} i^{-1} \right)^{1/2}, & \text{if } \gamma = -\frac{1+2q}{2}. \\ (n\tau_n^2)^{-\left(\frac{\beta}{1+2\alpha+2(q+\gamma)}\wedge 1\right)} + n^{-1/2}, & \text{if } \gamma < -\frac{1+2q}{2}, \end{cases}$$

ensures that the $\mathbb{E}_{\mu_0} \Pi_n(\{\mu : \|\mu - \mu_0\|_{H_1} \geq M_n \varepsilon_n | Y\}) \rightarrow 0$ for every $M_n \rightarrow \infty$.

Note,

$$\left(\sum_{i \leq (n\tau_n^2)^{(1/u)}} i^{-1} \right)^{1/2} \leq \left(\int_{i \leq (n\tau_n^2)^{(1/u)}} i^{-1} \right)^{1/2} = u^{-1/2} (\ln(n\tau_n^2))^{1/2},$$

thus we can further simplify ε_n . □

Next, let us see how one obtains the rates stated in the theorem.

Proof. • Setting $\tau_n = 1$,

$$\varepsilon_n = \begin{cases} n^{-\left(\frac{\beta}{1+2\alpha+2(q+\gamma)}\wedge 1\right)} + n^{-\frac{\alpha}{1+2\alpha+2(q+\gamma)}} = \mathcal{O}(n^{-\frac{\beta \wedge \alpha}{1+2\alpha+2(q+\gamma)}}), & \text{if } \gamma > -\frac{1+2q}{2}. \\ n^{-\left(\frac{\beta}{1+2\alpha+2(q+\gamma)}\wedge 1\right)} + n^{-1/2} (\ln n)^{1/2}, & \text{if } \gamma = -\frac{1+2q}{2}. \\ n^{-\left(\frac{\beta}{1+2\alpha+2(q+\gamma)}\wedge 1\right)} + n^{-1/2} = \mathcal{O}(n^{-\frac{\beta}{1+2\alpha+2(q+\gamma)} \wedge \frac{1}{2}}), & \text{if } -\frac{1+2q}{2} - \alpha < \gamma < -\frac{1+2q}{2}. \end{cases}$$

• If $\beta \leq 1 + 2\alpha + 2\tilde{q}$ then ε_n can be minimised by setting

$$\tau_n \asymp \begin{cases} n^{\frac{\alpha-\beta}{1+2\beta+2\tilde{q}}} \implies \varepsilon_n = \mathcal{O}(n^{-\frac{\beta \wedge \alpha}{1+2\beta+2\tilde{q}}}), & \text{if } \gamma > -\frac{1+2q}{2}. \\ n^{\frac{1+2\alpha+2(q+\gamma)}{4\beta} - \frac{1}{2}} \implies \varepsilon_n = \mathcal{O}(n^{-\frac{1}{2}} [\ln(n^{\frac{1+2\alpha+2(q+\gamma)}{2\beta}})]^{\frac{1}{2}}), & \text{if } \gamma = -\frac{1+2q}{2}. \\ n^{\frac{1+2\alpha+2\tilde{q}-2\beta}{4\beta}} \implies \varepsilon_n = \mathcal{O}(n^{-\frac{1}{2}}), & \text{if } -\frac{1+2q}{2} - \alpha < \gamma < -\frac{1+2q}{2}. \end{cases}$$

Note for the case when $\gamma = -\frac{1+2q}{2}$,

$$\varepsilon_n = (n\tau_n^2)^{-\frac{\beta}{1+2\alpha+2(q+\gamma)}} + n^{-1/2} u^{-1/2} (\ln(n\tau_n^2))^{1/2}.$$

This can be simplified by using the following order-preserving inequality,

$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}. \quad (18)$$

Consequently, ignoring constants,

$$\varepsilon_n \leq [(n\tau_n^2)^{-\frac{2\beta}{1+2\alpha+2(q+\gamma)}} + n^{-1} \ln(n\tau_n^2)]^{1/2}. \quad (19)$$

Subsequently, the minimum of the R.H.S. of (19) is attained when $\tau_n = \mathcal{O}(n^{\frac{1+2\alpha+2(q+\gamma)}{4\beta} - \frac{1}{2}})$.

- If $\beta > 1 + 2\alpha + 2\tilde{q}$ and $\gamma > -\frac{1+2q}{2}$, then the minimiser of the positive, increasing function $\varepsilon_n(\tau)$ i.e.

$$\tau^* = \mathcal{O}(n^{-\frac{1+\alpha+2\tilde{q}}{3+4\alpha+6\tilde{q}}}) \implies \varepsilon_n(\tau^*) = \mathcal{O}(n^{-\frac{1+2\alpha+2\tilde{q}}{3+4\alpha+6\tilde{q}}}).$$

Hence, we see that for any scaling τ_n , $\varepsilon_n(\tau_n) \gg n^{-\frac{\beta}{1+2\beta+2\tilde{q}}}$ if and only if

$$\frac{1+2\alpha+2\tilde{q}}{3+4\alpha+6\tilde{q}} < \frac{\beta}{1+2\beta+2\tilde{q}} \iff \frac{1}{2+\frac{1+2\tilde{q}}{1+2\alpha+2\tilde{q}}} < \frac{1}{2+\frac{1+2\tilde{q}}{\beta}} \iff 1+2\alpha+2\tilde{q} < \beta,$$

which is indeed the case.

- If $\beta > 1 + 2\alpha + 2\tilde{q}$ and $\gamma = -\frac{1+2q}{2}$, then using (18), we can show that the minimiser of $\varepsilon(\tau)$ is

$$\tau^* = \mathcal{O}(n^{-\frac{1}{4}}) \implies \varepsilon_n(\tau^*) = \mathcal{O}(n^{-\frac{1}{2}}[\ln(n^{\frac{1}{2}})]^{\frac{1}{2}}).$$

- If $\beta > 1 + 2\alpha + 2\tilde{q}$ and $-\frac{1+2q}{2} - \alpha < \gamma < -\frac{1+2q}{2}$, then the decreasing $\varepsilon_n(\tau) = \mathcal{O}(n^{-1/2})$ if $\tau_n \geq Cn^{-\frac{1}{4}}$.

□

Observe that when $\gamma > -\frac{1+2q}{2}$, the rates obtained are similar to the homogeneous case, albeit with a different degree of ill-posedness, specifically $\tilde{q} = q + \gamma$. Furthermore, when $\gamma = -\frac{1+2q}{2}$, the optimal parametric rate ($n^{-\frac{1}{2}}$) can be achieved, up to a logarithmic factor, $(\ln n)^{1/2}$. Please see Proposition 3.2 for more details about the optimal rates.

When $\tau_n = 1$, i.e. when the scaling is fixed, if $\gamma > -\frac{1+2q}{2}$, then the optimal rate can only be achieved if $\alpha = \beta$.

Note,

$$-\frac{1+2q}{2} - \alpha < \gamma \leq -\frac{1+2q}{2} \iff 0 < 1 + 2\alpha + 2(q + \gamma) \leq 2\alpha.$$

Therefore,

$$-\frac{1+2q}{2} - \alpha < \gamma \leq -\frac{1+2q}{2} \implies \frac{\beta}{2\alpha} < \frac{\beta}{1+2\alpha+2(q+\gamma)}.$$

Consequently, when $-\frac{1+2q}{2} - \alpha < \gamma \leq -\frac{1+2q}{2}$, we can achieve the optimal rates if $\alpha \leq \beta$, since

$$\alpha \leq \beta \implies \frac{1}{2} \leq \frac{\beta}{1+2\alpha+2(q+\gamma)}.$$

Thus, μ has to be, at most, as *smooth* as μ_0 in order to achieve said optimal rates.

When $\beta \leq 1 + 2\alpha + 2(q + \gamma)$ we see that the optimal rates can be achieved if the appropriate scaling is used. However, whilst $\alpha \geq \beta$ in order to achieve the optimal non-parametric rate (for $\gamma > -\frac{1+2q}{2}$), in the case when $-\frac{1+2q}{2} - \alpha < \gamma \leq -\frac{1+2q}{2}$, the optimal rate is achieved regardless of α 's relationship with β .

When $\beta > 1 + 2\alpha + 2(q + \gamma)$ and if $\gamma > -\frac{1+2q}{2}$, we see that the optimal rate can never be achieved regardless of the scaling used, just like in the homogeneous case. Conversely, if $-\frac{1+2q}{2} - \alpha < \gamma \leq -\frac{1+2q}{2}$ the optimal rate can be achieved if the appropriate scaling is used.

Therefore, unlike in the homogeneous case, if the scaling is not fixed and if $-\frac{1+2q}{2} - \alpha < \gamma \leq -\frac{1+2q}{2}$ then the optimal rates are achieved regardless of α 's relationship with β . This agrees with our intuition, since the problem is well-posed i.e. $\tilde{q} < 0$, when $-\frac{1+2q}{2} - \alpha < \gamma \leq -\frac{1+2q}{2}$.

3.5 Conclusion

To summarise, in the homogeneous case, we were able to obtain the contraction rates, (see Theorem 2.1) given the following assumptions:

- $\exists \{\phi_i\}_{i=1}^\infty$ an orthonormal basis of eigenfunctions for $K^T K$.
- $K^T K$ and Λ have the same eigenfunctions $\{\phi_i\}_{i=1}^\infty$, with eigenvalues (k_i^2) and (λ_i) , satisfying

$$\lambda_i = \tau_n^2 i^{-1-2\alpha}, \quad \text{and} \quad C^{-1} i^{-q} \leq k_i \leq C i^{-q},$$

for some $\alpha > 0, q \geq 0, C \geq 1$ and $\tau_n > 0$ such that $n\tau_n^2 \rightarrow \infty$.

- The true parameter $\mu_0 \in S^\beta$ for some $\beta > 0$.

Furthermore, for the heterogeneous case, not only did we require the assumptions *above*, but we needed additional ones regarding V , and its eigenvalues σ_i^2 , in order to obtain the contraction rates, (c.f. Theorem 3.2). These additional assumptions are as follows:

- $\exists \{\varphi_i\}_{i=1}^\infty$ an orthonormal basis of eigenfunctions for $V := WW^T$.
- KK^T and V have the same eigenfunctions $\{\varphi_i\}_{i=1}^\infty$, with V 's eigenvalues (σ_i^2) satisfying

$$C_2^{-1} i^\gamma \leq \sigma_i \leq C_2 i^\gamma, \quad \text{and} \quad \sum_i \frac{\lambda_i k_i^2}{\sigma_i^2} < \infty,$$

for some $C_2 \geq 1$.

Subsequently, in the heterogeneous case, we saw that the ill-posed inverse problem became well-posed for particular values of γ . What was surprising however, was our achievement of the optimal rate even when $\beta > 1 + 2\alpha + 2(q + \gamma)$, unlike in the homogeneous variance setting, (see Theorem 2.1). Consequently, understanding why this occurs could help us better understand the problem as a whole.

Another question we may ask ourselves (whilst still being in the mildly ill-posed setting) is whether it is possible for our posterior rates to converge to the optimal rates without *fine-tuning* our prior regularity parameter, i.e. α . This is indeed true in the homogeneous case, as proved in [Ray, 2013]'s Proposition 3.2 and [Knapik et al., 2016]'s

Theorem 2.5, where in some instances the rates achieved were $n^{-1/2}$, up to a *logarithmic factor*. Subsequently, investigating whether optimal, adaptive priors exist for the heterogeneous variance case would be one possible avenue worth investigating.

In the future, we would like to study inverse problems with Poisson noise, which are more complex than their Gaussian counterparts; such problems can typically be found in Tomography, such as Single Photon Emission Computed Tomography (SPECT) [Bauschke et al., 1999], and Emission Computed Tomography (ECT) [Gourion and Noll, 2002].

Another issue worth studying in the future, is when the covariance operator V , specifically σ_i , is observed with noise. The next section contains our first attempt at doing so.

3.6 Noise in Covariance Operator V

In this section we assume that the covariance operator V is unknown. Hence, we use a plug in estimator, \hat{V} , with singular values

$$\hat{\sigma}_i^2 \asymp \sigma_i^2 + \epsilon^{-2\delta}, \quad (20)$$

where $\sigma_i^2 \asymp i^{2\gamma}$, and $\delta \in (0, \frac{1}{2}]$.

Recall, the contraction rate will be affected by $\epsilon^2 \hat{\sigma}_i^2$ which, for large enough i , will be dominated by the noise $\epsilon^{2(1-\delta)}$. Consequently, we seek to investigate the effect this has on our contraction rates.

3.6.1 Contraction Rates for General Setting

Theorem 3.3. *Given Assumption 3.2 and a monotonically increasing sequence $\hat{\sigma}_i^2/[\lambda_i k_i^2]$, the $\mathbb{E}_{\mu_0} \Pi_n(\{\mu : \|\mu - \mu_0\|_{H_1} \geq M_n \varepsilon_n | Y\}) \rightarrow 0$, as $n \rightarrow \infty$, for every $M_n \rightarrow \infty$ where*

$$\varepsilon_n = \left[\epsilon^2 \sum_{i \leq i_\epsilon} \sigma_i^2 k_i^{-2} + i_\epsilon^{-2\beta} + \sum_{i > i_\epsilon} \lambda_i + \epsilon^4 \max_{i \leq i_\epsilon} \left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2 + \sum_{i \leq i_\delta} \frac{k_i^2 \lambda_i^2}{\epsilon^2 \sigma_i^2} + \sum_{i > i_\delta} \frac{k_i^2 \lambda_i^2}{\epsilon^{2(1-\delta)}} \right]^{1/2},$$

with $i_\epsilon = \max\{i : \epsilon^2 \sigma_i^2 \leq k_i^2 \lambda_i\}$ and $i_\delta = \max\{i : \sigma_i^2 \geq \epsilon^{-2\delta}\}$. The rate is uniform over μ_0 in balls in S^β .

Proof. Under Assumption 3.1, using Parseval's identity, we have that

$$\mathbb{E}(\|\mu - \mu_0\|^2 | Y) = \sum_i \mathbb{E}[(\mu_i - \mu_{0,i})^2 | Y] = \sum_i [\text{Var}(\mu_i | Y) + (\mathbb{E}[\mu_i | Y] - \mu_{0,i})^2].$$

Taking expected value with respect to the true distribution of the data and using the

explicit form of the posterior distribution (10), we have

$$\begin{aligned}\mathbb{E}_{\mu_0}[\text{Var}(\mu_i | Y)] + \mathbb{E}_{\mu_0}[(\mathbb{E}[\mu_i | Y] - \mu_{0,i})^2] &= \mathbb{E}_{\mu_0} \left[\frac{Y_i k_i \lambda_i}{\lambda_i k_i^2 + \epsilon^2 \hat{\sigma}_i^2} - \mu_{0,i} \right]^2 + \frac{\hat{\sigma}_i^2 \lambda_i}{\epsilon^{-2} \lambda_i k_i^2 + \hat{\sigma}_i^2} \\ &= \frac{\epsilon^2 \hat{\sigma}_i^2 k_i^2 \lambda_i^2}{[\lambda_i k_i^2 + \epsilon^2 \hat{\sigma}_i^2]^2} + \mu_{0,i}^2 \left[\frac{\epsilon^2 \hat{\sigma}_i^2}{\lambda_i k_i^2 + \epsilon^2 \hat{\sigma}_i^2} \right]^2 \\ &\quad + \frac{\hat{\sigma}_i^2 \lambda_i}{\epsilon^{-2} \lambda_i k_i^2 + \hat{\sigma}_i^2}.\end{aligned}$$

Note,

$$\lambda_i k_i^2 + \epsilon^2 \hat{\sigma}_i^2 \asymp \tau_n^2 i^{-1-2\alpha-2p} + \epsilon^2 [i^{2\gamma} + \epsilon^{-2\delta}] = \tau_n^2 i^{-1-2\alpha-2p} + \epsilon^2 i^{2\gamma} + \epsilon^{2(1-\delta)},$$

and $i^{2\gamma}$ is either constant or monotonic. Hence, denote $i_\epsilon = \max\{i : \lambda_i k_i^2 \geq \epsilon^2 \hat{\sigma}_i^2\}$ and $i_\delta = \max\{i : \sigma_i^2 \geq \epsilon^{-2\delta}\}$.

Consequently,

$$\begin{aligned}S_1 &= \sum_i \frac{\epsilon^2 \hat{\sigma}_i^2 k_i^2 \lambda_i^2}{[\lambda_i k_i^2 + \epsilon^2 \hat{\sigma}_i^2]^2} \asymp \sum_{i \leq i_\epsilon} \frac{\epsilon^2 \hat{\sigma}_i^2 k_i^2 \lambda_i^2}{[\lambda_i k_i^2]^2} + \sum_{i > i_\epsilon} \frac{k_i^2 \lambda_i^2}{\epsilon^2 \hat{\sigma}_i^2} \\ &\asymp \sum_{i \leq i_\epsilon} \epsilon^2 \hat{\sigma}_i^2 k_i^{-2} + \sum_{i > i_\epsilon}^{i \leq i_\delta} \frac{k_i^2 \lambda_i^2}{\epsilon^2 \sigma_i^2} + \sum_{i > i_\delta} \frac{k_i^2 \lambda_i^2}{\epsilon^{2(1-\delta)}}, \\ S_3 &= \sum_i \frac{\epsilon^2 \hat{\sigma}_i^2 \lambda_i}{\lambda_i k_i^2 + \epsilon^2 \hat{\sigma}_i^2} \asymp \sum_{i \leq i_\epsilon} \epsilon^2 \hat{\sigma}_i^2 k_i^{-2} + \sum_{i > i_\epsilon} \lambda_i.\end{aligned}$$

As for the remaining term,

$$\begin{aligned}S_2 &= \sum_i \mu_{0,i}^2 \left[\frac{\epsilon^2 \hat{\sigma}_i^2}{k_i^2 \lambda_i + \epsilon^2 \hat{\sigma}_i^2} \right]^2 = \|\mu_0\|_{S^\beta}^2 \sum_i \bar{\mu}_{0,i}^2 \left[\frac{\epsilon^2 \hat{\sigma}_i^2 i^{-\beta}}{k_i^2 \lambda_i + \epsilon^2 \hat{\sigma}_i^2} \right]^2 \\ &\asymp \|\mu_0\|_{S^\beta}^2 \epsilon^4 \sum_{i \leq i_\epsilon} \bar{\mu}_{0,i}^2 \left[\frac{\hat{\sigma}_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2 + \|\mu_0\|_{S^\beta}^2 \sum_{i > i_\epsilon} \bar{\mu}_{0,i}^2 i^{-2\beta} \\ &\leq \|\mu_0\|_{S^\beta}^2 \left[\epsilon^4 \sum_{i \leq i_\epsilon} \bar{\mu}_{0,i}^2 \left[\frac{\hat{\sigma}_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2 + i_\epsilon^{-2\beta} \right],\end{aligned}$$

where $\bar{\mu}_{0,i}^2 := \mu_{0,i}^2 i^{2\beta} / \|\mu_0\|_{S^\beta}^2$. The lower bound can be proved by taking μ_0 such that $\bar{\mu}_{0,i} = 1$ for $i \leq i_\gamma$ and $\bar{\mu}_{0,i} = 0$ for $i > i_\gamma$ to get the first term, and $\bar{\mu}_{0,i} = 0$ for $i \leq i_\gamma$ and $\bar{\mu}_{0,i} = 1$ for $i > i_\gamma$ to get the second term.

Hence,

$$S_2 \asymp \|\mu_0\|_{S^\beta}^2 \left[\epsilon^4 \max_{i \leq i_\epsilon} \left[\frac{\hat{\sigma}_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2 + i_\epsilon^{-2\beta} \right].$$

Combining these results together, we obtain

$$\mathbb{E}_{\mu_0} \mathbb{E}(\|\mu - \mu_0\|^2 \mid Y) \asymp \epsilon^2 \sum_{i \leq i_\epsilon} \hat{\sigma}_i^2 k_i^{-2} + i_\epsilon^{-2\beta} + \sum_{i > i_\epsilon} \lambda_i + \epsilon^4 \max_{i \leq i_\epsilon} \left[\frac{\hat{\sigma}_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2 + \sum_{i > i_\epsilon} \frac{k_i^2 \lambda_i^2}{\epsilon^2 \sigma_i^2} + \sum_{i > i_\delta} \frac{k_i^2 \lambda_i^2}{\epsilon^{2(1-\delta)}}.$$

□

Thus, we see that the effect of the plug in estimator on the contraction rate is described by the last two terms.

Subsequently, we conclude this section by providing an example of an ill-posed inverse problem with heterogeneous variance, using the Volterra operator.

3.7 Example using Simulated Data

We illustrate our results using simulated Y and the Volterra operator [Halmos, 1982]. We set $H_1 = H_2 = L^2[0, 1]$ and use the eigenbasis

$$\phi_i(x) = \sqrt{2} \cos\left(\left(i - \frac{1}{2}\right)\pi x\right).$$

For practicality we shall truncate $\mu_0(x)$, hence define

$$\mu_0^N(x) := \sum_i^N \mu_{0,i} \phi_i(x),$$

where N is the truncation parameter. Furthermore, we shall set

$$\mu_{0,i} := i^{-3/2} \sin(i).$$

Note, we require $\mu_0 \in S^\beta$, which is indeed the case when $\beta = 1$, (this can be shown using Dirichlet's test, [Voxman and Goetschel Jr, 1981]). Furthermore, we consider the Volterra operator, $K : L^2[0, 1] \rightarrow L^2[0, 1]$, where

$$K\mu(x) := \int_0^x \mu(s) ds, \quad \text{and} \quad K^T \mu(x) := \int_x^1 \mu(s) ds.$$

The eigenvalues of KK^T and the eigenbasis for the range of K are

$$k_i := \left[\left(i - \frac{1}{2}\right)^2 \pi^2\right]^{-\frac{1}{2}}, \quad \text{and} \quad \varphi_i(x) := \sqrt{2} \sin\left(\left(i - \frac{1}{2}\right)\pi x\right) \quad \text{for every } i \in \mathbb{N},$$

where $k_i \asymp i^{-q}$ with $q = 1$. Additionally, as discussed in Section 3, we define

$$\lambda_i := \tau_\epsilon^2 i^{-1-2\alpha}, \quad \text{and} \quad \sigma_i := 2i^\gamma,$$

with $\tau_\epsilon = 1$ and $\gamma = 1$.

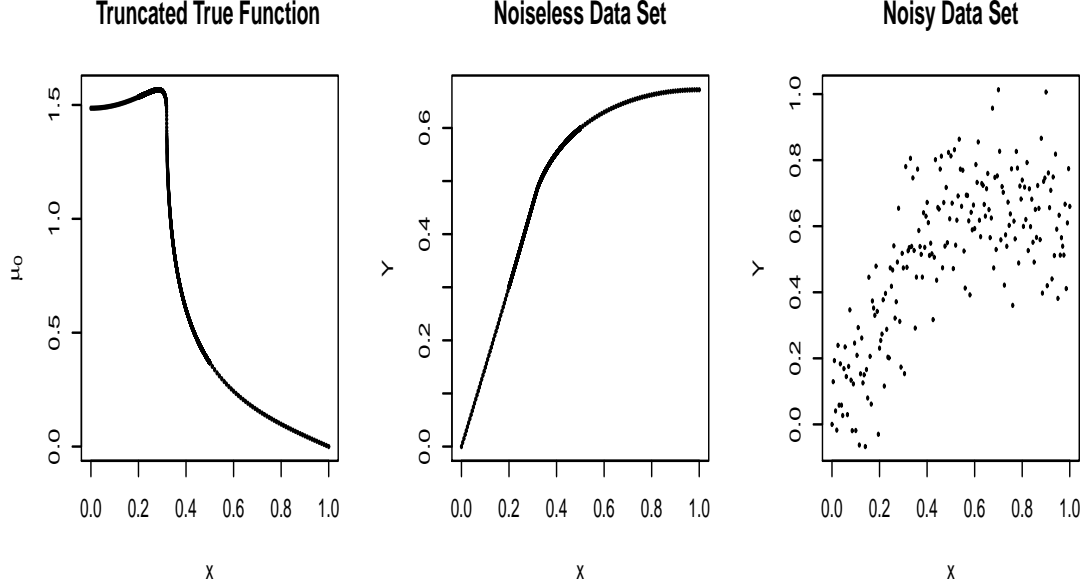


Figure 1: Graphs of $\mu_0^N(x)$, $Y_0(x)$ and $Y(x)$, (i.e. the truncated true function, the noiseless data set and a noisy data set, respectively), with $\epsilon = 10^{-4}$ and $N = 2000$.

Figure 1 displays the (truncated) true function $\mu_0^N(x)$, along with the observed function $Y_0(x) := K\mu_0^N(x)$ and its noisy counterpart $Y(x)$; the latter being simulated as follows:

$$Y_i | \mu_{0,i} \sim N(\mu_{0,i} k_i, \epsilon^2 \sigma_i^2).$$

Consequently, the posterior distribution will be

$$\hat{\mu}^N(x) | Y \sim N\left(\sum_i^N \frac{Y_i k_i \lambda_i \phi_i(x)}{\lambda_i k_i^2 + \epsilon^2 \sigma_i^2}, \sum_i^N \frac{\epsilon^2 \sigma_i^2 \lambda_i \phi_i^2(x)}{\lambda_i k_i^2 + \epsilon^2 \sigma_i^2}\right).$$

We can therefore obtain posterior pointwise credible bands for each x . Hence, by altering ϵ and α , we can dictate the degree of noise in our model, and the smoothness of our estimator, respectively.

Each of the panels in Figure 2 correspond to an independent realization of $Y(x)$, with $N = 2000$. The blue, red and green curves are the true curve ($\mu_0^N(x)$), the posterior mean and the posterior pointwise credible bands, respectively. The panels also show 500 realizations from the posterior distribution for various values of x . Note, the 6 panels correspond to the following 6 values of $\alpha = (0.5, 0.75, 1, 2, 3, 5)$.

There are several conclusions that can be drawn from the panels in Figure 2. For large values of α , i.e. a prior that is too smooth, not only do the confidence bands fail to contain the true curve, they also collapse to an incorrect curve. The optimal

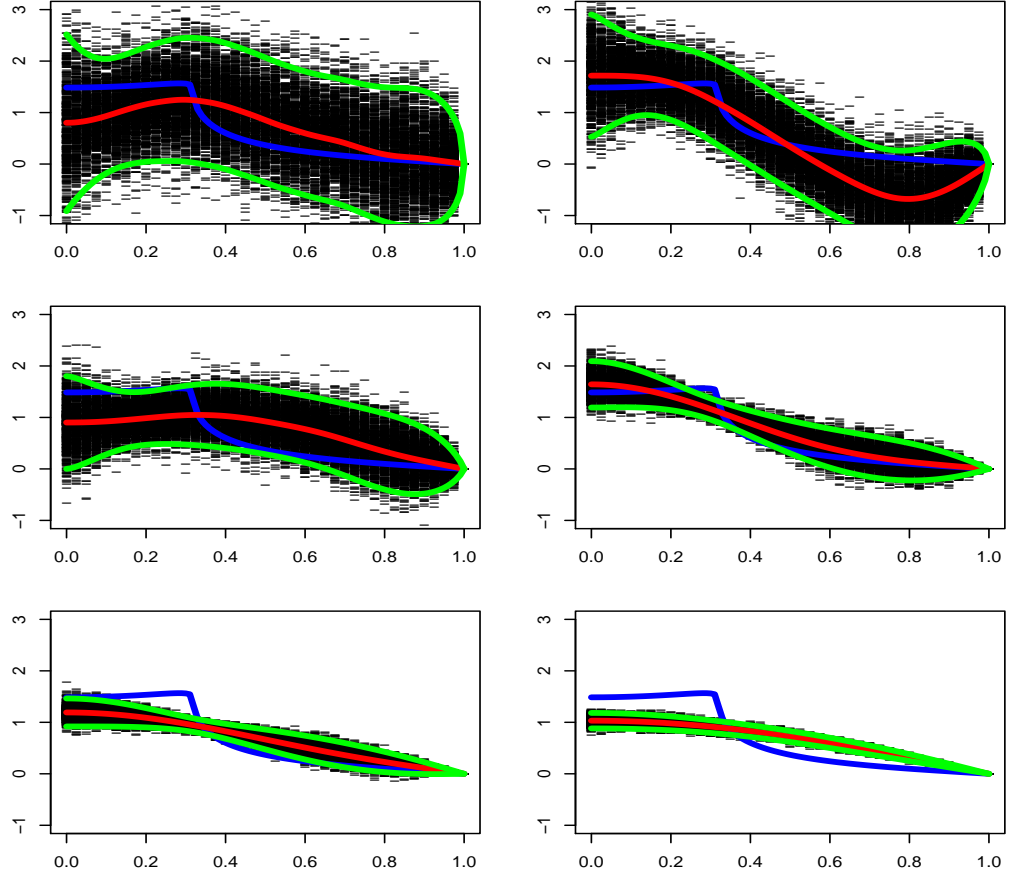


Figure 2: Plots of $\mu_0^N(x)$ (blue lines) along with the posterior mean (red line), 95% pointwise credible intervals (green curves) and 500 draws from the posterior (dashes) for $\alpha = (0.5, 0.75, 1, 2, 3, 5)$, respectively, with $\epsilon = 10^{-3/2}$ and $N = 2000$ in all cases.

alpha appears to be 0.75, which is less than β (the smoothness of $\mu_0(x)$). In addition, the posterior mean estimates the true curve ($\mu_0^N(x)$) poorly, even for small ϵ , i.e. for $\epsilon = 10^{-3/2}$. This is due to the very nature of ill-posed inverse problems, c.f. Figure 1 (note how different the graphs of $Y_0(x)$ and $Y(x)$ are, even though ϵ is small).

Recall that as $\epsilon \rightarrow 0$, the posterior mean will converge to $\mu_0(x)$ if the prior parameters satisfy conditions of Theorem 3.2, and the choice of the parameters affects the rate of convergence. To that end, consider Figure 3, which is constructed in exactly the same fashion as Figure 2, albeit with $\epsilon = 10^{-4}$ instead. Nonetheless, just as in Figure 2, an over-smooth prior is still inaccurate, even for a very small ϵ . However, unlike in Figure 2, the posterior mean has converged to the true function. Furthermore, the optimal alpha, ($\alpha = 0.75$), remains unchanged. In conclusion, our simulations show oversmooth priors remain inaccurate, and their posterior means continue to converge to the truth slowly, even as $\epsilon \rightarrow 0$.

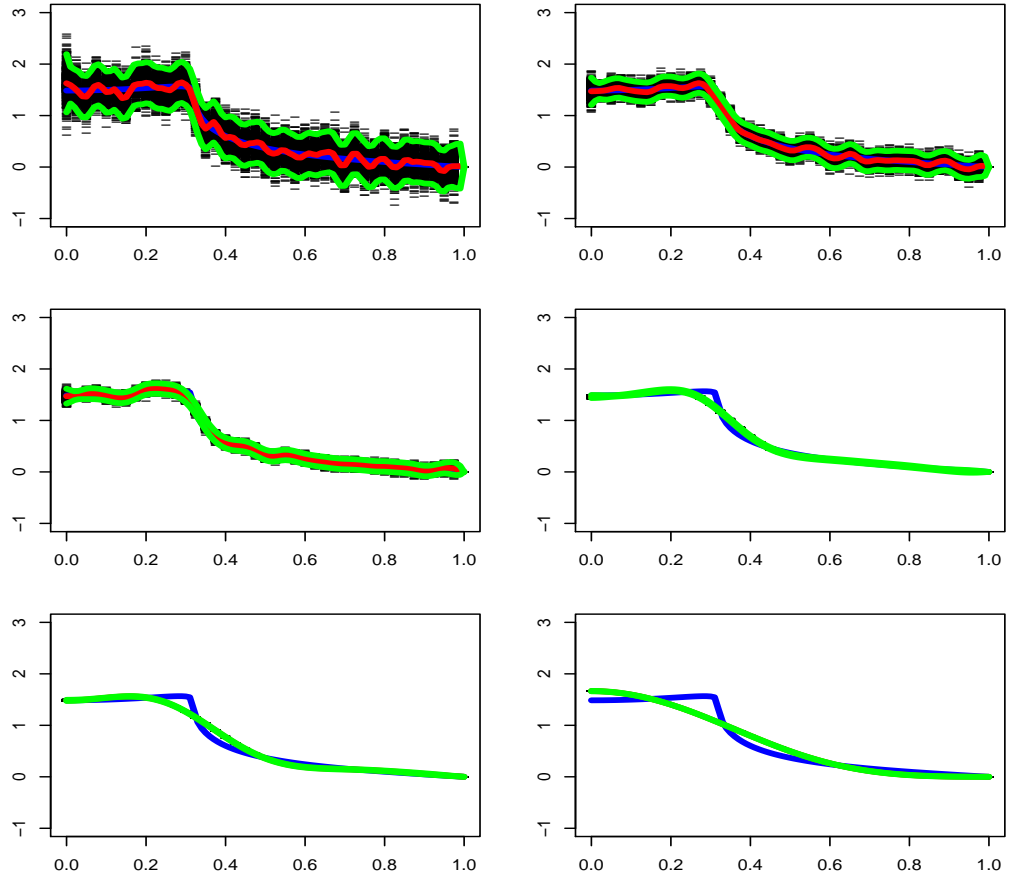


Figure 3: Plots of $\mu_0^N(x)$ (blue lines) along with the posterior mean (red line), 95% pointwise credible intervals (green curves) and 500 draws from the posterior (dashes) for $\alpha = (0.5, 0.75, 1, 2, 3, 5)$, respectively, with $\epsilon = 10^{-4}$ and $N = 2000$ in all cases.

4 Finite sample Bernstein - von Mises Theorems

Remark 4.1. *In this section, we review the paper: *Finite Sample Bernstein – von Mises Theorem for Semiparametric Problems (2015)*, by Panov and Spokoiny. Therefore, all of the results stated are from said paper, i.e. [Panov and Spokoiny, 2015]. However, some of the notation will be changed in order to maintain consistency. Additionally, we shall also return to using vector notation, hence given a vector x , x_i will refer its i^{th} component.*

We begin this section by discussing our motivation for studying Bernstein - von Mises Theorems. As mentioned in the conclusion of Section 3, we would like to study inverse problems with Poisson noise; such problems can typically be found in Tomography, such as Single Photon Emission Computed Tomography (SPECT), [Bauschke et al., 1999], and Emission Computed Tomography (ECT), [Gourion and Noll, 2002].

As seen in Section 1.3.9, using a Gaussian approximation for Poisson noise results in the variance of Y being dependent on μ as well. Inverse problems such as this are said to have *Signal Dependent noise*; the terminology arising from Signal Processing inverse problems wherein which μ refers to the signal one is interested in recovering.

Problems involving signal dependent noise are encountered in many fields, such as Biology [Delpretti et al., 2008], Medicine [Nichols et al., 2002] and Astronomy [Snyder et al., 1993]. In these applications, (as outlined in [Foi et al., 2008]), the noise will comprise of two components: the Poisson component models the signal-dependent part of the error (generally due to the photon-counting process), and the Gaussian component models the signal-independent part of the error (which can be caused by electric and thermal noise).

Generally, the techniques used to solve these models involve ignoring one of the error components, or approximating the model in some way. For instance, in [Foi et al., 2008] they use the fact that a Poisson Distribution, with parameter λ , can be approximated by a Normal distribution, with mean and variance equal to λ , just as we do in Section 1.3.9.

As stated in [Jezierska et al., 2012], these compromises are due to the inherent nature of the model, i.e. the probability distribution for the proposed model is continuous-discrete, and the log-likelihood involves an infinite sum. Consequently, (in [Jezierska et al., 2012]), they frame the problem in an optimisation setting, where they argue their cost function (the Poisson-Gaussian negative log-likelihood) is μ -Lipschitz differentiable, hence it can be expressed as a sum of simpler functions, which are easier to work with.

These difficulties therefore motivated our study of models where the variance of the (Gaussian) noise, \tilde{Z} , depends on μ . However, in our setting i.e. for the Bayesian approach, assuming signal dependent Gaussian noise means that we lose conjugacy, and hence are unable to derive a posterior distribution. Consequently, we use Bernstein-Von-Mises type theorems from [Panov and Spokoiny, 2015] to obtain the posterior distribution; specifically we obtain non-asymptotic bounds for it that are Gaussian in nature. Subsequently, we can use these bounds to derive our contraction rates.

The results in [Panov and Spokoiny, 2015] are for a finite dimensional parameter, are non-asymptotic (w.r.t. the dimension of the parameter of interest) and are stated for

the direct problem. Hence, we initially consider the finite dimensional, direct problem,

$$Y_i = \eta_i + \epsilon \tilde{Z}_i, \quad \text{for } i \leq p,$$

where ϵ is the noise level, and Y_i, η_i, \tilde{Z}_i are components of the vectors $Y, \eta, \tilde{Z} \in \mathbb{R}^p$, respectively. Furthermore, the \tilde{Z}_i are independent, zero mean Normal random variables, whose variance is dependent on η , (the latter will be touched upon further in Section 5). In addition, we will assume there exists some underlying true parameter $\eta_0 \in \mathbb{R}^p$; our goal being to recover said parameter.

Subsequently, we are able to derive corresponding results for the indirect problem by setting

$$\eta_i := k_i \mu_i,$$

and thus recover $\mu_0 \in \mathbb{R}^p$, by setting $\mu_{0,i} := \eta_{0,i}/k_i$.

Note that these results are non-asymptotic w.r.t. p . Therefore, by letting $p \rightarrow \infty$ we can in fact recover infinite dimensional versions of η_0 and μ_0 , which we shall denote as η_0^∞ and μ_0^∞ , respectively.

Hence, analogous to Sections 2 and 3, we shall assume $\eta_0^\infty \in S^{\beta_c}$ and $\mu_0^\infty \in S^\beta$, where the Sobolev space,

$$S^a = \{h : \|h\|_{S^a}^2 < \infty\}, \quad \text{where } \|h\|_{S^a}^2 = \sum_{i=1}^{\infty} h_i^2 i^{2a}.$$

Consequently, we shall set

$$\eta_{0,i}^\infty = \eta_{0,i} \quad \text{and} \quad \mu_{0,i}^\infty = \mu_{0,i}, \quad \text{for } i \leq p,$$

where the left hand side of each equation corresponds to a singular value, and the right hand side to a vector component. Thus, using the results from [Panov and Spokoiny, 2015] we will be able to recover the first p singular values of η_0^∞ and μ_0^∞ . Furthermore, we will be able to study the effect increasing p has on our results.

In summary, we discuss the assumptions stated in [Panov and Spokoiny, 2015]'s Section 2.1, and subsequently review Theorem 9 (Local Approximation), and Theorem 10 (Concentration). Consequently, we highlight how the previously mentioned theorems are used to derive the BvM results under a Uniform prior (Theorem 1), and a Gaussian prior (Theorem 2).

4.1 Formal description of the problem

The results of [Panov and Spokoiny, 2015] rely primarily on [Panov and Spokoiny, 2015]'s Theorem 9. Specifically, let η be the parameter of interest, and assume it lives in the parameter space $\Upsilon \subset \mathbb{R}^p$, (with η_0 denoting the true underlying parameter).

Remark 4.2. *The semi-parametric framework is used in [Panov and Spokoiny, 2015]; it is assumed one is interested in recovering only a subset, θ of η , with the target of estimation $\theta_0 := \Pi_0 \eta_0$, for some mapping $\Pi_0 : \Upsilon \rightarrow \mathbb{R}^a$, where $a \leq p$. We however are interested in recovering the entire η , hence we will set Π_0 to be the identity map, with $a = p$.*

We suspect there exists a neighbourhood around η_0 where η concentrates, i.e. $\Upsilon_0(r_0)$. Furthermore, inside $\Upsilon_0(r_0)$ we suspect η behaves almost like a normal random variable. Hence, we partition Υ , i.e.

$$\eta = \eta \mathbb{I}\{\eta \in \Upsilon_0(r_0)\} + \eta \mathbb{I}\{\eta \in \Upsilon \setminus \Upsilon_0(r_0)\} = \eta_{\Upsilon_0} + \eta_{\Upsilon_0^c}.$$

Thus, the goal is to show that the posterior distribution of $\eta_{\Upsilon_0^c}$ is negligible and that the posterior distribution of η_{Υ_0} is nearly normal.

Both of these results are dependent on the so called *Excess*,

$$L(\eta, \eta_0) := L(\eta) - L(\eta_0),$$

where $L(\eta)$ corresponds to the log-likelihood function.

Theorems 10 and 11 (from [Panov and Spokoiny, 2015]) address $\eta_{\Upsilon_0^c}$, while Theorems 12 and 13 address η_{Υ_0} , however all of these theorems rely on the excess being bounded. This is proved in [Panov and Spokoiny, 2015]'s Theorem 9, which states

- one can use a quadratic process $\mathbb{L}(\eta, \eta_0)$ instead of $L(\eta, \eta_0)$ for $\eta \in \Upsilon_0(r_0)$,
- and, $L(\eta, \eta_0)$ is bounded by a quadratic deterministic term outside of $\Upsilon_0(r_0)$.

Note that both of these bounds are *quadratic*, (and rely on D_0 which is defined in Section 4.2); see Remarks 4.3 and 4.4 for how this leads to Normality.

Remark 4.3. *Approximating $L(\eta, \eta_0)$: Local Normality: For $\eta \in \Upsilon_0(r_0)$ we will use*

$$\mathbb{L}(\eta, \eta_0) = \xi^T D_0(\eta - \eta_0) - \frac{\|D_0(\eta - \eta_0)\|^2}{2},$$

which is proportional to a Normal density. We can make this exact by considering,

$$m(\xi) := -\frac{\|\xi\|^2}{2} + \log(\det D_0) - p \log(\sqrt{2\pi}).$$

Subsequently, by noting $\|x\|^2 = x^T x$,

$$m(\xi) + \mathbb{L}(\eta, \eta_0) = -\frac{\|D_0(\eta - [\eta_0 + D_0^{-1}\xi])\|^2}{2} + \log(\det D_0) - p \log(\sqrt{2\pi}),$$

corresponds to the log-density of the normal law with mean $\eta^0 = \eta_0 + D_0^{-1}\xi$ and covariance matrix D_0^{-2} .

Remark 4.4. *Approximating $L(\eta, \eta_0)$: Global Normality: Note, for $\eta \notin \Upsilon_0(r_0)$, we will use the fact that*

$$L(\eta, \eta_0) \leq -\frac{b\|D_0(\eta - \eta_0)\|^2}{2},$$

where

$$-\frac{b\|D_0(\eta - \eta_0)\|^2}{2} + \log(\det[b^{1/2}D_0]) - p \log(\sqrt{2\pi}),$$

corresponds to the log-density of the normal law with mean η_0 and covariance matrix $(bD_0)^{-2}$.

Note, $L(\eta, \eta_0)$ is a random process, due to the data Y . Hence, its approximation (via $\mathbb{L}(\eta, \eta_0)$) holds only on a random set $\Omega_{r_0}(x)$. Specifically, Theorem 9 states that on $\Upsilon_0(r_0)$,

$$\mathbb{P}(|L(\eta, \eta_0) - \mathbb{L}(\eta, \eta_0)| \leq \Delta(r_0, x)) \geq 1 - e^{-x},$$

where $\Delta(r_0, x)$ denotes an upper bound for the approximation error. In addition, it also provides a deterministic upper bound for $L(\eta, \eta_0)$ outside of $\Upsilon_0(r_0)$ as well.

Consequently, all we need to do now is to prove the concentration properties of $\eta_{\Upsilon_0^c}$ (on $\Omega_{r_0}(x)$). Theorem 10 (and 11) from [Panov and Spokoiny, 2015] address this matter.

Specifically, Theorem 10 discusses the concentration properties of the posterior for the full parameter space Υ , while Theorem 11 does the same but for the targeted parameter space Θ_0 (we are however not interested in the latter). In Theorem 10, this concentration property is described by the random quantity

$$\rho^*(r_0) = \frac{\int_{\Upsilon \setminus \Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} d\eta}{\int_{\Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} d\eta}. \quad (21)$$

Observe, it can be shown that,

$$\mathbb{P}(\eta \notin \Upsilon_0(r_0) | Y) \leq \rho^*(r_0),$$

For instance, let

$$C = \int_{\Upsilon} \exp\{L(\eta, \eta_0)\} d\eta,$$

and assume it is finite. Then, by using C as a normalisation constant, we can treat $\exp\{L(\eta, \eta_0)\}$ as a density function hence,

$$\begin{aligned} \rho^*(r_0) &= \frac{C^{-1} \int_{\Upsilon \setminus \Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} d\eta}{C^{-1} \int_{\Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} d\eta} = \frac{\mathbb{E}(\mathbb{I}(\{\eta \notin \Upsilon_0(r_0)\}))}{\mathbb{E}(\mathbb{I}(\{\eta \in \Upsilon_0(r_0)\}))} \\ &= \frac{\mathbb{P}(\{\eta \notin \Upsilon_0(r_0)\})}{\mathbb{P}(\{\eta \in \Upsilon_0(r_0)\})} \\ &\geq \mathbb{P}(\{\eta \notin \Upsilon_0(r_0)\}). \end{aligned}$$

Furthermore, Theorems 12 and 13 provide local (Gaussian) upper and lower bounds for the Posterior Expectation, respectively. These in turn can be used to derive approximate posterior distributions for when the prior is Uniform (Theorem 1) and Gaussian (Theorem 2); the latter being the theorem we are most interested in.

Hence, in order to use Theorem 2, we must verify the conditions for Theorems 9 to 13, which are listed in the next section.

4.2 Assumptions

The conditions in [Panov and Spokoiny, 2015] consist of local and global assumptions on $L(\eta)$. The global conditions have to be fulfilled on the whole of $\Upsilon := \mathbb{R}^p$, for some $p > 0$. Conversely, the local conditions have to hold for $\eta \in \Upsilon_0(r_0)$ with some fixed value r_0 :

$$\Upsilon_0(r_0) := \{\eta \in \Upsilon : \|D_0(\eta - \eta_0)\| \leq r_0\}, \quad (22)$$

where

$$D_0^2 = -\nabla^2 \mathbb{E}L(\eta_0), \quad (23)$$

and

$$\eta_0 = \arg \max_{\eta \in \Upsilon} \mathbb{E}L(\eta). \quad (24)$$

The stochastic component of $L(\eta)$ is defined as,

$$\zeta(\eta) = L(\eta) - \mathbb{E}L(\eta), \quad (25)$$

the log-likelihood ratio as,

$$L(\eta, \eta_0) = L(\eta) - L(\eta_0), \quad (26)$$

the hessian of the expected log-likelihood as,

$$D_0^2(\eta) = -\nabla^2 \mathbb{E}L(\eta), \quad (27)$$

and the score as,

$$\xi = D_0^{-1} \nabla L(\eta_0). \quad (28)$$

Hence, the local conditions are as follows:

(ED_2): (Local Exponential Condition) There exists a constant $\nu_0 > 0$, a constant $\omega > 0$ and for each $r > 0$ a constant $\aleph(r) > 0$ such that for all $\eta \in \Upsilon_0(r)$:

$$\sup_{\psi_1, \psi_2 \in \mathbb{R}^p} \log \mathbb{E} \exp \left\{ \frac{\varkappa}{\omega} \frac{\psi_1^T \nabla^2 \zeta(\eta) \psi_2}{\|D_0 \psi_1\| \|D_0 \psi_2\|} \right\} \leq \frac{\nu_0^2 \varkappa^2}{2}, \quad |\varkappa| \leq \aleph(r). \quad (29)$$

The above condition describes the exponential moment of the gradient $\nabla \zeta(\cdot)$ inside the local neighbourhoods of η_0 , i.e. $\Upsilon_0(r)$.

Remark 4.5. Assumption (ED_2) is used in [Panov and Spokoiny, 2015]'s Theorem 9, and it only needs to hold for $r \leq r_0$ for some $r_0 > 0$. See Section 4.3.1 for more details.

(L_0): (Smoothness of Expected Log-Likelihood) There exists a constant $\delta(r)$ such that it holds on $\Upsilon_0(r)$ for all $r \leq r_0$

$$\|D_0^{-1} D_0^2(\eta) D_0^{-1} - I_p\| \leq \delta(r). \quad (30)$$

The above condition is needed to ensure the second order smoothness of the expected log-likelihood $\mathbb{E}L(\eta)$ inside $\Upsilon_0(r_0)$. Specifically, it ensures that $-\mathbb{E}L(\eta, \eta_0)$ can be approximated by a quadratic function of $\eta - \eta_0$ in the neighbourhood of η_0 .

(I): (Identifiability) There exists a constant $a > 0$ such that

$$a^2 D_0^2 \geq \Sigma_0^2. \quad (31)$$

The above condition relates the matrices D_0^2 and Σ_0^2 .

In fact, it is shown in [Spokoiny, 2012] that (L_0) will correspond to the Kullback-Leibler divergence between \mathbb{P}_η and \mathbb{P}_{η_0} . Hence, conditions (L_0) and (I) follow from the usual regularity conditions on the family $\{\mathbb{P}_\eta\}$, [Ibragimov and Has'minskii, 1981].

Similarly, the global conditions are:

(ED_0) : (Global Exponential Condition) There exists a constant $\nu_0 > 0$, a positive symmetric matrix Σ_0^2 satisfying $\text{Var}(\nabla\zeta(\eta_0)) \leq \Sigma_0^2$ and a constant $\aleph > 0$ such that

$$\sup_{\psi \in \mathbb{R}^p} \log \mathbb{E} \exp\left\{\varkappa \frac{\psi^T \nabla\zeta(\eta_0)}{\|\Sigma_0 \psi\|}\right\} \leq \frac{\nu_0^2 \varkappa^2}{2}, \quad |\varkappa| \leq \aleph.$$

The above condition describes the exponential moment of the gradient $\nabla\zeta(\cdot)$ at η_0 .

(L_r) : (Global Identification Condition) For any r there exists a value $b(r) > 0$ such that $rb(r) \rightarrow \infty$, $r \rightarrow \infty$ and

$$- \mathbb{E}L(\eta, \eta_0) \geq r^2 b(r), \quad \text{for all } \eta \text{ with } r = \|D_0(\eta - \eta_0)\|. \quad (32)$$

The above condition ensures that the deterministic component of the log likelihood $\mathbb{E}L(\eta, \eta_0)$ is competitive with its variance $\text{Var}(L(\eta, \eta_0))$.

Remark 4.6. *In our setting, Assumption (L_r) strictly speaking, does not hold, as $rb(r)$ does not go to infinity. However, as shown at the end of Section 5.8.9, this will not matter and the BvM result will still hold.*

Subsequently, let us briefly discuss how the above assumptions are connected to the quadratic process $\mathbb{L}(\eta, \eta_0)$ on $\Upsilon_0(r)$. Typically, $L(\eta)$ is decomposed into its deterministic and stochastic components:

$$L(\eta) = \mathbb{E}L(\eta) + \zeta(\eta).$$

Assumption (L_0) allows us to approximate the smooth deterministic function $\mathbb{E}L(\eta) - \mathbb{E}L(\eta_0)$ around a neighbourhood of η_0 by the quadratic term $- \|D_0(\eta - \eta_0)\|^2/2$.

Similarly, Assumptions (ED_0) and (ED_2) allow us to linearly approximate $\zeta(\eta) - \zeta(\eta_0) \approx (\eta - \eta_0)^T \nabla\zeta(\eta_0)$.

Thus, on $\Upsilon_0(r)$,

$$L(\eta, \eta_0) \approx \mathbb{L}(\eta, \eta_0) = (\eta - \eta_0)^T \nabla\zeta(\eta_0) - \|D_0(\eta - \eta_0)\|^2/2.$$

However, the error from this approximation (on $\Upsilon_0(r)$) grows quadratically with r and starts to dominate at some critical value of r . This can be circumvented by either shrinking or stretching $\|D_0(\eta - \eta_0)\|^2/2$ as required. This shrinking/stretching factor will be dependent on the constants defined in (L_0) , (ED_0) , (ED_2) and (I) .

Lastly, we need to show the process $L(\eta, \eta_0)$ concentrates around $\Upsilon_0(r)$, in order for its approximation by $\mathbb{L}(\eta, \eta_0)$ to be valid. Assumption (L_r) (along with Assumptions (ED_0) and (ED_2)) ensure that the concentration probability $\mathbb{P}(\eta_0 \notin \Upsilon_0(r_0))$ is indeed small enough, for some critical value r_0 .

4.3 Approximation Theorems

4.3.1 Local Approximation Theorem

Theorem ([Panov and Spokoiny, 2015]’s Theorem 9). *Suppose the Assumptions (ED_0) , (ED_2) , (L_0) and (I) from Section 4.2 hold for some $r_0 > 0$. Then on a random set $\Omega_{r_0}(x)$*

of dominating probability at least $1 - e^{-x}$

$$|L(\eta, \eta_0) - \mathbb{L}(\eta, \eta_0)| \leq \Delta(r_0, x), \quad \eta \in \Upsilon_0(r_0),$$

where

$$\begin{aligned} \Delta(r_0, x) &:= (\delta(r_0) + 6\nu_0 z_{\mathbb{H}}(x)\omega)r_0^2, \\ z_{\mathbb{H}}(x) &:= 2p^{1/2} + \sqrt{2x} + \aleph^{-1}(\aleph^{-2}x + 1)4p, \\ \Upsilon_0(r_0) &:= \{\eta : \|D_0(\eta - \eta_0)\| \leq r_0\}. \end{aligned}$$

Moreover the random vector $\xi = D_0^{-1}\nabla L(\eta_0)$ fulfills on a random set $\Omega_B(x)$ of dominating probability at least $1 - 2e^{-x}$

$$\|\xi\|^2 \leq z_B^2(x),$$

where $z_B^2(x) := p_B + 6\lambda_B x$, with

$$B := D_0^{-1}\Sigma_0^2 D_0^{-1}, \quad p_B := \text{tr}(B), \quad \lambda_B := \lambda_{\max}(B).$$

Furthermore, assume (L_r) with $b(r) \equiv b$ yielding

$$-\mathbb{E}L(\eta, \eta_0) \geq b\|D_0(\eta - \eta_0)\|^2,$$

for each $\eta \in \Upsilon \setminus \Upsilon_0(r_0)$. Let also

$$r \geq \frac{2}{b} \{z_B(x) + 6\nu_0 z_{\mathbb{H}}(x + \log(2r/r_0))\omega\}, \quad r \geq r_0.$$

Then,

$$L(\eta, \eta_0) \leq -\frac{b}{2}\|D_0(\eta - \eta_0)\|^2, \quad \eta \in \Upsilon \setminus \Upsilon_0(r_0),$$

holds on a random set $\Omega(x)$ of dominating probability at least $1 - 4e^{-x}$.

The spread value, $\Delta(r_0, x)$, controls the quality of the local approximation, with $\delta(r_0)r_0^2$ measuring the error of the quadratic approximation due to Assumption (L_0) and $6\nu_0 z_{\mathbb{H}}(x)\omega r_0^2$ controlling the stochastic term $\zeta(\eta)$. Additionally, the constraints on p are due to $z_{\mathbb{H}}(x)$; specifically its connection to the entropy of the parameter space. Consequently, we will need $\Delta(r_0, x)$ to be small in order for our approximation to be valid.

4.3.2 Concentration Theorem

Theorem 4.1 ([Panov and Spokoiny, 2015]’s Theorem 10). *Suppose the conditions of [Panov and Spokoiny, 2015]’s Theorem 9 hold. Then it holds on $\Omega_{r_0}(x)$*

$$\rho^*(r_0) \leq \exp\{2\Delta(r_0, x) + \nu(r_0)\}b^{-p/2}P(\|\mathcal{Z}\|^2 \geq br_0^2), \quad (33)$$

with

$$\nu(r_0) := -\log P(\|\mathcal{Z} + \xi\| \leq r_0 | Y).$$

If $r_0 \geq z_B(x) + z(p, x)$, then on $\Omega(x)$,

$$\nu(r_0) \leq 2e^{-x},$$

where $z^2(p, x) := p + [\sqrt{6.6px} \vee (6.6x)]$.

This theorem states conditions on r_0 for the posterior to concentrate on $\Upsilon_0(r_0)$. Furthermore, we can use [Panov and Spokoiny, 2015]’s Lemma 7 to simplify said conditions and obtain:

Corollary 4.1. *Assume the conditions of the above Theorem. Then the additional inequality $br_0^2 \geq z^2(p, x + \frac{p}{2} \log \frac{e}{b})$ ensures on a random set $\Omega(x)$ of probability at least $1 - 4e^{-x}$,*

$$\rho^*(r_0) \leq \exp(2\Delta(r_0, x) + 2e^{-x} - x).$$

4.3.3 BvM under Uniform Prior

Subsequently, using [Panov and Spokoiny, 2015]’s Theorems 9 and 10, (along with Corollaries 3 and 5), the BvM results for the posterior probability are derived, under a non-informative (a.k.a Uniform) prior.

Theorem 4.2 ([Panov and Spokoiny, 2015]’s Theorem 1). *Suppose the conditions of Section 4.2 hold. Let the prior be uniform on Υ . Then there exists a random set $\Omega(x)$ of dominating probability at least $1 - 4e^{-x}$ such that*

$$\begin{aligned} \|D_0(\bar{\eta} - \eta^0)\|^2 &\leq 4\Delta(r_0, x) + 16e^{-x}, \\ \|I_p - D_0\mathfrak{G}^2 D_0\| &\leq 4\Delta(r_0, x) + 16e^{-x}, \end{aligned}$$

where

$$\bar{\eta} := \mathbb{E}(\eta|Y), \quad \mathfrak{G}^2 := \text{Cov}(\eta|Y), \quad \eta^0 = \eta_0 + D_0^{-1}\xi.$$

Moreover, on $\Omega(x)$ for any measurable set $A \subset \mathbb{R}^p$

$$\begin{aligned} \exp(-2\Delta(r_0, x) - 8e^{-x})\mathbb{P}(\mathcal{Z} \in A) - e^{-x} \\ \leq \mathbb{P}(D_0(\eta - \eta^0) \in A|Y) \\ \leq \exp(2\Delta(r_0, x) + 5e^{-x})(\mathbb{P}(\mathcal{Z} \in A)), \end{aligned}$$

where \mathcal{Z} is a standard Gaussian vector in \mathbb{R}^p .

This theorem states the BvM result for the posterior under a *Uniform* prior. The random point η^0 can be viewed as a first order approximation of the MLE η_0 . Hence, the BvM result claims that $\bar{\eta}$ is close to η^0 , \mathfrak{G}^2 is nearly equal to D_0^{-2} , and $D_0(\eta - \eta^0)$ is nearly standard normal conditional on Y .

4.3.4 BvM under Gaussian Prior

The BvM results obtained under the Uniform prior can also be extended to the Gaussian prior, Π . Intuitively, this is because any smooth prior can be locally approximated by a Gaussian one.

Assumption 4.1. *Let the prior measure $\Pi = N(0, G^{-2})$ on \mathbb{R}^p satisfy the following conditions:*

$$\begin{aligned} \|D_0^{-1}G^2D_0^{-1}\| &\leq \delta_G \leq 1/2, \\ \text{tr}(D_0^{-1}G^2D_0^{-1})^2 &\leq \delta_{tr}^2, \\ \|D_G^{-1}G^2\eta_0\| &\leq \delta_\beta, \end{aligned}$$

where $D_G^2 := D_0^2 + G^2$.

Note, the uniform prior can be viewed as a limiting case of a normal prior as $G \rightarrow 0$. Hence, given Theorem 4.2, we can ask ourselves how small should G be to ensure the BvM result? The answer is given by [Panov and Spokoiny, 2015]’s Lemma 8, which requires δ_G, δ_{tr} and δ_β to be small. Consequently, we obtain the following theorem:

Theorem 4.3 ([Panov and Spokoiny, 2015]’s Theorem 2). *Suppose the conditions in Section 4.2 and Assumption 4.1 holds. Then it holds on a random set $\Omega(x)$ of dominating probability at least $1 - 5e^{-x}$*

$$\begin{aligned} \mathbb{P}(\check{D}_0(\eta - \theta^0) \in A|Y) &\geq \exp(-2\Delta(r_0, x) - 8e^{-x})[\mathbb{P}(\mathcal{Z} \in A) - \varrho] - e^{-x}, \\ \mathbb{P}(\check{D}_0(\eta - \theta^0) \in A|Y) &\leq \exp(2\Delta(r_0, x) + 5e^{-x})[\mathbb{P}(\mathcal{Z} \in A) + \varrho] + e^{-x}, \end{aligned}$$

where

$$\varrho := \frac{1}{2}[(1 + \delta_G)(3\delta_\beta + \delta_G z_B(x))^2 + \delta_{tr}^2]^{1/2}.$$

Subsequently, we can see that the error in our BvM result is controlled by $\Delta(r_0, x)$ and ϱ , with the latter dictating how well Π approximates the uniform prior from Theorem 4.2. Thus, we need to ensure both terms are $o(1)$ in order for the BvM result to hold.

5 Bernstein - von Mises Theorem and Contraction Rates for Direct and Indirect Inverse problems with Signal-Dependent Noise

Remark 5.1. *We would strongly recommend perusing Section 4 before continuing. Additionally, in this section we will use vector notation, hence given a vector x , x_i will refer its i^{th} component.*

As discussed in Section 4, we begin by considering the finite dimensional, direct problem,

$$Y_i = \eta_i + \epsilon \tilde{Z}_i, \quad \text{for } i \leq p, \quad (34)$$

where ϵ is the noise level, and Y_i, η_i, \tilde{Z}_i are components of the vectors $Y, \eta, \tilde{Z} \in \mathbb{R}^p$, respectively. The \tilde{Z}_i are independent, zero mean Normal random variables, whose variance is dependent on η , (the latter will be touched upon further in Section 5.1). In addition, we set

$$\epsilon = \frac{1}{\sqrt{n}}.$$

Furthermore, we will also assume there exists some underlying true parameter $\eta_0 \in \mathbb{R}^p$; our goal being to recover η_0 .

Note, we will be able to derive results for the indirect problem by setting

$$\eta_i := k_i \mu_i,$$

and thus recover $\mu_0 \in \mathbb{R}^p$, by setting $\mu_{0,i} := \eta_{0,i}/k_i$.

Nevertheless, in the Bayesian setting we lose conjugacy, and hence are unable to derive a posterior distribution. Therefore, we use Bernstein-Von-Mises type theorems from [Panov and Spokoiny, 2015] to obtain the posterior distribution; specifically we obtain non-asymptotic bounds for it that are Gaussian in nature. Subsequently, we can use these bounds to derive our contraction rates.

Note that their results are non-asymptotic w.r.t. p . Therefore, by letting $p \rightarrow \infty$ we can in fact recover infinite dimensional versions of η_0 and μ_0 , which we shall denote as η_0^∞ and μ_0^∞ , respectively.

Hence, analogous to Sections 2 and 3, we shall assume $\eta_0^\infty \in S^{\beta_c}$ and $\mu_0^\infty \in S^\beta$, where the Sobolev space,

$$S^a = \{h : \|h\|_{S^a}^2 < \infty\}, \quad \text{where } \|h\|_{S^a}^2 = \sum_{i=1}^{\infty} h_i^2 i^{2a}.$$

Consequently, we set

$$\eta_{0,i}^\infty = \eta_{0,i} \quad \text{and} \quad \mu_{0,i}^\infty = \mu_{0,i}, \quad \text{for } i \leq p,$$

where the left hand side of each equation corresponds to a singular value, and the right hand side to a vector component. Thus, using the results from [Panov and Spokoiny, 2015]

we will be able to recover the first p singular values of η_0^∞ and μ_0^∞ . Furthermore, we will be able to study the effect increasing p has on our results.

In summary, Theorems 5.1 and 5.2 provide Bernstein-von Mises results for the posterior distribution for the direct problem, which is defined in Section 5.1. Subsequently, Theorem 5.3 states the BvM results for the posterior distribution for the indirect problem, which is defined in Section 5.3. Note, these aforementioned theorems rely on a concentration result, which is proved in Theorem 5.4. Consequently, the contraction rates for the direct and indirect problems are derived in Theorems 5.6 and 5.7, respectively.

5.1 Formal description of the Direct problem

Assume Model (34), with $\eta_0^\infty \in S^{\beta_c}$. In this section, we set

$$\tilde{Z}_i \sim N(0, g^2(\eta_i)),$$

where $g(\eta_i) : \mathbb{R} \rightarrow \mathbb{R}$.

Consequently,

$$Y_i \sim N(\eta_i, \epsilon^2 g^2(\eta_i)). \quad (35)$$

In addition, let Υ and η_0 denote the parameter space and the true parameter, respectively. We define

$$\eta_0 := \arg \max_{\eta \in \Upsilon} \mathbb{E} L(\eta),$$

where $L(\eta)$ represents the log-likelihood.

Subsequently, we assume a Normal prior on η_i , i.e.

$$\eta_i \sim N(0, \lambda_i),$$

where $\lambda_i = \tau_n^2 i^{-(1+2\alpha_c)}$ and $n\tau_n^2 \rightarrow \infty$, with $\alpha_c > 0$.

Hence, our aim is to derive the posterior distribution of $\eta|Y$, which in turn will help us estimate $\{\eta_{0,i}\}_{i=1}^p$, i.e. the first p singular values of $\eta_0^\infty \in S^{\beta_c}$. However, because the variance is signal-dependent, (see (35)), there is no longer conjugacy. Therefore, we approximate the posterior distribution using the results from [Panov and Spokoiny, 2015]. Note, this will require us to study the behaviour of $L(\eta)$ around η_0 .

Thus, define a local neighbourhood around η_0 as follows:

$$\Upsilon_0(r_0) := \{\eta \in \Upsilon : \|D_0(\eta - \eta_0)\| \leq r_0\},$$

where

$$D_0^2 = -\nabla^2 \mathbb{E} L(\eta_0).$$

Consequently, having implemented the framework from [Panov and Spokoiny, 2015], (please see Section 4 for more details), we can now derive Bernstein-von Mises results for the direct case.

5.2 Bernstein-Von Mises Results: Direct Problem

Before we can state the results however, we need the following assumptions regarding $g^2(\cdot)$ to hold. The first corresponds to its global properties, while the second addresses its local behaviour on a compact set. Note, the latter effectively implies $g^2(\cdot)$ and its derivatives will always be bounded, but the nature of these bounds will change (i.e. whether they are dependent on r and n).

Assumption 5.1 (Global). *$g(x)$ is positive, continuous and injective. Additionally, $g(x)$ will be uniformly bounded from below by c , and $|g'(x)|$ will be uniformly bounded from above by C , where c and C are positive constants, which are independent of r , ϵ and p .*

Remark 5.2. *Note, the constants c and C are required to prove Proposition 5.7, and Theorem 5.4.*

Assumption 5.2 (Local). *For a given $g(\cdot)$ and any $r \leq \frac{1}{2} \min_{1 \leq i \leq p} (D_0)_{i,i}$, where $D_0^2 := -\nabla^2 \mathbb{E}L(\eta_0)$, the following exist for $1 \leq i \leq p$:*

$$\begin{aligned} \max_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq r^2 (D_0^{-2})_{i,i}} |g(\eta_i)| &:= m_{r,0,u,i}, & \min_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq r^2 (D_0^{-2})_{i,i}} |g(\eta_i)| &:= m_{r,0,l,i}, \\ \max_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq r^2 (D_0^{-2})_{i,i}} |g'(\eta_i)| &:= m_{r,1,u,i}, & \min_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq r^2 (D_0^{-2})_{i,i}} |g'(\eta_i)| &:= m_{r,1,l,i}, \\ \max_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq r^2 (D_0^{-2})_{i,i}} |g''(\eta_i)| &:= m_{r,2,u,i}, & \min_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq r^2 (D_0^{-2})_{i,i}} |g''(\eta_i)| &:= m_{r,2,l,i}, \\ \max_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq r^2 (D_0^{-2})_{i,i}} |g'''(\eta_i)| &:= m_{r,3,u,i}. \end{aligned}$$

Note, all these terms are separated away from 0 and ∞ by constants, which are independent of r , ϵ and p .

Note, in the classical Bernstein-von Mises Theorem, the posterior distribution is centered around the MLE. However, the posterior distribution given below is centered at $\eta^0 = \eta_0 + D_0^{-2} \nabla L(\eta_0)$, which can be viewed as the first order approximation to the MLE.

Theorem 5.1 (General Bounded Case). *Let $g^2(\eta_i)$ satisfy Assumptions 5.1 and 5.2. Under the setting described in Sections 5.1 and 5.2: on a random set $\Omega(x)$ with probability at least $1 - 6e^{-x}$, for $x \asymp p$, $\eta \in \Upsilon_0(r_0)$, and $r_0^2 \leq o(n)$, we have*

$$\begin{aligned} &\exp(-2\Delta(r_0, x) - 8e^{-x}) [\mathbb{P}(\mathcal{Z} \in A) - \varrho] - e^{-x} \\ &\leq \mathbb{P}(D_0(\eta - \eta^0) \in A | Y) \\ &\leq \exp(2\Delta(r_0, x) + 5e^{-x}) [\mathbb{P}(\mathcal{Z} \in A) + \varrho] + e^{-x}, \end{aligned}$$

as long as $\Delta(r_0, x) \leq 1/2$, where \mathcal{Z} is a standard Gaussian vector in \mathbb{R}^p ,

$$\begin{aligned} \Delta(r_0, x) &\asymp (p^{1/2} + x^{1/2}) n^{-1/2} r_0^2, \quad \text{and} \\ \varrho^2 &\asymp (n\tau_n^2)^{-2} p^{2(1+2\alpha_c)} \cdot \begin{cases} (\tau_n^{-4} p^{2+4[\alpha_c - \beta_c]} + p + x), & \text{if } 2 + 4\alpha_c - 2\beta_c > 0, \\ (\tau_n^{-4} p^{-2(1+2\alpha_c)} + p + x), & \text{if } 2 + 4\alpha_c - 2\beta_c \leq 0. \end{cases} \end{aligned}$$

Whilst for $\eta \notin \Upsilon_0(r_0)$ we have with probability at least $1 - 3e^{-x}$, on a random set $\Omega(x)$, for x large enough and $r_0^2 \geq Cp$,

$$\mathbb{P}(\{\eta \notin \Upsilon_0(r_0)|Y\}) \leq C \exp \left(-\frac{r_0^2}{4} + \sqrt{3xp} + \frac{p}{2} - \frac{p+1}{2} \log p + (p-2) \log r_0 + \Delta(r_0, x) + 2e^{-x} \right).$$

Remark 5.3. Specifically, in the general case, $\aleph \asymp n^{1/2}$, $z_{\mathbb{H}}(x) \asymp p^{1/2} + x^{1/2}$, $\nu_0 \asymp 1$, $\delta(r_0) \asymp o(1)$, $\omega \asymp n^{-1/2}$, $z_B(x) \asymp p^{1/2} + x^{1/2}$ and $z(p, x) \asymp [p+x]^{1/2}$. Furthermore, the condition, $x \asymp p$, is used to simplify results and isn't actually necessary.

Remark 5.4. As stated in Remark 4.6, in our setting $rb(r) \not\rightarrow \infty$. Hence we can't use the results obtained when Assumption (L_r) holds. Consequently, we derive our own upper bound on the tail posterior probability in Theorem 5.4, and we show that η^0 converges weakly to η_0 , for large n , in Proposition 5.1, (c.f. Z_i and its moments).

Corollary 5.1. Given Assumption 5.2, the results derived in Theorem 5.1 are unaffected by the nature of $g^2(\cdot)$, for n large enough.

When $g^2(\eta_i)$ is linear, we must consider a Truncated Normal prior on η_i to ensure $g^2(\eta_i)$ remains non-negative. Furthermore, the Gaussian approximation will hold if the elements of $\Upsilon_0(r_0)$ are separated away from zero, (c.f. Section 5.8.4). Thus the following additional assumption.

Assumption 5.3. We assume

$$\eta_i \sim TN(0, \lambda_i),$$

with support equal to $[0, \infty)$, $\lambda_i = \tau_n^2 i^{-(1+2\alpha_c)}$ and $n\tau_n^2 \rightarrow \infty$, where $\alpha_c > 0$. Additionally,

$$\min_{1 \leq i \leq p} \eta_{0,i} \geq c_0 > \max_{1 \leq i \leq p} \frac{r_0}{(D_0)_{i,i}}.$$

Consequently, we derive BvM results for the case considered in [Foi et al., 2008], i.e. when the variance of \tilde{Z}_i has an affine formulation based on η_i ,

$$g^2(\eta_i) = a\eta_i + b.$$

Theorem 5.2 (Linear Case). Let η_i satisfy Assumption 5.3 and $g^2(\eta_i) = a\eta_i + b$, (where $a, b > 0$), satisfy Assumptions 5.1 and 5.2. Under the setting described in Sections 5.1 and 5.2: on a random set $\Omega(x)$, with probability at least $1 - 6e^{-x}$, for $x \asymp p$, $\eta \in \Upsilon_0(r_0)$, and $r_0^2 \leq o(n)$, we have

$$\begin{aligned} & \exp(-2\Delta(r_0, x) - 8e^{-x})[\mathbb{P}(\mathcal{Z} \in A) - \varrho] - e^{-x} \\ & \leq \mathbb{P}(D_0(\eta - \eta^0) \in A|Y) \\ & \leq \exp(2\Delta(r_0, x) + 5e^{-x})[\mathbb{P}(\mathcal{Z} \in A) + \varrho] + e^{-x}, \end{aligned}$$

as long as $\Delta(r_0, x) \leq 1/2$, where \mathcal{Z} is a standard Gaussian vector in \mathbb{R}^p ,

$$\Delta(r_0, x) \asymp (p^{1/2} + x^{1/2})n^{-1/2}r_0^2, \quad \text{and}$$

$$\varrho^2 \asymp (n\tau_n^2)^{-2}p^{2(1+2\alpha_c)} \cdot \begin{cases} (\tau_n^{-4}p^{2+4[\alpha_c-\beta_c]} + p + x), & \text{if } 2 + 4\alpha_c - 2\beta_c > 0, \\ (\tau_n^{-4}p^{-2(1+2\alpha_c)} + p + x), & \text{if } 2 + 4\alpha_c - 2\beta_c \leq 0. \end{cases}$$

Whilst for $\eta \notin \Upsilon_0(r_0)$ we have with probability at least $1 - 3e^{-x}$, on a random set $\Omega(x)$, for x large enough and $r_0^2 \geq Cp$,

$$\mathbb{P}(\{\eta \notin \Upsilon_0(r_0)|Y\}) \leq C \exp \left(-\frac{r_0^2}{4} + \sqrt{3xp} + \frac{p}{2} - \frac{p+1}{2} \log p + (p-2) \log r_0 + \Delta(r_0, x) + 2e^{-x} \right).$$

Remark 5.5. Specifically, in the linear case, $\aleph \asymp n^{1/2}$, $z_{\mathbb{H}}(x) \asymp p^{1/2} + x^{1/2}$, $\nu_0 \asymp 1$, $\delta(r_0) \asymp o(1)$, $\omega \asymp n^{-1/2}$, $z_B(x) \asymp [p+x]^{1/2}$, and $z(p, x) \asymp [p+x]^{1/2}$. Furthermore, the condition, $x \asymp p$, is used to simplify results and isn't actually necessary.

Corollary 5.2. For $cp \leq r_0^2 \leq Cn$, $x \asymp p$, and n large enough,

- $\varrho^2 = o(1)$ if the following conditions hold:

$$p = o((n\tau_n^2)^{1/(1+2\alpha_c+1/2)}), \quad \text{and} \quad \begin{cases} p = o((\tau_n^2 n \tau_n^2)^{1/(2+4\alpha_c-2\beta_c)}), & \text{if } 2 + 4\alpha_c - 2\beta_c > 0, \\ n\tau_n^4 \rightarrow \infty, & o/w. \end{cases}$$

- If $p^3 n^{-1} = o(1)$, then $\Delta(r_0, x) = o(1)$.
- If $\Delta(r_0, x) = o(1)$, $\varrho^2 = o(1)$ and c is chosen sufficiently large, then $\mathbb{P}(\{\eta \notin \Upsilon_0(r_0)|Y\}) = o(1)$.

Remark 5.6. If the conditions stated in Corollary 5.2 hold, then Theorems 5.1 and 5.2 imply that BvM holds.

5.3 Formal description of the Indirect Problem

We now consider the indirect version of Model (34) by setting

$$\eta_i = k_i \mu_i,$$

with $k_i \asymp i^{-q}$ for $q > 0$, and $\mu_0^\infty \in S^\beta$.

Consequently,

$$Y_i \sim N(k_i \mu_i, \epsilon^2 g^2(\eta_i)). \quad (36)$$

In addition, let $\tilde{\Upsilon}$ and μ_0 denote the parameter space and the true parameter, respectively. We define

$$\mu_{0,i} := \frac{\eta_{0,i}}{k_i}.$$

Subsequently, we assume a Normal prior on $\mu_{0,i}$, i.e.

$$\mu_{0,i} \sim N(0, \tilde{\lambda}_i),$$

where $\tilde{\lambda}_i = \tau_n^2 i^{-(1+2\alpha)}$, with $n\tau_n^2 \rightarrow \infty$, and $\alpha > 0$.

Hence, our aim is to derive the posterior distribution of $\mu|Y$, which in turn will help us estimate $\{\mu_{0,i}\}_{i=1}^p$, i.e. the first p singular values of $\mu_0^\infty \in S^\beta$. However, as discussed in the direct case setting (c.f. Section 5.1), there is no longer conjugacy. Therefore, we approximate the posterior distribution using the results from [Panov and Spokoiny, 2015].

Note, if $\eta \in \Upsilon_0(r_0)$, then $\mu \in \tilde{\Upsilon}_0(r_0)$, where

$$(\tilde{D}_0^2)_{i,i} := (D_0^2)_{i,i} k_i^2, \quad \text{and} \quad \tilde{\Upsilon}_0(r) := \{\mu : \|\tilde{D}_0(\mu - \mu_0)\| \leq r\}.$$

Consequently, having implemented the framework from [Panov and Spokoiny, 2015], (please see Section 4 for more details), we can now derive Bernstein-von Mises results for the indirect case.

5.4 Bernstein-Von Mises Results: Indirect Problem

Note, the posterior distribution described below is now centered at $\mu_i^0 := \mu_{0,i} + (\tilde{D}_0^2)_{i,i}^{-1} (\nabla_\mu L(\mu_0))_i$.

Theorem 5.3 (General Bounded Case: Indirect Case). *Let $g^2(\eta_i)$ satisfy Assumptions 5.1 and 5.2. Under the setting described in Sections 5.1, 5.2, and 5.3, on a random set $\Omega(x)$ with probability at least $1 - 6e^{-x}$, for $\mu \in \tilde{\Upsilon}_0(r_0)$, and $r_0^2 \leq o(1)p^{-2q}n$, we have*

$$\begin{aligned} & \exp(-2\Delta(r_0, x) - 8e^{-x})[\mathbb{P}(\mathcal{Z} \in A) - \varrho] - e^{-x} \\ & \leq \mathbb{P}(\tilde{D}_0(\mu - \mu^0) \in A|Y) \\ & \leq \exp(2\Delta(r_0, x) + 5e^{-x})[\mathbb{P}(\mathcal{Z} \in A) + \varrho] + e^{-x}, \end{aligned}$$

as long as $\Delta(r_0, x) \leq 1/2$, where \mathcal{Z} is a standard Gaussian vector in \mathbb{R}^p ,

$$\begin{aligned} \Delta(r_0, x) & \asymp \left[p^{1/2} + x^{1/2} + (p^{2q}n^{-1}x + 1)4p^{q+1}n^{-1/2} \right] n^{-1/2}r_0^2, \quad \text{and} \\ \varrho^2 & \asymp (n\tau_n^2)^{-2}p^{2(1+2[\alpha+q])} \cdot \begin{cases} (\tau_n^{-4}p^{2+4[\alpha-\beta]} + p + x), & \text{if } 2 + 2[\alpha + q] + 2[\alpha - \beta] > 0, \\ (\tau_n^{-4}p^{-2(1+2[\alpha+q])} + p + x), & \text{if } 2 + 2[\alpha + q] + 2[\alpha - \beta] \leq 0. \end{cases} \end{aligned}$$

Whilst for $\mu \notin \tilde{\Upsilon}_0(r_0)$ we have with probability at least $1 - 3e^{-x}$, on a random set $\Omega(x)$, for x large enough and $r_0^2 \geq C[p + x]$,

$$\begin{aligned} \mathbb{P}(\{\eta \notin \Upsilon_0(r_0)|Y\}) & \leq C \exp \left(-\frac{r_0^2}{4} + \sqrt{3xp} + \frac{p}{2} \right. \\ & \quad \left. - \frac{p+1}{2} \log p + (p-2) \log r_0 + \Delta(r_0, x) + 2e^{-x} \right). \end{aligned}$$

Remark 5.7. Specifically, in the general case, $\aleph \asymp p^{-q}n^{1/2}$, $z_{\mathbb{H}}(x) \asymp p^{1/2} + x^{1/2} + (p^{2q}n^{-1}x + 1)4p^{q+1}n^{-1/2}$, $\nu_0 \asymp 1$, $\delta(r_0) \asymp o(1)$, $\omega \asymp n^{-1/2}$, $z_B(x) \asymp [p + x]^{1/2}$, and $z(p, x) \asymp [p + x]^{1/2}$.

Corollary 5.3. For $cp \leq r_0^2 \leq o(1)p^{-2q}n$, $x = \log n$, and n large enough,

- $\varrho^2 = o(1)$ if the following conditions hold:

$$p = o((n\tau_n^2)^{1/(1+2[\alpha+q]+1/2)}),$$

$$\text{and } \begin{cases} p = o((\tau_n^2 n \tau_n^2)^{1/(2+2[\alpha+q]+2[\alpha-\beta])}), & \text{if } 2 + 2[\alpha + q] + 2[\alpha - \beta] > 0, \\ n\tau_n^4 \rightarrow \infty, & o/w. \end{cases}$$

- If $p = o(n^{1/(2q+1)})$, $p = o(n^{1/3})$, $p = o([\frac{n^2}{\log n}]^{1/(3q+2)})$, and $p = o(n^{1/(q+2)})$, then $\Delta(r_0, x) = o(1)$.
- If $\Delta(r_0, x) = o(1)$, $\varrho^2 = o(1)$ and c is chosen sufficiently large, then $\mathbb{P}(\{\eta \notin \Upsilon_0(r_0)|Y\}) = o(1)$.

5.5 Concentration Theorem: General Bounded Case

Theorem 5.4 (Concentration: General Bounded Case). Suppose the conditions (ED_0) , (ED_2) , (L_0) and (I) from Section 4.2 hold for some $r_0 > 0$. Then on a random set $\Omega_{r_0}(x)$ of dominating probability at least $1 - 3e^{-x}$, for $r_0 \geq 4\sqrt{x} \vee [z_B(x) + z(p, x)]$,

$$\begin{aligned} \rho^*(r_0) \leq C \exp \left(-\frac{r_0^2}{4} + \sqrt{3xp} + \frac{p}{2} \right. \\ \left. - \frac{p+1}{2} \log p + (p-2) \log r_0 + \Delta(r_0, x) + 2e^{-x} \right), \end{aligned} \quad (37)$$

where C is an absolute constant and

$$\begin{aligned} z_B^2(x) &:= p_B + 6\lambda_B x, \\ \text{with } B &:= D_0^{-1} \Sigma_0^2 D_0^{-1}, \quad p_B := \text{tr}(B), \quad \lambda_B := \lambda_{\max}(B), \\ z^2(p, x) &:= p + [\sqrt{6.6px} \vee (6.6x)]. \end{aligned}$$

Furthermore, this result will also hold when a flat normal prior is used, assuming Assumption 4.1 holds.

Corollary 5.4. Given the conditions of Theorem 5.4 and the setting described in Section 5.11.1, $\rho^*(r_0)$ is of the same form as that in the direct setting (37), assuming $\|r\tilde{D}_0^{-1}\| = o(1)$.

Remark 5.8. In the direct setting, we assumed $\|rD_0^{-1}\| = o(1)$ instead.

5.6 Contraction Rates: General Bounded Case

5.6.1 The Setup

Recall, using the results from Sections 5.2 and 5.4, we can derive the posterior distribution of $\eta|Y$ and $\mu|Y$, which in turn will help us estimate $\{\eta_{0,i}\}_{i=1}^p$ and $\{\mu_{0,i}\}_{i=1}^p$, which by definition coincide with the first p singular values of η_0^∞ and μ_0^∞ . Therefore, in this section, we gauge how effective η and μ are at recovering $\eta_0^\infty \in S^{\beta_c}$ and $\mu_0^\infty \in S^\beta$, respectively.

We do so by trivially extending η and μ to the infinite dimensional setting. Specifically, we construct η^∞ , such that for $i \leq p$, $\eta_i^\infty := \eta_i$, while for $i > p$, $\eta_i^\infty = 0$. Similarly, we construct μ^∞ , such that for $i \leq p$, $\mu_i^\infty := \mu_i$, while for $i > p$, $\mu_i^\infty := 0$.

Furthermore, since we only recover the first p singular values, we can trivially consider an infinite dimensional prior on η^∞ , such that for $i \leq p$, $\eta_i^\infty := \eta_i \sim N(0, \lambda_i)$, while for $i > p$, $\eta_i^\infty = 0$ with probability 1. Similarly, we consider a prior on μ^∞ , such that for $i \leq p$, $\mu_i^\infty := \mu_i \sim N(0, \tilde{\lambda}_i)$, while for $i > p$, $\mu_i^\infty = 0$ with probability 1.

Thus, η^∞ and μ^∞ are well-defined estimators of $\eta_0^\infty \in S^{\beta_c}$ and $\mu_0^\infty \in S^\beta$, respectively. Consequently, we can derive the contraction rates of η^∞ and μ^∞ , under the setting described in Sections 5.1 and 5.3.

Observe,

$$\Pi_n(\{\eta^\infty : \|\eta^\infty - \eta_0^\infty\|^2 \geq M_n^2 \varepsilon_n^2 | Y\}) \leq \frac{\mathbb{E}(\|\eta - \eta_0\|^2) + \mathbb{E}(\|\eta_0^\infty\|_{\mathbb{R}^{p,\infty}}^2)}{M_n^2 \varepsilon_n^2},$$

and

$$\Pi_n(\{\mu^\infty : \|\mu^\infty - \mu_0^\infty\|^2 \geq M_n^2 \varepsilon_n^2 | Y\}) \leq \frac{\mathbb{E}(\|\mu - \mu_0\|^2) + \mathbb{E}(\|\mu_0^\infty\|_{\mathbb{R}^{p,\infty}}^2)}{M_n^2 \varepsilon_n^2},$$

where

$$\|x\|_{\mathbb{R}^{p,\infty}}^2 := \sum_{i>p} x_i^2.$$

5.6.2 Results

Hence, we list the approximated posterior distribution, its moments, and finally the Mean Squared Error. All of the results are for the setting described in Section 5.6.1:

Lemma 5.5. (*Approximated Posterior Distribution*) *Given Assumption 4.1, Theorem 5.1 implies the centered scaled posterior distribution is approximately standard normal, i.e. $D_0(\eta - \tilde{\eta}_0) \sim N(0, I)$, on a set $\Omega(x)$ of probability $1 - 5e^{-x}$, where*

$$\begin{aligned} \tilde{\eta}_0 &= \eta_0 + D_0^{-2} \nabla L(\eta_0), \\ (D_0^{-2})_{i,i} &= \frac{n^{-1} g^2(\eta_{0,i})}{1 + 2n^{-1} [g'(\eta_{0,i})]^2}, \\ (\nabla L(\eta_0))_i &= \frac{g'(\eta_{0,i})}{n^{-1} g^3(\eta_{0,i})} (Y_i - \eta_{0,i})^2 + \frac{1}{n^{-1} g^2(\eta_{0,i})} (Y_i - \eta_{0,i}) - \frac{g'(\eta_{0,i})}{g(\eta_{0,i})}. \end{aligned}$$

Consequently, for the direct case, we obtain the following proposition:

Proposition 5.1 (General Bounded Function: Direct Case). *Let the conditions given in Lemma 5.5 hold for $i = 1, \dots, p$. Then,*

$$\begin{aligned}\mathbb{E}(\eta_i|Y_i) &\asymp \eta_{0,i} + [A_i(Y_i - \eta_{0,i})^2 + B_i(Y_i - \eta_{0,i}) + C_i], \\ \text{Var}(\eta_i|Y_i) &\asymp (D_0)_{i,i}^{-2},\end{aligned}$$

where

$$A_i = \frac{g'(\eta_{0,i})}{g(\eta_{0,i})(1 + 2n^{-1}[g'(\eta_{0,i})]^2)}, \quad B_i = \frac{1}{1 + 2n^{-1}[g'(\eta_{0,i})]^2}, \quad \text{and} \quad C_i = -\frac{n^{-1}g(\eta_{0,i})g'(\eta_{0,i})}{1 + 2n^{-1}[g'(\eta_{0,i})]^2}.$$

Furthermore, if we define $X_i := A_i(Y_i - \eta_{0,i})^2 + B_i(Y_i - \eta_{0,i}) + C_i$ and assume $Y_i|\eta_{0,i} \sim N(\eta_{0,i}, n^{-1}g^2(\eta_{0,i}))$, then

$$\begin{aligned}\mathbb{E}_{\eta_{0,i}}(X_i) &= 0, \\ \text{Var}_{\eta_{0,i}}(X_i) &= A_i^2 2n^{-2}g^4(\eta_{0,i}) + B_i^2 n^{-1}g^2(\eta_{0,i}).\end{aligned}$$

Please see Appendix C.7 for details regarding the proof.

Thus, the MSE is as follows:

Corollary 5.5 (Mean Square Error). *Let the conditions given in Proposition 5.1 hold. Then,*

$$\mathbb{E}_{\eta_0} \mathbb{E} \|\eta - \eta_0\|^2 \lesssim \sum_{i=1}^p (D_0^{-2})_{i,i} + A_i^2 2n^{-2}g^4(\eta_{0,i}) + B_i^2 n^{-1}g^2(\eta_{0,i}).$$

Furthermore, for the indirect case, i.e. when $\eta_i = k_i \mu_i$

$$\mathbb{E}_{\mu_0} \mathbb{E} \|\mu - \mu_0\|^2 \lesssim \sum_{i=1}^p k_i^{-2} (D_0^{-2})_{i,i} + k_i^{-2} [A_i^2 2n^{-2}g^4(k_i \mu_{0,i}) + B_i^2 n^{-1}g^2(k_i \mu_{0,i})].$$

Please see Appendix C.9 for details regarding the proof.

Corollary 5.6 (Excess Bias). *Let the conditions given in Proposition 5.1 hold. Assume $\eta_0^\infty \in S^{\beta_c}$. Then, the excess bias*

$$\|\eta_0^\infty\|_{\mathbb{R}^{p,\infty}}^2 = \mathcal{O}(p^{-2\beta_c}).$$

Furthermore, for the indirect case, where $\mu_0^\infty \in S^\beta$,

$$\|\mu_0^\infty\|_{\mathbb{R}^{p,\infty}}^2 = \mathcal{O}(p^{-2\beta}).$$

Proof. Note, for $x \in S^a$,

$$\begin{aligned} \|x\|_{\mathbb{R}^{p,\infty}}^2 &:= \sum_{i>p} x_i^2 = \sum_{i>p} x_i^2 i^{2a} i^{-2a} \leq p^{-2a} \sum_{i>p} x_i^2 i^{2a} \\ &\leq p^{-2a} \|x\|_{S^a}^2. \end{aligned}$$

□

Hence, we can now derive the contraction rates and show that it satisfies the BvM conditions.

Theorem 5.6 (Contraction Rates: Direct Case). *Given Assumptions 5.1 and 5.2, for the setting described in Section 5.6.1, the $\mathbb{E}_{\eta_0} \Pi_n(\{\eta^\infty : \|\eta^\infty - \eta_0^\infty\|^2 \geq M_n \varepsilon_n |Y\}) \rightarrow 0$, as $n \rightarrow \infty$, for every $M_n \rightarrow \infty$, where*

$$\varepsilon_n = p^{-\beta_c} + n^{-\frac{1}{2}} p^{\frac{1}{2}}.$$

Furthermore, optimality is achieved when $p^* = n^{\frac{1}{2\beta_c+1}}$, with

$$\varepsilon_n^* : n^{\frac{-\beta_c}{2\beta_c+1}}.$$

Additionally, p^* will satisfy the assumptions in Corollary 5.2 if

- when no scaling is used, i.e. $\tau_n^2 = 1$, we have the following condition:

$$\beta_c > [(\alpha_c + \frac{1}{4}) \vee 1].$$

- otherwise, i.e. when $\tau_n^2 \neq 1$, we have the following conditions:

$$\beta_c > 1, \text{ and } o(\tau_n^2) = \begin{cases} n^{\frac{2[\alpha_c - \beta_c] + 1/2}{2\beta_c + 1}}, & \text{if } \beta_c < 2\alpha_c + 1, \\ n^{-1/2}, & \text{if } \beta_c \geq 2\alpha_c + 1. \end{cases}$$

Remark 5.9. The conditions $\beta_c > [(\alpha_c + \frac{1}{4}) \vee 1]$ and $o(\tau_n^2) = n^{\frac{2[\alpha_c - \beta_c] + 1/2}{2\beta_c + 1}}$, come from Assumption 4.1, specifically $\|D_G^{-1} G^2 \eta_0\|^2 = o(1)$. Said assumption is used in [Panov and Spokoiny, 2015]’s Theorem 2, and could possibly be sub-optimal. Note, in their more recent work [Spokoiny and Panov, 2020], they no longer have this condition however their setting doesn’t apply to ours.

Proof. [Deriving the Contraction Rates:]

Using Corollaries 5.5 and 5.6, along with Propositions 5.8, 5.9 and 5.10 imply

$$\begin{aligned}
& \mathbb{E}_{\eta_0} \Pi_n(\{\eta^\infty : \|\eta^\infty - \eta_0^\infty\| \geq M_n \varepsilon_n | Y\}) \\
& \leq \frac{C}{M_n^2 \varepsilon_n^2} (p^{-2\beta_c} + \sum_{i=1}^p (D_0^{-2})_{i,i} + [A_i^2 2n^{-2} g^4(\eta_{0,i}) + B_i^2 n^{-1} g^2(\eta_{0,i})]) \\
& \leq \frac{C}{M_n^2 \varepsilon_n^2} \left[p^{-2\beta_c} + n^{-1}p + n^{-2}p + n^{-1}p \right] \\
& \leq \frac{C}{M_n^2 \varepsilon_n^2} [p^{-2\beta_c} + (2n^{-1} + n^{-2})p].
\end{aligned}$$

Hence, setting

$$\varepsilon_n = p^{-\beta_c} + n^{-\frac{1}{2}} p^{\frac{1}{2}},$$

ensures that the $\mathbb{E}_{\eta_0} \Pi_n(\{\eta^\infty : \|\eta^\infty - \eta_0^\infty\| \geq M_n \varepsilon_n | Y\}) \rightarrow 0$ for every $M_n \rightarrow \infty$.

Furthermore,

$$p^{-\beta_c} \asymp n^{-\frac{1}{2}} p^{\frac{1}{2}} \iff n^{\frac{1}{2}} \asymp p^{\beta_c + \frac{1}{2}} \iff p \asymp n^{\frac{1}{2\beta_c + 1}}.$$

Thus for $p^* = n^{\frac{1}{2\beta_c + 1}}$,

$$\varepsilon_n^* : n^{\frac{-\beta_c}{2\beta_c + 1}}.$$

[Proving p^* satisfies BvM conditions:]

Recall we have the following conditions from Corollary 5.2:

For $cp \leq r_0^2 \leq Cn$, $x \asymp p$, and n large enough,

$$p^3 n^{-1} = o(1), \quad p = o((n\tau_n^2)^{1/(1+2\alpha_c+1/2)}), \quad \text{and} \quad \begin{cases} p = o((\tau_n^2 n \tau_n^2)^{1/(2+4\alpha_c-2\beta_c)}), & \text{if } 2+4\alpha_c-2\beta_c > 0, \\ n\tau_n^4 \rightarrow \infty, & \text{o/w.} \end{cases}$$

Let us consider each of the bounds separately, initially for the case where the tuning parameter isn't used and later when it is.

For $\tau_n^2 = 1$, and $p^3 n^{-1} = o(1)$ we have

$$\begin{aligned}
[p^*]^3 n^{-1} = n^{\frac{3}{2\beta_c + 1} - 1} = o(1) & \iff \frac{3}{2\beta_c + 1} - 1 < 0 \\
& \iff 3 < 2\beta_c + 1 \iff 1 < \beta_c.
\end{aligned}$$

For $\tau_n^2 = 1$, and $p = o((n\tau_n^2)^{1/(1+2\alpha_c+1/2)})$ we have

$$\begin{aligned}
p^* = n^{\frac{1}{2\beta_c + 1}} = o(n^{\frac{1}{1+2\alpha_c+1/2}}) & \iff \frac{1}{2\beta_c + 1} < \frac{1}{1+2\alpha_c+1/2} \\
& \iff 1+2\alpha_c+1/2 < 2\beta_c + 1 \iff \alpha_c + \frac{1}{4} < \beta_c.
\end{aligned}$$

For $\tau_n^2 = 1$, and $[p = o((\tau_n^2 n \tau_n^2)^{1/(2+4\alpha_c-2\beta_c)})]$, and $2 + 4\alpha_c - 2\beta_c > 0$ or equivalently $\beta_c < 2\alpha_c + 1]$,

$$p^* = n^{\frac{1}{2\beta_c+1}} = o(n^{\frac{1}{2+4\alpha_c-2\beta_c}}) \iff 2 + 4\alpha_c - 2\beta_c < 2\beta_c + 1 \iff \alpha_c + \frac{1}{4} < \beta_c.$$

For $\tau_n^2 = 1$ and $2 + 4\alpha_c - 2\beta_c \leq 0$ (or equivalently $\beta_c \geq 2\alpha_c + 1$), $n\tau_n^4 \rightarrow \infty$ will always be true.

Hence, for $\tau_n^2 = 1$: $\beta_c > [\alpha_c + \frac{1}{4} \vee 1]$ satisfies all cases.

Next we consider the cases when τ_n^2 is used.

For $p^3 n^{-1} = o(1)$ we have

$$[p^*]^3 n^{-1} = n^{\frac{3}{2\beta_c+1}-1} = o(1) \iff 1 < \beta_c.$$

For $p = o((n\tau_n^2)^{1/(1+2\alpha_c+1/2)})$ we have,

$$\begin{aligned} p^* = n^{\frac{1}{2\beta_c+1}} = o((n\tau_n^2)^{\frac{1}{1+2\alpha_c+1/2}}) &\iff n^{\frac{1+2\alpha_c+1/2}{2\beta_c+1}} = o(n\tau_n^2) \iff n^{\frac{1+2\alpha_c+1/2}{2\beta_c+1}-1} = o(\tau_n^2) \\ &\iff n^{\frac{1+4[\alpha_c-\beta_c]}{2\beta_c+1}} = o(\tau_n^4) \iff \tau_n^{-4} = o(n^{-\frac{1+4[\alpha_c-\beta_c]}{2\beta_c+1}}). \end{aligned}$$

We also have the condition (from the definition of λ_i): $n\tau_n^2 \rightarrow \infty$, which is equivalent to assuming $\tau_n^{-2} = o(n)$. Thus, we require

$$\tau_n^{-4} = o(n^{-\frac{1+4[\alpha_c-\beta_c]}{2\beta_c+1}} \wedge n^2).$$

For $p = o((\tau_n^2 n \tau_n^2)^{1/(2+4\alpha_c-2\beta_c)})$, and $2 + 4\alpha_c - 2\beta_c > 0$ (or equivalently $\beta_c < 2\alpha_c + 1$, or $\frac{1+4[\alpha_c-\beta_c]}{2\beta_c+1} > -1$),

$$\begin{aligned} p^* = n^{\frac{1}{2\beta_c+1}} = o((\tau_n^2 n \tau_n^2)^{1/(2+4\alpha_c-2\beta_c)}) &\iff n^{\frac{2+4\alpha_c-2\beta_c}{2\beta_c+1}-1} = o(\tau_n^4) \\ &\iff n^{\frac{1+4[\alpha_c-\beta_c]}{2\beta_c+1}} = o(\tau_n^4). \end{aligned}$$

Thus our conditions overlap. Furthermore, $\frac{1+4[\alpha_c-\beta_c]}{2\beta_c+1} > -1$, thus,

$$\tau_n^{-4} = o(n^{-\frac{1+4[\alpha_c-\beta_c]}{2\beta_c+1}} \wedge n^2) = o(n^{-\frac{1+4[\alpha_c-\beta_c]}{2\beta_c+1}}).$$

Hence, it remains to be shown that for $2 + 4\alpha_c - 2\beta_c \leq 0$ (or equivalently $\beta_c \geq 2\alpha_c + 1$ or $\frac{1+4[\alpha_c-\beta_c]}{2\beta_c+1} \leq -1$), $n\tau_n^4 \rightarrow \infty$, which is equivalent to assuming $(n\tau_n^4)^{-1} = o(1)$. Our conditions overlap if

$$\tau_n^{-4} = o(n^{-\frac{1+4[\alpha_c-\beta_c]}{2\beta_c+1}} \wedge n^2 \wedge n) = o(n),$$

since $\frac{1+4[\alpha_c-\beta_c]}{2\beta_c+1} \leq -1$.

Thus,

$$\tau_n^{-4} = \begin{cases} o(n^{-\frac{1+4[\alpha_c-\beta_c]}{2\beta_c+1}}), & \text{if } \beta_c < 2\alpha_c + 1, \\ o(n), & \text{if } \beta_c \geq 2\alpha_c + 1, \end{cases}$$

or equivalently,

$$o(\tau_n^2) = \begin{cases} n^{\frac{2[\alpha_c-\beta_c]+1/2}{2\beta_c+1}}, & \text{if } \beta_c < 2\alpha_c + 1, \\ n^{-1/2}, & \text{if } \beta_c \geq 2\alpha_c + 1. \end{cases}$$

□

Theorem 5.7 (Contraction Rates: Indirect Case). *Given Assumptions 5.1 and 5.2, for the setting described in Section 5.6.1, the $\mathbb{E}_{\mu_0}\Pi_n(\{\mu^\infty : \|\mu^\infty - \mu_0^\infty\|^2 \geq M_n\varepsilon_n|Y\}) \rightarrow 0$, as $n \rightarrow \infty$, for every $M_n \rightarrow \infty$, where*

$$\varepsilon_n = p^{-\beta} + n^{-\frac{1}{2}}p^{q+\frac{1}{2}}. \quad (38)$$

Furthermore, optimality is achieved when $p^* = n^{1/(2[\beta+q]+1)}$, with

$$\varepsilon_n^* = n^{\frac{-\beta}{2(\beta+q)+1}}.$$

Additionally, p^* , will satisfy the assumptions in Corollary 5.3 if

- When no scaling is used, i.e. $\tau_n^2 = 1$, we have the following condition:

$$\beta > \max\{\alpha + \frac{1}{4}, 1 - q\}.$$

- Otherwise, i.e. when $\tau_n^2 \neq 1$, we have the following conditions:

$$\beta > 1 - q, \text{ and } o(\tau_n^2) = \begin{cases} n^{(2[\alpha-\beta]+1/2)/(2\beta+2q+1)}, & \text{if } 2 + 2[\alpha + q] + 2[\alpha - \beta] > 0, \\ n^{-1/2}, & \text{if } 2 + 2[\alpha + q] + 2[\alpha - \beta] \leq 0. \end{cases}$$

Proof. [**Deriving the Contraction Rates:**]

Using Corollaries 5.5 and 5.6, along with Propositions 5.8, 5.9 and 5.10 imply

$$\mathbb{E}_{\mu_0}\Pi_n(\{\mu^\infty : \|\mu^\infty - \mu_0^\infty\| \geq M_n\varepsilon_n|Y\}) \leq \frac{C}{M_n^2\varepsilon_n^2}[p^{-2\beta} + (2n^{-1} + n^{-2})pp^{2q}].$$

Hence,

$$\varepsilon_n = p^{-\beta} + n^{-\frac{1}{2}}p^{q+\frac{1}{2}},$$

ensures that the $\mathbb{E}_{\mu_0}\Pi_n(\{\mu^\infty : \|\mu^\infty - \mu_0^\infty\| \geq M_n\varepsilon_n|Y\}) \rightarrow 0$ for every $M_n \rightarrow \infty$.

Furthermore, we see

$$p^{-\beta} \asymp n^{-\frac{1}{2}}p^{q+\frac{1}{2}} \iff n^{\frac{1}{2}} \asymp p^{\beta+q+\frac{1}{2}} \iff p \asymp n^{\frac{1}{2(\beta+q)+1}}.$$

Thus, when $p^* = n^{\frac{1}{2(\beta+q)+1}}$,

$$\varepsilon_n^* : n^{\frac{-\beta}{2(\beta+q)+1}}.$$

[Proving p^* satisfies BvM conditions:]

Recall we have the following conditions from Corollary 5.3:

For $cp \leq r_0^2 \leq o(1)p^{-2q}n$, $x = \log n$, and n large enough we need to show:

$$p = o(n^{1/(2q+1)}), \quad p = o(n^{1/3}), \quad p = o(\lfloor \frac{n^2}{\log n} \rfloor^{1/(3q+2)}), \quad p = o(n^{1/(q+2)}),$$

$$p = o((n\tau_n^2)^{1/(1+2[\alpha+q]+1/2)}), \text{ and } \begin{cases} p = o((\tau_n^2 n \tau_n^2)^{1/(2+2[\alpha+q]+2[\alpha-\beta])}), & \text{if } 2 + 2[\alpha + q] + 2[\alpha - \beta] > 0, \\ n\tau_n^4 \rightarrow \infty, & \text{o/w.} \end{cases}$$

Hence, we have 7 conditions, and the last 3 of them involve τ_n , therefore they will need to be derived for $\tau_n^2 = 1$, and $\tau_n^2 \neq 1$.

For the first condition, i.e.

$$\begin{aligned} p^* = o(n^{1/(2q+1)}) &\iff n^{1/(2\beta+2q+1)} = o(n^{1/(2q+1)}) \iff n^{(2q+1)/(2\beta+2q+1)} = o(n) \\ &\iff n^{(2q+1)/(2\beta+2q+1)-1} = o(1) \\ &\iff (2q+1)/(2\beta+2q+1) - 1 < 0 \iff 2q+1 < 2\beta+2q+1 \\ &\iff \beta > 0. \end{aligned}$$

For the second condition, i.e.

$$\begin{aligned} p^* = o(n^{1/3}) &\iff n^{1/(2\beta+2q+1)} = o(n^{1/3}) \iff n^{3/(2\beta+2q+1)} = o(n) \\ &\iff n^{3/(2\beta+2q+1)-1} = o(1) \\ &\iff 3/(2\beta+2q+1) - 1 < 0 \iff 3 < 2\beta+2q+1 \\ &\iff \beta > 1 - q. \end{aligned}$$

For the third condition, i.e.

$$\begin{aligned} p^* = o(\lfloor \frac{n^2}{\log n} \rfloor^{1/(3q+2)}) &\iff n^{1/(2\beta+2q+1)} = o(\lfloor \frac{n^2}{\log n} \rfloor^{1/(3q+2)}) \iff n^{(3q+2)/(2\beta+2q+1)} = o(\frac{n^2}{\log n}) \\ &\iff n^{(3q+2)/(2\beta+2q+1)-2} \log n = o(1) \\ &\iff n^{(3q+2-2[2\beta+2q+1])/(2\beta+2q+1)} \log n = o(1) \\ &\iff n^{-[4\beta+q]/(2\beta+2q+1)} \log n = o(1), \end{aligned}$$

which will always hold, since $\beta, q > 0$.

For the fourth condition, i.e.

$$\begin{aligned} p^* = o(n^{1/(q+2)}) &\iff n^{1/(2\beta+2q+1)} = o(n^{1/(q+2)}) \iff n^{(q+2)/(2\beta+2q+1)} = o(n) \\ &\iff n^{(q+2)/(2\beta+2q+1)-1} = o(1) \\ &\iff (q+2)/(2\beta+2q+1) - 1 < 0 \iff q+2 < 2\beta+2q+1 \\ &\iff \beta > \frac{1}{2} - \frac{q}{2}. \end{aligned}$$

For the fifth condition, with $\tau_n^2 = 1$, i.e.

$$\begin{aligned}
p^* = o(n^{1/(1+2[\alpha+q]+1/2)}) &\iff n^{1/(2\beta+2q+1)} = o(n^{1/(2[\alpha+q]+3/2)}) \iff n^{(2[\alpha+q]+3/2)/(2\beta+2q+1)} = o(n) \\
&\iff n^{(2[\alpha+q]+3/2)/(2\beta+2q+1)-1} = o(1) \\
&\iff (2[\alpha+q] + 3/2)/(2\beta + 2q + 1) - 1 < 0 \\
&\iff 2[\alpha+q] + 3/2 < 2\beta + 2q + 1 \\
&\iff \beta > \alpha + \frac{1}{4}.
\end{aligned}$$

For the sixth condition, with $\tau_n^2 = 1$ and $2 + 2[\alpha + q] + 2[\alpha - \beta] > 0$, i.e.

$$\begin{aligned}
p^* = o(n^{1/(2+2[\alpha+q]+2[\alpha-\beta])}) &\iff n^{1/(2\beta+2q+1)} = o(n^{1/(2+2[\alpha+q]+2[\alpha-\beta])}) \iff n^{(2+2[\alpha+q]+2[\alpha-\beta])/(2\beta+2q+1)} = o(n) \\
&\iff n^{(2+2[\alpha+q]+2[\alpha-\beta])/(2\beta+2q+1)-1} = o(1) \\
&\iff ((2 + 2[\alpha + q] + 2[\alpha - \beta])/(2\beta + 2q + 1) - 1 < 0 \\
&\iff 2 + 2[\alpha + q] + 2[\alpha - \beta] < 2\beta + 2q + 1 \\
&\iff \beta > \alpha + \frac{1}{4}.
\end{aligned}$$

For the seventh condition, with $\tau_n^2 = 1$ and $2 + 2[\alpha + q] + 2[\alpha - \beta] \leq 0$, i.e. $n\tau_n^4 \rightarrow \infty$ will always be true.

Thus, for all 7 conditions to be true, with $\tau_n^2 = 1$, we require

$$\beta > \max\{\alpha + \frac{1}{4}, \frac{1}{2} - \frac{q}{2}, 1 - q\} = \max\{\alpha + \frac{1}{4}, 1 - q\}.$$

Next, let us re-evaluate conditions 5-7 for when $\tau_n^2 \neq 1$:

For the fifth condition, with $\tau_n^2 \neq 1$, i.e.

$$\begin{aligned}
p^* = o((n\tau_n^2)^{1/(1+2[\alpha+q]+1/2)}) &\iff n^{1/(2\beta+2q+1)} = o((n\tau_n^2)^{1/(2[\alpha+q]+3/2)}) \\
&\iff n^{(2[\alpha+q]+3/2)/(2\beta+2q+1)} = o(n\tau_n^2) \\
&\iff n^{(2[\alpha+q]+3/2)/(2\beta+2q+1)-1} = o(\tau_n^2) \\
&\iff n^{(2[\alpha-\beta]+1/2)/(2\beta+2q+1)} = o(\tau_n^2).
\end{aligned}$$

For the sixth condition, with $\tau_n^2 \neq 1$ and $2 + 2[\alpha + q] + 2[\alpha - \beta] > 0$ (or equivalently $(2[\alpha - \beta] + 1/2)/(2\beta + 2q + 1) > -1/2$), i.e.

$$\begin{aligned}
p^* = o((\tau_n^2 n\tau_n^2)^{1/(2+2[\alpha+q]+2[\alpha-\beta])}) &\iff n^{1/(2\beta+2q+1)} = o((n\tau_n^4)^{1/(2+2[\alpha+q]+2[\alpha-\beta])}) \\
&\iff n^{(2+2[\alpha+q]+2[\alpha-\beta])/(2\beta+2q+1)} = o(n\tau_n^4) \\
&\iff n^{(2+2[\alpha+q]+2[\alpha-\beta])/(2\beta+2q+1)-1} = o(\tau_n^4) \\
&\iff n^{(4[\alpha-\beta]+1)/(2\beta+2q+1)} = o(\tau_n^4) \\
&\iff n^{(2[\alpha-\beta]+1/2)/(2\beta+2q+1)} = o(\tau_n^2).
\end{aligned}$$

For the seventh condition, with $\tau_n^2 \neq 1$ and $2 + 2[\alpha + q] + 2[\alpha - \beta] \leq 0$ (or equivalently $(2[\alpha - \beta] + 1/2)/(2\beta + 2q + 1) \leq -1/2$), i.e.

$$n\tau_n^4 \rightarrow \infty \iff (n\tau_n^4)^{-1} = o(1) \iff n^{-1} = o(\tau_n^4) \iff n^{-1/2} = o(\tau_n^2).$$

Thus, for $\tau_n^2 \neq 1$ and $2 + 2[\alpha + q] + 2[\alpha - \beta] > 0$: we require $\beta > \frac{1}{2} - \frac{q}{2} \vee 0$, and

$$o(\tau_n^2) = n^{(2[\alpha - \beta] + 1/2)/(2\beta + 2q + 1)},$$

whilst for $\tau_n^2 \neq 1$ and $2 + 2[\alpha + q] + 2[\alpha - \beta] \leq 0$ (or equivalently $(2[\alpha - \beta] + 1/2)/(2\beta + 2q + 1) \leq -1/2$): we require $\beta > \frac{1}{2} - \frac{q}{2} \vee 0$ and

$$o(\tau_n^2) = n^{(2[\alpha - \beta] + 1/2)/(2\beta + 2q + 1)} \vee n^{-1/2} = n^{-1/2}.$$

Note, we also have an extra condition from $\tilde{\lambda}_i$, which requires $n\tau_n^2 \rightarrow \infty$. This condition however will always be dominated by conditions 6 or 7, and hence is irrelevant. \square

5.7 Conclusion

We begin by discussing the Bernstein-von Mises Theorems for the direct and indirect case. Note, the results in these theorems will be valid only if the approximation errors $\Delta(r_0, x)$ and ϱ are small. The first two theorems, (Theorems 5.1 and 5.2), address the direct problem. Antithetically, Theorem 5.3 addresses the indirect problem.

Specifically, Theorem 5.1 claims that $D_0(\eta - \eta^0)$ is nearly standard normal conditional on Y , for some general bounded $g^2(\cdot)$. In addition, as stated in Corollary 5.1, these results are unaffected by the nature of $g^2(\cdot)$.

Similarly, Theorem 5.2 claims that $D_0(\eta - \eta^0)$ is nearly standard normal conditional on Y , assuming a truncated prior and a linear $g^2(\cdot)$. Furthermore, Corollary 5.2 provides conditions for the aforementioned approximation errors to be $o(1)$.

Additionally, note how the posterior distribution given by Theorems 5.1 and 5.2 is centered at $\eta^0 := \eta_0 + D_0^{-2} \nabla L(\eta_0)$, which can be viewed as the first order approximation to the MLE. This is similar to the classical Bernstein-von Mises Theorem, where the posterior distribution is centered around the MLE.

Subsequently, analogous to the direct case theorems, Theorem 5.3 claims that $\tilde{D}_0(\mu - \mu^0)$ is nearly standard normal conditional on Y , for some general bounded $g^2(\cdot)$. Furthermore, Corollary 5.2 provides conditions for the corresponding approximation errors to be $o(1)$.

In addition, note that once again the posterior distribution is centered at $\mu_i^0 := \mu_{0,i} + (\tilde{D}_0^2)_{i,i}^{-1} (\nabla_\mu L(\mu_0))_i$, which can be viewed as the first order approximation to the MLE. Thus, each of the Bernstein-von Mises theorems is similar to its classical counterpart.

Some of our conditions are also similar to the ones obtained in [Panov and Spokoiny, 2015]. Specifically, in Section 4 of [Panov and Spokoiny, 2015], for the i.i.d case, it was shown that the condition $p = o(n^{1/3})$ had to be satisfied for the Bernstein-Von Mises result to hold. Furthermore, it could not be dropped or relaxed in a general situation, hence

it was referred to as the *critical dimension*. As seen in Corollaries 5.2 and 5.3 we too obtain the same exact condition.

However, as discussed in Remark 5.9, there is no guarantee that the conditions stated in [Panov and Spokoiny, 2015] are optimal. Thus, for instance, it might still be possible to improve our contraction rates. A possible method for gauging the optimality of [Panov and Spokoiny, 2015]’s conditions would be to consider the case where $g^2(\cdot)$ is a constant function, and compare the corresponding rates with those derived in Section 2.

Nevertheless, when considering the indirect case, the Bernstein-von Mises results were derived under the mildly ill-posed setting (i.e. $k_i \asymp p$). However, Bernstein-von Mises results for the severely ill-posed setting $k_i = \mathcal{O}(e^{-iq})$, (see Definition 1.8), would also follow under the framework used in Sections 5.1 and 5.3.

Furthermore, [Panov and Spokoiny, 2015] considered a semi-parametric framework where the model could be mis-specified. In our setting, a mis-specified model would affect the variance as well, since the noise considered is signal dependent. Consequently, one could derive the corresponding Bernstein-Von Mises results, and ascertain the effect $g^2(\cdot)$ has on the approximation errors.

Note, we delay the discussion on how the Bernstein-Von Mises results in this section can be applied to linear Poisson inverse problems to Section 6.

5.8 Preliminaries for Proofs of Theorem 5.4, Theorem 5.1 and Theorem 5.3

5.8.1 Outline for Proofs

We begin by deriving the likelihood and its derivatives for the direct model defined in Section 5.1. Subsequently, we verify the assumptions from Section 4.2 in order to use the paper's theorems, stated verbatim in 4.3.

Specifically, we use [Panov and Spokoiny, 2015]'s Theorem 2 to obtain local bounds on the posterior, and [Panov and Spokoiny, 2015]'s Theorem 9 to obtain our own bounds on the tail posterior probability, c.f. Theorem 5.4. We present the proof of Theorem 5.4 in Section 5.9, the proof of Theorem 5.1 in Section 5.10 and the proof of Theorem 5.3 in Section 5.11.

Furthermore, the proof of Corollary 5.2 is provided in Section 5.10 as well.

5.8.2 Likelihood Derivations

We begin by deriving the likelihood and its derivatives for a general $g^2(\cdot)$, with the true value $\eta_0 \in \mathbb{R}^p$. Note these derivations are only valid if Assumption 5.1 holds.

Let $\eta_0 = (\eta_{0,1}, \dots, \eta_{0,p})$ and $\eta = (\eta_1, \dots, \eta_p)$, where $Y_i \sim N(\eta_i, n^{-1}g^2(\eta_i))$. Consequently,

$$\begin{aligned} f(Y_i|\eta_i) &= \frac{1}{\sqrt{2\pi n^{-1}g^2(\eta_i)}} \exp\left(-\frac{(Y_i - \eta_i)^2}{2n^{-1}g^2(\eta_i)}\right), \\ L(\eta) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(\eta_i)) - \frac{n}{2} \sum_{i=1}^p \frac{(Y_i - \eta_i)^2}{g^2(\eta_i)}, \\ \mathbb{E}_{\eta_0} L(\eta) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(\eta_i)) - \frac{1}{2} \sum_{i=1}^p \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - \frac{n}{2} \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{g^2(\eta_i)}, \\ \mathbb{E}_{\eta_0} L(\eta_0) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(\eta_{0,i})) - \frac{p}{2}. \end{aligned}$$

Given Assumption 5.1, (see C.6 for further details), η_0 is the unique maximum of

$\mathbb{E}_{\eta_0} L(\eta)$, and $\nabla \mathbb{E}_{\eta_0} L(\eta) = 0$ at η_0 . Subsequently,

$$\begin{aligned}
\frac{\partial}{\partial \eta_i} L(\eta) &= -\frac{g'(\eta_i)}{g(\eta_i)} + \frac{(Y_i - \eta_i)}{n^{-1}g^2(\eta_i)} + \frac{(Y_i - \eta_i)^2 g'(\eta_i)}{n^{-1}g^3(\eta_i)}, \\
\frac{\partial}{\partial \eta_i} L(\eta_{0,i}) &= -\frac{g'(\eta_{0,i})}{g(\eta_{0,i})} + \frac{(Y_i - \eta_{0,i})}{n^{-1}g^2(\eta_{0,i})} + \frac{(Y_i - \eta_{0,i})^2 g'(\eta_{0,i})}{n^{-1}g^3(\eta_{0,i})}, \\
\frac{\partial}{\partial \eta_i} \mathbb{E}_{\eta_0} L(\eta) &= \frac{(\eta_{0,i} - \eta_i)}{n^{-1}g^2(\eta_i)} + \frac{(\eta_{0,i} - \eta_i)^2 g'(\eta_i)}{n^{-1}g^3(\eta_i)} + \frac{g'(\eta_i)}{g^3(\eta_i)} (g^2(\eta_{0,i}) - g^2(\eta_i)), \\
\frac{\partial}{\partial \eta_i} \mathbb{E}_{\eta_0} L(\eta_{0,i}) &= 0, \\
\frac{\partial^2}{\partial \eta_i^2} \mathbb{E}_{\eta_0} L(\eta) &= -\frac{1}{n^{-1}g^2(\eta_i)} - g'(\eta_i) \frac{4(\eta_{0,i} - \eta_i)}{n^{-1}g^3(\eta_i)} \\
&\quad + g''(\eta_i) \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^3(\eta_i)} \\
&\quad - [g'(\eta_i)]^2 \frac{3(\eta_{0,i} - \eta_i)^2 + 2n^{-1}g^2(\eta_i) + 3n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^4(\eta_i)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\zeta(\eta) &= L(\eta) - \mathbb{E}_{\eta_0} L(\eta) \\
&= \sum_{i=1}^p \frac{n^{-1}g^2(\eta_{0,i}) + (\eta_{0,i} - \eta_i)^2 - (Y_i - \eta_i)^2}{2n^{-1}g^2(\eta_i)}, \\
(\nabla \zeta(\eta))_i &= \frac{Y_i - \eta_{0,i}}{n^{-1}g^2(\eta_i)} + \frac{g'(\eta_i)}{n^{-1}g^3(\eta_i)} [(Y_i - \eta_i)^2 - (\eta_{0,i} - \eta_i)^2 - n^{-1}g^2(\eta_{0,i})], \\
(\nabla \zeta(\eta_0))_i &= \frac{Y_i - \eta_{0,i}}{n^{-1}g^2(\eta_{0,i})} + \frac{g'(\eta_{0,i})}{n^{-1}g^3(\eta_{0,i})} [(Y_i - \eta_{0,i})^2 - n^{-1}g^2(\eta_{0,i})], \\
(\nabla^2 \zeta(\eta))_{i,i} &= -\frac{2g'(\eta_i)(Y_i - \eta_{0,i})}{n^{-1}g^3(\eta_i)} + \frac{g''(\eta_i)}{n^{-1}g^3(\eta_i)} [(Y_i - \eta_i)^2 - (\eta_{0,i} - \eta_i)^2 - n^{-1}g^2(\eta_{0,i})] \\
&\quad + \frac{g'(\eta_i)}{n^{-1}g^3(\eta_i)} [-2(Y_i - \eta_i) + 2(\eta_{0,i} - \eta_i)] \\
&\quad - \frac{3[g'(\eta_i)]^2}{n^{-1}g^4(\eta_i)} [(Y_i - \eta_i)^2 - (\eta_{0,i} - \eta_i)^2 - n^{-1}g^2(\eta_{0,i})], \\
(\nabla^2 \zeta(\eta))_{ij} &= 0, \quad \text{when } i \neq j, \\
(\nabla^2 \zeta(\eta_0))_{i,i} &= -\frac{4g'(\eta_{0,i})(Y_i - \eta_{0,i})}{n^{-1}g^3(\eta_{0,i})} + \frac{g''(\eta_{0,i})}{n^{-1}g^3(\eta_{0,i})} [(Y_i - \eta_{0,i})^2 - n^{-1}g^2(\eta_{0,i})] \\
&\quad - \frac{3[g'(\eta_{0,i})]^2}{n^{-1}g^4(\eta_{0,i})} [(Y_i - \eta_{0,i})^2 - n^{-1}g^2(\eta_{0,i})].
\end{aligned}$$

Note, in our model $\nabla \zeta(\eta_0) = \nabla L(\eta_0) - \nabla \mathbb{E}_{\eta_0} L(\eta_0) = \nabla L(\eta_0)$, since $\nabla \mathbb{E}_{\eta_0} L(\eta_0) = 0$.

Furthermore,

$$\mathbb{E}_{\eta_0}(\nabla\zeta(\eta_0))_i = \frac{\mathbb{E}_{\eta_0}(Y_i) - \eta_{0,i}}{n^{-1}g^2(\eta_{0,i})} + \frac{g'(\eta_{0,i})}{n^{-1}g^3(\eta_{0,i})} [\mathbb{E}_{\eta_0}(Y_i - \eta_{0,i})^2 - n^{-1}g^2(\eta_{0,i})] = 0,$$

since $Y_i|\eta_{0,i} \sim N(\eta_{0,i}, n^{-1}g^2(\eta_{0,i}))$.

Denote $(\Sigma_0)_{i,j}^2 = \text{Cov}((\nabla\zeta(\eta_0))_i, (\nabla\zeta(\eta_0))_j)$, where $\Sigma_0^2 \in \mathbb{R}^{p \times p}$. Since the Y_i are independent, $\text{Cov}((\nabla\zeta(\eta_0))_i, (\nabla\zeta(\eta_0))_j) = 0$.

Furthermore, the results from C.10 (with $X = Y_i - \eta_{0,i}$, $\sigma^2 = n^{-1}g^2(\eta_{0,i})$, $a = \frac{1}{n^{-1}g^2(\eta_{0,i})}$ and $b = \frac{g'(\eta_{0,i})}{n^{-1}g^3(\eta_{0,i})}$) imply

$$\text{Var}((\nabla\zeta(\eta_0))_i) = a^2\sigma^2 + 2b^2\sigma^4 = \frac{1}{n^{-1}g^2(\eta_{0,i})} + 2\left[\frac{g'(\eta_{0,i})}{g(\eta_{0,i})}\right]^2. \quad (39)$$

Therefore, Σ_0^2 is a positive-symmetric, diagonal matrix and $(\nabla\zeta(\eta_0))_i$ are independent random variables. Specifically,

$$(\Sigma_0)_{i,j}^2 = \begin{cases} \frac{1}{n^{-1}g^2(\eta_{0,i})} + 2\left[\frac{g'(\eta_{0,i})}{g(\eta_{0,i})}\right]^2, & \text{if } i = j, \\ 0, & \text{o/w.} \end{cases} \quad (40)$$

5.8.3 Fisher Information Matrix: D_0^2

Recall, $D_0^2 := -\nabla^2 \mathbb{E}L(\eta_0)$, i.e the Fisher Information matrix, specifically,

$$\begin{aligned} (D_0^2)_{i,j} &= -\frac{\partial^2}{\partial\eta_i\partial\eta_j} \mathbb{E}_{\eta_0}L(\eta_0) = 0, \\ (D_0^2)_{i,i} &= -\frac{\partial^2}{\partial\eta_i^2} \mathbb{E}_{\eta_0}L(\eta_0) = \frac{1}{n^{-1}g^2(\eta_{0,i})} + 2\left[\frac{g'(\eta_{0,i})}{g(\eta_{0,i})}\right]^2 \\ &= \frac{1 + 2n^{-1}[g'(\eta_{0,i})]^2}{n^{-1}g^2(\eta_{0,i})} \\ &= \frac{n + 2[g'(\eta_{0,i})]^2}{g^2(\eta_{0,i})} \\ &= (\Sigma_0^2)_{i,i}. \end{aligned}$$

Furthermore, using Assumption 5.2 at $r = 0$, one can show

$$(D_0^2)_{i,i} \asymp n.$$

5.8.4 Local Neighbourhoods: $\Upsilon_0(r_0)$

We shall need to show the local conditions regarding $L(\eta)$ hold for $\eta \in \Upsilon_0(r_0)$. Recall,

$$\Upsilon_0(r_0) := \{\eta \in \Upsilon : \|D_0(\eta - \eta_0)\| \leq r_0\},$$

where $D_0^2 = -\nabla^2 \mathbb{E}L(\eta_0)$. Observe,

$$\begin{aligned} \|D_0(\eta - \eta_0)\|^2 &= \sum_{i=1}^p (D_0^2)_{i,i}(\eta_i - \eta_{0,i})^2, \quad \text{and} \\ (D_0^2)_{i,i}(\eta_i - \eta_{0,i})^2 &= \frac{1}{n^{-1}g^2(\eta_{0,i})}(\eta_i - \eta_{0,i})^2 + 2\left[\frac{g'(\eta_{0,i})}{g(\eta_{0,i})}\right]^2(\eta_i - \eta_{0,i})^2. \end{aligned}$$

Furthermore, for $\eta \in \Upsilon_0(r_0)$,

$$\sum_{i=1}^p (D_0^2)_{i,i}(\eta_i - \eta_{0,i})^2 \leq r_0^2 \implies (\eta_i - \eta_{0,i})^2 \leq \frac{r_0^2}{(D_0^2)_{i,i}}, \quad (41)$$

therefore we obtain the following corollary:

Corollary 5.7. *Given Assumption 5.2, the following exist for $1 \leq i \leq p$:*

$$\begin{aligned} \max_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq r_0^2 (D_0^{-2})_{i,i}} \left| \frac{g(\eta_{0,i})}{g(\eta_i)} \right| &\leq g(\eta_{0,i}) m_{r_0,0,l,i}^{-1} := m_{r_0,1,i}, \\ \max_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq r_0^2 (D_0^{-2})_{i,i}} \left| \frac{g'(\eta_i)}{g(\eta_i)} \right| &\leq m_{r_0,1,u,i} m_{r_0,0,l,i}^{-1} := m_{r_0,2,i}, \\ \max_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq r_0^2 (D_0^{-2})_{i,i}} \left| \frac{g''(\eta_i)}{g(\eta_i)} \right| &\leq m_{r_0,2,u,i} m_{r_0,0,l,i}^{-1} := m_{r_0,3,i}, \\ \max_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq r_0^2 (D_0^{-2})_{i,i}} |g^2(\eta_{0,i}) - g^2(\eta_i)| &\leq g^2(\eta_{0,i}) + m_{r_0,0,u,i}^2 := m_{r_0,4,i} \end{aligned}$$

This corollary was only used once, and that was to obtain a rough bound for the term $\delta(r)$ in Assumption L_0 , (c.f. C.4) which in turn motivated the results in Section 5.8.8.

5.8.5 Identifiability Condition: I

In the parametric case, the identifiability condition from [Panov and Spokoiny, 2015] and reduces to those stated in Spokoiny (2012), i.e.

$$\exists a > 0 \quad \text{s.t.} \quad a^2 D_0^2 \geq \Sigma_0^2,$$

where $D_0^2 := -\nabla^2 \mathbb{E}L(\eta_0)$, i.e the Fisher Information matrix. Sections 5.8.2 and 5.8.3 show that in fact, $\Sigma_0^2 = D_0^2$, hence this condition holds with $a = 1$.

5.8.6 Assumption ED_0

Given $\varkappa \in \mathbb{R}$ and $\psi \in \mathbb{R}^p$, consider

$$\mathbb{E} \exp\left(\varkappa \frac{\psi^T \nabla \zeta(\eta_0)}{\|\Sigma_0 \psi\|}\right) = \prod_{i=1}^p \mathbb{E} \exp\left(\varkappa \frac{\psi_i (\nabla \zeta(\eta_0))_i}{\|\Sigma_0 \psi\|}\right),$$

due to the independence of $(\nabla\zeta(\eta_0))_i$.

Note,

$$\begin{aligned} (\nabla\zeta(\eta_0))_i &= \frac{Y_i - \eta_{0,i}}{n^{-1}g^2(\eta_{0,i})} + \frac{g'(\eta_{0,i})}{n^{-1}g^3(\eta_{0,i})}[(Y_i - \eta_{0,i})^2 - n^{-1}g^2(\eta_{0,i})] \\ &= \frac{1}{n^{-1/2}g(\eta_{0,i})}X_i + \frac{g'(\eta_{0,i})}{g(\eta_{0,i})}[X_i^2 - 1] \\ &= \frac{g'(\eta_{0,i})}{g(\eta_{0,i})}X_i^2 + \frac{1}{n^{-1/2}g(\eta_{0,i})}X_i - \frac{g'(\eta_{0,i})}{g(\eta_{0,i})}, \end{aligned}$$

where

$$X_i = \frac{Y_i - \eta_{0,i}}{n^{-1/2}g(\eta_{0,i})} \sim N(0, 1).$$

Lemma 5.8. *Given $X \sim N(0, 1)$, and $a < 1/2$*

$$aX^2 + bX + c = -\frac{1}{2A^2}(x - B)^2 + C,$$

where

$$A^2 = -\frac{1}{2a}, \quad B = -\frac{b}{2a}, \quad C = c - \frac{b^2}{4a}.$$

Furthermore,

$$\log \mathbb{E} \exp(aX^2 + bX + c) = -\frac{1}{2} \log(1 - 2a) - a + [c + a - \frac{b^2}{4a - 2}].$$

Proof. Note, when $a < 0$,

$$\begin{aligned} aX^2 + bX + c &= a(X + \frac{b}{2a})^2 + (c - \frac{b^2}{4a}) \\ &= -\frac{1}{2A^2}(X - B)^2 + C, \end{aligned}$$

where

$$A^2 = -\frac{1}{2a}, \quad B = -\frac{b}{2a}, \quad C = c - \frac{b^2}{4a}$$

Consequently,

$$\begin{aligned} \mathbb{E} \exp(aX^2 + bX + c) &= \int \frac{1}{\sqrt{2\pi}} \exp(ax^2 + bx + c) \exp(-\frac{x^2}{2}) dx \\ &= \int \frac{1}{\sqrt{2\pi}} \exp(a_1x^2 + bx + c) dx \\ &= A_1 \exp(C_1) \int \frac{1}{\sqrt{2\pi A_1^2}} \exp(-\frac{1}{2A_1^2}(x - B_1)^2) dx \\ &= A_1 \exp(C_1), \end{aligned}$$

where

$$A_1^2 = -\frac{1}{2a_1}, \quad B_1 = -\frac{b}{2a_1}, \quad C_1 = c - \frac{b^2}{4a_1}, \quad a_1 = a - \frac{1}{2}.$$

Note, the requirement on $a_1 < 0$, rather than a .

Thus,

$$\begin{aligned} \log \mathbb{E} \exp(aX^2 + bX + c) &= \log A_1 + C_1 = -\frac{1}{2} \log A_1^{-2} + C_1 \\ &= -\frac{1}{2} \log[-2(a - \frac{1}{2})] + c - \frac{b^2}{4a - 2} \\ &= -\frac{1}{2} \log(1 - 2a) + c - \frac{b^2}{4a - 2} - a + a \\ &= -\frac{1}{2} \log(1 - 2a) - a + c + a - \frac{b^2}{4a - 2}. \end{aligned}$$

□

Corollary 5.8. For $|a| < \frac{1}{2}$,

$$-\frac{1}{2} \log(1 - 2a) - a = a^2 + 2a^2 \sum_{k=1}^{\infty} \frac{(2a)^k}{k+2}.$$

Furthermore, for $|a| < \frac{1}{4}$,

$$-\frac{1}{2} \log(1 - 2a) - a \leq 3a^2.$$

Proof. Note, using Taylor series (with $|2a| < 1$)

$$-\frac{1}{2} \log(1 - 2a) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(2a)^k}{k}.$$

Thus,

$$\begin{aligned} -\frac{1}{2} \log(1 - 2a) - a &= a + a^2 + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(2a)^{k+2}}{k+2} - a \\ &= a^2 + \frac{(2a)^2}{2} \sum_{k=1}^{\infty} \frac{(2a)^k}{k+2} = a^2 + 2a^2 \sum_{k=1}^{\infty} \frac{(2a)^k}{k+2} \\ &\leq a^2 + 2a^2 \sum_{k=1}^{\infty} (2a)^k = a^2 + 2a^2 \left[\frac{2a}{1 - 2a} \right] \\ &\leq a^2 + 2a^2 \left[\left| \frac{2a}{1 - 2a} \right| \right]. \end{aligned}$$

Note, by the modulus' definition,

$$|\frac{2a}{1-2a}| \leq 1 \iff \frac{2a}{1-2a} \leq 1 \text{ and, } \frac{2a}{1-2a} \geq -1.$$

However,

$$\frac{2a}{1-2a} = \frac{2a-1+1}{1-2a} = -1 + \frac{1}{1-2a}.$$

Thus,

$$|\frac{2a}{1-2a}| \leq 1 \iff \frac{1}{1-2a} \leq 2 \text{ and, } \frac{1}{1-2a} \geq 0.$$

By comparing both these cases with x^{-1} , where $x := 1 - 2a$, we see

$$\begin{aligned} x^{-1} \leq 2 \text{ and, } x^{-1} \geq 0 &\iff x \geq \frac{1}{2} \text{ and, } x \geq 0 \\ &\iff \frac{1}{4} \geq a \text{ and, } \frac{1}{2} \geq a. \end{aligned}$$

In conclusion, for $|a| \leq \frac{1}{4}$,

$$\begin{aligned} -\frac{1}{2} \log(1-2a) - a &\leq a^2 + 2a^2 \left[\left| \frac{2a}{1-2a} \right| \right] \\ &\leq a^2 + 2a^2 = 3a^2. \end{aligned}$$

□

Proposition 5.2. For $|\varkappa| \leq (4H)^{-1}$,

$$\sup_{\psi \in \mathbb{R}^p} \log \mathbb{E} \left\{ \varkappa \frac{\psi^T \nabla \zeta(\eta_0)}{\|\Sigma_0 \psi\|} \right\} \leq 2\varkappa^2,$$

where,

$$H := \max_{1 \leq i \leq p} \left| \frac{1}{(\Sigma_0)_{i,i}} \frac{g'(\eta_{0,i})}{g(\eta_{0,i})} \right|.$$

Proof. Using Lemma 5.8 and Corollary 5.8, we can show

$$\begin{aligned} \log \mathbb{E} \exp \left(\varkappa \frac{\psi_i (\nabla \zeta(\eta_0))_i}{\|\Sigma_0 \psi\|} \right) &= \log \mathbb{E} \exp(a_i X^2 + b_i X + c_i) \leq 3a_i^2 + [c_i + a_i - \frac{b_i^2}{4a_i - 2}] \\ &\iff |a_i| \leq \frac{1}{4}, \end{aligned}$$

where

$$a_i = h_i \frac{g'(\eta_{0,i})}{g(\eta_{0,i})}, \quad b_i = h_i \frac{1}{n^{-1/2} g(\eta_{0,i})}, \quad c_i = -a_i \text{ and, } h_i = \varkappa \frac{\psi_i}{\|\Sigma_0 \psi\|}.$$

Note,

$$|a_i| = |\varkappa \frac{\psi_i}{\|\Sigma_0 \psi\|} \frac{g'(\eta_{0,i})}{g(\eta_{0,i})}| = |\frac{\varkappa}{(\Sigma_0)_{i,i}} \frac{(\Sigma_0 \psi)_{i,i}}{\|\Sigma_0 \psi\|} \frac{g'(\eta_{0,i})}{g(\eta_{0,i})}| \leq |\varkappa| \max_{1 \leq i \leq p} |\frac{1}{(\Sigma_0)_{i,i}} \frac{g'(\eta_{0,i})}{g(\eta_{0,i})}|.$$

Therefore, for all i ,

$$|\varkappa|H \leq \frac{1}{4} \implies |a_i| \leq \frac{1}{4},$$

where

$$H := \max_{1 \leq i \leq p} |\frac{1}{(\Sigma_0)_{i,i}} \frac{g'(\eta_{0,i})}{g(\eta_{0,i})}|.$$

Thus, indeed

$$\begin{aligned} \log \mathbb{E} \exp(\varkappa \frac{\psi_i(\nabla \zeta(\eta_0))_i}{\|\Sigma_0 \psi\|}) &\leq 3a_i^2 + [c_i + a_i - \frac{b_i^2}{4a_i - 2}] = 3a_i^2 + \frac{b_i^2}{2 - 4a_i} \\ &\leq 3a_i^2 + b_i^2, \end{aligned}$$

where the last inequality follows from $|a_i| \leq \frac{1}{4}$.

Finally, since

$$\begin{aligned} h_i^2 &= \frac{\varkappa^2}{(\Sigma_0)_{i,i}^2} \frac{(\Sigma_0 \psi)_{i,i}^2}{\|\Sigma_0 \psi\|^2}, \\ a_i^2 &= h_i^2 [\frac{g'(\eta_{0,i})}{g(\eta_{0,i})}]^2, \\ b_i^2 &= h_i^2 \frac{1}{n^{-1}g^2(\eta_{0,i})}, \end{aligned}$$

we can deduce

$$b_i^2 + 2a_i^2 = h_i^2 [\frac{1}{n^{-1}g^2(\eta_{0,i})} + 2[\frac{g'(\eta_{0,i})}{g(\eta_{0,i})}]^2] = h_i^2 (\Sigma_0)_{i,i}^2.$$

Therefore,

$$\begin{aligned} \sum_{i=1}^p \log \mathbb{E} \exp(\varkappa \frac{\psi_i(\nabla \zeta(\eta_0))_i}{\|\Sigma_0 \psi\|}) &\leq \sum_{i=1}^p 3a_i^2 + b_i^2 \\ &\leq \sum_{i=1}^p 2(2a_i^2 + b_i^2) \\ &\leq 2 \sum_{i=1}^p \left[\frac{\varkappa^2}{(\Sigma_0)_{i,i}^2} \frac{(\Sigma_0 \psi)_{i,i}^2}{\|\Sigma_0 \psi\|^2} (\Sigma_0)_{i,i}^2 \right] \\ &\leq 2\varkappa^2 \sum_{i=1}^p \left[\frac{(\Sigma_0 \psi)_{i,i}^2}{\|\Sigma_0 \psi\|^2} \right] \\ &\leq 2\varkappa^2. \end{aligned}$$

□

Note, using the results from Section 5.8.3, along with Assumption 5.2,

$$H := \max_{1 \leq i \leq p} \left| \frac{1}{(\Sigma_0)_{i,i}} \frac{g'(\eta_{0,i})}{g(\eta_{0,i})} \right| \asymp \max_{1 \leq i \leq p} \left| \frac{1}{(\Sigma_0)_{i,i}} \right| \asymp n^{-1/2}.$$

Thus

$$\aleph = (4H)^{-1} \asymp n^{1/2}.$$

Hence, the terms in Assumption ED_0 are:

$$\begin{aligned} \nu_0 &\asymp 1, \\ \aleph &\asymp n^{1/2}. \end{aligned}$$

5.8.7 Assumption ED_2

Lemma 5.9. *Note,*

$$\begin{aligned}
(\nabla^2 \zeta(\eta))_{i,i} &= a_i X_i^2 + b_i X_i + c_i, \quad \text{where} \\
X_i &= \frac{Y_i - \eta_{0,i}}{n^{-1/2} g(\eta_{0,i})} \sim N(0, 1), \\
a_i &= g^2(\eta_{0,i}) \left[\frac{g''(\eta_i)}{g^3(\eta_i)} - \frac{3[g'(\eta_i)]^2}{g^4(\eta_i)} \right], \\
b_i &= n^{-1/2} g(\eta_{0,i}) \left[-\frac{4g'(\eta_i)}{n^{-1} g^3(\eta_i)} + 2(\eta_{0,i} - \eta_i) \frac{a_i}{n^{-1} g^2(\eta_{0,i})} \right], \\
c_i &= -a_i.
\end{aligned}$$

Furthermore, given Assumption 5.2 (and Remark 4.5), the following exist for $1 \leq i \leq p$ and $r \leq r_0$:

$$\begin{aligned}
\max_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq \frac{r^2}{(D_0^2)_{i,i}}} |a_i| &\leq a_{u,r,i} := [m_{r,2,u,i} m_{r,0,l,i}^{-3} + 3m_{r,1,u,i}^2 m_{r,0,l,i}^{-4}] g^2(\eta_{0,i}), \\
\max_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq \frac{r^2}{(D_0^2)_{i,i}}} |b_i| &\leq b_{u,r,i} := [4m_{r,1,u,i} n m_{r,0,l,i}^{-3} + 2 \frac{r a_{u,r,i}}{(D_0)_{i,i} n^{-1} g^2(\eta_{0,i})}] n^{-1/2} g(\eta_{0,i}), \\
\max_{\eta_i: (\eta_i - \eta_{0,i})^2 \leq \frac{r^2}{(D_0^2)_{i,i}}} |c_i| &\leq a_{u,r,i}.
\end{aligned}$$

The proof for the lemma can be found in C.3.1.

Proposition 5.3. *For $|\varkappa| \leq \omega(4H)^{-1}$,*

$$\sup_{\psi_1, \psi_2 \in \mathbb{R}^p} \log \mathbb{E} \exp \left\{ \frac{\varkappa}{\omega} \frac{\psi_1^T \nabla^2 \zeta(\eta) \psi_2}{\|D_0 \psi_1\| \cdot \|D_0 \psi_2\|} \right\} \leq \frac{\varkappa^2}{\omega^2} \sup_{i \leq p} \sup_{\eta_i \in \Upsilon(r)} \left(\frac{a_i^2 + b_i^2}{(D_0^4)_{i,i}} \right),$$

where,

$$H := \max_{1 \leq i \leq p} \left| \frac{a_{u,r,i}}{(D_0^2)_{i,i}} \right|.$$

Proof. Using Lemma 5.8 and Corollary 5.8, we can show

$$\begin{aligned}
\log \mathbb{E} \exp \left\{ \frac{\varkappa}{\omega} \frac{\psi_{1,i} (\nabla^2 \zeta(\eta))_{i,i} \psi_{2,i}}{\|D_0 \psi_1\| \cdot \|D_0 \psi_2\|} \right\} &= \log \mathbb{E} \exp(A_i X^2 + B_i X + C_i) \leq 3A_i^2 + [C_i + A_i - \frac{B_i^2}{4A_i - 2}], \\
&\iff |A_i| \leq \frac{1}{4},
\end{aligned}$$

where

$$A_i = h_i a_i, \quad B_i = h_i b_i, \quad C_i = h_i c_i \quad \text{and} \quad h_i = \frac{\varkappa}{\omega} \frac{\psi_{1,i} \psi_{2,i}}{\|D_0 \psi_1\| \cdot \|D_0 \psi_2\|}.$$

Note,

$$\begin{aligned}
|A_i| &= |h_i| |a_i| \leq \left| \frac{\varkappa}{\omega} \frac{\psi_{1,i} \psi_{2,i}}{\|D_0 \psi_1\| \cdot \|D_0 \psi_2\|} \right| |a_{u,r,i}| \\
&\leq \left| \frac{\varkappa}{\omega (D_0^2)_{i,i}} \frac{|(D_0)_{i,i} \psi_{1,i}| |(D_0)_{i,i} \psi_{2,i}|}{\|D_0 \psi_1\| \cdot \|D_0 \psi_2\|} \right| |a_{u,r,i}| \\
&\leq \left| \frac{a_{u,r,i}}{(D_0^2)_{i,i}} \right| \frac{|\varkappa|}{\omega} \\
&\leq \left\| \frac{a_{u,r}}{\text{diag}(D_0^2)} \right\|_\infty \frac{|\varkappa|}{\omega}.
\end{aligned}$$

Therefore,

$$H \frac{|\varkappa|}{\omega} \leq \frac{1}{4} \implies |A_i| \leq \frac{1}{4},$$

where

$$H := \left\| \frac{a_{u,r}}{\text{diag}(D_0^2)} \right\|_\infty.$$

Thus, indeed

$$\begin{aligned}
\log \mathbb{E} \exp \left\{ \frac{\varkappa}{\omega} \frac{\psi_{1,i} (\nabla^2 \zeta(\eta))_{i,i} \psi_{2,i}}{\|D_0 \psi_1\| \cdot \|D_0 \psi_2\|} \right\} &\leq 3A_i^2 + [C_i + A_i - \frac{B_i^2}{4A_i - 2}] = 3A_i^2 + \frac{B_i^2}{2 - 4A_i} \\
&\leq 3A_i^2 + B_i^2 \\
&\leq 3 \left(\frac{\varkappa^2}{\omega^2} \frac{\psi_{1,i}^2 \psi_{2,i}^2}{\|D_0 \psi_1\|^2 \cdot \|D_0 \psi_2\|^2} \right) (a_i^2 + b_i^2) \\
&\leq 3 \left(\frac{\varkappa^2}{\omega^2} \frac{(D_0^2)_{i,i} \psi_{1,i}^2 (D_0^2)_{i,i} \psi_{2,i}^2}{\|D_0 \psi_1\|^2 \cdot \|D_0 \psi_2\|^2} \right) \left(\frac{a_i^2 + b_i^2}{(D_0^4)_{i,i}} \right)
\end{aligned}$$

where the second inequality follows from $|A_i| \leq \frac{1}{4}$.

Recall,

$$\begin{aligned}
(D_0^2)_{i,i} &= \frac{1}{n^{-1} g^2(\eta_{0,i})} + 2 \left[\frac{g'(\eta_{0,i})}{g(\eta_{0,i})} \right]^2 \\
&= \frac{1 + 2n^{-1} [g'(\eta_{0,i})]^2}{n^{-1} g^2(\eta_{0,i})} \\
&\asymp n.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
a_{u,r,i} &= [m_{r,2,u,i} m_{r,0,l,i}^{-3} + 3m_{r,1,u,i}^2 m_{r,0,l,i}^{-4}] g^2(\eta_{0,i}), \\
&\asymp m_{r,0,l,i}^{-4} [m_{r,2,u,i} m_{r,0,l,i} + m_{r,1,u,i}^2].
\end{aligned}$$

Therefore,

$$H \asymp \frac{m_{r,0,l,i}^{-4} [m_{r,2,u,i} m_{r,0,l,i} + m_{r,1,u,i}^2]}{n}.$$

□

Using the results and assumptions from Section 5.8.8, as well as Assumption 5.2, we obtain the following bounds,

$$\begin{aligned}
a_i &\leq C_0, \\
b_i &\leq C_0 n^{1/2}, \\
H &\asymp n^{-1}, \\
\sup_{i \leq p} \sup_{\eta_i \in \Upsilon(r)} \frac{a_i^2 + b_i^2}{(D_0^4)_{i,i}} &\leq C_0 \sup_{i \leq p} \frac{n}{(D_0^4)_{i,i}} \\
&\leq C_0 n^{-1}.
\end{aligned}$$

where C_0 is a constant dependent on η_0 and independent of r and n .

Therefore, since

$$\omega^{-2} \sup_{i \leq p} \sup_{\eta_i \in \Upsilon(r)} \frac{a_i^2 + b_i^2}{(D_0^4)_{i,i}} \leq (\omega^2 n)^{-1},$$

we can show $\nu_0 \asymp 1$, if $\omega^2 n \asymp 1$.

Additionally,

$$|\varkappa| \leq \omega(4H)^{-1} \asymp n^{-1/2} n \asymp n^{1/2}.$$

Consequently, the terms in Assumption ED_2 are as follows

$$\begin{aligned}
\omega^2 n &\asymp 1, \\
\nu_0 &\asymp 1, \\
g(r) &\asymp n^{1/2}.
\end{aligned}$$

Finally, note ν_0 and $g(r)$ are of the same order as the corresponding terms in Assumption ED_0 .

5.8.8 Assumption L_0

We wish to show $\|D_0^{-1}D_0^2(\eta)D_0^{-1} - I_p\| \leq 1/2$ for all $\eta \in \Upsilon_0(r)$, where $r \leq r_0$. Recall,

$$\begin{aligned}
\|D_0^{-1}D_0^2(\eta)D_0^{-1} - I_p\| &= \max_{1 \leq i \leq p} \left((D_0^2)_{i,i}^{-1} (D_0^2(\eta))_{i,i} - 1 \right)^2 \\
&= \max_{1 \leq i \leq p} \left(\left(\frac{1 + 2n^{-1}[g'(\eta_{0,i})]^2}{n^{-1}g^2(\eta_{0,i})} \right)^{-1} \left[\frac{1}{n^{-1}g^2(\eta_i)} + g'(\eta_i) \frac{4(\eta_{0,i} - \eta_i)}{n^{-1}g^3(\eta_i)} \right. \right. \\
&\quad \left. \left. - g''(\eta_i) \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^3(\eta_i)} \right. \right. \\
&\quad \left. \left. + [g'(\eta_i)]^2 \frac{3(\eta_{0,i} - \eta_i)^2 + 2n^{-1}g^2(\eta_i) + 3n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^4(\eta_i)} \right] - 1 \right)^2 \\
&= \max_{1 \leq i \leq p} \left(\frac{1 + 2n^{-1}[g'(\eta_i)]^2}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - 1 \right. \\
&\quad \left. + \left(\frac{1 + 2n^{-1}[g'(\eta_{0,i})]^2}{n^{-1}g^2(\eta_{0,i})} \right)^{-1} \left[g'(\eta_i) \frac{4(\eta_{0,i} - \eta_i)}{n^{-1}g^3(\eta_i)} \right. \right. \\
&\quad \left. \left. - g''(\eta_i) \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^3(\eta_i)} \right. \right. \\
&\quad \left. \left. + [g'(\eta_i)]^2 \frac{3(\eta_{0,i} - \eta_i)^2 + 3n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^4(\eta_i)} \right] \right)^2.
\end{aligned}$$

Note, how the first term corresponds to a ratio which one hopes would be close to 1. Similarly, the other terms are dependent on the neighbourhood around $\eta_{0,i}$. Now that we've gained some insight as to why L_0 should hold, consider the following proposition:

Proposition 5.4. *Given Assumptions 5.1 and 5.2:*

$$\max_{i \leq p} |(D_0^{-2})_{i,i} (D_0^2(\eta))_{i,i} - 1| \leq \max_{i \leq p} \tilde{M}_i,$$

where

$$\tilde{M}_i = \frac{r}{(D_0)_{i,i}} \sup_{\eta_i \in \Upsilon(r_0)} |[\log((D_0^2(\eta))_{i,i})]'|,$$

and

$$\|rD_0^{-1}\| = o(1) \implies \tilde{M}_i = o(1).$$

Proof. Let us assume $g \in C^3(\mathbb{R})$ and let $y := \eta_i$. Consequently, on any compact domain, g and its derivatives are bounded. Furthermore, since $g > 0$ everywhere, and $(D_0^2(\eta))_{i,i}$ is diagonal,

$$\begin{aligned}
f_i(y) := (D_0^2(\eta))_{i,i} &= \frac{1 + 2n^{-1}[g'(y)]^2}{n^{-1}g^2(y)} + g'(y) \frac{4(\eta_{0,i} - y)}{n^{-1}g^3(y)} \\
&\quad - g''(y) \frac{(\eta_{0,i} - y)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(y))}{n^{-1}g^3(y)} \\
&\quad + [g'(y)]^2 \frac{3(\eta_{0,i} - y)^2 + 3n^{-1}(g^2(\eta_{0,i}) - g^2(y))}{n^{-1}g^4(y)},
\end{aligned}$$

will be a continuous function. Consequently, there must exist a compact neighbourhood around $\eta_{0,i}$, such that

$$|(D_0^2(\eta))_{i,i} - (D_0^2)_{i,i}| \leq M_i \iff (D_0^{-2})_{i,i} |(D_0^2(\eta))_{i,i} - (D_0^2)_{i,i}| = |(D_0^{-2})_{i,i} (D_0^2(\eta))_{i,i} - 1| \leq M_i (D_0^{-2})_{i,i}.$$

Subsequently, choosing $M_i = \frac{(D_0^2)_{i,i}}{2}$ and $r_0^2 := \min_{i \leq p} (D_0^2)_{i,i} |\eta_{0,i} - \eta_i|^2$ ensures $\|D_0^{-1} D_0^2(\eta) D_0^{-1} - I_p\|^2 \leq 1/2$.

Next, we will explicitly derive the terms $\tilde{M}_i := M_i (D_0^{-2})_{i,i}$ and r_0^2 , as well as the conditions required for $\tilde{M}_i \rightarrow 0$, as $n \rightarrow \infty$.

Let $Q(\eta_i) = D_0^2(\eta_i)$, then using the Mean Value Theorem,

$$\begin{aligned} \frac{|Q(\eta_i) - Q(\eta_{0,i})|}{(D_0^2)_{i,i}} &\leq \sup_{\eta_i \in \Upsilon(r_0)} (D_0)_{i,i} |\eta_i - \eta_{0,i}| \frac{|Q'|}{(D_0^3)_{i,i}} \\ &\leq \frac{r}{(D_0)_{i,i}} \sup_{\eta_i \in \Upsilon(r_0)} \frac{|Q'|}{(D_0^2)_{i,i}}. \end{aligned}$$

Thus $\tilde{M}_i := \frac{r}{(D_0)_{i,i}} \sup_{\eta_i \in \Upsilon(r_0)} \frac{|Q'|}{(D_0^2)_{i,i}}$. Note, for $n \rightarrow \infty$,

$$\tilde{M}_i \rightarrow 0 \implies |(D_0^{-2})_{i,i} (D_0^2(\eta))_{i,i} - 1| \rightarrow 0.$$

Furthermore,

$$r_0^2 := \min_{i \leq p} (D_0^2)_{i,i} |\eta_{0,i} - \eta_i|^2 \leq \|D_0(\eta - \eta_0)\|^2.$$

Finally, recall,

$$\begin{aligned} \frac{\partial^2}{\partial \eta_i^2} \mathbb{E}_{\eta_0} L(\eta) &= -\frac{1}{n^{-1} g^2(\eta_i)} - g'(\eta_i) \frac{4(\eta_{0,i} - \eta_i)}{n^{-1} g^3(\eta_i)} \\ &\quad + g''(\eta_i) \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1} (g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1} g^3(\eta_i)} \\ &\quad - [g'(\eta_i)]^2 \frac{3(\eta_{0,i} - \eta_i)^2 + 2n^{-1} g^2(\eta_i) + 3n^{-1} (g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1} g^4(\eta_i)}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial^3}{\partial \eta_i^3} \mathbb{E}_{\eta_0} L(\eta) &= 2 \frac{g'(\eta_i)}{n^{-1}g^3(\eta_i)} - \frac{4g''(\eta_i)(\eta_{0,i} - \eta_i) - 4g'(\eta_i)}{n^{-1}g^3(\eta_i)} + 3[g'(\eta_i)]^2 \frac{4(\eta_{0,i} - \eta_i)}{n^{-1}g^4(\eta_i)} \\
&\quad + g'''(\eta_i) \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^3(\eta_i)} \\
&\quad + g''(\eta_i) \frac{-2(\eta_{0,i} - \eta_i) - n^{-1}(2g(\eta_i)g'(\eta_i))}{n^{-1}g^3(\eta_i)} \\
&\quad - 3g''(\eta_i)g'(\eta_i) \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^4(\eta_i)} \\
&\quad - 2[g'(\eta_i)g''(\eta_i)] \frac{3(\eta_{0,i} - \eta_i)^2 + 2n^{-1}g^2(\eta_i) + 3n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^4(\eta_i)} \\
&\quad - [g'(\eta_i)]^2 \frac{-6(\eta_{0,i} - \eta_i) + 4n^{-1}g(\eta_i)g'(\eta_i) - 6n^{-1}(g(\eta_i)g'(\eta_i))}{n^{-1}g^4(\eta_i)} \\
&\quad + 4[g'(\eta_i)]^3 \frac{3(\eta_{0,i} - \eta_i)^2 + 2n^{-1}g^2(\eta_i) + 3n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^5(\eta_i)} \\
&= 6 \frac{g'(\eta_i)}{n^{-1}g^3(\eta_i)} + g'''(\eta_i) \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^3(\eta_i)} \\
&\quad - g''(\eta_i) \frac{6(\eta_{0,i} - \eta_i)}{n^{-1}g^3(\eta_i)} \\
&\quad - [g'(\eta_i)g''(\eta_i)] \frac{9(\eta_{0,i} - \eta_i)^2 + 6n^{-1}g^2(\eta_i) + 9n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^4(\eta_i)} \\
&\quad + [g'(\eta_i)]^2 \frac{18(\eta_{0,i} - \eta_i)}{n^{-1}g^4(\eta_i)} \\
&\quad + [g'(\eta_i)]^3 \frac{12(\eta_{0,i} - \eta_i)^2 + 10n^{-1}g^2(\eta_i) + 12n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^5(\eta_i)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|Q'| &\leq \left| 6 \frac{ng'(\eta_i)}{g^3(\eta_i)} \right| + \left| g'''(\eta_i) \frac{n(\eta_{0,i} - \eta_i)^2 + (g^2(\eta_{0,i}) - g^2(\eta_i))}{g^3(\eta_i)} \right| \\
&\quad + \left| g''(\eta_i) \frac{6n(\eta_{0,i} - \eta_i)}{g^3(\eta_i)} \right| \\
&\quad + \left| [g'(\eta_i)g''(\eta_i)] \frac{9n(\eta_{0,i} - \eta_i)^2 + 6g^2(\eta_i) + 9(g^2(\eta_{0,i}) - g^2(\eta_i))}{g^4(\eta_i)} \right| \\
&\quad + \left| [g'(\eta_i)]^2 \frac{18n(\eta_{0,i} - \eta_i)}{g^4(\eta_i)} \right| \\
&\quad + \left| [g'(\eta_i)]^3 \frac{12n(\eta_{0,i} - \eta_i)^2 + 10g^2(\eta_i) + 12(g^2(\eta_{0,i}) - g^2(\eta_i))}{g^5(\eta_i)} \right|.
\end{aligned}$$

Note, since $\eta \in \Upsilon(r)$

$$|\eta_i - \eta_{0,i}| \leq \frac{r}{(D_0)_{i,i}}.$$

Thus, for $r \leq r_0$ and $\eta_{ci} \in (\eta_{0,i}, \eta_i) \cup (\eta_i, \eta_{0,i})$,

$$\begin{aligned} |g^2(\eta_{0,i}) - g^2(\eta_i)| &\leq (D_0)_{i,i} |\eta_i - \eta_{0,i}| \cdot \frac{|[g^2]'(\eta_{ci})|}{(D_0)_{i,i}} \\ &\leq \frac{r}{(D_0)_{i,i}} |[g^2]'(\eta_{ci})| \\ &\leq \frac{r}{(D_0)_{i,i}} \sup_{\eta_i \in \Upsilon(r)} |[g^2]'(\eta_i)| \\ &\leq \frac{r}{(D_0)_{i,i}} \sup_{\eta_i \in \Upsilon(r_0)} |[g^2]'(\eta_i)|. \end{aligned}$$

Observe,

$$\begin{aligned} g^2(\eta_i) &= g^2(\eta_i) - g^2(\eta_{0,i}) + g^2(\eta_{0,i}) \\ &\leq |g^2(\eta_{0,i}) - g^2(\eta_i)| + g^2(\eta_{0,i}). \end{aligned}$$

Hence, any term in $|Q'|$ whose numerator is dependent on $g^2(\eta_i)$ or $|\eta_i - \eta_{0,i}|$ can be bounded. In addition, Assumption 5.2 can be used to bound the denominator with a constant independent of n and r .

Similarly,

$$\begin{aligned} g'(\eta_i) &= g'(\eta_i) - g'(\eta_{0,i}) + g'(\eta_{0,i}) \\ &\leq |g'(\eta_{0,i}) - g'(\eta_i)| + g'(\eta_{0,i}) \\ &\leq \frac{r}{(D_0)_{i,i}} \sup_{\eta_i \in \Upsilon(r_0)} |g''(\eta_i)|. \end{aligned}$$

Thus, since $(D_0)_{i,i}^2 \asymp n$ (c.f. Section 5.8.3),

$$\begin{aligned} \frac{|Q'|}{(D_0)_{i,i}^2} &\leq c[|g'(\eta_i)| + (\eta_{0,i} - \eta_i) + |(\eta_{0,i} - \eta_i)^2| + n^{-1}C_{r_0}] \\ &\leq c\left[\frac{r}{(D_0)_{i,i}} + \frac{r^2}{(D_0)_{i,i}^2} + n^{-1}C_{r_0}\right], \end{aligned}$$

where c and C_{r_0} are constants, the latter being dependent on r_0 .

Consequently,

$$\frac{r}{(D_0)_{i,i}} \rightarrow 0 \implies \frac{r}{(D_0)_{i,i}} \sup_{\eta_i \in \Upsilon(r_0)} \frac{|Q'|}{(D_0)_{i,i}^2} \rightarrow 0.$$

Note, we need the above to hold for all i , i.e.

$$\max_{i \leq p} \frac{r}{(D_0)_{i,i}} \rightarrow 0.$$

□

5.8.9 Assumption L_r

Note,

$$\begin{aligned}
-\mathbb{E}L(\eta, \eta_0) &= -\mathbb{E}[L(\eta) - L(\eta_0)] \\
&= -\left[\left(\frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{2n^{-1}g^2(\eta_i)} - \sum_{i=1}^p \log(g(\eta_i)) - \sum_{i=1}^p \frac{g^2(\eta_{0,i})}{2g^2(\eta_i)} \right) \right. \\
&\quad \left. - \left(\frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(\eta_{0,i})) - \frac{p}{2} \right) \right] \\
&= \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{2n^{-1}g^2(\eta_i)} + \frac{1}{2} \sum_{i=1}^p \left[\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - \log\left(\frac{g^2(\eta_{0,i})}{g^2(\eta_i)}\right) - 1 \right] \\
&\geq \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{2n^{-1}g^2(\eta_i)},
\end{aligned}$$

since $\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - \log\left(\frac{g^2(\eta_{0,i})}{g^2(\eta_i)}\right) - 1 \geq 0$ for all $g^2(\eta_i)$, (c.f. C.5). This bound helps us derive the results below, and is used in Theorem 5.4 as well.

Proposition 5.5. *Assume that for any $r = \|D_0(\eta - \eta_0)\| \geq r_0$*

$$\max_{\eta_i: \eta \in \Upsilon_0(r)} g(\eta_i) = M_i(\eta_0, g, n) \in (0, \infty), \quad i = 1, \dots, p, \quad (42)$$

where $M_i(\eta_0, g, n)$ can depend on η_0 , n and g (and possibly derivatives of g) but not on η (and not on r).

Then, for any $r = \|D_0(\eta - \eta_0)\| \geq r_0$,

$$\frac{-2\mathbb{E}L(\eta, \eta_0)}{\|D_0(\eta - \eta_0)\|^2} \geq b = \min_i \left[\frac{g^2(\eta_{0,i})}{M_i^2[1 + 2n^{-1}(g'(\eta_{0,i}))^2]} \right], \quad \text{for all } \eta \in \Upsilon_0(r). \quad (43)$$

Note that $rb(r) = br$ is non-decreasing.

In particular, condition (42) holds if

1. g is bounded from above, e.g. $g(x) \leq C_g$ for all x , then $M_i = C_g$.

Proof. Observe,

$$\frac{-2\mathbb{E}L(\eta, \eta_0)}{\|D_0(\eta - \eta_0)\|^2} \geq \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{2n^{-1}g^2(\eta_i)\|D_0(\eta - \eta_0)\|^2} = \sum_{i=1}^p \frac{(D_0^2)_{i,i}(\eta_{0,i} - \eta_i)^2}{\|D_0(\eta - \eta_0)\|^2} [n^{-1}g^2(\eta_i)(D_0^2)_{i,i}]^{-1}.$$

Furthermore,

$$[n^{-1}g^2(\eta_i)(D_0^2)_{i,i}]^{-1} = \left[\frac{g(\eta_{0,i})}{g(\eta_i)} \right]^2 [1 + 2n^{-1}(g'(\eta_{0,i}))^2]^{-1}.$$

For $\eta \in \Upsilon_0(r)$, using assumption (42),

$$\begin{aligned} [n^{-1}g^2(\eta_i)(D_0^2)_{i,i}]^{-1} &\geq \min_i \left[\frac{g(\eta_{0,i})}{\max_{\eta_i: \eta \in \Upsilon_0(r)} g(\eta_i)} \right]^2 [1 + 2n^{-1}(g'(\eta_{0,i}))^2]^{-1} \\ &\geq \min_i \left[\frac{g^2(\eta_{0,i})}{M_i^2(\eta_0, g, n)} [1 + 2n^{-1}(g'(\eta_{0,i}))^2]^{-1} \right] = b. \end{aligned}$$

□

Proposition 5.6. Assume that for any $r = \|D_0(\eta - \eta_0)\| \geq r_0$

$$\max_{\eta_i: \eta \in \Upsilon_0(r)} |(g^2)'(\eta_i)| = \tilde{M}_i(\eta_0, g, n) \in (0, \infty), \quad i = 1, \dots, p, \quad (44)$$

where $\tilde{M}_i(\eta_0, g, n)$ can depend on η_0 , n and g and its derivative(s) but not on η (and not on r).

Then, for any $r = \|D_0(\eta - \eta_0)\| \geq r_0$,

$$\frac{-2\mathbb{E}L(\eta, \eta_0)}{\|D_0(\eta - \eta_0)\|^2} \geq b(r) = \min_i \left[\frac{1}{g^2(\eta_{0,i}) + r\tilde{M}_i/[(D_0)_{i,i}]} \right] \left[\frac{n}{(D_0^2)_{i,i}} \right] \quad \text{for all } \eta \in \Upsilon_0(r). \quad (45)$$

Note that $rb(r)$ is of the form $r/(c + ar)$ with $a, c > 0$ which is non-decreasing for $r > 0$.

In particular, if $g^2(x) = a + cx$, then $\tilde{M}_i(\eta_0, g, n) = c$.

Proof of Proposition 5.6. Observe,

$$\begin{aligned} \frac{-2\mathbb{E}L(\eta, \eta_0)}{\|D_0(\eta - \eta_0)\|^2} &\geq \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{2n^{-1}g^2(\eta_i)\|D_0(\eta - \eta_0)\|^2} = \sum_{i=1}^p \frac{(D_0^2)_{i,i}(\eta_{0,i} - \eta_i)^2}{\|D_0(\eta - \eta_0)\|^2} [n^{-1}g^2(\eta_i)(D_0^2)_{i,i}]^{-1} \\ &\geq \min_{1 \leq i \leq p} [n^{-1}g^2(\eta_i)(D_0^2)_{i,i}]^{-1}. \end{aligned}$$

Using the Mean Value Theorem,

$$g^2(\eta_i) = g^2(\eta_{0,i}) + (\eta_i - \eta_{0,i})[g^2]'(\eta_{ci}),$$

where $\eta_{ci} \in \langle \eta_{0,i}, \eta_i \rangle$. Note that if $\eta_i \in \Upsilon_0(r)$, then $\eta_{ci} \in \Upsilon_0(r)$. Hence for $\eta_i \in \Upsilon_0(r)$,

$$g^2(\eta_i) \leq g^2(\eta_{0,i}) + (D_0)_{i,i}|\eta_i - \eta_{0,i}| \cdot |[g^2]'(\eta_{ci})|/(D_0)_{i,i} \leq g^2(\eta_{0,i}) + r\tilde{M}_i/(D_0)_{i,i},$$

using assumption (44) and $(D_0)_{i,i}|\eta_i - \eta_{0,i}| \leq \|D_0(\eta - \eta_0)\| \leq r$.

Hence for $\eta \in \Upsilon_0(r)$,

$$\min_{1 \leq i \leq p} [n^{-1}g^2(\eta_i)(D_0^2)_{i,i}]^{-1} \geq \min_i \left[\frac{1}{g^2(\eta_{0,i}) + r\tilde{M}_i/[(D_0)_{i,i}]} \right] \left[\frac{n}{(D_0^2)_{i,i}} \right] := b(r).$$

□

Proposition 5.7 (General Bounded Case). *For any $r = \|D_0(\eta - \eta_0)\| \geq r_0$,*

$$\frac{-2\mathbb{E}L(\eta, \eta_0)}{\|D_0(\eta - \eta_0)\|^2} \geq b(r) := \min_{i \leq p} [1 + \frac{r}{(D_0)_{i,i}} c]^{-1} \quad \text{for all } \eta \in \Upsilon_0(r). \quad (46)$$

with c being independent of n , r and p . Note that $rb(r)$ is of the form $r/(1 + ar)$ with $a > 0$, which is non-decreasing for $r > 0$.

Proof. Observe,

$$\begin{aligned} \frac{-2\mathbb{E}L(\eta, \eta_0)}{\|D_0(\eta - \eta_0)\|^2} &\geq \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{2n^{-1}g^2(\eta_i)\|D_0(\eta - \eta_0)\|^2} = \sum_{i=1}^p \frac{(D_0^2)_{i,i}(\eta_{0,i} - \eta_i)^2}{\|D_0(\eta - \eta_0)\|^2} [n^{-1}g^2(\eta_i)(D_0^2)_{i,i}]^{-1} \\ &\geq \min_{1 \leq i \leq p} [n^{-1}g^2(\eta_i)(D_0^2)_{i,i}]^{-1}. \end{aligned}$$

Now,

$$[n^{-1}g^2(\eta_i)(D_0^2)_{i,i}]^{-1} \geq b(r) \iff \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} \geq b(r)[1 + 2n^{-1}(g'(\eta_{0,i}))^2].$$

We evaluate $\frac{g^2(\eta_{0,i})}{g^2(\eta_i)}$, by employing the Mean Value Theorem, i.e.

$$\begin{aligned} g^2(\eta_i) &= g^2(\eta_{0,i}) + (\eta_i - \eta_{0,i})[g^2]'(\eta_{ci}) \\ &\leq g^2(\eta_{0,i}) + |(\eta_i - \eta_{0,i})|[g^2]'(\eta_{ci})| \\ &\leq g^2(\eta_{0,i}) \cdot (1 + \frac{r}{(D_0)_{i,i}} c). \end{aligned} \quad (47)$$

Note that if $\eta_i \in \Upsilon_0(r)$, then $\eta_{ci} \in \Upsilon_0(r)$, hence we use Assumption 5.1 to obtain the last inequality, with c being independent of n , r and p .

Hence,

$$\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} \geq \min_{i \leq p} \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} \geq \min_{i \leq p} [1 + \frac{r}{(D_0)_{i,i}} c]^{-1} = b(r).$$

□

As stated in Remark 4.6, in our setting $rb(r) \not\rightarrow \infty$. Hence we can't use the results obtained when Assumption (L_r) holds: i.e. an upper bound for the tail posterior probability, $\rho(r)$ (defined in (21)), and proving that the centering parameter in the BvM result, η^0 , is close to η_0 with high probability.

Consequently, we derive our own upper bound on the tail posterior probability in Theorem 5.4, and we show that η^0 converges weakly to η_0 , for large n , in Proposition 5.1, (c.f. X_i and its moments).

5.8.10 Assumption 4.1

Recall, $\eta \in \mathbb{R}^p$, and $\eta_0^\infty \in S^{\beta_c}$, where the Sobolev space,

$$S^{\beta_c} = \{h : \|h\|_{S^{\beta_c}}^2 < \infty\}, \text{ where } \|h\|_{S^{\beta_c}}^2 = \sum_{i=1}^{\infty} h_i^2 i^{2\beta_c}.$$

Furthermore, we consider a prior on $\eta_i \sim N(0, \lambda_i)$, where $\lambda_i = \tau_n^2 i^{-(1+2\alpha_c)}$ and $n\tau_n^2 \rightarrow \infty$, with $\alpha_c > 0$.

Consequently, the precision matrix G^2 from Assumption 4.1 will be defined as

$$G_{ii}^2 = \lambda_i^{-1} = \tau_n^{-2} i^{1+2\alpha_c}.$$

In addition, let

$$g_{0,p}^2 := \max_i g^2(\eta_{0,i}).$$

Note, given Assumption 5.2, this constant can be ignored. However, we chose to track it for a possible future application.

Thus,

$$(D_G^2)_{i,i} = (D_0^2)_{i,i} + (G^2)_{i,i} = \frac{1 + 2n^{-1}[g'(\eta_{0,i})]^2}{n^{-1}g^2(\eta_{0,i})} + \lambda_i^{-1} = \frac{1 + 2n^{-1}[g'(\eta_{0,i})]^2 + \lambda_i^{-1}n^{-1}g^2(\eta_{0,i})}{n^{-1}g^2(\eta_{0,i})}.$$

Furthermore,

$$\begin{aligned} \|D_0^{-1}G^2D_0^{-1}\| &= \max_i \left| \frac{n^{-1}g^2(\eta_{0,i})\lambda_i^{-1}}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} \right| \asymp \max_i |n^{-1}g^2(\eta_{0,i})\tau_n^{-2}i^{1+2\alpha_c}| \\ &\asymp (n\tau_n^2)^{-1}p^{1+2\alpha_c}g_{0,p}^2, \\ \text{tr}(D_0^{-1}G^2D_0^{-1})^2 &= \sum_{i=1}^p \left[\frac{n^{-1}g^2(\eta_{0,i})\lambda_i^{-1}}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} \right]^2 \asymp (n\tau_n^2)^{-2} \sum_{i=1}^p i^{2(1+2\alpha_c)} \max_i g^4(\eta_{0,i}) \\ &\asymp (n\tau_n^2)^{-2}p^{2(1+2\alpha_c)+1}g_{0,p}^4. \end{aligned}$$

Subsequently,

$$\begin{aligned} \|D_0^{-1}G^2D_0^{-1}\| \rightarrow 0 &\iff p = o\left(\left(\frac{n\tau_n^2}{g_{0,p}^2}\right)^{1/(1+2\alpha_c)}\right). \\ \text{tr}(D_0^{-1}G^2D_0^{-1})^2 \rightarrow 0 &\iff p = o\left(\left(\frac{n\tau_n^2}{g_{0,p}^2}\right)^{1/(1+2\alpha_c+1/2)}\right). \end{aligned}$$

Thus, all cases are satisfied when $p = o\left(\left(\frac{n\tau_n^2}{g_{0,p}^2}\right)^{1/(1+2\alpha_c+1/2)}\right)$. Let us assume the latter holds from now on.

Then by observing,

$$g_{0,p}^{-2}(n\tau_n^2)i^{-(1+2\alpha_c)} \geq 1 \iff i \leq (g_{0,p}^{-2}n\tau_n^2)^{1/(1+2\alpha_c)},$$

and letting $i_N = (g_{0,p}^{-2} n \tau_n^2)^{1/(1+2\alpha_c)}$, we can deduce

$$p = o\left(\left(\frac{n \tau_n^2}{g_{0,p}^2}\right)^{1/(1+2\alpha_c+1/2)}\right) = o(i_N).$$

Thus,

$$\begin{aligned} \|D_G^{-1} G^2 \eta_0\|^2 &= \sum_{i=1}^p \left[\left(\frac{1 + 2n^{-1} [g'(\eta_{0,i})]^2 + \lambda_i^{-1} n^{-1} g^2(\eta_{0,i})}{n^{-1} g^2(\eta_{0,i})} \right)^{-1/2} \lambda_i^{-1} \eta_{0,i} \right]^2 \\ &= \sum_{i=1}^p \left(\frac{n^{-1} g^2(\eta_{0,i}) \lambda_i^{-2} \eta_{0,i}^2}{1 + 2n^{-1} [g'(\eta_{0,i})]^2 + \lambda_i^{-1} n^{-1} g^2(\eta_{0,i})} \right) \\ &= \sum_{i=1}^p \left(\frac{n^{-1} \lambda_i^{-2} \eta_{0,i}^2}{g^{-2}(\eta_{0,i}) + 2n^{-1} [g'(\eta_{0,i})/g(\eta_{0,i})]^2 + \lambda_i^{-1} n^{-1}} \right) \\ &\leq C \sum_{i=1}^p \frac{n^{-1} \tau_n^{-4} i^{2+4\alpha_c} \eta_{0,i}^2}{g_{0,p}^{-2} + i^{1+2\alpha_c} (n \tau_n^2)^{-1}} \\ &\leq C n^{-1} \tau_n^{-4} \sum_{i=1}^p \frac{i^{2(1+2\alpha_c-\beta_c)} \eta_{0,i}^2 i^{2\beta_c}}{g_{0,p}^{-2} + i^{1+2\alpha_c} (n \tau_n^2)^{-1}} \\ &= C \tau_n^{-2} \sum_{i=1}^p \frac{i^{1+2\alpha_c-2\beta_c} \eta_{0,i}^2 i^{2\beta_c}}{g_{0,p}^{-2} (n \tau_n^2) i^{-(1+2\alpha_c)} + 1} \\ &\leq C \tau_n^{-2} \sum_{i=1}^{p \wedge i_N} \frac{i^{1+2\alpha_c-2\beta_c} \eta_{0,i}^2 i^{2\beta_c}}{g_{0,p}^{-2} (n \tau_n^2) i^{-(1+2\alpha_c)}} \\ &\leq C n^{-1} \tau_n^{-4} g_{0,p}^2 p^{2+4\alpha_c-2\beta_c \vee 0} \sum_{i=1}^{p \wedge i_N} \eta_{0,i}^2 i^{2\beta_c} \\ &\leq C n^{-1} \tau_n^{-4} g_{0,p}^2 p^{2+4\alpha_c-2\beta_c \vee 0} \|\eta_0^\infty\|_{S^{\beta_c}}^2. \end{aligned}$$

Subsequently, to show $\|D_G^{-1} G^2 \eta_0\|^2 = o(1)$ we need to consider 2 cases.

Case 1: Assume $2 + 4\alpha_c - 2\beta_c > 0$, then

$$\begin{aligned} \|D_G^{-1} G^2 \eta_0\|^2 = o(1), &\iff g_{0,p}^2 \tau_n^{-2} (n \tau_n^2)^{-1} p^{(2+4\alpha_c-2\beta_c)} = o(1) \\ &\iff p = o\left(\left(\frac{\tau_n^2 n \tau_n^2}{g_{0,p}^2}\right)^{1/(2+4\alpha_c-2\beta_c)}\right). \end{aligned}$$

Case 2: Assume $2 + 4\alpha_c - 2\beta_c \leq 0$, then

$$\|D_G^{-1} G^2 \eta_0\|^2 = o(1), \iff \left(\frac{n \tau_n^4}{g_{0,p}^2}\right)^{-1} = o(1).$$

However, by Assumption 5.2 $g_{0,p}^2 \asymp 1$, thus we require

$$n\tau_n^4 \rightarrow \infty.$$

Consequently, the conditions are as follows:

$$p = o\left(\left(\frac{n\tau_n^2}{g_{0,p}^2}\right)^{1/1+2\alpha_c+1/2}\right), \text{ and } \begin{cases} p = o\left(\left(\frac{\tau_n^2 n\tau_n^2}{g_{0,p}^2}\right)^{1/(2+4\alpha_c-2\beta_c)}\right), & \text{if } 2 + 4\alpha_c - 2\beta_c > 0, \\ n\tau_n^4 \rightarrow \infty, & \text{o/w.} \end{cases}$$

Furthermore,

$$\begin{aligned} \|D_0^{-1}G^2D_0^{-1}\| &\leq Cg_{0,p}^{-2}(n\tau_n^2)^{-1}p^{(1+2\alpha_c)} = \delta_G \\ \text{tr}(D_0^{-1}G^2D_0^{-1})^2 &\leq Cg_{0,p}^{-2}(n\tau_n^2)^{-2}p^{2(1+2\alpha_c)+1} = \delta_{tr}^2 \\ \|D_G^{-1}G^2\eta_0\|^2 &\leq Cg_{0,p}^{-2}\tau_n^{-2}(n\tau_n^2)^{-1}p^{(2+4\alpha_c-2\beta_c)\vee 0} = \delta_\beta^2, \end{aligned}$$

where all 3 terms are $o(1)$, given the conditions hold.

Note, due to Assumption 5.2, $g_{0,p}^2 \asymp 1$, and therefore can be viewed as negligible. Regardless, one could use the above derivations to investigate the effect of relaxing the local assumptions on $g(\cdot)$. Note, this would lead to additional (non-trivial) terms in the bounds that depend on $g^2(\eta_0)$ as well, and their asymptotics would need to be tracked.

5.9 Proof for Theorem 5.4

The outline for the proof is as follows:

1. We need to bound the tail posterior probability $\mathbb{P}(\{\eta \notin \Upsilon_0(r_0)|Y\})$.
2. This can be done by bounding $\rho^*(r_0)$ (with high probability) since $\mathbb{P}(\{\eta \notin \Upsilon_0(r_0)|Y\}) \leq \rho^*(r_0)$. Recall, $\rho^*(r_0)$ is a random quantity which describes the concentration properties of the posterior.
3. Note: [Panov and Spokoiny, 2015] provides a probability bound for $\rho^*(r_0)$, c.f. Section 4.3's [Theorem 10]. However this bound is only valid if Assumption L_r holds - and in our setting it does not. (See Section 5.8.9 for further discussion on Assumption L_r .)
4. Hence, we need to derive a new probability bound for $\rho^*(r_0)$, where recall

$$\rho^*(r_0) = \frac{\int_{\Upsilon \setminus \Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} \pi(\eta) d\eta}{\int_{\Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} \pi(\eta) d\eta},$$

5. First, we obtain this new bound for $\rho^*(r_0)$ under a uniform prior, and later we show how we can use this result to obtain a bound for $\rho^*(r_0)$ under a Normal prior.
6. Thus we need to consider $\exp\{L(\eta, \eta_0)\}$ over the set $\Upsilon \setminus \Upsilon_0(r_0)$ and $\Upsilon_0(r_0)$, for the numerator and denominator, respectively.
7. For the numerator term on $\Upsilon \setminus \Upsilon_0(r_0)$: We aim to bound $L(\eta, \eta_0)$ from above, with large probability. Next, we evaluate the relevant integral:

(a) Bounding $L(\eta, \eta_0)$ from above, with large probability:

- i. We begin by centering $L(\eta, \eta_0)$ i.e.

$$L(\eta, \eta_0) = \mathbb{E}L(\eta, \eta_0) + [L(\eta, \eta_0) - \mathbb{E}L(\eta, \eta_0)] = \mathbb{E}L(\eta, \eta_0) + \zeta(\eta, \eta_0).$$

Hence, we've split $L(\eta, \eta_0)$ into 2 terms: $\mathbb{E}L(\eta, \eta_0)$ - a deterministic term (which we will bound by DT, defined in (48)) and $\zeta(\eta, \eta_0)$ - a random term (which we will denote by RT). Consequently, we seek to bound DT and RT from above, (the latter with high probability).

- ii. We can bound DT by using Proposition 5.7 and Assumption 5.2, which for $r \geq r_0$ implies

$$\mathbb{E}L(\eta, \eta_0) \leq \frac{-b(r) \|D_0(\eta - \eta_0)\|^2}{2} = \frac{-b(r)r^2}{2} \leq -\frac{r^2}{2[1 + c_{D_{0p}}^r]},$$

where $D_{0p} := \min_{i \leq p} (D_0)_{i,i}$.

- iii. We can bound RT with high probability over the set $\Upsilon \setminus \Upsilon_0(r_0)$ using [Laurent and Massart, 2000].
- iv. Before we use the lemma however we need to show that its prerequisite condition (50) holds. Specifically, (50) requires us to bound $\log \mathbb{E} \exp(aX^2 + bX + c)$, for $X \sim N(0, 1)$. However, this condition is similar to Assumptions $ED0$ and $ED2$. Hence, we can use Lemma 5.8 and Corollary 5.8 to bound it, (just as we did for $ED0$ and $ED2$).
- v. Hence we can use the result from [Laurent and Massart, 2000], i.e. (51), to show

$$RT \leq c\sqrt{x}(r + \sqrt{3p}),$$

with large probability (at least $1 - e^{-x}$).

- vi. Thus the integrand of the numerator term in $\rho^*(r_0)$ will be bounded from above by $\exp(DT + RT) \leq \exp\left(-\frac{r^2}{2[1+c\frac{r}{D_{0p}}]} + \sqrt{x}[r + \sqrt{3p}]\right) := f(r, x)$.

(b) Evaluating the integral of $f(r, x)$ over $\Upsilon \setminus \Upsilon_0(r_0)$:

- i. We first change variables to obtain a sphere in p -dimensions, i.e.

$$x_i := (D_0)_{i,i}(\eta_i - \eta_{0,i}) \implies \Upsilon \setminus \Upsilon_0(r_0) = \{x : \|x\| \geq r_0\}.$$

Consequently, we can express x_i using spherical co-ordinates, and integrate with respect to dr . Furthermore, the integration will take place over the following interval: (r_0, ∞) . During this process, we will assume $r_0 \geq 4\sqrt{x}$ in order to simplify our bounds.

- ii. Finally, we will use the asymptotic bound for the Upper Incomplete Gamma Function, [Abramowitz et al., 1988] and Stirling's Formula to simplify the obtained bound on the numerator of $\rho^*(r_0)$.

8. For the denominator term on $\Upsilon_0(r_0)$:

- (a) On the set $\Upsilon_0(r_0)$, we've proved conditions (ED_0) , (ED_2) , (L_0) and (I) hold for some $r_0 > 0$. Thus,

$$|L(\eta, \eta_0) - \mathbb{L}(\eta, \eta_0)| \leq \Delta(r_0, x), \quad \eta \in \Upsilon_0(r_0),$$

on a random set $\Omega_{r_0}(x)$ with probability at least $1 - e^{-x}$, c.f. Section 4.3's [Theorem 9].

- (b) Thus for the denominator term in $\rho^*(r_0)$, we can use Remark 4.4 from Section 4, along with Equations (46) and (47) to bound it. Specifically, since $\mathbb{L}(\eta, \eta_0)$ is quadratic, $\exp\{\mathbb{L}(\eta, \eta_0)\}$ will be a normal density (up to some normalising constants and the error term $\Delta(r_0, x)$).

9. Hence, we obtain a bound on $\rho^*(r_0)$ with probability at least $1 - 3e^{-x}$, (c.f. (37)), under the assumption: $r_0 \geq 4\sqrt{x} \vee [z_B(x) + z(p, x)]$.

10. However this is a bound under a uniform prior, and as stated in Section 5.6.1, we need a bound under a normal prior.
11. Fortunately it was proved, in [Panov and Spokoiny, 2015]: Theorem 2 (c.f. [Panov and Spokoiny, 2015] (55) and Lemma 8 for more details), that the posterior distribution obtained under a flat Normal prior, $N(0, G^{-2})$, can be approximated via the uniform prior, if Assumption 4.1 holds.

Proving 7(a)i: Recall,

$$\begin{aligned}
L(\eta) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(\eta_i)) - \frac{n}{2} \sum_{i=1}^p \frac{(Y_i - \eta_i)^2}{g^2(\eta_i)}, \\
\mathbb{E}_{\eta_0} L(\eta) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(\eta_i)) - \frac{1}{2} \sum_{i=1}^p \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - \frac{n}{2} \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{g^2(\eta_i)}, \\
\mathbb{E}_{\eta_0} L(\eta_0) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(\eta_{0,i})) - \frac{p}{2}.
\end{aligned}$$

Let

$$X_i = \frac{Y_i - \eta_{0,i}}{n^{-1/2}g(\eta_{0,i})} \sim N(0, 1).$$

Then,

$$\begin{aligned}
L(\eta, \eta_0) &:= L(\eta) - L(\eta_0) \\
&= \left(\left[\frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(\eta_i)) - \frac{n}{2} \sum_{i=1}^p \frac{(Y_i - \eta_i)^2}{g^2(\eta_i)} \right] \right. \\
&\quad \left. - \left[\frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(\eta_{0,i})) - \frac{n}{2} \sum_{i=1}^p \frac{(Y_i - \eta_{0,i})^2}{g^2(\eta_{0,i})} \right] \right) \\
&= \left(\sum_{i=1}^p \log\left(\frac{g(\eta_{0,i})}{g(\eta_i)}\right) + \frac{1}{2} \sum_{i=1}^p X_i^2 - \frac{n}{2} \sum_{i=1}^p \frac{(Y_i - \eta_i)^2}{g^2(\eta_i)} \right).
\end{aligned}$$

Note,

$$\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - \log\left(\frac{g^2(\eta_{0,i})}{g^2(\eta_i)}\right) - 1 \geq 0,$$

for all $g^2(\eta_i)$, (c.f. C.5).

Furthermore,

$$\frac{(Y_i - \eta_i)^2}{n^{-1}g^2(\eta_{0,i})} = \frac{([Y_i - \eta_{0,i}] + [\eta_{0,i} - \eta_i])^2}{n^{-1}g^2(\eta_{0,i})} = X_i^2 + \frac{(\eta_{0,i} - \eta_i)^2}{n^{-1}g^2(\eta_{0,i})} + 2X_i \frac{(\eta_{0,i} - \eta_i)}{n^{-1/2}g(\eta_{0,i})}.$$

Therefore,

$$\begin{aligned}
L(\eta, \eta_0) &= - \left(\sum_{i=1}^p -\frac{1}{2} \log\left(\frac{g^2(\eta_{0,i})}{g^2(\eta_i)}\right) + \left[\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - 1\right] - \left[\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - 1\right] \right) \\
&\quad \left(\frac{1}{2} \sum_{i=1}^p X_i^2 - \sum_{i=1}^p \frac{g^2(\eta_{0,i})}{2g^2(\eta_i)} \left[X_i^2 + \frac{(\eta_{0,i} - \eta_i)^2}{n^{-1}g^2(\eta_{0,i})} + 2X_i \frac{(\eta_{0,i} - \eta_i)}{n^{-1/2}g(\eta_{0,i})} \right] \right) \\
&\leq \frac{1}{2} \sum_{i=1}^p \left[\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - 1 \right] - \frac{1}{2} \sum_{i=1}^p X_i^2 \left[\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - 1 \right] \\
&\quad - \frac{1}{2} \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{n^{-1}g^2(\eta_i)} - n^{1/2} \sum_{i=1}^p X_i (\eta_{0,i} - \eta_i) \frac{g(\eta_{0,i})}{g^2(\eta_i)} \\
&= -\frac{1}{2} \sum_{i=1}^p (X_i^2 - 1) \left[\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - 1 \right] \\
&\quad - \frac{1}{2} \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{n^{-1}g^2(\eta_i)} - n^{1/2} \sum_{i=1}^p X_i (\eta_{0,i} - \eta_i) \frac{g(\eta_{0,i})}{g^2(\eta_i)}.
\end{aligned}$$

Hence, we can set

$$DT := \frac{1}{2} \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{n^{-1}g^2(\eta_i)}. \quad (48)$$

Note, Section 5.8.9 corroborates that indeed

$$\mathbb{E}L(\eta, \eta_0) \leq DT.$$

In addition, we can set

$$RT := -\frac{1}{2} \sum_{i=1}^p (X_i^2 - 1) \left[\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - 1 \right] - n^{1/2} \sum_{i=1}^p X_i (\eta_{0,i} - \eta_i) \frac{g(\eta_{0,i})}{g^2(\eta_i)}. \quad (49)$$

□

Proving 7(a)iv. We wish to use the following result from [Laurent and Massart, 2000] (Pg 1326):

Lemma. *Let Z denote a random variable, $v > 0$, and $c \geq 0$. If*

$$\log(\mathbb{E}[e^{uZ}]) \leq \frac{vu^2}{2(1 - cu)}, \quad (50)$$

then for any positive x ,

$$P(Z \geq cx + \sqrt{2vx}) \leq e^{-x}. \quad (51)$$

Note that via Lemma 5.8 and Corollary 5.8, for $|a_i| \leq \frac{1}{4}$,

$$\begin{aligned} \log \mathbb{E} \exp(a_i X^2 + b_i X + c_i) &\leq 3a_i^2 + [c_i + a_i - \frac{b_i^2}{4a_i - 2}] \\ &= 3a_i^2 + [c_i + a_i + \frac{b_i^2}{2 - 4a_i}] \\ &\leq 3a_i^2 + b_i^2 + [c_i + a_i], \end{aligned}$$

where the last inequality follows from $|a_i| \leq \frac{1}{4}$.

Thus, by setting $c_i = -a_i$,

$$\begin{aligned} \log \mathbb{E} \exp(a_i(X^2 - 1) + b_i X) &= \log \mathbb{E} \exp(a_i X^2 + b_i X - a_i) \\ &\leq 3a_i^2 + b_i^2. \end{aligned}$$

Consequently, for $\varkappa|a_i| \leq \frac{1}{4}$,

$$\log \mathbb{E} \exp(\varkappa \sum_{i \leq p} a_i(X^2 - 1) + b_i X) \leq \varkappa^2 \sum_{i \leq p} 3a_i^2 + b_i^2, \quad (52)$$

since the X_i are independent.

Note, the coefficients of X_i in RT are as follows:

$$\begin{aligned} a_i &= \frac{1}{2} [1 - \frac{g^2(\eta_{0,i})}{g^2(\eta_i)}], \\ b_i &= n^{1/2} \frac{(\eta_i - \eta_{0,i})}{g(\eta_{0,i})} \frac{g^2(\eta_{0,i})}{g^2(\eta_i)}. \end{aligned}$$

Using Assumption 5.1,

$$\begin{aligned} |a_i| &\leq \frac{1}{2} [1 + |\frac{g^2(\eta_{0,i})}{g^2(\eta_i)}|] \leq \frac{1}{2} [1 + C_1], \\ |b_i| &= n^{1/2} \frac{|\eta_i - \eta_{0,i}|}{g(\eta_{0,i})} \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} \leq C_2(D_0)_{i,i} |\eta_i - \eta_{0,i}|. \end{aligned}$$

where C_1 and C_2 are independent of r , n and p .

Hence, setting $\varkappa = [2(1 + C_1)]^{-1}$, implies $\varkappa|a_i| \leq \frac{1}{4}$, and therefore (52) holds.

Additionally,

$$\varkappa^2 \sum_{i \leq p} 3a_i^2 + b_i^2 \leq c \frac{\varkappa^2}{2} (p + r^2),$$

where c_1 is independent of r , n and p .

Consequently, (52) implies

$$\log(\mathbb{E}[e^{\varkappa RT}]) \leq c_1 \frac{\varkappa^2}{2} (p + r^2).$$

□

Proving: $\gamma(a)v$. Consequently, we can use (51), with $Z = RT$, and $c = 0$, to show

$$P(RT < \sqrt{2vx}) = 1 - P(RT \geq \sqrt{2vx}) \geq 1 - e^{-x},$$

where

$$v = \sum_{i \leq p} 3a_i^2 + b_i^2 \leq c_1 \frac{(p + r^2)}{2}.$$

Note, for $A, B > 0$

$$\begin{aligned} v &\leq A^2 + B^2 \leq (A + B)^2 \\ \implies \sqrt{v} &\leq \sqrt{A^2 + B^2} \leq A + B. \end{aligned}$$

Thus,

$$RT < \sqrt{x}(r + \sqrt{p}),$$

with at least $1 - e^{-x}$.

□

Proving $\gamma(b)i$. Using the following substitution

$$x_i = (D_0)_{i,i}(\eta_i - \eta_{0,i}) \implies \Upsilon \setminus \Upsilon_0(r_0) = \{x : \|x\| \geq r_0\},$$

where

$$dx_i = (D_0)_{i,i} d\eta_i.$$

Furthermore, since D_0 is a diagonal matrix

$$\prod_{i=1}^p (D_0)_{i,i} = \det(D_0).$$

Subsequently, for $x \in \mathbb{R}^p$, we can represent cartesian co-ordinates as spherical co-ordinates using the following change of variables:

$$\begin{aligned} x_1 &= r \cos(\theta_1), \\ x_2 &= r \sin(\theta_1) \cos(\theta_2), \\ x_3 &= r \sin(\theta_1) \sin(\theta_2) \cos(\theta_3), \\ &\vdots \\ x_{p-1} &= r \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{p-2}) \cos(\theta_{p-1}), \\ x_p &= r \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{p-1}), \end{aligned}$$

where $\theta_1, \dots, \theta_{n-2} \in (0, \pi]$ and $\theta_{n-1} \in (0, 2\pi]$.

Consequently,

$$\int_{\Upsilon \setminus \Upsilon_0(r_0)} d\eta = \int_{\|x\| > r_0} \det(D_0^{-1}) dx = \int_{r > r_0} \det(D_0^{-1}) s(p) r^{p-1} dr, \quad (53)$$

where $\|x\|$ is the Euclidean norm, and $s(p)$ is the surface area of the p -dimensional unit sphere, i.e.

$$s(p) = \frac{2\pi^{p/2}}{\Gamma(p/2)}.$$

Using (53)

$$\begin{aligned} \int_{\Upsilon \setminus \Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} d\eta &= \int_{r>r_0} f(r, x) \det(D_0^{-1}) s(p) r^{p-1} dr \\ &= \int_{r>r_0} \exp\left(-\frac{r^2}{2[1+c\frac{r}{D_{0p}}]} + \sqrt{x}[r + \sqrt{3p}]\right) \det(D_0^{-1}) s(p) r^{p-1} dr \\ &= \det(D_0^{-1}) s(p) e^{\sqrt{3xp}} \int_{r>r_0} \exp\left(-\frac{r^2}{2[1+c\frac{r}{D_{0p}}]} + \sqrt{x}r\right) r^{p-1} dr. \end{aligned}$$

Recall, we require $\|rD_0^{-1}\| = o(1)$ (for $r \leq r_0$) in order for Assumption L_0 to hold, (c.f. Proposition 5.4). Thus, for $D_{0p} := \min_{i \leq p} (D_0)_{i,i}$, we must have that $r_0 \leq D_{0p}$.

Therefore, for $r \geq r_0$,

$$\begin{aligned} -\frac{r^2}{2[1+c\frac{r}{D_{0p}}]} &\leq -\frac{1}{2} \min\{r^2, c^{-1}rD_{0p}\} \\ &\leq -\frac{r}{2} \min\{r, c^{-1}D_{0p}\} \\ &\leq -\frac{r}{2} \min\{r_0, c^{-1}D_{0p}\} \\ &\leq -\frac{r_0}{2}r. \end{aligned}$$

Assume,

$$r_0 \geq 4\sqrt{x}.$$

Then, this implies

$$\frac{r_0}{4} - \sqrt{x} \geq 0 \iff \frac{r_0}{4} \left(+\frac{r_0}{4} - \frac{r_0}{4}\right) - \sqrt{x} \geq 0 \iff \frac{r_0}{2} - \sqrt{x} \geq \frac{r_0}{4}.$$

Hence,

$$\exp\left(-\frac{r^2}{2[1+c\frac{r}{D_{0p}}]} + \sqrt{x}r\right) \leq \exp\left(-\frac{r_0}{2}r + \sqrt{x}r\right) \leq \exp\left(-r\left(\frac{r_0}{2} - \sqrt{x}\right)\right) \leq \exp\left(-r\left(\frac{r_0}{4}\right)\right).$$

Thus,

$$\int_{\Upsilon \setminus \Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} d\eta \leq \det(D_0^{-1}) s(p) e^{\sqrt{3xp}} \int_{r>r_0} e^{-\frac{r_0}{4}r} r^{p-1} dr.$$

Using the following substitution,

$$x = \frac{r_0}{4}r \implies dx = \frac{r_0}{4} dr,$$

leads to

$$\begin{aligned} \int_{\Upsilon \setminus \Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} d\eta &\leq \det(D_0^{-1}) s(p) e^{\sqrt{3xp}} \left(\frac{4}{r_0}\right)^p \int_{x > \frac{r_0^2}{4}} e^{-x} x^{p-1} dx \\ &\leq \det(D_0^{-1}) s(p) e^{\sqrt{3xp}} \left(\frac{4}{r_0}\right)^p \Gamma(p, \frac{r_0^2}{4}), \end{aligned}$$

where $\Gamma(\cdot, \cdot)$ refers to the Upper Incomplete Gamma Function, which is defined as follows:

$$\Gamma(p, a) := \int_{x>a} e^{-x} x^{p-1} dx.$$

□

Proving 7(b)ii. We use the following result from [Abramowitz et al., 1988] (c.f. Equation 6.5.32) to bound $\Gamma(\cdot, \cdot)$: For $z \in \mathbb{C}$

$$\Gamma(a, z) \sim z^{a-1} e^{-z} \left[1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} \dots \right], \quad (54)$$

as $z \rightarrow \infty$ in $|\arg z| < \frac{3\pi}{2}$.

In our case $z \in \mathbb{R}$, therefore $|\arg z| = 0$ will indeed be less than $\frac{3\pi}{2}$, and hence

$$\Gamma(a, z) \leq C z^{a-1} e^{-z},$$

for large z .

Therefore,

$$\begin{aligned} \int_{\Upsilon \setminus \Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} d\eta &\leq C \det(D_0^{-1}) s(p) e^{\sqrt{3xp}} \left(\frac{4}{r_0}\right)^p \left(\frac{r_0^2}{4}\right)^{p-1} e^{-\frac{r_0^2}{4}} \\ &= C \det(D_0^{-1}) \frac{2\pi^{p/2}}{\Gamma(p/2)} 4r_0^{p-2} e^{-\frac{r_0^2}{4} + \sqrt{3xp}}. \end{aligned}$$

Finally, recall Stirling's Formula, (which holds for large p):

$$p! \sim \sqrt{2\pi p} \left(\frac{p}{e}\right)^p.$$

Thus,

$$\begin{aligned} \int_{\Upsilon \setminus \Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} d\eta &\leq C_1 \det(D_0^{-1}) 2\pi^{p/2} [\sqrt{2\pi p/2} \left(\frac{p/2}{e}\right)^{p/2}]^{-1} 4r_0^{p-2} e^{-\frac{r_0^2}{4} + \sqrt{3xp}} \\ &\leq C_2 \det(D_0^{-1}) 8\pi^{(p-1)/2} [p^{(p+1)/2} \left(\frac{1}{2e}\right)^{p/2}]^{-1} r_0^{p-2} e^{-\frac{r_0^2}{4} + \sqrt{3xp}} \\ &= C_2 \det(D_0^{-1}) 2^{p/2} 8\pi^{(p-1)/2} e^{-\frac{r_0^2}{4} + \sqrt{3xp} + \frac{p}{2} - \frac{p+1}{2} \log p + (p-2) \log r_0}. \end{aligned}$$

□

Proving 8b. Thus for the denominator term in $\rho^*(r_0)$, we can use Remark 4.4 from Section 4, along with Equations (46) and (47) to bound it. Specifically, inside $\Upsilon_0(r_0)$, on $\Omega(x)$ with probability at least $1 - e^{-x}$,

$$\begin{aligned} \int_{\Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} d\eta &\geq \exp(-\Delta(r_0, x) - m(\xi)) \int_{\Upsilon_0(r_0)} \exp(\mathbb{L}(\eta, \eta_0) + m(\xi)) d\eta \\ &\geq \exp(-\Delta(r_0, x) - m(\xi) - \nu(r_0)), \end{aligned}$$

where

$$\begin{aligned} \nu(r_0) &:= -\log(\mathbb{P}(\|\mathcal{Z} + \xi\| \leq r_0 | Y)), \\ m(\xi) &:= m(\xi) := -\frac{\|\xi\|^2}{2} + \log(\det D_0) - p \log(\sqrt{2\pi}). \end{aligned}$$

Recall, $\xi = D_0^{-1} \nabla L(\eta_0)$ fulfils on a random set $\Omega_B(x)$ of dominating probability at least $1 - 2e^{-x}$

$$\|\xi\|^2 \leq z_B^2(x).$$

Hence, assume

$$r_0 \geq z_B(x) + z(p, x),$$

where

$$\begin{aligned} z_B^2(x) &:= p_B + 6\lambda_B x, \\ \text{with } B &:= D_0^{-1} \Sigma_0^2 D_0^{-1}, \quad p_B := \text{tr}(B), \quad \lambda_B := \lambda_{\max}(B). \\ z^2(p, x) &:= p + [\sqrt{6.6px} \vee (6.6x)]. \end{aligned}$$

Then using [Panov and Spokoiny, 2015]'s Lemma 7, on a random set $\Omega_B(x)$ of dominating probability at least $1 - 3e^{-x}$,

$$\begin{aligned} \exp(m(\xi)) &= \exp(-\|\xi\|^2/2)(2\pi)^{-p/2} \det(D_0) \\ &\leq (2\pi)^{-p/2} \det(D_0), \\ \nu(r_0) &= -\log(\mathbb{P}(\|\mathcal{Z} + \xi\| \leq r_0 | Y)) \\ &\leq -\log(\mathbb{P}(\|\mathcal{Z}\| + \|\xi\| \leq r_0 | Y)) \\ &\leq -\log(\mathbb{P}(\|\mathcal{Z}\| \leq z(p, x) | Y)) \\ &\leq -\log[1 - (\mathbb{P}(\|\mathcal{Z}\| > z(p, x) | Y))] \\ &\leq 2e^{-x}, \end{aligned}$$

where the latter follows from

$$-\log(1 - a) \leq 2a.$$

Thus,

$$\begin{aligned} \left[\int_{\Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} d\eta \right]^{-1} &\leq \exp(\Delta(r_0, x) + m(\xi) + \nu(r_0)) \\ &\leq (2\pi)^{-p/2} \det(D_0) \exp(\Delta(r_0, x) + 2e^{-x}). \end{aligned}$$

□

Proving 9. Hence, with probability at least $1 - 3e^{-x}$ and assuming $r_0 \geq 4\sqrt{x} \vee [z_B(x) + z(p, x)]$

$$\begin{aligned}\rho^*(r_0) &\leq C(2\pi)^{-p/2} \det(D_0) \exp(\Delta(r_0, x) + 2e^{-x}) \\ &\quad \det(D_0^{-1}) 2^{p/2} 8\pi^{(p-1)/2} e^{-\frac{r_0^2}{4} + \sqrt{3xp} + \frac{p}{2} - \frac{p+1}{2} \log p + (p-2) \log r_0} \\ &= C 8\pi^{-1/2} e^{-\frac{r_0^2}{4} + \sqrt{3xp} + \frac{p}{2} - \frac{p+1}{2} \log p + (p-2) \log r_0 + \Delta(r_0, x) + 2e^{-x}}.\end{aligned}$$

□

5.9.1 Proof of Corollary 5.4

Let $\mu_0 = (\mu_{0,1}, \dots, \mu_{0,p})$ and $\mu = (\mu_1, \dots, \mu_p)$, where $Y_i \sim N(k_i \mu_i, n^{-1} g^2(k_i \mu_i))$. Consequently,

$$\begin{aligned}f(Y_i | \mu_i) &= \frac{1}{\sqrt{2\pi n^{-1} g^2(k_i \mu_i)}} \exp\left(-\frac{(Y_i - k_i \mu_i)^2}{2n^{-1} g^2(k_i \mu_i)}\right), \\ L(\mu) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(k_i \mu_i)) - \frac{n}{2} \sum_{i=1}^p \frac{(Y_i - k_i \mu_i)^2}{g^2(k_i \mu_i)}, \\ L(\mu_0) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(k_i \mu_{0,i})) - \frac{n}{2} \sum_{i=1}^p \frac{(Y_i - k_i \mu_{0,i})^2}{g^2(k_i \mu_{0,i})}, \\ \mathbb{E}_{\mu_0} L(\mu) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(k_i \mu_i)) - \frac{1}{2} \sum_{i=1}^p \frac{g^2(k_i \mu_{0,i})}{g^2(k_i \mu_i)} - \frac{n}{2} \sum_{i=1}^p \frac{(k_i \mu_{0,i} - k_i \mu_i)^2}{g^2(k_i \mu_i)}, \\ \mathbb{E}_{\mu_0} L(\mu_0) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(k_i \mu_{0,i})) - \frac{p}{2}.\end{aligned}$$

Let

$$X_i = \frac{Y_i - k_i \mu_{0,i}}{n^{-1/2} g(k_i \mu_{0,i})} \sim N(0, 1).$$

Then,

$$\begin{aligned}L(\mu, \mu_0) &:= L(\mu) - L(\mu_0) \\ &= \left(\sum_{i=1}^p \log\left(\frac{g(k_i \mu_{0,i})}{g(k_i \mu_i)}\right) + \frac{1}{2} \sum_{i=1}^p X_i^2 - \frac{n}{2} \sum_{i=1}^p \frac{(Y_i - k_i \mu_i)^2}{g^2(k_i \mu_i)} \right)\end{aligned}$$

Note,

$$\frac{g^2(k_i \mu_{0,i})}{g^2(k_i \mu_i)} - \log\left(\frac{g^2(k_i \mu_{0,i})}{g^2(k_i \mu_i)}\right) - 1 \geq 0,$$

for all $g^2(\eta_i)$, (c.f. C.5).

Furthermore,

$$\frac{(Y_i - k_i \mu_i)^2}{n^{-1} g^2(k_i \mu_{0,i})} = \frac{([Y_i - k_i \mu_{0,i}] + [k_i \mu_{0,i} - k_i \mu_i])^2}{n^{-1} g^2(k_i \mu_{0,i})} = X_i^2 + \frac{(k_i \mu_{0,i} - k_i \mu_i)^2}{n^{-1} g^2(k_i \mu_{0,i})} + 2X_i \frac{(k_i \mu_{0,i} - k_i \mu_i)}{n^{-1/2} g(k_i \mu_{0,i})}.$$

Thus,

$$\begin{aligned} L(\mu, \mu_0) &\leq -\frac{1}{2} \sum_{i=1}^p (X_i^2 - 1) \left[\frac{g^2(k_i \mu_{0,i})}{g^2(k_i \mu_i)} - 1 \right] \\ &\quad - \frac{1}{2} \sum_{i=1}^p \frac{(k_i \mu_{0,i} - k_i \mu_i)^2}{n^{-1} g^2(k_i \mu_i)} - n^{1/2} \sum_{i=1}^p X_i (k_i \mu_{0,i} - k_i \mu_i) \frac{g(k_i \mu_{0,i})}{g^2(k_i \mu_i)}. \end{aligned}$$

Hence, we can set

$$\begin{aligned} DT &:= \frac{1}{2} \sum_{i=1}^p \frac{k_i^2 (\mu_{0,i} - \mu_i)^2}{n^{-1} g^2(k_i \mu_i)}, \\ RT &:= -\frac{1}{2} \sum_{i=1}^p (X_i^2 - 1) \left[\frac{g^2(k_i \mu_{0,i})}{g^2(k_i \mu_i)} - 1 \right] - n^{1/2} \sum_{i=1}^p X_i k_i (\mu_{0,i} - \mu_i) \frac{g(k_i \mu_{0,i})}{g^2(k_i \mu_i)}. \end{aligned}$$

Recall, the indirect setting and the definition of \tilde{D}_0 can be found in Section 5.3. Subsequently, we can bound DT by using Proposition 5.7 and Assumption 5.2. Note, that

$$\frac{-2\mathbb{E}L(\mu, \mu_0)}{\|\tilde{D}_0(\mu - \mu_0)\|^2} \geq \sum_{i=1}^p \frac{(k_i \mu_{0,i} - k_i \mu_i)^2}{2n^{-1} g^2(k_i \mu_i) \|\tilde{D}_0(\mu - \mu_0)\|^2} = \sum_{i=1}^p \frac{k_i^2 (D_0^2)_{i,i} (\mu_{0,i} - \mu_i)^2}{\|\tilde{D}_0(\mu - \mu_0)\|^2} [n^{-1} g^2(k_i \mu_i) (D_0^2)_{i,i}]^{-1},$$

remains unchanged from the direct setting. However, we need to apply the Mean Value Theorem (for the indirect case) when deriving $b(r)$, specifically, when bounding $[n^{-1} g^2(k_i \mu_i) (D_0^2)_{i,i}]^{-1}$ from below. Thus, we obtain:

$$\begin{aligned} g^2(k_i \mu_i) &= g^2(k_i \mu_{0,i}) + (\mu_i - \mu_{0,i}) [g^2(k_i \mu_{ci})]' k_i \\ &\leq g^2(k_i \mu_{0,i}) + |k_i (\mu_i - \mu_{0,i})| |[g^2]'(k_i \mu_{ci})| \\ &\leq g^2(k_i \mu_{0,i}) \cdot \left(1 + \frac{k_i r}{(\tilde{D}_0)_{i,i}} c\right). \end{aligned}$$

Therefore,

$$\mathbb{E}L(\mu, \mu_0) \leq \frac{-b(r) \|\tilde{D}_0(\mu - \mu_0)\|^2}{2} = \frac{-b(r) r^2}{2} \leq -\frac{r^2}{2} \min_{i \leq p} \left[1 + c \frac{k_i r}{(\tilde{D}_0)_{i,i}}\right]^{-1},$$

where

$$\tilde{D}_{0p} := \min_{i \leq p} (\tilde{D}_0)_{i,i} \asymp p^{-q} n^{1/2}.$$

Subsequently, we proceed to bound RT. Note, the coefficients of X_i in RT are as follows:

$$a_i = \frac{1}{2} \left[1 - \frac{g^2(k_i \mu_{0,i})}{g^2(k_i \mu_i)} \right],$$

$$b_i = n^{1/2} \frac{k_i(\mu_{0,i} - \mu_i)}{g(k_i \mu_{0,i})} \frac{g^2(k_i \mu_{0,i})}{g^2(k_i \mu_i)}.$$

Using Assumption 5.1,

$$|a_i| \leq \frac{1}{2} \left[1 + \left| \frac{g^2(k_i \mu_{0,i})}{g^2(k_i \mu_i)} \right| \right] \leq \frac{1}{2} [1 + C_1],$$

$$|b_i| = n^{1/2} \frac{k_i |\mu_{0,i} - \mu_i|}{g(k_i \mu_{0,i})} \frac{g^2(k_i \mu_{0,i})}{g^2(k_i \mu_i)} \leq C_2 k_i (D_0)_{i,i} |\mu_{0,i} - \mu_i|$$

$$\leq C_2 (\tilde{D}_0)_{i,i} |\mu_{0,i} - \mu_i|,$$

where C_1 and C_2 are independent of r , n and p .

Hence, setting $\varkappa = [2(1 + C_1)]^{-1}$, implies $\varkappa |a_i| \leq \frac{1}{4}$, and therefore (52) holds.

Additionally,

$$\varkappa^2 \sum_{i \leq p} 3a_i^2 + b_i^2 \leq c \frac{\varkappa^2}{2} (p + r^2),$$

where c_1 is independent of r , n and p .

Consequently, (52) implies

$$\log(\mathbb{E}[e^{\varkappa RT}]) \leq c_1 \frac{\varkappa^2}{2} (p + r^2).$$

Thus,

$$RT < \sqrt{x}(r + \sqrt{p}),$$

with probability at least $1 - e^{-x}$.

Subsequently, the integrand of the numerator term in $\rho^*(r_0)$ will be bounded from above by $\exp(DT + RT) \leq \exp\left(-\frac{r^2}{2[1+c\frac{r}{\tilde{D}_{0p}}]} + \sqrt{x}[r + \sqrt{3p}]\right) := f(r, x)$.

Summarizing for the indirect case: The bound for DT has changed, while the RT remains unaffected.

Using the following substitution

$$x_i = (\tilde{D}_0)_{i,i}(\mu_i - \mu_{0,i}) \implies \tilde{\Upsilon} \setminus \tilde{\Upsilon}_0(r_0) = \{x : \|x\| \geq r_0\},$$

where

$$dx_i = (\tilde{D}_0)_{i,i} d\mu_i,$$

implies

$$\int_{\tilde{\Upsilon} \setminus \tilde{\Upsilon}_0(r_0)} \exp\{L(\mu, \mu_0)\} d\mu = \det(\tilde{D}_0^{-1}) s(p) e^{\sqrt{3xp}} \int_{r > r_0} \exp\left(-\frac{r^2}{2[1+c\frac{r}{\tilde{D}_{0p}}]} + \sqrt{x}r\right) r^{p-1} dr.$$

Recall, for Assumption L_0 we need to bound

$$\begin{aligned} \max_{i \leq p} |(D_0^{-2})_{i,i} (D_0^2(\eta))_{i,i} - 1| &\equiv \max_{i \leq p} |([k_i D_0]^{-2})_{i,i} (k_i^2 D_0^2(\eta))_{i,i} - 1| \\ &\equiv \max_{i \leq p} |(\tilde{D}_0^{-2})_{i,i} (\tilde{D}_0^2(\eta))_{i,i} - 1|. \end{aligned}$$

Thus we need $\|r \tilde{D}_0^{-1}\| = o(1)$ (for $r \leq r_0$) in order for Assumption L_0 to hold, (c.f. Proposition 5.4). Thus, for $D_{0p} := \min_{i \leq p} (\tilde{D}_0)_{i,i}$, we must have that $r_0 \leq c \tilde{D}_{0,p}$ for any $c > 0$, (assuming n is large enough).

Therefore, for $r \geq r_0$,

$$\begin{aligned} -\frac{r^2}{2} \min_{i \leq p} [1 + c \frac{k_i r}{(\tilde{D}_0)_{i,i}}]^{-1} &= -\frac{r^2}{2} \frac{1}{\max_{i \leq p} [1 + c \frac{k_i r}{(\tilde{D}_0)_{i,i}}]} = -\frac{r}{2} \frac{1}{\max_{i \leq p} [r^{-1} + c \frac{k_i}{(\tilde{D}_0)_{i,i}}]} \\ &\leq -\frac{r}{2} \frac{1}{\max_{i \leq p} [r^{-1}, c \frac{k_i}{(\tilde{D}_0)_{i,i}}]} = -\frac{r}{2} \min_{i \leq p} [r, c^{-1} \frac{(\tilde{D}_0)_{i,i}}{k_i}] \\ &\leq -\frac{r}{2} \min_{i \leq p} [r_0, c^{-1} \frac{(\tilde{D}_0)_{i,i}}{k_i}] \\ &\leq -\frac{r}{2} \min_{i \leq p} [r_0, c^{-1} (\tilde{D}_0)_{i,i}] = -\frac{r}{2} \min\{r_0, c^{-1} \tilde{D}_{0,p}\} \\ &\leq -\frac{r_0}{2} r, \end{aligned}$$

since $k_i = i^{-q} \geq 1$ for all i .

Consequently, for $\|r \tilde{D}_0^{-1}\| = o(1)$,

$$\int_{\Upsilon \setminus \Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} d\eta \leq \det(\tilde{D}_0^{-1}) s(p) e^{\sqrt{3xp}} \left(\frac{4}{r_0}\right)^p \Gamma(p, \frac{r_0^2}{4}).$$

Similarly, for the denominator, noting the change of variables,

$$\begin{aligned} \int_{\Upsilon_0(r_0)} \exp\{L(\eta, \eta_0)\} d\eta &\geq \exp(-\Delta(r_0, x) - \tilde{m}(\xi)) \int_{\Upsilon_0(r_0)} \exp(\mathbb{L}(\eta, \eta_0) + \tilde{m}(\xi)) d\mu \\ &\geq \exp(-\Delta(r_0, x) - \tilde{m}(\xi) - \nu(r_0)), \end{aligned}$$

where

$$\begin{aligned} \nu(r_0) &:= -\log(\mathbb{P}(\|\mathcal{Z} + \xi\| \leq r_0 | Y)) \\ \tilde{m}(\xi) &:= -\frac{\|\xi\|^2}{2} + \log(\det \tilde{D}_0) - p \log(\sqrt{2\pi}). \end{aligned}$$

Hence, with probability at least $1 - 3e^{-x}$ and assuming $r_0 \geq 4\sqrt{x} \vee [z_B(x) + z(p, x)]$,

$$\rho^*(r_0) \leq C 8\pi^{-1/2} e^{-\frac{r_0^2}{4} + \sqrt{3xp} + \frac{p}{2} - \frac{p+1}{2} \log p + (p-2) \log r_0 + \Delta(r_0, x) + 2e^{-x}}.$$

In conclusion, $\rho^*(r_0)$ is of the same form as that in the direct setting.

5.10 Proof of Theorem 5.1

Proof. Recall, the pertinent terms from Section 4.3's [Theorem 2] are

$$\begin{aligned}\Delta(r_0, x) &:= (\delta(r_0) + 6\nu_0 z_{\mathbb{H}}(x)\omega)r_0^2, \\ z_{\mathbb{H}}(x) &:= 2p^{1/2} + \sqrt{2x} + \aleph^{-1}(\aleph^{-2}x + 1)4p, \\ \varrho &:= \frac{1}{2}[(1 + \delta_G)(3\delta_\beta + \delta_G z_B(x))^2 + \delta_{tr}^2]^{1/2}.\end{aligned}$$

The terms ν_0, ω and \aleph come from Assumptions ED_0 and ED_2 . The term, ν_0 , is present in both of the aforementioned assumptions; however, as shown in their respective sections, both are asymptotically a constant. In Assumption ED_2 , this is achieved by choosing ω^2 appropriately. Hence, $\nu_0^2 \asymp 1$ in both assumptions.

The term $\delta(r_0)$ and the condition on r_0 come from Assumption L_0 ; the latter will rely on the asymptotic bound for D_0^2 derived in Section 5.8.3.

Thus, we have

$$\begin{aligned}\aleph &= (4H)^{-1} \asymp \left[\max_{1 \leq i \leq p} \left| \frac{1}{(\Sigma_0)_{i,i}} \frac{g'(\eta_{0,i})}{g(\eta_{0,i})} \right| \right]^{-1} \asymp [n^{-1/2}]^{-1} \\ &= n^{1/2}, \\ z_{\mathbb{H}}(x) &= 2p^{1/2} + \sqrt{2x} + \aleph^{-1}(\aleph^{-2}x + 1)4p \\ &\asymp p^{1/2} + x^{1/2} + (n^{-1}x + 1)4pn^{-1/2} \\ &\asymp p^{1/2} + x^{1/2}, \\ \omega &\asymp n^{-1/2}, \\ \nu_0^2 &\asymp 1, \\ r_0^2 &= o(n), \\ \delta(r_0) &= o(1), \\ \Delta(r_0, x) &= (\delta(r_0) + 6\nu_0 z_{\mathbb{H}}(x)\omega)r_0^2.\end{aligned}$$

Furthermore, (as derived in Section 5.8.10), when

$$p = o\left(\left(\frac{n\tau_n^2}{g_{0,p}^2}\right)^{1/1+2\alpha_c+1/2}\right), \text{ and } \begin{cases} p = o\left(\left(\frac{\tau_n^2 n \tau_n^2}{g_{0,p}^2}\right)^{1/(2+4\alpha_c-2\beta_c)}\right), & \text{if } 2 + 4\alpha_c - 2\beta_c > 0, \\ n\tau_n^4 \rightarrow \infty, & \text{o/w,} \end{cases}$$

the terms δ_G, δ_{tr} , and δ_β are $o(1)$, where

$$\begin{aligned}\|D_0^{-1}G^2D_0^{-1}\| &\leq \delta_G = Cg_{0,p}^{-2}(n\tau_n^2)^{-1}p^{(1+2\alpha_c)}, \\ \text{tr}(D_0^{-1}G^2D_0^{-1})^2 &\leq \delta_{tr}^2 = Cg_{0,p}^{-2}(n\tau_n^2)^{-2}p^{2(1+2\alpha_c)+1}, \\ \|D_G^{-1}G^2\eta_0\|^2 &\leq \delta_\beta^2 = Cg_{0,p}^{-2}\tau_n^{-2}(n\tau_n^2)^{-1}p^{(2+4\alpha_c-2\beta_c) \vee 0}.\end{aligned}$$

The terms in ϱ relate to the matrices Σ_0 and G . Specifically, $z_B^2(x) := p_B + 6\lambda_B x$, with

$$B := D_0^{-1} \Sigma_0^2 D_0^{-1}, \quad p_B := \text{tr}(B), \quad \lambda_B := \lambda_{\max}(B).$$

In our case $B = I$, therefore $p_B = p$ and $\lambda_B = 1$.

Consequently,

$$\begin{aligned} z_B(x) &= [p_B + 6\lambda_B x]^{1/2} \\ &\asymp [p + x]^{1/2}, \\ \varrho &= \frac{1}{2} [(1 + \delta_G)(3\delta_\beta + \delta_G z_B(x))^2 + \delta_{tr}^2]^{1/2} \\ &\asymp [1 \cdot (\delta_\beta^2 + \delta_G^2 z_B^2(x)) + \delta_{tr}^2]^{1/2} = [(\delta_\beta^2 + \delta_G^2 z_B^2(x)) + \delta_{tr}^2]^{1/2} \\ &\asymp [\delta_\beta^2 + \delta_G^2 [p + x]]^{1/2}. \end{aligned}$$

Note $\delta_G^2 p \asymp \delta_{tr}^2$, and the latter is $o(1)$ (given the conditions on p stated above). Hence, $\varrho = o(1)$.

If, $2 + 4\alpha_c - 2\beta_c > 0$,

$$\begin{aligned} \frac{\delta_\beta}{\delta_G} &= \frac{\tau_n^{-2} (n\tau_n^2)^{-1} p^{1+2\alpha_c+1+2[\alpha_c-\beta_c]}}{(n\tau_n^2)^{-1} p^{(1+2\alpha_c)}} \\ &= \tau_n^{-2} p^{1+2[\alpha_c-\beta_c]}. \end{aligned}$$

Else, if $2 + 4\alpha_c - 2\beta_c \leq 0$,

$$\begin{aligned} \frac{\delta_\beta}{\delta_G} &= \frac{\tau_n^{-2} (n\tau_n^2)^{-1}}{(n\tau_n^2)^{-1} p^{(1+2\alpha_c)}} \\ &= \tau_n^{-2} p^{-(1+2\alpha_c)}. \end{aligned}$$

Therefore,

$$\varrho^2 \asymp g_{0,p}^{-4} (n\tau_n^2)^{-2} p^{2(1+2\alpha_c)} \begin{cases} (\tau_n^{-4} p^{2+4[\alpha_c-\beta_c]} + p + x), & \text{if } 2 + 4\alpha_c - 2\beta_c > 0, \\ (\tau_n^{-4} p^{-2(1+2\alpha_c)} + p + x), & \text{if } 2 + 4\alpha_c - 2\beta_c \leq 0. \end{cases}$$

Note, (as stated in Section 5.8.10), Assumption 5.2 implies $g_{0,p}^{-2} \asymp 1$, and therefore its negligible. Regardless, one could use the above derivations to investigate the effect of relaxing the aforementioned assumption. This would lead to additional (non-trivial) terms in the bounds that depend on $g^2(\eta_0)$ as well.

For $\eta \notin \Upsilon_0(r_0)$: We will use Theorem 5.4, which contains the following terms:

$$\begin{aligned} z(p, x) &= [p + \sqrt{6.6px} \vee (6.6x)]^{1/2} \\ &\asymp [p + x]^{1/2}, \\ r_0^2 &\geq [\sqrt{x} \vee z_B(x) + z(p, x)]^2 \\ &= c(p + x). \end{aligned}$$

Hence, for simplicity assume $x \asymp p$, which implies $r_0^2 \geq Cp$.

□

5.10.1 Proof for Corollary 5.2

Let us derive the conditions needed in order for these probability bounds to be meaningful. By construction $\varrho = o(1)$, for $p = o((\frac{n\tau_n^2}{g_{0,p}^2})^{1/(1+2\alpha_c+1/2)} \wedge (\frac{\tau_n^2 n\tau_n^2}{g_{0,p}^2})^{1/(3+4\alpha_c-2\beta_c)})$.

Furthermore, wrt $\Delta(r_0, x)$, (and assuming large enough p)

$$\begin{aligned}\Delta(r_0, x) &\asymp (p^{1/2} + x^{1/2})\omega r_0^2 \\ &\asymp p^{1/2}\omega r_0^2.\end{aligned}$$

Hence, since $r_0^2 \geq Cp$, we can deduce

$$\Delta(r_0, x) = o(1) \iff p^{1/2}\omega r_0^2 = o(1) \iff \frac{p^3}{n} = o(1).$$

Finally, for $\rho^*(r_0) = o(1)$, (defined in Theorem 5.4), it suffices to show that

$$-\frac{r_0^2}{4} + \sqrt{xp} + \frac{p}{2} - \frac{p}{2} \log p + p \log r_0 \rightarrow -\infty$$

and $\Delta(r_0, x) = o(1)$. The latter condition already holds therefore let us focus our attention on proving the former condition.

Recall, we assumed $x \asymp p$, therefore $x \leq c_2 p$. Furthermore, we assumed $r_0^2 \geq c_1 p$, thus set $r_0^2 = c_1 p$. Hence,

$$\begin{aligned}&-\frac{r_0^2}{4} + \sqrt{xp} + \frac{p}{2} - \frac{p}{2} \log p + p \log r_0 \\ &\leq -\frac{c_1 p}{4} + c_2 p + \frac{p}{2} - \frac{p}{2} \log \left[\frac{p}{r_0^2} \right] \\ &\leq -\frac{p}{4} (c_1 - 4c_2 - 2 + 2 \log \left(\frac{1}{c_1} \right)) \\ &= -\frac{p}{4} (c_1 - 4c_2 - 2 - 2 \log(c_1)).\end{aligned}$$

Thus, since x grows faster than $2 \log(x)$, choosing c_1 sufficiently large ensures $\rho^*(r_0) = o(1)$.

5.11 Proof of Theorem 5.3

5.11.1 Preliminaries

Recall, in the indirect case, we assume $\eta_i := k_i \mu_i$, with $k_i \asymp i^{-q}$, $\mu \in \mathbb{R}^p$, and $\mu_0^\infty \in S^\beta$, where the Sobolev space,

$$S^\beta = \{h : \|h\|_{S^\beta}^2 < \infty\}, \text{ where } \|h\|_{S^\beta}^2 = \sum_{i=1}^{\infty} h_i^2 i^{2\beta}.$$

Furthermore, we set a prior on $\mu_i \sim N(0, \tilde{\lambda}_i)$, where $\tilde{\lambda}_i = \tau_n^2 i^{-(1+2\alpha)}$ and $n\tau_n^2 \rightarrow \infty$, with $\alpha > 0$. In addition, by comparing the indirect and direct settings, we see that $\beta_c = \beta + q$, and $\alpha_c = \alpha + q$.

Consequently, if $\eta \in \Upsilon_0(r_0)$, then

$$\begin{aligned} \sum_{i=1}^p (D_0^2)_{i,i} (\eta_i - \eta_{0,i})^2 \leq r_0^2 &\implies \sum_{i=1}^p (D_0^2)_{i,i} k_i^2 (\mu_i - \mu_{0,i})^2 \leq r_0^2 \\ &\implies \sum_{i=1}^p (\tilde{D}_0^2)_{i,i} (\mu_i - \mu_{0,i})^2 \leq r_0^2. \end{aligned}$$

Therefore, if $\eta \in \Upsilon_0(r_0)$, then $\mu \in \tilde{\Upsilon}_0(r_0)$, where

$$(\tilde{D}_0^2)_{i,i} := (D_0^2)_{i,i} k_i^2, \text{ and } \tilde{\Upsilon}_0(r) := \{\mu : \|\tilde{D}_0(\mu - \mu_0)\| \leq r\}.$$

Furthermore, $\nabla_{\mu_i}(\cdot) = k_i \nabla_{\eta_i}(\cdot)$. Thus,

$$\begin{aligned} \eta_i^0 &= \eta_{0,i} + (D_0^2)_{i,i}^{-1} (\nabla_{\eta} L(\eta_0))_i \\ &= k_i \mu_{0,i} + k_i^2 (k_i^2 D_0^2)_{i,i}^{-1} k_i^{-1} (\nabla_{\mu} L(\eta_0))_i \\ &= k_i [\mu_{0,i} + (\tilde{D}_0^2)_{i,i}^{-1} (\nabla_{\mu} L(\mu_0))_i] \\ &= k_i \mu_i^0, \end{aligned}$$

where

$$\mu_i^0 = \mu_{0,i} + (\tilde{D}_0^2)_{i,i}^{-1} (\nabla_{\mu} L(\mu_0))_i.$$

Subsequently,

$$D_0(\eta - \eta^0) = \tilde{D}_0(\mu - \mu^0).$$

Thus, we now state the proof.

5.11.2 Proof

Plugging in our expressions for α_c, β_c and \tilde{D}_0 into the terms from Section 4.3's [Theorem 2] lead to:

$$\begin{aligned}\aleph &= (4H)^{-1} \asymp \left[\max_{1 \leq i \leq p} \left| \frac{1}{(\tilde{D}_0)_{i,i}} \frac{g'(\eta_{0,i})}{g(\eta_{0,i})} \right| \right]^{-1} \asymp \left[\max_{1 \leq i \leq p} i^q n^{-1/2} \right]^{-1} \\ &= p^{-q} n^{1/2}, \\ z_{\mathbb{H}}(x) &= 2p^{1/2} + \sqrt{2x} + \aleph^{-1}(\aleph^{-2}x + 1)4p \\ &\asymp p^{1/2} + x^{1/2} + (p^{2q}n^{-1}x + 1)4pp^q n^{-1/2} \\ &\asymp p^{1/2} + x^{1/2} + (p^{2q}n^{-1}x + 1)4p^{q+1}n^{-1/2}.\end{aligned}$$

Note, Assumption ED_2 depends on

$$(\nabla_{\mu}^2 \zeta(\eta))_{i,i} = (\nabla_{\eta}^2 \zeta(\eta))_{i,i} \cdot (\nabla_{\mu} \eta)^2 = (\nabla_{\eta}^2 \zeta(\eta))_{i,i} \cdot (k_i)^2.$$

Hence,

$$\begin{aligned}b_i &\asymp n^{1/2} k_i^2 \implies \\ \omega^{-2} \sup_{i \leq p} \frac{b_i^2}{(\tilde{D}_0^4)_{i,i}} &\asymp \omega^{-2} \sup_{i \leq p} \frac{(n^{1/2} k_i^2)^2}{(k_i^2 n)^2} = \omega^{-2} n^{-1} \asymp 1 \implies \nu_0^2 \asymp 1,\end{aligned}$$

where b_i and ω are derived in Section 5.8.7.

Therefore,

$$\begin{aligned}\omega^2 &\asymp n^{-1} \implies \\ \nu_0^2 &\asymp 1.\end{aligned}$$

Furthermore,

$$\|r_0 \tilde{D}_0^{-1}\| = o(1) \iff \frac{r_0 p^q}{n^{1/2}} = o(1) \iff r_0^2 = o(p^{-2q}n).$$

Consequently,

$$\begin{aligned}\delta(r_0) &= o(1), \\ \Delta(r_0, x) &= (\delta(r_0) + 6\nu_0 z_{\mathbb{H}}(x)\omega)r_0^2 \\ &\asymp z_{\mathbb{H}}(x)\omega r_0^2 \\ &\asymp \left[p^{1/2} + x^{1/2} + (p^{2q}n^{-1}x + 1)4p^{q+1}n^{-1/2} \right] n^{-1/2} r_0^2.\end{aligned}$$

Furthermore, for

$$\begin{aligned} z_B(x) &= [p_B + 6\lambda_B x]^{1/2} \\ &\asymp [p + x]^{1/2}, \end{aligned}$$

and

$$p = o((n\tau_n^2)^{1/1+2[\alpha+q]+1/2}), \text{ and } \begin{cases} p = o((\tau_n^2 n \tau_n^2)^{1/(2+2[\alpha+q]+2[\alpha-\beta])}), & \text{if } 2 + 2[\alpha + q] + 2[\alpha - \beta] > 0, \\ n\tau_n^4 \rightarrow \infty, & \text{o/w,} \end{cases}$$

we know (just as in the direct case) $\varrho^2 = o(1)$, where

$$\varrho^2 \asymp (n\tau_n^2)^{-2} p^{2(1+2[\alpha+q])} \begin{cases} (\tau_n^{-4} p^{2+4[\alpha-\beta]} + p + x), & \text{if } 2 + 2[\alpha + q] + 2[\alpha - \beta] > 0, \\ (\tau_n^{-4} p^{-2(1+2[\alpha+q])} + p + x), & \text{if } 2 + 2[\alpha + q] + 2[\alpha - \beta] \leq 0. \end{cases}$$

Finally, for $\eta \notin \Upsilon_0(r_0)$ (and equivalently for $\mu \notin \tilde{\Upsilon}_0(r_0)$), we use Corollary 5.4. Note, the assumption in the corollary will hold, since we've already assumed $\|r_0 \tilde{D}_0^{-1}\| = o(1)$.

Thus, we conclude the proof by noting that

$$z(p, x) \asymp [p + x]^{1/2}.$$

5.12 Proof of Corollary 5.3

We derive the conditions needed in order for the bounds in Theorem 5.3 to be meaningful. Naturally we require $x \leq Cp$, and $x \rightarrow \infty$, therefore setting $x = \log n$ seems reasonable.

Hence, we need show that for $x = \log n$ and $cp \leq r_0^2 \leq o(1)p^{-2q}n$: $\varrho^2 = o(1)$ and $\Delta(r_0, x) = o(1)$. Consequently, set $r_0^2 = cp$. Note, this would imply

$$p = o(n^{1/(2q+1)}).$$

Recall, by definition $\varrho^2 = o(1)$, if

$$p = o((n\tau_n^2)^{1/1+2[\alpha+q]+1/2}), \text{ and } \begin{cases} p = o((\tau_n^2 n \tau_n^2)^{1/(2+2[\alpha+q]+2[\alpha-\beta])}), & \text{if } 2 + 2[\alpha + q] + 2[\alpha - \beta] > 0, \\ n\tau_n^4 \rightarrow \infty, & \text{o/w.} \end{cases}$$

Furthermore,

$$\begin{aligned} \Delta(r_0, x) &\asymp \left[p^{1/2} + x^{1/2} + (p^{2q} n^{-1} x + 1) 4p^{q+1} n^{-1/2} \right] n^{-1/2} r_0^2 \\ &\asymp \left[\left[\frac{p}{n} \right]^{1/2} + \left[\frac{x}{n} \right]^{1/2} + \frac{p^{3q+1} x^{1/2}}{n^{3/2}} \frac{x^{1/2}}{n^{1/2}} + \frac{p^{q+1}}{n} \right] r_0^2 \\ &\asymp \left[\frac{p^{3/2}}{n^{1/2}} + \frac{p^{3q+2} x}{n^2} + \frac{p^{q+2}}{n} \right]. \end{aligned}$$

Hence, $\Delta(r_0, x) = o(1)$ if the following holds:

$$p = o(n^{1/3}), \quad p = o\left(\left[\frac{n^2}{\log n}\right]^{1/(3q+2)}\right), \quad \text{and} \quad p = o(n^{1/(q+2)}).$$

Additionally, for the tail posterior probability, (c.f. Corollary 5.4):
if $\Delta(r_0, x) = o(1)$, $\varrho^2 = o(1)$ and c is chosen sufficiently large, then $\mathbb{P}(\{\eta \notin \Upsilon_0(r_0)|Y\}) = o(1)$; just as in Corollary 5.2, albeit with a different upper bound on r_0 .

5.13 Preliminaries for Proof of Contraction Rate Theorems in Section 5.6

We can bound each of the terms in Corollary 5.5 using Propositions 5.8, 5.9 and 5.10.

Proposition 5.8 (Sum 1). *For the direct case,*

$$\sum_{i=1}^p (D_0^{-2})_{i,i} = \mathcal{O}(n^{-1}p).$$

For the indirect case, assuming $k_i \asymp i^{-q}$, where $q > 0$, and using Lemma 5.10,

$$\sum_{i=1}^p k_i^{-2} (D_0^{-2})_{i,i} = \mathcal{O}(n^{-1}p^{2q+1}).$$

Proof. For the direct case, recall $(D_0^2)_{i,i} \asymp n$ as shown in Section 5.8.3. Hence,

$$\sum_{i=1}^p (D_0^{-2})_{i,i} \asymp n^{-1} \sum_{i=1}^p 1 = \mathcal{O}(n^{-1}p).$$

For the indirect case, since $q > 0$,

$$\begin{aligned} \sum_{i=1}^p k_i^{-2} (D_0^{-2})_{i,i} &\asymp \sum_{i=1}^p i^{2q} \frac{n^{-1}g^2(\eta_{0,i})}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} = \sum_{i=1}^p i^{2q} \frac{4g^4(\eta_{0,i})}{4ng^2(\eta_{0,i}) + 2a^2} \\ &\leq \sum_{i=1}^p i^{2q} \frac{4g^4(\eta_{0,i})}{4ng^2(\eta_{0,i})} = n^{-1} \sum_{i=1}^p i^{2q} g^2(\eta_{0,i}) \\ &= \mathcal{O}(n^{-1}p^{2q+1}), \end{aligned}$$

since Assumption 5.2 implies $g^2(\eta_{0,i}) \asymp 1$. □

Proposition 5.9 (Sum 2). *Given $A_i = \frac{g'(\eta_{0,i})}{g(\eta_{0,i})(1+2n^{-1}[g'(\eta_{0,i})]^2)}$, one can show*

$$A_i^2 2n^{-2} g^4(\eta_{0,i}) = \mathcal{O}(n^{-2}).$$

For the direct case,

$$\sum_{i=1}^p A_i^2 2n^{-2} g^4(\eta_{0,i}) = \mathcal{O}(n^{-2}p).$$

For the indirect case, assuming $k_i \asymp i^{-q}$, where $q > 0$, and using Lemma 5.10,

$$\sum_{i=1}^p k_i^{-2} A_i^2 2n^{-2} g^4(\eta_{0,i}) = \mathcal{O}(n^{-2}p^{2q+1}).$$

Proof. Note,

$$\begin{aligned}
A_i^2 2n^{-2} g^4(\eta_{0,i}) &= \left[\frac{g'(\eta_{0,i})}{g(\eta_{0,i})(1 + 2n^{-1}[g'(\eta_{0,i})]^2)} \right]^2 2n^{-2} g^4(\eta_{0,i}) \\
&= \frac{2}{n^2} \left[\frac{g'(\eta_{0,i})g(\eta_{0,i})}{(1 + 2n^{-1}[g'(\eta_{0,i})]^2)} \right]^2 = \frac{a^2}{2n^2} \left[\frac{4ng^2(\eta_{0,i})}{4ng^2(\eta_{0,i}) + 2a^2} \right]^2 \\
&\leq \frac{a^2}{2n^2} \left[\frac{4ng^2(\eta_{0,i})}{4ng^2(\eta_{0,i})} \right]^2 = \mathcal{O}(n^{-2}).
\end{aligned}$$

Hence, using Lemma 5.10,

$$\begin{aligned}
\sum_{i=1}^p A_i^2 2n^{-2} g^4(\eta_{0,i}) &\leq n^{-2} \sum_{i=1}^p 1 = \frac{n^{-2}p}{2q+1} (1 + o(1)), \\
\sum_{i=1}^p k_i^{-2} A_i^2 2n^{-2} g^4(\eta_{0,i}) &\leq n^{-2} \sum_{i=1}^p i^{2q} = \frac{n^{-2}p^{2q+1}}{2q+1} (1 + o(1)).
\end{aligned}$$

□

Proposition 5.10 (Sum 3). *Let $B_i := \frac{1}{1+2n^{-1}[g'(\eta_{0,i})]^2}$, then*

$$B_i^2 n^{-1} g^2(\eta_{0,i}) = \mathcal{O}(c_0 n^{-1}).$$

For the direct case,

$$\sum_{i=1}^p B_i^2 n^{-1} g^2(\eta_{0,i}) = \mathcal{O}(n^{-1}p).$$

Furthermore, for the indirect case, assuming $k_i \asymp i^{-q}$ where $q > 0$, and using Lemma 5.10 imply

$$\sum_{i=1}^p k_i^{-2} B_i^2 n^{-1} g^2(\eta_{0,i}) = \mathcal{O}(n^{-1}p^{2q+1}).$$

Proof. Note,

$$\begin{aligned}
B_i^2 n^{-1} g^2(\eta_{0,i}) &= \frac{n^{-1} g^2(\eta_{0,i})}{[1 + 2n^{-1}[g'(\eta_{0,i})]^2]^2} = \frac{n^{-1} g^2(\eta_{0,i}) [4ng^2(\eta_{0,i})]^2}{[4ng^2(\eta_{0,i}) + 2a^2]^2} \\
&\leq \frac{n^{-1} g^2(\eta_{0,i}) [4ng^2(\eta_{0,i})]^2}{[4ng^2(\eta_{0,i})]^2} = \mathcal{O}(n^{-1}),
\end{aligned}$$

using Assumption 5.2.

Therefore for the direct case,

$$\sum_{i=1}^p B_i^2 n^{-1} g^2(\eta_{0,i}) \asymp n^{-1} \sum_{i=1}^p 1 = \mathcal{O}(n^{-1}p).$$

Hence, assuming $k_i \asymp i^{-q}$, where $q > 0$, and using Lemma 5.10 imply

$$\sum_{i=1}^p k_i^{-2} B_i^2 n^{-1} g^2(\eta_{0,i}) \asymp n^{-1} \sum_{i=1}^p i^{2q} = \frac{n^{-1} p^{2q+1}}{2q+1} (1 + o(1)).$$

□

The following lemma is used to bound the terms in Propositions 5.8, 5.9 and 5.10.

Lemma 5.10. *Partial sums of the sequence $\{i^k\}_{k=1}^\infty$, where $k \in \mathbb{R}$, can be approximated as follows:*

$$\sum_{i=1}^N i^\kappa = \frac{N^{\kappa+1}}{\kappa+1} (1 + o(1)), \quad \text{as } N \rightarrow \infty, \quad \text{if } \kappa > -1.$$

$$\sum_{i=1}^N i^\kappa = \ln N (1 + o(1)), \quad \text{as } N \rightarrow \infty, \quad \text{if } \kappa = -1.$$

$$\sum_{i=1}^N i^{-\kappa} = \frac{1}{\kappa-1} (1 + o(1)), \quad \text{as } N \rightarrow \infty, \quad \text{if } \kappa > 1.$$

5.14 Preliminaries for Proof of Theorem 5.2

5.14.1 Proof outline

We begin by confirming the global and local assumptions regarding $g(\cdot)$ hold. Hence, we derive the likelihood and its derivatives for our model. Subsequently, we verify the assumptions from Section 4.2 in order to use the paper's theorems, stated verbatim in 4.3.

Specifically, we use [Panov and Spokoiny, 2015]'s Theorem 2 to obtain local bounds on the posterior, and [Panov and Spokoiny, 2015]'s Theorem 9 to obtain our own bounds on the tail posterior probability, c.f. Theorem 5.4. Finally, we present the proof in Section 5.15.

5.14.2 Verifying Assumptions 5.1 and 5.2

Recall, $g^2(\eta_i) = a\eta_i + b$, where $\eta_i, a, b > 0$. Furthermore, $\eta_0 = (\eta_{0,1}, \dots, \eta_{0,p})$ and $\eta = (\eta_1, \dots, \eta_p)$, with $Y_i \sim N(\eta_i, n^{-1}g^2(\eta_i))$. Hence, let us confirm the global and local assumptions regarding $g(\cdot)$ hold.

- Assumption 5.1 is satisfied, if for all $i \leq p$: $\eta_i > -\frac{b}{a} + c$, for some $c > 0$.
- Assumption 5.2 is satisfied, since $g(\cdot)$ is bounded over compact sets and monotonic increasing.

Lemma 5.11. *The derivatives of $g(\eta_i)$ can be expressed using $g(\eta_i)$ itself, i.e.*

$$g'(\eta_i) = \frac{a}{2g(\eta_i)}, \quad \text{and} \quad g''(\eta_i) = -\frac{a^2}{4g^3(\eta_i)}.$$

Corollary 5.9. *The constants defined in Assumption 5.2 exist, and are as follows:*

$$\begin{aligned} m_{r,0,u,i}^2 &= a(\eta_{0,i} + r(D_0^{-1})_{i,i}) + b, & m_{r,0,l,i}^2 &= a(\eta_{0,i} - r(D_0^{-1})_{i,i}) + b, \\ m_{r,1,u,i} &= \frac{a}{2m_{r,0,l,i}}, & m_{r,1,l,i} &= \frac{a}{2m_{r,0,u,i}}, \\ m_{r,2,u,i} &= \frac{a^2}{4m_{r,0,l,i}^3}, & m_{r,2,l,i} &= \frac{a^2}{4m_{r,0,u,i}^3}. \end{aligned}$$

Proof. Note, $g \rightarrow g^2$ is bijective and strictly increasing, therefore $(\max g)^2 = \max g^2$. Additionally, the derivatives of g can be expressed using g itself, c.f. Lemma 5.11.

Hence, the maximum and minimum of our functions over the set $\{\eta_i : (\eta_i - \eta_{0,i})^2 \leq r^2(D_0^{-2})_{i,i}\}$ will lie on the boundary, and note D is easy to manipulate since it's a diagonal matrix.

□

5.14.3 Likelihood Derivations

In this section we show that η_0 is the unique maximum of $\mathbb{E}_{\eta_0} L(\eta)$, and $\nabla \mathbb{E}_{\eta_0} L(\eta) = 0$ at η_0 .

If needed, the likelihood and its derivatives are detailed below, along with the uniqueness' proof which follows from Assumption 5.1 and C.6.

Specifically,

$$\begin{aligned} f(Y_i|\eta_i) &= \frac{1}{\sqrt{2\pi n^{-1}(a\eta_i + b)}} \exp\left(-\frac{(Y_i - \eta_i)^2}{2n^{-1}(a\eta_i + b)}\right), \\ L(\eta) = L(Y|\eta) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \frac{1}{2} \sum_{i=1}^p \log((a\eta_i + b)) - \sum_{i=1}^p \frac{(Y_i - \eta_i)^2}{2n^{-1}(a\eta_i + b)}, \\ \mathbb{E}_{\eta_0} L(\eta) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \frac{n}{2} \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{(a\eta_i + b)} - \frac{1}{2} \sum_{i=1}^p \log(a\eta_i + b) - \frac{1}{2} \sum_{i=1}^p \frac{(a\eta_{0,i} + b)}{(a\eta_i + b)}. \end{aligned}$$

Subsequently,

$$\begin{aligned} \frac{\partial}{\partial \eta_i} L(\eta) &= -\frac{a}{2(a\eta_i + b)} + \frac{(Y_i - \eta_i)}{n^{-1}(a\eta_i + b)} + \frac{a(Y_i - \eta_i)^2}{n^{-1}(a\eta_i + b)^2}, \\ \frac{\partial}{\partial \eta_i} \mathbb{E}_{\eta_0} L(\eta) &= \frac{(\eta_{0,i} - \eta_i)}{n^{-1}(a\eta_i + b)} + \frac{a(\eta_{0,i} - \eta_i)^2}{2n^{-1}(a\eta_i + b)^2} + \frac{a^2(\eta_{0,i} - \eta_i)}{2(a\eta_i + b)^2}. \end{aligned}$$

The function, $g(\eta_i)$, satisfies the condition stated in C.6. Thus, η_0 is the unique maximum of $\mathbb{E}_{\eta_0} L(\eta)$, and $\nabla \mathbb{E}_{\eta_0} L(\eta) = 0$ at η_0 .

5.14.4 Fisher Information Matrix: D_0^2

In this section we show that the Fisher Information Matrix, i.e. $D_0^2 := -\nabla^2 \mathbb{E} L(\eta_0)$, is a positive, symmetric diagonal matrix, where

$$\begin{aligned} (D_0^2)_{i,i} &= \frac{2ng^2(\eta_{0,i}) + a^2}{2g^4(\eta_{0,i})} \\ &= \frac{2n(a\eta_{0,i} + b) + a^2}{2(a\eta_{0,i} + b)^2} \\ &\asymp n, \\ (D_0^2)_{i,j} &= 0. \end{aligned}$$

If needed, the intermediary results (along with the proofs) are stated below:

$$\begin{aligned} \frac{\partial}{\partial \eta_i} \mathbb{E}_{\eta_0} L(\eta) &= \frac{(\eta_{0,i} - \eta_i)}{n^{-1}(a\eta_i + b)} + \frac{a(\eta_{0,i} - \eta_i)^2}{2n^{-1}(a\eta_i + b)^2} + \frac{a^2(\eta_{0,i} - \eta_i)}{2(a\eta_i + b)^2}, \\ \frac{\partial^2}{\partial \eta_i \partial \eta_j} \mathbb{E}_{\eta_0} L(\eta) &= 0, \\ (D_0^2)_{i,j} &:= -\frac{\partial^2}{\partial \eta_i \partial \eta_j} \mathbb{E}_{\eta_0} L(\eta_0) = 0. \end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{\partial^2}{\partial \eta_i^2} \mathbb{E}_{\eta_0} L(\eta) &= \frac{(\eta_{0,i} - \eta_i)}{n^{-1}g^2(\eta_i)} + \frac{(\eta_{0,i} - \eta_i)^2 g'(\eta_i)}{n^{-1}g^3(\eta_i)} + \frac{g'(\eta_i)}{g^3(\eta_i)} (g^2(\eta_{0,i}) - g^2(\eta_i)) \\
&= -\frac{1}{n^{-1}(a\eta_i + b)} - \frac{a(\eta_{0,i} - \eta_i)}{n^{-1}(a\eta_i + b)^2} - \frac{a(\eta_{0,i} - \eta_i)}{n^{-1}(a\eta_i + b)^2} \\
&\quad - \frac{a^2(\eta_{0,i} - \eta_i)^2}{n^{-1}(a\eta_i + b)^3} - \frac{a^2}{2(a\eta_i + b)^2} - \frac{a^3(\eta_{0,i} - \eta_i)}{(a\eta_i + b)^3}, \\
(D_0^2)_{i,i} &:= -\frac{\partial^2}{\partial \eta_i^2} \mathbb{E}_{\eta_0} L(\eta_0) = \frac{1}{n^{-1}(a\eta_{0,i} + b)} + \frac{a^2}{2(a\eta_{0,i} + b)^2} \\
&= \frac{2n(a\eta_{0,i} + b) + a^2}{2(a\eta_{0,i} + b)^2} \\
&\asymp n.
\end{aligned}$$

5.14.5 Likelihood's Stochastics: $\nabla \zeta(\eta_0)$

In this section we discuss the stochastic part of the Likelihood. Recall, $\zeta(\eta) := L(\eta) - \mathbb{E}_{\eta_0} L(\eta)$, and in this case $(\nabla \zeta(\eta_0))_i$ are independent (zero mean) random variables, whose covariance matrix (Σ_0^2) is a positive-symmetric, diagonal matrix. Specifically,

$$\begin{aligned}
\nabla \zeta(\eta_0) &:= \nabla L(\eta_0) - \nabla \mathbb{E}_{\eta_0} L(\eta_0) = \nabla L(\eta_0), \\
\mathbb{E}_{\eta_0}(\nabla \zeta(\eta_0))_i &= 0, \\
\text{Cov}((\nabla \zeta(\eta_0))_i, (\nabla \zeta(\eta_0))_j) &:= (\Sigma_0^2)_{i,j} = 0, \\
\text{Var}((\nabla \zeta(\eta_0))_i) &:= (\Sigma_0^2)_{i,i} = \frac{2n(a\eta_{0,i} + b) + a^2}{2(a\eta_{0,i} + b)^2}.
\end{aligned}$$

If needed, the intermediary results, along with the proofs, are stated below:

$$\begin{aligned}
\zeta(\eta) &= L(\eta) - \mathbb{E}_{\eta_0} L(\eta), \\
(\nabla \zeta(\eta))_i &= \frac{Y_i - \eta_{0,i}}{n^{-1}(a\eta_i + b)} + \frac{a}{n^{-1}(a\eta_i + b)^{3/2}} [(Y_i - \eta_i)^2 - (\eta_{0,i} - \eta_i)^2 - n^{-1}(a\eta_{0,i} + b)], \\
(\nabla \zeta(\eta_0))_i &= \frac{Y_i - \eta_{0,i}}{n^{-1}(a\eta_{0,i} + b)} + \frac{a}{n^{-1}(a\eta_{0,i} + b)^{3/2}} [(Y_i - \eta_{0,i})^2 - n^{-1}(a\eta_{0,i} + b)].
\end{aligned}$$

Note, in our model $\nabla \zeta(\eta_0) = \nabla L(\eta_0) - \nabla \mathbb{E}_{\eta_0} L(\eta_0) = \nabla L(\eta_0)$, since $\nabla \mathbb{E}_{\eta_0} L(\eta_0) = 0$. Furthermore,

$$\mathbb{E}_{\eta_0}(\nabla \zeta(\eta_0))_i = \frac{\mathbb{E}_{\eta_0}(Y_i) - \eta_{0,i}}{n^{-1}g^2(\eta_{0,i})} + \frac{g'(\eta_{0,i})}{n^{-1}g^3(\eta_{0,i})} [\mathbb{E}_{\eta_0}(Y_i - \eta_{0,i})^2 - n^{-1}g^2(\eta_{0,i})] = 0,$$

since $Y_i | \eta_{0,i} \sim N(\eta_{0,i}, n^{-1}g^2(\eta_{0,i}))$.

Denote $(\Sigma_0^2)_{i,j}^2 = \text{Cov}((\nabla \zeta(\eta_0))_i, (\nabla \zeta(\eta_0))_j)$, where $\Sigma_0^2 \in \mathbb{R}^{p \times p}$. Since the Y_i are independent, $\text{Cov}((\nabla \zeta(\eta_0))_i, (\nabla \zeta(\eta_0))_j) = 0$.

Furthermore, the results from C.10 (with $X = Y_i - \eta_{0,i}$, $\sigma^2 = n^{-1}g^2(\eta_{0,i})$, $\tilde{a} = \frac{1}{n^{-1}g^2(\eta_{0,i})}$ and $\tilde{b} = \frac{g'(\eta_{0,i})}{n^{-1}g^3(\eta_{0,i})}$) imply

$$\begin{aligned}\text{Var}((\nabla\zeta(\eta_0))_i) &= \tilde{a}^2\sigma^2 + 2\tilde{b}^2\sigma^4 = \frac{1}{n^{-1}g^2(\eta_{0,i})} + 2\left[\frac{g'(\eta_{0,i})}{g(\eta_{0,i})}\right]^2 \\ &= \frac{1}{n^{-1}(a\eta_{0,i} + b)} + \frac{a^2}{2(a\eta_{0,i} + b)^2} = \frac{2n(a\eta_{0,i} + b) + a^2}{2(a\eta_{0,i} + b)^2}.\end{aligned}$$

Therefore, Σ_0^2 is a positive-symmetric, diagonal matrix and $(\nabla\zeta(\eta_0))_i$ are independent random variables.

5.14.6 Local Neighbourhoods: $\Upsilon_0(r_0)$

In this section we show that the local conditions regarding $L(\eta)$ hold for $\eta \in \Upsilon_0(r_0)$. Subsequently, we obtain the following bound for $\eta \in \Upsilon_0(r_0)$,

$$(\eta_i - \eta_{0,i})^2 \leq \frac{r_0^2}{(D_0^2)_{i,i}} = r_0^2 \frac{2(a\eta_{0,i} + b)^2}{2n(a\eta_{0,i} + b) + a^2}. \quad (55)$$

If needed, the intermediary results, along with the proofs, are stated below.

Recall,

$$\Upsilon_0(r_0) := \{\eta \in \Upsilon : \|D_0(\eta - \eta_0)\| \leq r_0\},$$

where $D_0^2 = -\nabla^2 \mathbb{E}L(\eta_0)$. Observe,

$$\begin{aligned}\|D_0(\eta - \eta_0)\|^2 &= \sum_{i=1}^p (D_0^2)_{i,i} (\eta_i - \eta_{0,i})^2, \quad \text{and} \\ (D_0^2)_{i,i} (\eta_i - \eta_{0,i})^2 &= \frac{2(a\eta_{0,i} + b) + n^{-1}a^2}{2n^{-1}(a\eta_{0,i} + b)^2} (\eta_i - \eta_{0,i})^2.\end{aligned}$$

Furthermore, for $\eta \in \Upsilon_0(r_0)$,

$$(\eta_i - \eta_{0,i})^2 \leq \frac{r_0^2}{(D_0^2)_{i,i}} = r_0^2 \frac{2(a\eta_{0,i} + b)^2}{2n(a\eta_{0,i} + b) + a^2}. \quad (56)$$

5.14.7 Assumption ED_0

In this section we derive the terms stated in Proposition 5.2.

Observe,

$$\begin{aligned}H &= \max_{1 \leq i \leq p} \left| \frac{1}{(\Sigma_0)_{i,i}} \frac{g'(\eta_{0,i})}{g(\eta_{0,i})} \right| = \max_{1 \leq i \leq p} \left| \frac{1}{(\Sigma_0)_{i,i}} \frac{a}{2g^2(\eta_{0,i})} \right| \\ &= \left(\frac{2g^4(\eta_{0,i})}{2ng^2(\eta_{0,i}) + a^2} \right)^{1/2} \frac{a}{2g^2(\eta_{0,i})} \\ &= \frac{\sqrt{2}a}{(2ng^2(\eta_{0,i}) + a^2)^{1/2}} \\ &\asymp n^{-1/2},\end{aligned}$$

where the penultimate expression is bounded using Assumption 5.2.

Thus

$$\aleph = (4H)^{-1} \asymp n^{1/2}.$$

Hence, the terms in Assumption ED_0 are:

$$\begin{aligned}\nu_0 &\asymp 1, \\ \aleph &\asymp n^{1/2}.\end{aligned}$$

5.14.8 Assumption ED_2

Corollary 5.10. *Using the results from Section 5.8.7, we show*

$$\sup_{i \leq p} \sup_{\eta_i \in \Upsilon(r)} \left(\frac{a_i^2 + b_i^2}{(D_0^4)_{i,i}} \right) = \mathcal{O}(n^{-1}).$$

Consequently, setting

$$\omega^2 n \asymp 1,$$

implies $\nu_0 \asymp 1$.

Furthermore,

$$|\varkappa| \leq \aleph(r) := \omega(4H)^{-1} \asymp n^{1/2}.$$

Proof. Recall,

$$g^2(\eta_i) = a\eta_i + b, \quad g'(\eta_i) = \frac{a}{2g(\eta_i)}, \quad g''(\eta_i) = -\frac{a^2}{4g^3(\eta_i)},$$

where $\eta_i, a, b > 0$.

Hence,

$$\begin{aligned}
a_i &= n^{-1}g^2(\eta_{0,i})\left[\frac{g''(\eta_i)}{n^{-1}g^3(\eta_i)} - \frac{3[g'(\eta_i)]^2}{n^{-1}g^4(\eta_i)}\right] \\
&= n^{-1}g^2(\eta_{0,i})\left[\frac{-a^2}{4n^{-1}g^6(\eta_i)} - \frac{3a^2}{4n^{-1}g^6(\eta_i)}\right] \\
&= g^2(\eta_{0,i})\left[-\frac{a^2}{g^6(\eta_i)}\right] \\
&= -a^2\left[\frac{g^2(\eta_{0,i})}{g^6(\eta_i)}\right], \\
b_i &= n^{-1/2}g(\eta_{0,i})\left[-\frac{4g'(\eta_i)}{n^{-1}g^3(\eta_i)} + 2(\eta_{0,i} - \eta_i)\frac{a_i}{n^{-1}g^2(\eta_{0,i})}\right], \\
&= n^{-1/2}g(\eta_{0,i})\left[-\frac{2a}{n^{-1}g^4(\eta_i)} + 2(\eta_{0,i} - \eta_i)\frac{-a^2}{n^{-1}g^6(\eta_i)}\right], \\
&= -n^{1/2}\frac{g(\eta_{0,i})}{g^6(\eta_i)}[2ag^2(\eta_i) + 2a^2(\eta_{0,i} - \eta_i)], \\
&= -n^{1/2}\frac{g(\eta_{0,i})}{g^6(\eta_i)}[2a[g^2(\eta_i) - g^2(\eta_{0,i}) + g^2(\eta_{0,i})] + 2a^2(\eta_{0,i} - \eta_i)], \\
&= -n^{1/2}\frac{g(\eta_{0,i})}{g^6(\eta_i)}[2a[a(\eta_i - \eta_{0,i}) + g^2(\eta_{0,i})] + 2a^2(\eta_{0,i} - \eta_i)], \\
&= -n^{1/2}\frac{g(\eta_{0,i})}{g^6(\eta_i)}[2ag^2(\eta_{0,i})], \\
&= -2an^{1/2}\frac{g^3(\eta_{0,i})}{g^6(\eta_i)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
a_i^2 + b_i^2 &= a^4\left[\frac{g^4(\eta_{0,i})}{g^{12}(\eta_i)}\right] + 4a^2n\frac{g^6(\eta_{0,i})}{g^{12}(\eta_i)} \\
&= \frac{a^2g^4(\eta_{0,i})}{g^{12}(\eta_i)}[a^2 + 4ng^2(\eta_{0,i})] \\
&\leq \frac{2a^2g^4(\eta_{0,i})}{g^{12}(\eta_i)}[a^2 + 2ng^2(\eta_{0,i})].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
(D_0^2)_{i,i} &= \frac{1}{n^{-1}g^2(\eta_{0,i})} + 2\left[\frac{g'(\eta_{0,i})}{g(\eta_{0,i})}\right]^2 \\
&= \frac{1}{n^{-1}g^2(\eta_{0,i})} + \frac{a^2}{2g^4(\eta_{0,i})} \\
&= \frac{2ng^2(\eta_{0,i}) + a^2}{2g^4(\eta_{0,i})}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{a_i^2 + b_i^2}{(D_0^4)_{i,i}} &\leq \frac{2a^2 g^4(\eta_{0,i})}{g^{12}(\eta_i)} [a^2 + 2ng^2(\eta_{0,i})] \left[\frac{2g^4(\eta_{0,i})}{2ng^2(\eta_{0,i}) + a^2} \right]^2 \\
&\leq 8a^2 \frac{g^{12}(\eta_{0,i})}{g^{12}(\eta_i)} \left[\frac{1}{2ng^2(\eta_{0,i}) + a^2} \right] \\
&\leq C \frac{1}{n} \left[\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} \right]^6.
\end{aligned}$$

Subsequently, using (57) and (58), and letting $G_i = \frac{g(\eta_{0,i})}{g(\eta_i)}$, implies

$$\begin{aligned}
G_i^2 &\leq |G_i^2 - 1| + 1 \\
&\leq (1 - \epsilon_i)^{-1} r \left[\frac{1}{2} + R_i \right]^{-1/2} + 1 \\
&\leq \frac{\epsilon_i}{1 - \epsilon_i} + 1 \\
&= \mathcal{O}(1),
\end{aligned}$$

since $\epsilon_i \in (0, 1/2]$.

Thus, $\frac{a_i^2 + b_i^2}{(D_0^4)_{i,i}} = \mathcal{O}(n^{-1})$, and consequently we can choose ω s.t. $\omega^2 n = \mathcal{O}(1)$ as well.

Additionally, using Corollary 5.9,

$$H := \max_{1 \leq i \leq p} \left| \frac{a_{u,r,i}}{(D_0^2)_{i,i}} \right| \asymp n^{-1}.$$

□

5.14.9 Assumption L_0

Corollary 5.11. *Let $\eta \in \Upsilon_0(r_0)$, where*

$$r_0 = o\left(\min_{1 \leq i \leq p} \left[\frac{1}{2} + \frac{n(a\eta_{0,i} + b)}{a^2} \right]^{1/2}\right).$$

Then,

$$\max_{1 \leq i \leq p} |(D_0^{-1} D_0^2(\eta) D_0^{-1} - I_p)_{i,i}| \leq o(1).$$

Proof. Assume $\eta_i = \eta_{0,i} \pm \delta_i$. Then, we must find a bound for $|(D_0^{-2})_{i,i}(D_0^2(\eta))_{i,i} - 1|$, which holds for all $\eta \in \Upsilon_0(r)$, where r is fixed.

As stated at the beginning of Section 5.8.8, $|(D_0^{-2})_{i,i}(D_0^2(\eta))_{i,i} - 1|$ can be seen as the combination of ratio and neighbourhood terms.

We begin the proof by bounding the ratio:

Recall,

$$\begin{aligned}
|(D_0^{-2})_{i,i}(D_0^2(\eta))_{i,i} - 1| &\leq \left| \frac{1 + 2n^{-1}[g'(\eta_i)]^2}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - 1 \right| \\
&\quad + \left(\frac{1 + 2n^{-1}[g'(\eta_{0,i})]^2}{n^{-1}g^2(\eta_{0,i})} \right)^{-1} \left| [g'(\eta_i) \frac{4(\eta_{0,i} - \eta_i)}{n^{-1}g^3(\eta_i)}] \right| \\
&\quad + |g''(\eta_i) \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^3(\eta_i)}| \\
&\quad + |[g'(\eta_i)]^2 \frac{3(\eta_{0,i} - \eta_i)^2 + 3n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^4(\eta_i)}|.
\end{aligned}$$

Observe,

$$\begin{aligned}
\frac{1 + 2n^{-1}[g'(\eta_i)]^2}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} &= \frac{1 + 2n^{-1}[\frac{a}{2g(\eta_i)}]^2}{1 + 2n^{-1}[\frac{a}{2g(\eta_{0,i})}]^2} = \frac{g^2(\eta_{0,i}) + c[\frac{g(\eta_{0,i})}{g(\eta_i)}]^2}{g^2(\eta_{0,i}) + c} \\
&= \frac{g^2(\eta_{0,i}) + c[\frac{g(\eta_{0,i})}{g(\eta_i)}]^2 + c - c}{g^2(\eta_{0,i}) + c} \\
&= 1 + \frac{c[\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - 1]}{g^2(\eta_{0,i}) + c},
\end{aligned}$$

where $c = \frac{a^2}{2n}$.

Let $G_i = \frac{g(\eta_{0,i})}{g(\eta_i)}$, hence,

$$\begin{aligned}
\frac{1 + 2n^{-1}[g'(\eta_i)]^2}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - 1 &= [1 + \frac{c[G_i^2 - 1]}{g^2(\eta_{0,i}) + c}]G_i^2 - 1 \\
&= G_i^2 + \frac{c}{g^2(\eta_{0,i}) + c}[G_i^2 - 1](G_i^2 - 1 + 1) - 1 \\
&= (G_i^2 - 1) + \frac{c}{g^2(\eta_{0,i}) + c}[(G_i^2 - 1)^2 + (G_i^2 - 1)].
\end{aligned}$$

Note,

$$\begin{aligned}
G_i^2 - 1 &= \frac{g^2(\eta_{0,i}) - g^2(\eta_i)}{g^2(\eta_i)} = \frac{a(\eta_{0,i} - \eta_i)}{g^2(\eta_i)} = \frac{a(\eta_{0,i} - \eta_i)}{g^2(\eta_i) - g^2(\eta_{0,i}) + g^2(\eta_{0,i})} \\
&= \frac{a(\eta_{0,i} - \eta_i)}{a(\eta_i - \eta_{0,i}) + g^2(\eta_{0,i})}.
\end{aligned}$$

Therefore using the Reverse Triangle Inequality,

$$|G_i^2 - 1| \leq \left| \frac{a(\eta_{0,i} - \eta_i)}{g^2(\eta_{0,i}) + a(\eta_i - \eta_{0,i})} \right| = \frac{a|\delta_i|}{|g^2(\eta_{0,i}) - a|\delta_i||} = \frac{d_i|\delta_i|}{|1 - d_i|\delta_i||},$$

where $d_i = \frac{a}{g^2(\eta_{0,i})}$.

Since $\eta_i \in \Upsilon_0(r)$,

$$|\delta_i| \leq r(D_0^2)_{i,i}^{-1/2}.$$

Furthermore,

$$d_i(D_0^2)_{i,i}^{-1/2} = \frac{a}{g^2(\eta_{0,i})} \left[\frac{2g^4(\eta_{0,i})}{2ng^2(\eta_{0,i}) + a^2} \right]^{1/2} = \left[\frac{2a^2}{2ng^2(\eta_{0,i}) + a^2} \right]^{1/2} = \left[\frac{1}{2} + \frac{ng^2(\eta_{0,i})}{a^2} \right]^{-1/2} = \left[\frac{1}{2} + R_i \right]^{-1/2},$$

where $R_i = \frac{ng^2(\eta_{0,i})}{a^2}$.

Hence,

$$d_i|\delta_i| \leq r d_i(D_0^2)_{i,i}^{-1/2} = r \left[\frac{1}{2} + R_i \right]^{-1/2}.$$

Let

$$\epsilon_i := r \left[\frac{1}{2} + R_i \right]^{-1/2}, \quad \text{and assume } \epsilon_i \in (0, 1/2]. \quad (57)$$

Consequently,

$$|1 - d_i|\delta_i|| \geq 1 - \epsilon_i,$$

and

$$|G_i^2 - 1| \leq (1 - \epsilon_i)^{-1} r \left[\frac{1}{2} + R_i \right]^{-1/2}. \quad (58)$$

Note,

$$\frac{c}{g^2(\eta_{0,i}) + c} = \frac{1}{\frac{g^2(\eta_{0,i})}{c} + 1} = \frac{1}{\frac{2ng^2(\eta_{0,i})}{a^2} + 1} = \frac{1}{2R_i + 1},$$

and

$$\frac{1}{2R_i + 1} \leq \begin{cases} 1, & \text{if } R_i = \mathcal{O}(1), \\ 0, & \text{if } R_i \rightarrow \infty. \end{cases}$$

Therefore,

$$\begin{aligned} \left| \frac{1 + 2n^{-1}[g'(\eta_i)]^2}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - 1 \right| &\leq |G_i^2 - 1| + \frac{c}{g^2(\eta_{0,i}) + c} [|G_i^2 - 1|^2 + |G_i^2 - 1|] \\ &= |G_i^2 - 1| + \frac{1}{2R_i + 1} [|G_i^2 - 1|^2 + |G_i^2 - 1|] \\ &\leq (1 + \frac{2}{2R_i + 1}) |G_i^2 - 1| \\ &\leq 3(1 - \epsilon_i)^{-1} r \left[\frac{1}{2} + R_i \right]^{-1/2}. \end{aligned}$$

The next part of the proof consists of bounding each of the neighbourhood terms:

Considering each of the neighbourhood terms:

$$\begin{aligned}
|(D_0^2)_{i,i}^{-1} g'(\eta_i) \frac{4(\eta_{0,i} - \eta_i)}{n^{-1}g^3(\eta_i)}| &\leq (D_0^2)_{i,i}^{-1} \frac{2na|\delta_i|}{g^4(\eta_i)} = (D_0^2)_{i,i}^{-1} \frac{2na|\delta_i|}{g^4(\eta_{0,i})} \frac{g^4(\eta_{0,i})}{g^4(\eta_i)} \\
&= (D_0^2)_{i,i}^{-1} \frac{2nd_i^2|\delta_i|}{a} |G_i^4| \\
&= d_i^2 (D_0^2)_{i,i}^{-1} \frac{2n}{d_i a} |G_i^4| d_i |\delta_i|.
\end{aligned}$$

$$\begin{aligned}
|(D_0^2)_{i,i}^{-1} g''(\eta_i) \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^3(\eta_i)}| &\leq (D_0^2)_{i,i}^{-1} \frac{a^2 n}{4g^6(\eta_i)} [\delta_i^2 + n^{-1}a|\delta_i|] \\
&= (D_0^2)_{i,i}^{-1} \frac{a^2 n}{4g^6(\eta_{0,i})} |G_i^6| [\delta_i^2 + n^{-1}a|\delta_i|] \\
&= (D_0^2)_{i,i}^{-1} \frac{n}{4a} d_i^3 |G_i^6| [\delta_i^2 + n^{-1}a|\delta_i|] \\
&= d_i^2 (D_0^2)_{i,i}^{-1} \frac{n}{4d_i a} |G_i^6| [d_i^2 \delta_i^2 + n^{-1}ad_i^2 |\delta_i|].
\end{aligned}$$

$$\begin{aligned}
|(D_0^2)_{i,i}^{-1} [g'(\eta_i)]^2 \frac{3(\eta_{0,i} - \eta_i)^2 + 3n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^4(\eta_i)}| &\leq (D_0^2)_{i,i}^{-1} \frac{3a^2 n}{4g^6(\eta_i)} [\delta_i^2 + n^{-1}|\delta_i|] \\
&= (D_0^2)_{i,i}^{-1} \frac{3n}{4a} d_i^3 |G_i^6| [\delta_i^2 + n^{-1}|\delta_i|] \\
&= d_i^2 (D_0^2)_{i,i}^{-1} \frac{3n}{4d_i a} |G_i^6| [d_i^2 \delta_i^2 + n^{-1}ad_i^2 |\delta_i|].
\end{aligned}$$

Hence,

$$\begin{aligned}
&|(D_0^2)_{i,i}^{-1} g''(\eta_i) \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^3(\eta_i)}| \\
&+ |(D_0^2)_{i,i}^{-1} [g'(\eta_i)]^2 \frac{3(\eta_{0,i} - \eta_i)^2 + 3n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^4(\eta_i)}| \\
&\leq d_i^2 (D_0^2)_{i,i}^{-1} \frac{n}{d_i a} |G_i^6| [d_i^2 \delta_i^2 + n^{-1}ad_i^2 |\delta_i|].
\end{aligned}$$

Note,

$$\begin{aligned}
d_i^2 (D_0^2)_{i,i}^{-1} &\leq [\frac{1}{2} + R_i]^{-1}, \\
\frac{n}{d_i a} &= \frac{ng^2(\eta_{0,i})}{a^2} = R_i.
\end{aligned}$$

Consequently,

$$d_i^2 (D_0^2)_{i,i}^{-1} \frac{n}{d_i a} \leq [\frac{1}{2} + R_i]^{-1} R_i \leq 1.$$

Furthermore,

$$\begin{aligned}
G_i^4 &= (G_i^2)^2 = (G_i^2 - 1 + 1)^2 \\
&= 1 + 2(G_i^2 - 1) + (G_i^2 - 1)^2, \\
G_i^6 &= G_i^4 G_i^2 = [1 + 2(G_i^2 - 1) + (G_i^2 - 1)^2](G_i^2 - 1 + 1) \\
&= 1 + 2(G_i^2 - 1) + (G_i^2 - 1)^2 + (G_i^2 - 1) + 2(G_i^2 - 1)^2 + (G_i^2 - 1)^3 \\
&= 1 + 3(G_i^2 - 1) + 3(G_i^2 - 1)^2 + (G_i^2 - 1)^3,
\end{aligned}$$

and since

$$|G_i^2 - 1| \leq (1 - \epsilon_i)^{-1} r \left[\frac{1}{2} + R_i \right]^{-1/2} \leq \frac{\epsilon_i}{1 - \epsilon_i} \leq 1,$$

we can deduce

$$\begin{aligned}
|G_i^4| &\leq 1 + 2|G_i^2 - 1| + |G_i^2 - 1|^2 \leq 1 + 3|G_i^2 - 1| \\
&\leq 1 + 3(1 - \epsilon_i)^{-1} r \left[\frac{1}{2} + R_i \right]^{-1/2}, \\
|G_i^6| &\leq 1 + 3|G_i^2 - 1| + 3|G_i^2 - 1|^2 + |G_i^2 - 1|^3 \leq 1 + 7|G_i^2 - 1| \\
&\leq 1 + 7(1 - \epsilon_i)^{-1} r \left[\frac{1}{2} + R_i \right]^{-1/2}.
\end{aligned}$$

In addition, recall

$$d_i |\delta_i| \leq r \left[\frac{1}{2} + R_i \right]^{-1/2}.$$

Hence,

$$\begin{aligned}
|(D_0^2)_{i,i}^{-1} g'(\eta_i) \frac{4(\eta_{0,i} - \eta_i)}{n^{-1} g^3(\eta_i)}| &= d_i^2 (D_0^2)_{i,i}^{-1} \frac{2n}{d_i a} |G_i^4| d_i |\delta_i| \\
&\leq 2 \left[1 + 3(1 - \epsilon_i)^{-1} r \left[\frac{1}{2} + R_i \right]^{-1/2} \right] \left(r \left[\frac{1}{2} + R_i \right]^{-1/2} \right) \\
&\leq 2 \left[r \left[\frac{1}{2} + R_i \right]^{-1/2} + 3(1 - \epsilon_i)^{-1} r^2 \left[\frac{1}{2} + R_i \right]^{-1} \right] \\
&\leq 2(1 + 3(1 - \epsilon_i)^{-1}) \left[r \left[\frac{1}{2} + R_i \right]^{-1/2} \right].
\end{aligned}$$

To bound the remaining terms, note that

$$\begin{aligned}
d_i |\delta_i| &\leq r \left[\frac{1}{2} + R_i \right]^{-1/2} \leq \epsilon_i \implies \\
d_i^2 \delta_i^2 + n^{-1} d_i^2 |\delta_i| &\leq d_i \delta_i + n^{-1} d_i^2 |\delta_i| = (1 + n^{-1} d_i) d_i |\delta_i|.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
d_i^2 (D_0^2)_{i,i}^{-1} \frac{n}{d_i a} (1 + n^{-1} d_i) &\leq \left[\frac{1}{2} + R_i \right]^{-1} \left(\frac{n}{d_i a} + \frac{1}{a} \right) = \left[\frac{1}{2} + R_i \right]^{-1} \left[R_i + \frac{1}{a} \right] \\
&= \left[\frac{1}{2} + R_i \right]^{-1} \left[R_i + \frac{1}{a} + \frac{1}{2} - \frac{1}{2} \right] = 1 + \left[\frac{1}{2} + R_i \right]^{-1} \left[\frac{1}{a} - \frac{1}{2} \right] \\
&\leq 1 + 2 \left[\frac{1}{a} - \frac{1}{2} \right] = \frac{2}{a}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& |(D_0^2)_{i,i}^{-1} g''(\eta_i) \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^3(\eta_i)}| \\
& + |(D_0^2)_{i,i}^{-1} [g'(\eta_i)]^2 \frac{3(\eta_{0,i} - \eta_i)^2 + 3n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^4(\eta_i)}| \\
& \leq d_i^2 (D_0^2)_{i,i}^{-1} \frac{n}{d_i a} |G_i^6| (1 + n^{-1}d_i) d_i |\delta_i| \\
& \leq \frac{2}{a} \left[1 + 7(1 - \epsilon_i)^{-1} r \left[\frac{1}{2} + R_i \right]^{-1/2} \right] \left(r \left[\frac{1}{2} + R_i \right]^{-1/2} \right) \\
& \leq \frac{2}{a} \left[r \left[\frac{1}{2} + R_i \right]^{-1/2} + 7(1 - \epsilon_i)^{-1} r^2 \left[\frac{1}{2} + R_i \right]^{-1} \right] \\
& \leq \frac{2}{a} (1 + 7(1 - \epsilon_i)^{-1}) \left[r \left[\frac{1}{2} + R_i \right]^{-1/2} \right].
\end{aligned}$$

In conclusion,

$$\begin{aligned}
|(D_0^{-1} D_0^2(\eta) D_0^{-1} - I_p)_{i,i}| & \leq \left[r \left[\frac{1}{2} + R_i \right]^{-1/2} \right] (3(1 - \epsilon_i)^{-1} + 2(1 + 3(1 - \epsilon_i)^{-1}) + \frac{2}{a} (1 + 7(1 - \epsilon_i)^{-1})) \\
& = \left[r \left[\frac{1}{2} + \frac{ng^2(\eta_{0,i})}{a^2} \right]^{-1/2} \right] \left(2 \left[1 + \frac{1}{a} \right] + \left[9 + \frac{14}{a} \right] (1 - \epsilon_i)^{-1} \right).
\end{aligned}$$

Note, throughout this proof we've assumed (57). This leads to the following result,

$$\begin{aligned}
\max_{1 \leq i \leq p} |(D_0^{-1} D_0^2(\eta) D_0^{-1} - I_p)_{i,i}| & \leq \max_{1 \leq i \leq p} \left[r \left[\frac{1}{2} + \frac{ng^2(\eta_{0,i})}{a^2} \right]^{-1/2} \left(2 \left[1 + \frac{1}{a} \right] + \left[9 + \frac{14}{a} \right] (1 - \epsilon_i)^{-1} \right) \right] \\
& = \epsilon_i \left(2 \left[1 + \frac{1}{a} \right] + \left[9 + \frac{14}{a} \right] \left[\frac{1}{2} \right]^{-1} \right) \max_{1 \leq i \leq p} \epsilon_i \\
& = \left(20 + \frac{30}{a} \right) \max_{1 \leq i \leq p} \epsilon_i.
\end{aligned}$$

Let,

$$\epsilon := \max_{1 \leq i \leq p} \epsilon_i$$

and note, $\epsilon \in (0, 1/2]$. Furthermore, for all i ,

$$r \left[\frac{1}{2} + R_i \right]^{-1/2} = o(1) \iff r = o \left(\min_{1 \leq i \leq p} \left[\frac{1}{2} + \frac{ng^2(\eta_{0,i})}{a^2} \right]^{1/2} \right).$$

Consequently,

$$\begin{aligned}
\max_{1 \leq i \leq p} |(D_0^{-1} D_0^2(\eta) D_0^{-1} - I_p)_{i,i}| & \leq \epsilon \left(20 + \frac{30}{a} \right) \\
& \iff r = o \left(\min_{1 \leq i \leq p} \left[\frac{1}{2} + \frac{n(a\eta_{0,i} + b)}{a^2} \right]^{1/2} \right),
\end{aligned}$$

where $\epsilon \in (0, 1/2]$.

□

5.14.10 Assumption L_r

In this section, using Proposition 5.6, we can show

$$b(r) \asymp [1 + \frac{a}{ac_{0,p} + b} rn^{-1/2}]^{-1} \asymp [1 + rn^{-1/2}]^{-1},$$

where $c_{0,p} = \min_{i=1,\dots,p} \eta_{0,i}$.

The calculations are stated below.

$$\begin{aligned} b(r) &= \min_i [g^2(\eta_{0,i}) + \frac{r}{(D_0)_{i,i}} a]^{-1} \frac{n}{(D_0^2)_{i,i}} \\ &\asymp \min_i [g^2(\eta_{0,i}) + \frac{r}{(D_0)_{i,i}} a]^{-1} \\ &\asymp \min_i [1 + rn^{-1/2} \frac{a}{g^2(\eta_{0,i})}]^{-1} \\ &= [1 + \frac{a}{ac_{0,p} + b} rn^{-1/2}]^{-1}. \end{aligned}$$

Furthermore, we can bound the last expression using Assumption 5.2.

5.14.11 Assumption 4.1

Recall, $\eta \in \mathbb{R}^p$, and $\eta_0^\infty \in S^{\beta_c}$, where the Sobolev space,

$$S^{\beta_c} = \{h : \|h\|_{S^{\beta_c}}^2 < \infty\}, \text{ where } \|h\|_{S^{\beta_c}}^2 = \sum_{i=1}^{\infty} h_i^2 i^{2\beta_c}.$$

Furthermore, we consider a truncated prior on $\eta_i \sim TN(0, \lambda_i)$, where $\lambda_i = \tau_n^2 i^{-(1+2\alpha_c)}$ and $n\tau_n^2 \rightarrow \infty$, with $\alpha_c > 0$.

Consequently, the precision matrix G^2 from Assumption 4.1 will be defined as follows: $(G^2)_{i,i} = \lambda_i^{-1} = \tau_n^{-2} i^{1+2\alpha_c}$. In addition, let $g_{0,p}^2 := \max_i g^2(\eta_{0,i})$. Note, given Assumption 5.2, this constant can be ignored. However, we chose to track it for a possible future application.

Thus,

$$(D_G^2)_{i,i} = (D_0^2)_{i,i} + (G^2)_{i,i} = \frac{1 + 2n^{-1}[g'(\eta_{0,i})]^2}{n^{-1}g^2(\eta_{0,i})} + \lambda_i = \frac{1 + 2n^{-1}[g'(\eta_{0,i})]^2 + \lambda_i^{-1}n^{-1}g^2(\eta_{0,i})}{n^{-1}g^2(\eta_{0,i})}.$$

Hence,

$$\begin{aligned} \|D_0^{-1}G^2D_0^{-1}\| &= \max_i \left| \frac{n^{-1}g^2(\eta_{0,i})\lambda_i^{-1}}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} \right| \asymp \max_i \frac{4g^4(\eta_{0,i})}{4ng^2(\eta_{0,i}) + 2a^2} \tau_n^{-2} i^{(1+2\alpha_c)} \\ &= \max_i \left| \frac{4(\tau_n^2)^{-1}(a\eta_{0,i} + b)^2 i^{1+2\alpha_c}}{4n(a\eta_{0,i} + b) + 2a^2} \right|. \end{aligned}$$

Observe,

$$b \leq |a\eta_{0,i} + b| \leq aM + b \implies |a\eta_{0,i} + b| \asymp 1,$$

$$4nb \leq |4n(a\eta_{0,i} + b) + 2a^2| \leq [4(aM + b) + 2a^2]n \implies |4n(a\eta_{0,i} + b) + 2a^2| \asymp n.$$

Hence,

$$\begin{aligned} \|D_0^{-1}G^2D_0^{-1}\| &\leq C \max_i \left| \frac{(\tau_n^2)^{-1}i^{1+2\alpha_c}}{n} \right| \\ &\leq C(n\tau_n^2)^{-1}p^{1+2\alpha_c}. \end{aligned}$$

If required, note,

$$\begin{aligned} \|D_0^{-1}G^2D_0^{-1}\| = o(1), &\iff (n\tau_n^2)^{-1}p^{1+2\alpha_c} = o(1) \\ &\iff p = o((n\tau_n^2)^{1/(1+2\alpha_c)}). \end{aligned}$$

Furthermore,

$$\begin{aligned} \text{tr}(D_0^{-1}G^2D_0^{-1})^2 &= \sum_{i=1}^p \left[\frac{n^{-1}g^2(\eta_{0,i})\lambda_i^{-1}}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} \right]^2 \\ &\leq C \sum_{i=1}^p \left[\left| \frac{(\tau_n^2)^{-1}i^{1+2\alpha_c}}{n} \right| \right]^2 \\ &\leq C(n\tau_n^2)^{-2} \sum_{i=1}^p i^{2(1+2\alpha_c)} \\ &\leq C(n\tau_n^2)^{-2} p^{2(1+2\alpha_c)+1}. \end{aligned}$$

Again, note,

$$\begin{aligned} \text{tr}(D_0^{-1}G^2D_0^{-1})^2 = o(1), &\iff (n\tau_n^2)^{-2}p^{2(1+2\alpha_c)+1} = o(1) \\ &\iff p = o((n\tau_n^2)^{1/(1+2\alpha_c+1/2)}). \end{aligned}$$

Consequently, from this point on let us assume

$$p = o((n\tau_n^2)^{1/(1+2\alpha_c)} \wedge (n\tau_n^2)^{1/(1+2\alpha_c+1/2)}) = o((n\tau_n^2)^{1/(1+2\alpha_c+1/2)}).$$

Subsequently,

$$\begin{aligned} \|D_G^{-1}G^2\eta_0\|^2 &= \sum_{i=1}^p \left[\left(\frac{1 + 2n^{-1}[g'(\eta_{0,i})]^2 + \lambda_i^{-1}n^{-1}g^2(\eta_{0,i})}{n^{-1}g^2(\eta_{0,i})} \right)^{-1/2} \lambda_i^{-1}\eta_{0,i} \right]^2 \\ &= \sum_{i=1}^p \frac{n^{-1}g^2(\eta_{0,i})\lambda_i^{-2}\eta_{0,i}^2}{1 + 2n^{-1}[g'(\eta_{0,i})]^2 + \lambda_i^{-1}n^{-1}g^2(\eta_{0,i})} \\ &= \sum_{i=1}^p \frac{4g^4(\eta_{0,i})\eta_{0,i}^2}{4n\lambda_i^2g^2(\eta_{0,i}) + 2a^2\lambda_i^2 + 4\lambda_i g^4(\eta_{0,i})} \\ &= \sum_{i=1}^p \frac{4g^4(\eta_{0,i})\eta_{0,i}^2\lambda_i^{-1}}{\lambda_i[4ng^2(\eta_{0,i}) + 2a^2] + 4g^4(\eta_{0,i})}. \end{aligned}$$

Observe

$$\lambda_i[4ng^2(\eta_{0,i}) + 2a^2] + 4g^4(\eta_{0,i}) \asymp \lambda_i[n + 1] + 1 \asymp \lambda_i n + 1.$$

Note,

$$\lambda_i n = n\tau_n^2 i^{-(1+2\alpha_c)} \geq 1 \iff i \leq (n\tau_n^2)^{1/(1+2\alpha_c)}.$$

Thus, let $i_N = (n\tau_n^2)^{1/(1+2\alpha_c)}$. Hence,

$$p = o((n\tau_n^2)^{1/(1+2\alpha_c+1/2)}) = o(i_N).$$

Consequently,

$$\begin{aligned} \|D_G^{-1}G^2\eta_0\|^2 &= \sum_{i=1}^p \frac{4g^4(\eta_{0,i})\eta_{0,i}^2(\lambda_i)^{-1}}{\lambda_i[4ng^2(\eta_{0,i}) + 2a^2] + 4g^4(\eta_{0,i})} \\ &\asymp \sum_{i=1}^{i_N \wedge p} \frac{4g^4(\eta_{0,i})\eta_{0,i}^2(\lambda_i)^{-1}}{\lambda_i n} + \sum_{i=i_N}^p \eta_{0,i}^2(\lambda_i)^{-1} \\ &\asymp \sum_{i=1}^{i_N \wedge p} \frac{\eta_{0,i}^2(\lambda_i)^{-1}}{\lambda_i n} \\ &\asymp \sum_{i=1}^{i_N \wedge p} \eta_{0,i}^2 \lambda_i^{-2} n^{-1} \\ &\asymp \sum_{i=1}^{i_N \wedge p} \tau_n^{-4} i^{(2+4\alpha_c)} n^{-1} \eta_{0,i}^2 i^{2\beta_c} i^{-2\beta_c} \\ &\leq C n^{-1} \tau_n^{-4} p^{2+4\alpha_c-2\beta_c \vee 0} \sum_{i=1}^{i_N \wedge p} \eta_{0,i}^2 i^{2\beta_c} \\ &\leq C n^{-1} \tau_n^{-4} p^{2+4\alpha_c-2\beta_c \vee 0} \|\eta_0\|_{\beta_c}^2. \end{aligned}$$

Thus,

$$\|D_G^{-1}G^2\eta_0\|^2 \leq C n^{-1} \tau_n^{-4} p^{2+4\alpha_c-2\beta_c \vee 0}.$$

Subsequently, to show $\|D_G^{-1}G^2\eta_0\|^2 = o(1)$ we need to consider 2 cases.

Case 1: Assume $2 + 4\alpha_c - 2\beta_c > 0$, then

$$\begin{aligned} \|D_G^{-1}G^2\eta_0\|^2 = o(1), &\iff \tau_n^{-2} (n\tau_n^2)^{-1} p^{(2+4\alpha_c-2\beta_c)} = o(1) \\ &\iff p = o((\tau_n^2 n\tau_n^2)^{1/(2+4\alpha_c-2\beta_c)}). \end{aligned}$$

Case 2: Assume $2 + 4\alpha_c - 2\beta_c \leq 0$, then

$$\|D_G^{-1}G^2\eta_0\|^2 = o(1) \iff (n\tau_n^4)^{-1} = o(1).$$

Thus, we require

$$n\tau_n^4 \rightarrow \infty.$$

Consequently, the conditions are as follows:

$$p = o((n\tau_n^2)^{1/1+2\alpha_c+1/2}), \text{ and } \begin{cases} p = o((\tau_n^2 n\tau_n^2)^{1/(2+4\alpha_c-2\beta_c)}), & \text{if } 2 + 4\alpha_c - 2\beta_c > 0, \\ n\tau_n^4 \rightarrow \infty, & \text{o/w.} \end{cases}$$

Hence,

$$\begin{aligned} \|D_0^{-1}G^2D_0^{-1}\| &\leq C(n\tau_n^2)^{-1}p^{(1+2\alpha_c)} = \delta_G, \\ \text{tr}(D_0^{-1}G^2D_0^{-1})^2 &\leq C(n\tau_n^2)^{-2}p^{2(1+2\alpha_c)+1} = \delta_{tr}^2, \\ \|D_G^{-1}G^2\eta_0\|^2 &\leq C\tau_n^{-2}(n\tau_n^2)^{-1}p^{(2+4\alpha_c-2\beta_c)\vee 0} = \delta_\beta, \end{aligned}$$

where all 3 terms are $o(1)$, if the above conditions hold.

5.15 Proof of Theorem 5.2

Proof. Recall, the pertinent terms from Section 4.3's [Theorem 2] are

$$\begin{aligned} \Delta(r_0, x) &:= (\delta(r_0) + 6\nu_0 z_{\mathbb{H}}(x)\omega)r_0^2, \\ z_{\mathbb{H}}(x) &:= 2p^{1/2} + \sqrt{2x} + \aleph^{-1}(\aleph^{-2}x + 1)4p, \\ \varrho &:= \frac{1}{2}[(1 + \delta_G)(3\delta_\beta + \delta_G z_B(x))^2 + \delta_{tr}^2]^{1/2}. \end{aligned}$$

The terms ν_0, ω and \aleph come from Assumptions ED_0 and ED_2 . The term, ν_0 , is present in both of the aforementioned assumptions; however, as shown in their respective sections, both are asymptotically a constant.

The term ω^2 is from ED_2 and satisfies $\omega^2 n = \mathcal{O}(1)$.

The term $\aleph = (4H)^{-1}$, where

$$H := \max_{1 \leq i \leq p} \left| \frac{1}{(D_0)_{i,i}} \frac{g'(\eta_{0,i})}{g(\eta_{0,i})} \right|.$$

The term $\delta(r)$ and the corresponding condition on r_0 come from Assumption L_0 . Specifically, $\delta(r) = o(1)$, assuming

$$r_0 = o\left(\min_{1 \leq i \leq p} \left[\frac{1}{2} + \frac{n(a\eta_{0,i} + b)}{a^2}\right]^{1/2}\right).$$

The terms in ϱ relate to the matrices Σ_0 and G . Specifically, $z_B^2(x) := p_B + 6\lambda_B x$, with

$$B := D_0^{-1}\Sigma_0^2D_0^{-1}, \quad p_B := \text{tr}(B), \quad \lambda_B := \lambda_{\max}(B).$$

In our case $B = I$, therefore $p_B = p$ and $\lambda_B = 1$.

Furthermore, we assume $\Pi = N(0, G^{-2})$, where

$$\begin{aligned} \|D_0^{-1}G^2D_0^{-1}\| &\leq \delta_G \leq 1/2, \\ \text{tr}(D_0^{-1}G^2D_0^{-1})^2 &\leq \delta_{tr}^2, \\ \|D_G^{-1}G^2\eta_0\| &\leq \delta_\beta, \end{aligned}$$

where $D_G^2 := D_0^2 + G^2$. These terms are evaluated in Section 5.14.11.

Writing it out explicitly: For $\eta \in \Upsilon_0(r_0)$ we have

$$\begin{aligned} \aleph &= (4H)^{-1} \asymp \left[\max_{1 \leq i \leq p} \left| \frac{1}{(D_0)_{i,i}} \frac{g'(\eta_{0,i})}{g(\eta_{0,i})} \right| \right]^{-1} \asymp [n^{-1/2}]^{-1} \\ &= n^{1/2}, \\ z_{\mathbb{H}}(x) &= 2p^{1/2} + \sqrt{2x} + \aleph^{-1}(\aleph^{-2}x + 1)4p \\ &\asymp p^{1/2} + x^{1/2} + (n^{-1}x + 1)4pn^{-1/2} \\ &\asymp p^{1/2} + x^{1/2}, \\ \omega &\asymp n^{-1/2}, \\ \nu_0 &\asymp 1, \\ r_0^2 &= o\left(\left[\min_{1 \leq i \leq p} \left[\frac{1}{2} + \frac{n(a\eta_{0,i} + b)}{a^2}\right]^{1/2}\right]^2\right) \\ &= o(n), \\ \delta(r_0) &= \epsilon\left(20 + \frac{30}{a}\right) \asymp \epsilon \\ &\asymp r_0 \max_{1 \leq i \leq p} \left[\frac{1}{2} + \frac{n(a\eta_{0,i} + b)}{a^2}\right]^{-1/2} \\ &\asymp r_0 n^{-1/2} \\ &\asymp o(1), \\ \Delta(r_0, x) &= (\delta(r_0) + 6\nu_0 z_{\mathbb{H}}(x)\omega)r_0^2 \\ &\asymp (o(1) + (p^{1/2} + x^{1/2})\omega)r_0^2 \\ &\asymp (p^{1/2} + x^{1/2})\omega r_0^2. \end{aligned}$$

Furthermore, for

$$p = o((n\tau_n^2)^{1/1+2\alpha_c+1/2}), \text{ and } \begin{cases} p = o((\tau_n^2 n \tau_n^2)^{1/(2+4\alpha_c-2\beta_c)}), & \text{if } 2 + 4\alpha_c - 2\beta_c > 0, \\ n\tau_n^4 \rightarrow \infty, & \text{o/w,} \end{cases}$$

the terms δ_G, δ_{tr} , and δ_β are $o(1)$, where

$$\begin{aligned}
\|D_0^{-1}G^2D_0^{-1}\| &\leq \delta_G = C(n\tau_n^2)^{-1}p^{(1+2\alpha_c)}, \\
\text{tr}(D_0^{-1}G^2D_0^{-1})^2 &\leq \delta_{tr}^2 = C(n\tau_n^2)^{-2}p^{2(1+2\alpha_c)+1}, \\
\|D_G^{-1}G^2\eta_0\|^2 &\leq \delta_\beta^2 = \tau_n^{-2}(n\tau_n^2)^{-1}p^{(2+4\alpha_c-2\beta_c)\vee 0}, \\
z_B(x) &= [p_B + 6\lambda_B x]^{1/2} \\
&\asymp [p + x]^{1/2}, \\
\varrho &= \frac{1}{2}[(1 + \delta_G)(3\delta_\beta + \delta_G z_B(x))^2 + \delta_{tr}^2]^{1/2} \\
&\asymp [1 \cdot (\delta_\beta^2 + \delta_G^2 z_B^2(x)) + \delta^2]^{1/2} = [(\delta_\beta^2 + \delta_G^2 z_B^2(x)) + \delta_{tr}^2]^{1/2} \\
&\asymp [\delta_\beta^2 + \delta_G^2 [p + x]]^{1/2}.
\end{aligned}$$

If, $2 + 4\alpha_c - 2\beta_c > 0$,

$$\begin{aligned}
\frac{\delta_\beta}{\delta_G} &= \frac{\tau_n^{-2}(n\tau_n^2)^{-1}p^{1+2\alpha_c+1+2[\alpha_c-\beta_c]}}{(n\tau_n^2)^{-1}p^{(1+2\alpha_c)}} \\
&= \tau_n^{-2}p^{1+2[\alpha_c-\beta_c]}.
\end{aligned}$$

Else, if $2 + 4\alpha_c - 2\beta_c \leq 0$,

$$\begin{aligned}
\frac{\delta_\beta}{\delta_G} &= \frac{\tau_n^{-2}(n\tau_n^2)^{-1}}{(n\tau_n^2)^{-1}p^{(1+2\alpha_c)}} \\
&= \tau_n^{-2}p^{-(1+2\alpha_c)}.
\end{aligned}$$

Therefore,

$$\tau^2 \asymp (n\tau_n^2)^{-2}p^{2(1+2\alpha_c)} \begin{cases} (\tau_n^{-4}p^{2+4[\alpha_c-\beta_c]} + p + x), & \text{if } 2 + 4\alpha_c - 2\beta_c > 0, \\ (\tau_n^{-4}p^{-2(1+2\alpha_c)} + p + x), & \text{if } 2 + 4\alpha_c - 2\beta_c \leq 0. \end{cases}$$

For $\eta \notin \Upsilon_0(r_0)$:

We use Theorem 5.4, where

$$\begin{aligned}
z(p, x) &= [p + \sqrt{6.6px} \vee (6.6x)]^{1/2} \\
&\asymp [p + x]^{1/2}, \\
r_0^2 &\geq [\sqrt{x} \vee (z_B(x) + z(p, x))]^2 \\
&= c(p + x).
\end{aligned}$$

Hence, for simplicity assume $x \asymp p$, which implies $r_0^2 \geq Cp$. □

6 Discussion

The main contributions of this thesis are in Sections 3 and 5. Thus, we begin by discussing Section 3.

Recall, the minimax and contraction rates (Proposition 3.2 and Theorem 3.2, respectively) in Section 3 were derived for the mildly ill posed setting $k_i \asymp i^{-p}$, see Assumption 3.1. However, these results can also be derived for the severely ill-posed setting $k_i = \mathcal{O}(e^{-iq})$, (see Definition 1.8). Severely ill-posed problems can be found for instance in [Thanh et al., 2008], which uses Infrared Thermography to detect buried landmines. Note, [Agapiou et al., 2014] derive posterior contraction rates for the severely ill-posed setting as well. Thus, we could use our results to ascertain whether their rates are minimax optimal.

We could also assume there exists some error in the forward operator, such as where the singular functions of K are known but not its singular values. Consequently, we could formulate this as a Plug-in estimator problem, as discussed in Section 3.6 for the covariance operator V . Such issues arise in various problems, such as in statistical inference for econometric problems with instruments [Florens and Simoni, 2016], and are discussed in [Cavalier and Hengartner, 2005], and [Hoffmann and Reiss, 2008]. Furthermore, we don't have to assume a deterministic condition as done in Section 3.6, (c.f. Equation (20)). Instead we could assume the existence of a *consistent* estimator of the singular values (k_i or σ_i). Subsequently, we could explicitly describe the effects these Plug in Estimators have on the contraction rates.

Note, we could also assume that the eigenfunctions of K or V are unknown. Fortunately, there exists methods for estimating unknown eigenfunctions as well. For instance, we could implement the work done in [Koltchinskii and Lounici, 2017], who proposed a way of estimating eigenvectors based on the sample covariance matrix.

Another question one may ask is whether it is possible for our posterior rates to converge to the optimal rates without *fine-tuning* our prior regularity parameter, i.e. α . This is indeed true in the homogeneous case for the mildly ill-posed setting, as proved in [Ray, 2013]'s Proposition 3.2 and [Knapik et al., 2016]'s Theorem 2.5, where in some instances the rates achieved were parametric, up to a logarithmic factor. For our setting (i.e. the inhomogeneous case), we could construct the empirical Bayes posterior with a plugged-in maximum marginal likelihood estimator of the prior scale under an appropriate Gaussian prior, as was done for the direct problem by [Szabó et al., 2013], and study whether it achieves the minimax rate.

Note, all of these proposed extensions will be implemented in our upcoming paper, Bochkina and Rodrigues (2021).

Subsequently, let us discuss Section 5. As mentioned previously, we would like to study inverse problems with Poisson noise; such problems can typically be found in Tomography, and some advances have been made in this field. For instance, [Cavalier and Koo, 2002] studied Poisson linear inverse models and used an approximation lemma ([Cavalier and Koo, 2002]'s Lemma V.1) to bound their Poisson estimation risk by a corresponding Gaussian estimation risk. Additionally, [Reynaud-Bouret, 2003] used penalized projection esti-

mators to estimate the intensity of inhomogeneous Poisson processes. Furthermore, [Bochkina and Green, 2014] derived Bernstein-von Mises results for well-posed Poisson linear models.

Hence, we could apply the results from Section 5 to linear inverse problems with Poisson noise, and derive the corresponding Bernstein-von Mises results and contraction rates. The latter would extend the work done in [Cavalier and Koo, 2002] from the Frequentist setting to the Bayesian setting. In addition, our Bernstein-von Mises results would be for Poisson linear inverse problems, unlike [Bochkina and Green, 2014] whose Bernstein-von Mises results are for well-posed Poisson linear models.

A Linear Operator Theory and Gaussian Distributions: Hilbert Spaces

A.1 Linear Operators

In this section we shall list certain theorems regarding Hilbert spaces, as they will be essential when describing Gaussian processes. Whilst the proofs of these theorems will not be presented, they can be found quite easily, for instance in [Kuo, 1975].

Theorem A.1. *A Hilbert space, H , is separable iff it has a countable orthonormal basis.*

Recall that a Hilbert space, H , is called *separable* if it contains a countable, dense subset i.e, there exists a sequence $\{h_n\}_{n=1}^\infty$, where $h_n \in H$, such that every non-empty open subset of H contains at least one element of the sequence. Separability provides us with a notion of a space's *size*. Note that if a space is *finite* or *countably infinite* it is separable, since we can define our sequence to be the set itself. Furthermore, note that even though \mathbb{R} is uncountable, it is separable as we can define $\{h_n\}_{n=1}^\infty$.

Theorem A.2. *Let A be any linear operator of a separable Hilbert space H . Then for any two orthonormal bases of H , $\{e_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$, the*

$$\sum_{n=1}^{\infty} \|Ae_n\|_H^2 = \sum_{n=1}^{\infty} \|Ad_n\|_H^2.$$

Note that the above theorem implies that if $\sum_{n=1}^{\infty} \|Ae_n\|_H^2$ converges (or diverges) for some $\{e_n\}_{n=1}^\infty$ then it does so for all the other bases. Consequently, we can define a new type of operator.

Definition A.1. *A linear operator A , of H , is called a Hilbert-Schmidt operator if for some orthonormal basis $\{e_n\}_{n=1}^\infty$ of H , the $\sum_{n=1}^{\infty} \|Ae_n\|_H^2 < \infty$, where $\|\cdot\|_H$ is the norm on H . Furthermore, the Hilbert-Schmidt norm of A is*

$$\|A\|_{HS}^2 = \sum_{n=1}^{\infty} \|Ae_n\|_H^2.$$

One can show that the Hilbert-Schmidt norm is the same regardless of the orthonormal basis chosen and that it bounds the operator norm of A . Consequently, denoting $L(H)$ as the collection of bounded linear operators on H and $L_S(H)$ as the collection of Hilbert-Schmidt operators on H , we see that $L_S(H) \subset L(H)$.

On a side note, if H is finite dimensional then $L_S(H) = L(H)$, whilst if H is infinite dimensional then $L_S(H) \neq L(H)$. For instance, the identity operator of H is in $L(H)$ but not in $L_S(H)$. Subsequently, let us discuss how one can *compare* two H-S operators.

Definition A.2. *Having defined the H-S norm, we can define the Hilbert-Schmidt inner product of A and B as follows,*

$$\langle A, B \rangle_{HS} := \sum_{n=1}^{\infty} \langle Ae_n, Be_n \rangle_H.$$

We now have the tools to discuss *trace-class* operators. However, before doing so we shall need to discuss *compactness* and compact operators. Recall, that the *closure* of a set $B \subset H$, denoted as $\text{cl}(B)$, consists of B along with its limit points, or alternatively, $\text{cl}(B) = B \cup \partial B$, where ∂B is the boundary of the set B .

Definition A.3. An open cover of a set A in a metric space \mathcal{X} is a collection of open subsets of \mathcal{X} , $\{G_i\}$, such that $A \subset \bigcup_i G_i$

Definition A.4. A subset E of a metric space \mathcal{X} is compact if every open cover of E contains a finite subcover i.e for every open cover $\{G_i\}$, $A \subset \bigcup_{j=1}^n G_j$.

Definition A.5. An operator of H is called compact if it maps any bounded subset of H into a set whose closure is compact.

Note: Every Hilbert-Schmidt operator is compact. Additionally, if A is a compact operator then $A^T A$ is as well, where A^T is the adjoint of A . Furthermore, note that $A^T A$ is self-adjoint as well. Hence, consider the following theorem:

Theorem A.3. If A is a self-adjoint compact operator, then there exists an orthonormal basis $\{e_n\}_{n=1}^\infty$ of H such that

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle_H e_n,$$

where $\lambda_n \in \mathbb{R}$ and $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$.

The λ_n s are called *eigenvalues*, and the e_n s *eigenvectors*. In addition, the $\lambda_n \geq 0$, when A is positive definite. Recall that A is *positive definite* if for every non-zero $x \in H$, $\langle Ax, x \rangle_H > 0$. Thus one can show $A^T A$ is positive definite as well.

Consequently, for any compact operator A , one can apply the above theorem to $A^T A$, which subsequently helps form the following definition:

Definition A.6. A compact operator A of H is called *trace-class* if $\sum_{n=1}^{\infty} \lambda_n < \infty$, where the λ_n s are the eigenvalues of $(A^T A)^{1/2}$.

One can show that A is a Hilbert-Schmidt operator iff $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$, where the λ_n s are the eigenvalues of $(A^T A)^{1/2}$. In this case, $\|A\|_{HS}^2 = \sum_{n=1}^{\infty} \lambda_n^2$.

Subsequently, we can define the collection of H 's trace-class operators as $L_T(H)$, and if $A \in L_T(H)$ the *trace-class norm* of A as,

$$\|A\|_T = \sum_{n=1}^{\infty} \lambda_n.$$

Definition A.7. If $A \in L_T(H)$ the trace of A is defined as

$$\text{tr}(A) := \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle_H,$$

where $\{e_n\}_{n=1}^{\infty}$ is any orthonormal basis of H .

Subsequently, we can now discuss Gaussian processes in a Hilbert space.

A.2 Gaussian Distributions

Remark: In this subsection, we shall always assume any norm and inner product is of H , unless otherwise stated.

Definition A.8. Let H denote a real, separable Hilbert space. A H -isonormal Gaussian process is a family of real-valued random variables, $W = \{W_h : h \in H\}$, defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ such that W_h is a centered Gaussian random variable $\forall h \in H$ and for $h, g \in H$, the $\mathbb{E}(W_h W_g) = \langle h, g \rangle$.

Definition A.9. A Gaussian distribution on the borel sets of a Hilbert space, $N(\nu, \Lambda)$, has a mean $\nu \in H$ and a covariance operator, $\Lambda : H \mapsto H$, which is non-negative definite, self-adjoint and linear. The operator is also of trace class, i.e. it is a compact operator whose eigenvalues (λ_i) are summable, $(\sum_{i=1}^{\infty} \lambda_i < \infty)$.

Consequently, a random variable $G \in H$ has a $N(\nu, \Lambda)$ distribution if and only if $\{G_h := \langle G, h \rangle : h \in H\}$ is a Gaussian process with the following mean and covariance functions:

$$\mathbb{E}G_h = \langle \nu, h \rangle, \text{ and } \mathbb{Cov}(G_h, G_g) = \langle \Lambda h, g \rangle.$$

Recall that, w.r.t Λ 's orthonormal eigenbasis $\{\phi_i\}_{i=1}^{\infty}$, $\Lambda\phi_i = \lambda_i$, hence the $G_i := G_{\phi_i}$ are independent, univariate $N(\nu_i, \lambda_i)$, where $\nu_i := \langle \nu, \phi_i \rangle$.

Note that the Identity element of H would not be a trace class operator, hence we must define standard normal distributions separately i.e. using isonormal Gaussian processes. Regardless, the iso-normal Gaussian process, W , could be *thought* of as a random variable with a $N(0, I)$ distribution.

Finally, the following proposition should elucidate the importance of trace class operators w.r.t. Gaussian distributions. Note that $\langle \cdot, \cdot \rangle_{H_i}$ and $\|\cdot\|_{H_i}$ refers to the inner product and norm of H_i , respectively.

Proposition A.1. Assume we have two separable Hilbert spaces H_1 and H_2 , where the former has the orthonormal basis $\{\phi_i\}_{i=1}^{\infty}$. Let $\mu|Y \sim N(AY, S_n)$ and $Y|\mu \sim (K\mu, n^{-1}V)$, where $A : H_2 \mapsto H_1$, $K : H_1 \mapsto H_2$ and $V : H_2 \mapsto H_2$ are continuous linear operators. Consequently, we can show that the

$$\mathbb{E}\|\mu - \mu_0\|_{H_1}^2 = \|AY - \mu_0\|_{H_1}^2 + \text{tr}(S_n), \text{ and} \quad (59)$$

$$\mathbb{E}_{\mu_0}\|AY - \mu_0\|_{H_1}^2 = \|AK\mu_0 - \mu_0\|_{H_1}^2 + n^{-1}\text{tr}(AVA^T). \quad (60)$$

Recall that when we take $\mathbb{E}_{\mu_0}Y$ we assume the data, $Y|\mu \sim (K\mu, n^{-1}V)$, has been generated from $\mu = \mu_0$.

Proof. (59:) We shall use the following implications:

$$\begin{aligned}\mu \in H_1 &\implies \mu = \sum_{i=1}^{\infty} \mu_i \phi_i, \\ \mu|Y \sim N(AY, S_n) &\implies \mu_i|Y \sim N(\langle AY, \phi_i \rangle_{H_1}, \langle S_n \phi_i, \phi_i \rangle_{H_1}), \text{ and} \\ \mathbb{V}ar(\mu_i) &= \mathbb{E}\mu_i^2 - (\mathbb{E}\mu_i)^2.\end{aligned}$$

where $\mu_i := \langle \mu, \phi_i \rangle_{H_1}$. Thus, we can show that the

$$\begin{aligned}\mathbb{E}\|\mu - \mu_0\|_{H_1}^2 &= \mathbb{E}\left\|\sum_{i=1}^{\infty} (\mu_i - \mu_{0,i}) \phi_i\right\|_{H_1}^2 = \mathbb{E}\sum_{i=1}^{\infty} (\mu_i - \mu_{0,i})^2 = \sum_{i=1}^{\infty} \mathbb{E}\mu_i^2 + \mu_{0,i}^2 - 2\mu_{0,i}\mathbb{E}\mu_i \\ &= \sum_{i=1}^{\infty} \mathbb{V}ar(\mu_i) + (\mathbb{E}\mu_i)^2 + \mu_{0,i}^2 - 2\mu_{0,i}\mathbb{E}\mu_i = \sum_{i=1}^{\infty} \mathbb{V}ar(\mu_i) + (\mathbb{E}\mu_i - \mu_{0,i})^2 \\ &= \sum_{i=1}^{\infty} \langle S_n \phi_i, \phi_i \rangle_{H_1} + (\langle AY, \phi_i \rangle_{H_1} - \langle \mu_0, \phi_i \rangle_{H_1})^2 = \text{tr}(S_n) + \|AY - \mu_0\|_{H_1}^2.\end{aligned}$$

□

Proof. (60:) We shall use the following implications:

$$\begin{aligned}Y|\mu \sim (K\mu, n^{-1}V) &\implies AY \sim (AK\mu, n^{-1}AV A^T), \\ AY \in H_1 &\implies AY = \sum_{i=1}^{\infty} (AY)_i \phi_i,\end{aligned}$$

where $(AY)_i := \langle AY, \phi_i \rangle_{H_1}$. Subsequently, proceeding in the same way as we did in the previous proof,

$$\begin{aligned}\mathbb{E}_{\mu_0}\|AY - \mu_0\|_{H_1}^2 &= \sum_{i=1}^{\infty} \mathbb{V}ar_{\mu_0}((AY)_i) + (\mathbb{E}_{\mu_0}(AY)_i - \mu_{0,i})^2 \\ &= n^{-1}\text{tr}(AV A^T) + \|AK\mu_0 - \mu_0\|_{H_1}^2.\end{aligned}$$

□

Note the above proposition only holds if the distributions of $\mu|Y$ and $AY|\mu$ are *proper*, i.e. if their covariance operators have *finite trace*.

B Proofs for Section 2

B.1 Proof for Proposition 2.1

Note that computing the joint distribution was not necessary for my proof, but those following the proof described in [Knapik et al., 2011] might find the following useful.

Joint Distribution: Assuming $Y|\mu \sim N(K\mu, n^{-1}I)$ and $\mu \sim N(0, \Lambda)$ we must prove their joint probability is

$$Y, \mu \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} n^{-1}I + K\Lambda K^T & K\Lambda \\ \Lambda K^T & \Lambda \end{pmatrix}\right) \iff Y_i, \mu_i \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} n^{-1} + k_i^2\lambda_i & k_i\lambda_i \\ \lambda_i k_i & \lambda_i \end{pmatrix}\right).$$

Recall that if the normal random variables X_1, X_2 have the joint density

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]\right) \\ &\implies X_1, X_2 \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_2\sigma_1 & \sigma_2^2 \end{pmatrix}\right). \end{aligned}$$

Hence, since $Y_i|\mu_i \sim N(k_i\mu_i, n^{-1})$ and $\mu_i \sim N(0, \lambda_i)$, we know

$$\begin{aligned} f(y_i, \mu_i) &= f(y_i|\mu_i)f(\mu_i) \propto \exp\left(\frac{-n}{2}[y_i - k_i\mu_i]^2\right) \exp\left(\frac{-1}{2\lambda_i}\mu_i^2\right) = \exp\left(\frac{-n}{2}[y_i^2 + k_i^2\mu_i^2 - 2y_ik_i\mu_i + \frac{\mu_i^2}{n\lambda_i}]\right) \\ &= \exp\left(\frac{-(n\lambda_ik_i^2 + 1)}{2}\left[\frac{y_i^2}{k_i^2\lambda_i + n^{-1}} + \frac{\mu_i^2}{\lambda_i} - 2\frac{k_i\lambda_i^{1/2}}{(\lambda_ik_i^2 + n^{-1})^{1/2}}\frac{y_i\mu_i}{(\lambda_ik_i^2 + n^{-1})^{1/2}\lambda_i^{1/2}}\right]\right), \end{aligned}$$

which gives us our desired result. Note that $\rho := \frac{k_i\lambda_i^{1/2}}{(\lambda_ik_i^2 + n^{-1})^{1/2}}$ and $1 - \rho^2 = (n\lambda_ik_i^2 + 1)^{-1}$ as required. \square

Posterior distribution: Singular Values: Since $Y_i \sim N(k_i\mu_i, \frac{1}{n})$ and $\mu \sim N(0, \Lambda)$, (which implies $\mu_i := \langle \mu, \phi_i \rangle_{H_1} \sim N(0, \lambda_i)$), we can find the posterior distribution of $\mu_i|Y_i$ i.e.

$$\begin{aligned} f(\mu_i|y_i) &\propto f(y_i|\mu_i)f(\mu_i) \propto \exp\left(\frac{-n}{2}[y_i - k_i\mu_i]^2\right) \exp\left(\frac{-1}{2\lambda_i}\mu_i^2\right) \\ &= \exp\left(\frac{-n}{2}[y_i^2 + k_i^2\mu_i^2 - 2y_ik_i\mu_i + \frac{\mu_i^2}{n\lambda_i}]\right) \\ &\propto \exp\left(\frac{-(n\lambda_ik_i^2 + 1)}{2\lambda_i}\left[\mu_i^2 - 2\frac{y_ik_in\lambda_i}{n\lambda_ik_i^2 + 1}\mu_i\right]\right) \\ &\propto \exp\left(\frac{-(n\lambda_ik_i^2 + 1)}{2\lambda_i}\left[\mu_i - \frac{ny_ik_i\lambda_i}{n\lambda_ik_i^2 + 1}\right]^2\right), \end{aligned}$$

which implies

$$\mu_i|Y_i \sim N\left(\frac{nY_ik_i\lambda_i}{n\lambda_ik_i^2 + 1}, \frac{\lambda_i}{n\lambda_ik_i^2 + 1}\right).$$

\square

Claim: We can use the distribution of the singular values above to infer that $\mu|Y \sim N(AY, S_n)$.

Proof:

Note, given an orthonormal basis, $\{b_i\}_{i=1}^\infty$ and a linear operator M , if $Mb_i = m_i b_i$ then $M^{-1}Mb_i = b_i = M^{-1}(m_i b_i)$ thus $M^{-1}b_i = \frac{b_i}{m_i}$. Hence,

$$\begin{aligned} AY &= A \sum_i Y_i \varphi_i = \sum_i Y_i A \varphi_i = \sum_i Y_i \Lambda K^T (n^{-1}I + K \Lambda K^T)^{-1} \varphi_i \\ &= \sum_i Y_i \Lambda K^T \frac{1}{n^{-1} + k_i \lambda_i k_i} \varphi_i = \sum_i \frac{n Y_i \lambda_i k_i}{1 + n k_i^2 \lambda_i} \phi_i. \end{aligned}$$

Therefore,

$$\mathbb{E}(\mu_i|Y) = \langle AY, \phi_i \rangle_{H_1} = \frac{n Y_i \lambda_i k_i}{1 + n k_i^2 \lambda_i},$$

as required. Similarly, (keeping in mind that Λ is symmetric),

$$\begin{aligned} S_n \phi_j &= \Lambda \phi_j - A(n^{-1}I + K \Lambda K^T) A^T \phi_j = \Lambda \phi_j - A K \Lambda^T \phi_j \\ &= \lambda_j \phi_j - A k_j \lambda_j \phi_j = \lambda_j \phi_j - \frac{n \lambda_j k_j}{1 + n k_j^2 \lambda_j} k_j \lambda_j \phi_j = \frac{\lambda_j}{1 + n k_j^2 \lambda_j} \phi_j. \end{aligned}$$

Hence,

$$\text{Cov}(\mu_i, \mu_j|Y) = \langle \phi_i, S_n \phi_j \rangle_{H_1} = \langle \phi_i, \frac{\lambda_j}{1 + n k_j^2 \lambda_j} \phi_j \rangle_{H_1} = \frac{\lambda_j}{1 + n k_j^2 \lambda_j} \delta_{i,j},$$

as needed. □

Simplifying A: Note that for any compact linear operator $B : H_1 \mapsto H_2$ the following identity holds:

$$(I + BB^T)^{-1}B = B(I + B^T B)^{-1}. \quad (61)$$

Thus defining $\Omega := n^{-1}I$, we see that

$$\Omega + BB^T = \Omega(I + \Omega^{-1}BB^T) = \Omega^{1/2}(I + \Omega^{-1/2}BB^T\Omega^{-1/2})\Omega^{1/2}.$$

The last equality following from Ω 's symmetry. Consequently, setting $B := \Lambda^{1/2}K^T$,

$$\begin{aligned} A &= \Lambda^{1/2}(\Omega + BB^T)^{-1}B = \Lambda^{1/2}\Omega^{-1/2}(I + \Omega^{-1/2}BB^T\Omega^{-1/2})^{-1}\Omega^{-1/2}B \\ &= \Lambda^{1/2}\Omega^{-1/2}(I + \tilde{B}\tilde{B}^T)^{-1}\tilde{B}, \end{aligned}$$

where $\tilde{B} = \Omega^{-1/2}B = n^{1/2}\Lambda^{1/2}K^T$. Hence, we must prove \tilde{B} is a compact linear operator, in order to use the identity (61) and simplify A .

Note that \tilde{B} is a *Hilbert-Schmidt* operator, i.e $\text{tr}(\tilde{B}\tilde{B}^T) < \infty$, since

$$\text{tr}(\tilde{B}\tilde{B}^T) = \text{tr}(n\Lambda^{1/2}K^TK\Lambda^{1/2}) = n \sum_i \lambda_i k_i^2 \leq n \sum_i (\lambda_i C_1^2) = n C_1^2 \left(\sum_i \lambda_i \right) < \infty,$$

where the penultimate equality follows from the fact that

$$k_i \asymp \mathcal{O}(i^{-p}) \iff C_1^{-1}i^{-p} \leq k_i \leq C_1 i^{-p} \text{ for some } C_1 > 0.$$

This concludes the proof since we know that every H-S operator is compact, c.f. Appendix A.1. \square

S_n is proper: First note that $\Lambda - S_n$ is non-negative definite i.e. $\langle (\Lambda - S_n)h, h \rangle_{H_1} \geq 0$ since

$$(\Lambda - S_n)h = \sum_j h_j^2 \frac{n\lambda_j^2 k_j^2}{1 + nk_j^2 \lambda_j} \phi_j \implies \langle (\Lambda - S_n)h, h \rangle_{H_1} = \sum_j h_j^2 \frac{n\lambda_j^2 k_j^2}{1 + nk_j^2 \lambda_j} \geq 0,$$

where $h_j := \langle h, \phi_j \rangle_{H_1}$. Hence S_n is bounded above by Λ and since Λ is of trace class so is S_n . \square

B.2 Proof for Theorem 2.1

Proof. Given a probability measure \mathbb{P} , Markov's Inequality dictates that

$$\mathbb{P}(\{x \in X : |f(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_X |f| d\mathbb{P}.$$

Subsequently, given the posterior distribution $\Pi_n(\cdot|Y)$, Markov's Inequality applied on the second moment of $\mu - \mu_0$ implies

$$\Pi_n(\{\mu \in H_1 : \|\mu - \mu_0\|_{H_1}^2 \geq M_n^2 \varepsilon_n^2 |Y\}) \leq \frac{1}{M_n^2 \varepsilon_n^2} \int_{H_1} \|\mu - \mu_0\|_{H_1}^2 d\Pi_n(\mu|Y).$$

However, since $\mu|Y \sim N(AY, S_n)$, we know (using Proposition A.1) the

$$\begin{aligned} \mathbb{E} \|\mu - \mu_0\|_{H_1}^2 &= \int_{H_1} \|\mu - \mu_0\|_{H_1}^2 d\Pi_n(\mu|Y) = \|AY - \mu_0\|_{H_1}^2 + \text{tr}(S_n) \implies \\ \mathbb{E}_{\mu_0} \mathbb{E} \|\mu - \mu_0\|_{H_1}^2 &= \|AK\mu_0 - \mu_0\|_{H_1}^2 + n^{-1} \text{tr}(AA^T) + \text{tr}(S_n). \end{aligned}$$

Note that when we take $\mathbb{E}_{\mu_0} Y$ we assume the data, $Y|\mu \sim (K\mu, n^{-1}I)$, has been generated from $\mu = \mu_0$. Furthermore, recall that the *trace* of an operator is equal to the sum of its spectral coefficients. Consequently,

$$\begin{aligned} \bullet \quad \|AK\mu_0 - \mu_0\|_{H_1}^2 &= \|\sum_i k_i \mu_{0,i} A\varphi_i - \sum_i \mu_{0,i} \phi_i\|_{H_1}^2 \\ &= \|\sum_i k_i \mu_{0,i} \frac{n\lambda_i k_i}{1 + nk_i^2 \lambda_i} \phi_i - \sum_i \mu_{0,i} \phi_i\|_{H_1}^2 = \sum_i \frac{\mu_{0,i}^2}{(1 + nk_i^2 \lambda_i)^2}. \end{aligned}$$

The last equality following from *Parseval's Identity*.

- $\text{tr}(AA^T) = \text{tr}(\Lambda K^T(n^{-1}I + K\Lambda K^T)^{-2}K\Lambda^T)$

$$= \sum_i \lambda_i k_i \left(\frac{1}{n^{-1} + k_i \lambda_i k_i} \right)^2 k_i \lambda_i = \sum_i \frac{n^2 \lambda_i^2 k_i^2}{(1 + n \lambda_i k_i^2)^2}.$$
- $\text{tr}(S_n) = \text{tr}(\Lambda - A(n^{-1}I + K\Lambda K^T)A^T)$

$$= \sum_j \frac{\lambda_j}{1 + n k_j^2 \lambda_j}.$$

Hence, given Assumption 2.1, we know k_i 's bounds and thus

$$\begin{aligned} 1 + n \lambda_i C^{-2} i^{-2p} &\leq 1 + n k_i^2 \lambda_i \leq 1 + n \lambda_i C^2 i^{-2p} \implies \\ C_1(1 + n \lambda_i i^{-2p}) &\leq 1 + n k_i^2 \lambda_i \leq C_2(1 + n \lambda_i i^{-2p}) \implies \\ 1 + n k_i^2 \lambda_i &\asymp 1 + n \lambda_i i^{-2p}, \end{aligned}$$

where $C_1 = \min(1, C^{-2})$ and $C_2 = \max(1, C^2)$. Consequently, via Assumption 2.1 again,

- $\|AK\mu_0 - \mu_0\|_{H_1}^2 =$

$$\begin{aligned} \sum_i \frac{\mu_{0,i}^2}{(1 + n k_i^2 \lambda_i)^2} &\asymp \sum_i \frac{\mu_{0,i}^2}{(1 + n \tau_n^2 i^{-1-2\alpha-2p})^2} = \|\mu_0\|_{S^\beta}^2 \sum_i \frac{\mu_{0,i}^2 / \|\mu_0\|_{S^\beta}^2}{(1 + n \tau_n^2 i^{-1-2\alpha-2p})^2} \\ &= \|\mu_0\|_{S^\beta}^2 \sum_i \frac{\tilde{\mu}_{0,i}^2}{(1 + n \tau_n^2 i^{-1-2\alpha-2p})^2} \leq \|\mu_0\|_{S^\beta}^2 \sup_{\|\tilde{\mu}\|_{S^\beta} \leq 1} \sum_i \frac{\tilde{\mu}_i^2}{(1 + n \tau_n^2 i^{-1-2\alpha-2p})^2}, \end{aligned}$$

where $\tilde{\mu}_{0,i}^2 := \mu_{0,i}^2 / \|\mu_0\|_{S^\beta}^2$.

- $\text{tr}(AA^T) =$

$$\sum_i \frac{n^2 \lambda_i^2 k_i^2}{(1 + n \lambda_i k_i^2)^2} \asymp \sum_i \frac{n^2 \tau_n^4 i^{-2-4\alpha-2p}}{(1 + n \tau_n^2 i^{-1-2\alpha-2p})^2}.$$
- $\text{tr}(S_n) =$

$$\sum_i \frac{\lambda_j}{1 + n k_j^2 \lambda_j} \asymp \sum_i \frac{\tau_n^2 i^{-1-2\alpha}}{1 + n \tau_n^2 i^{-1-2\alpha-2p}}.$$

Finally using Lemma 8.1 from [Knapik et al., 2011], i.e. Lemma B.1, one obtains

$$\sup_{\|\tilde{\mu}\|_{S^\beta} \leq 1} \sum_i \frac{\tilde{\mu}_i^2}{(1 + n \tau_n^2 i^{-1-2\alpha-2p})^2} \asymp (n \tau_n^2)^{-\left(\frac{2\beta}{1+2\alpha+2p} \wedge 2\right)},$$

by setting $q = \beta, t = 0, u = 1 + 2\alpha + 2p, v = 2$ and $N = n \tau_n^2$. Consequently $\|AK\mu_0 - \mu_0\|_{H_1}^2 \asymp \|\mu_0\|_{S^\beta}^2 (n \tau_n^2)^{-\left(\frac{2\beta}{1+2\alpha+2p} \wedge 2\right)}$.

Similarly using Lemma 8.2 from [Knapik et al., 2011], i.e. Lemma B.2, along with setting $S(i) = 1, q = -1/2, t = 2 + 4\alpha + 2p, u = 1 + 2\alpha + 2p, v = 2$ and $N = n\tau_n^2$, we obtain

$$\sum_i \frac{i^{-2-4\alpha-2p}}{(1 + n\tau_n^2 i^{-1-2\alpha-2p})^2} \asymp (n\tau_n^2)^{-\frac{1+4\alpha+2p}{1+2\alpha+2p}}.$$

Thus $n^{-1}\text{tr}(AA^T) \asymp \tau_n^2(n\tau_n^2)^{-\frac{1+4\alpha+2p}{1+2\alpha+2p}+1} = \tau_n^2(n\tau_n^2)^{-\frac{2\alpha}{1+2\alpha+2p}}.$

Furthermore, setting $S(i) = 1, q = -1/2, t = 1 + 2\alpha, u = 1 + 2\alpha + 2p, v = 1$ and $N = n\tau_n^2$ in Lemma B.2, we obtain

$$\sum_i \frac{i^{-1-2\alpha}}{1 + n\tau_n^2 i^{-1-2\alpha-2p}} \asymp (n\tau_n^2)^{-\frac{2\alpha}{1+2\alpha+2p}}.$$

Subsequently, we see that the $\text{tr}(S_n) \asymp \tau_n^2(n\tau_n^2)^{-\frac{2\alpha}{1+2\alpha+2p}}$. Thus, the

$$\begin{aligned} \mathbb{E}_{\mu_0} \Pi_n(\{\mu : \|\mu - \mu_0\| \geq M_n \varepsilon_n | Y\}) &\leq \frac{1}{M_n^2 \varepsilon_n^2} (\|AK\mu_0 - \mu_0\|_{H_1}^2 + n^{-1}\text{tr}(AA^T) + \text{tr}(S_n)) \\ &\asymp \frac{1}{M_n^2 \varepsilon_n^2} (\|\mu\|_{S_\beta}^2 (n\tau_n^2)^{-(\frac{2\beta}{1+2\alpha+2p} \wedge 2)} + 2\tau_n^2 (n\tau_n^2)^{-\frac{2\alpha}{1+2\alpha+2p}}) \end{aligned}$$

Hence, setting $\varepsilon_n := (n\tau_n^2)^{-(\frac{\beta}{1+2\alpha+2p} \wedge 1)} + \tau_n(n\tau_n^2)^{-\frac{\alpha}{1+2\alpha+2p}}$ ensures that the $\mathbb{E}_{\mu_0} \Pi_n(\{\mu : \|\mu - \mu_0\|_{H_1} \geq M_n \varepsilon_n | Y\}) \rightarrow 0$ for every $M_n \rightarrow \infty$. \square

Next, let us see how one obtains the rates stated in the theorem.

Proof. • Setting $\tau_n = 1$ implies

$$\varepsilon_n = n^{-(\frac{\beta}{1+2\alpha+2p} \wedge 1)} + n^{-\frac{\alpha}{1+2\alpha+2p}} = \mathcal{O}(n^{-\frac{\beta \wedge \alpha}{1+2\alpha+2p}}),$$

since $\frac{\alpha}{1+2\alpha+2p} < 1$.

- If $\beta \leq 1 + 2\alpha + 2p$, then ε_n can be minimised by setting $\tau_n \asymp n^{\frac{\alpha-\beta}{1+2\beta+2p}}$ as this ensures both terms in ε_n are of the same order. To see this, define $c_\alpha := 1 + 2\alpha + 2p$ and $c_\beta := 1 + 2\beta + 2p$, and note that

$$\varepsilon_n \asymp n^{(\frac{2(\alpha-\beta)}{c_\beta}+1)(\frac{-\beta}{c_\alpha})} + n^{(\frac{2(\alpha-\beta)}{c_\beta}+1)(\frac{-\alpha}{c_\alpha})+\frac{\alpha-\beta}{c_\beta}} = n^{\frac{-\beta}{c_\beta}} + n^{\frac{-\beta}{c_\beta}} = \mathcal{O}(n^{\frac{-\beta}{c_\beta}}),$$

as required.

- If $\beta > 1 + 2\alpha + 2p$, then $\varepsilon_n = (n\tau_n^2)^{-1} + \tau_n(n\tau_n^2)^{-\frac{\alpha}{1+2\alpha+2p}}$. Note that this is a positive, increasing function wrt τ . Consequently, we can find its minimiser, τ_* , as follows:

$$\begin{aligned} \varepsilon_n(\tau) &= n^{-1}\tau^{-2} + n^{\frac{-\alpha}{c_\alpha}} \tau^{\frac{-2\alpha}{c_\alpha}+1} \quad \therefore \\ \varepsilon'_n(\tau) &= 0 \implies \tau_* = \frac{2c_\alpha}{1+2p} n^{-\frac{1+\alpha+2p}{3+4\alpha+6p}} = \mathcal{O}(n^{-\frac{1+\alpha+2p}{3+4\alpha+6p}}). \end{aligned}$$

Thus,

$$\varepsilon_n(\tau_*) = C_1 n^{-\frac{c\alpha}{3+4\alpha+6p}} + C_2 n^{-\frac{c\alpha}{3+4\alpha+6p}} = \mathcal{O}(n^{-\frac{1+2\alpha+2p}{3+4\alpha+6p}}).$$

Hence, we see that for any scaling τ_n , $\varepsilon_n(\tau_n) \gg n^{-\frac{\beta}{1+2\beta+2p}}$ if and only if

$$\frac{1+2\alpha+2p}{3+4\alpha+6p} < \frac{\beta}{1+2\beta+2p} \iff \frac{1}{2+\frac{1+2p}{1+2\alpha+2p}} < \frac{1}{2+\frac{1+2p}{\beta}} \iff 1+2\alpha+2p < \beta,$$

which is indeed the case. Note that, just as in the previous case, the minimiser of ε_n was achieved when the terms where of the same order. \square

B.3 Technical Lemmas from [Knapik et al., 2011]

The following lemmas are used in the proofs for Theorems 2.1 and 3.2. The first is Lemma 8.1 from [Knapik et al., 2011]:

Lemma B.1. *For any $r \geq 0, t \geq -2r, u > 0$ and $v \geq 0$, as $N \rightarrow \infty$,*

$$\sup_{\|\xi\|_{S^r} \leq 1} \sum_i \frac{\xi_i^2 i^{-t}}{(1 + N i^{-u})^v} \asymp N^{-(t+2r)/u \wedge v}.$$

Moreover, for every fixed $\xi \in S^r$, as $N \rightarrow \infty$,

$$N^{((t+2r)/u) \wedge v} \sum_i \frac{\xi_i^2 i^{-t}}{(1 + N i^{-u})^v} \rightarrow \begin{cases} 0, & \text{if } (t+2r)/u < v. \\ \|\xi\|_{S^{(uv-t)/2}}^2, & \text{if } (t+2r)/u \geq v. \end{cases}$$

The last assertion remains true if the sum is limited to the terms $i \leq cN^{1/u}$, for any $c > 0$.

Finally, Lemma 8.2 from [Knapik et al., 2011] is as follows:

Lemma B.2. *For any $t, v \geq 0, u > 0$ and (ξ_i) such that $|\xi_i| = i^{-r-1/2} \mathcal{S}(i)$ for $r > -t/2$ and a slowly varying function $S : (0, \infty) \rightarrow (0, \infty)$, as $N \rightarrow \infty$,*

$$\sum_i \frac{\xi_i^2 i^{-t}}{(1 + N i^{-u})^v} \asymp \begin{cases} N^{-(t+2r)/u} \mathcal{S}^2(N^{1/u}), & \text{if } (t+2r)/u < v. \\ N^{-v} \sum_{i \leq N^{1/u}} \mathcal{S}^2(i)/i, & \text{if } (t+2r)/u = v. \\ N^{-v}, & \text{if } (t+2r)/u > v. \end{cases}$$

Moreover, for every $c > 0$, the sum on the left is asymptotically equivalent to the same sum restricted to the terms $i \leq cN^{1/u}$ if and only if $(t+2r)/u \geq v$.

B.4 P-series

A p-series,

$$\sum_{i \in \mathbb{N}} \frac{1}{i^p},$$

converges if $p > 1$ and diverges if $p \leq 1$. Note,

$$\sum_{i \in \mathbb{N}} i^a = \sum_{i \in \mathbb{N}} \frac{1}{i^{-a}},$$

will converge if $a < -1$ and diverge otherwise, (since $-a = p > 1 \iff a < -1$).

C Proofs for Section 5

C.1 Proof for derivations involving $L(\eta)$

$$\begin{aligned}
f(Y_i|\eta_i) &= \frac{1}{\sqrt{2\pi n^{-1}g^2(\eta_i)}} \exp\left(-\frac{(Y_i - \eta_i)^2}{2n^{-1}g^2(\eta_i)}\right), \\
L(\eta) = L(Y|\eta) &= \log\left[\prod_{i=1}^p \frac{1}{\sqrt{2\pi n^{-1}g^2(\eta_i)}} \exp\left(-\frac{(Y_i - \eta_i)^2}{2n^{-1}g^2(\eta_i)}\right)\right] \\
&= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(\eta_i)) - \frac{n}{2} \sum_{i=1}^p \frac{(Y_i - \eta_i)^2}{g^2(\eta_i)}, \\
\mathbb{E}_{\eta_0} L(\eta) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(\eta_i)) - \frac{n}{2} \sum_{i=1}^p \frac{n^{-1}g^2(\eta_{0,i}) + (\eta_{0,i} - \eta_i)^2}{g^2(\eta_i)} \\
&= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \sum_{i=1}^p \log(g(\eta_i)) - \frac{1}{2} \sum_{i=1}^p \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - \frac{n}{2} \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{g^2(\eta_i)}.
\end{aligned}$$

Given Assumption 5.1, (see C.6 for further details), η_0 is the unique maximum of $\mathbb{E}_{\eta_0} L(\eta)$, and $\nabla \mathbb{E}_{\eta_0} L(\eta) = 0$ at η_0 . Subsequently,

$$\begin{aligned}
\frac{\partial}{\partial \eta_i} L(\eta) &= -\frac{g'(\eta_i)}{g(\eta_i)} + \frac{(Y_i - \eta_i)}{n^{-1}g^2(\eta_i)} + \frac{(Y_i - \eta_i)^2 g'(\eta_i)}{n^{-1}g^3(\eta_i)}, \\
\frac{\partial}{\partial \eta_i} L(\eta_{0,i}) &= -\frac{g'(\eta_{0,i})}{g(\eta_{0,i})} + \frac{(Y_i - \eta_{0,i})}{n^{-1}g^2(\eta_{0,i})} + \frac{(Y_i - \eta_{0,i})^2 g'(\eta_{0,i})}{n^{-1}g^3(\eta_{0,i})}, \\
\frac{\partial}{\partial \eta_i} \mathbb{E}_{\eta_0} L(\eta) &= \frac{(\eta_{0,i} - \eta_i)}{n^{-1}g^2(\eta_i)} + \frac{(\eta_{0,i} - \eta_i)^2 g'(\eta_i)}{n^{-1}g^3(\eta_i)} + \frac{g'(\eta_i)}{g^3(\eta_i)} (g^2(\eta_{0,i}) - g^2(\eta_i)), \\
\frac{\partial}{\partial \eta_i} \mathbb{E}_{\eta_0} L(\eta_{0,i}) &= \frac{(\eta_{0,i} - \eta_{0,i})}{n^{-1}g^2(\eta_{0,i})} + \frac{(\eta_{0,i} - \eta_{0,i})^2 g'(\eta_{0,i})}{n^{-1}g^3(\eta_{0,i})} + \frac{g'(\eta_{0,i})}{g^3(\eta_{0,i})} (g^2(\eta_{0,i}) - g^2(\eta_{0,i})) \\
&= 0.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\zeta(\eta) &= L(\eta) - \mathbb{E}_{\eta_0} L(\eta) = \sum_{i=1}^p \frac{n^{-1}g^2(\eta_{0,i}) + (\eta_{0,i} - \eta_i)^2 - (Y_i - \eta_i)^2}{2n^{-1}g^2(\eta_i)}, \\
(\nabla \zeta(\eta))_i &= \frac{Y_i - \eta_i}{n^{-1}g^2(\eta_i)} + \frac{g'(\eta_i)}{n^{-1}g^3(\eta_i)} [(Y_i - \eta_i)^2 - (\eta_{0,i} - \eta_i)^2 - n^{-1}g^2(\eta_{0,i})], \\
(\nabla^2 \zeta(\eta))_{i,i} &= \frac{Y_i}{n^{-1}g^2(\eta_i)} - \frac{2g'(\eta_i)(Y_i - \eta_i)}{n^{-1}g^3(\eta_i)} + \frac{g''(\eta_i)}{n^{-1}g^3(\eta_i)} [(Y_i - \eta_i)^2 - (\eta_{0,i} - \eta_i)^2 - n^{-1}g^2(\eta_{0,i})] \\
&\quad + \frac{g'(\eta_i)}{n^{-1}g^3(\eta_i)} [-2(Y_i - \eta_i) + 2(\eta_{0,i} - \eta_i)] \\
&\quad - \frac{3[g'(\eta_i)]^2}{n^{-1}g^4(\eta_i)} [(Y_i - \eta_i)^2 - (\eta_{0,i} - \eta_i)^2 - n^{-1}g^2(\eta_{0,i})], \\
(\nabla^2 \zeta(\eta))_{ij} &= 0, \quad \text{when } i \neq j, \\
(\nabla \zeta(\eta_0))_i &= \frac{Y_i - \eta_{0,i}}{n^{-1}g^2(\eta_{0,i})} + \frac{g'(\eta_{0,i})}{n^{-1}g^3(\eta_{0,i})} [(Y_i - \eta_{0,i})^2 - n^{-1}g^2(\eta_{0,i})], \\
(\nabla^2 \zeta(\eta_0))_{i,i} &= \frac{Y_i}{n^{-1}g^2(\eta_{0,i})} - \frac{4g'(\eta_{0,i})(Y_i - \eta_{0,i})}{n^{-1}g^3(\eta_{0,i})} + \frac{g''(\eta_{0,i})}{n^{-1}g^3(\eta_{0,i})} [(Y_i - \eta_{0,i})^2 - n^{-1}g^2(\eta_{0,i})] \\
&\quad - \frac{3[g'(\eta_{0,i})]^2}{n^{-1}g^4(\eta_{0,i})} [(Y_i - \eta_{0,i})^2 - n^{-1}g^2(\eta_{0,i})].
\end{aligned}$$

C.2 Proof for deriving Fisher's Information Matrix, i.e D_0^2

Note,

$$\begin{aligned}
\frac{\partial}{\partial \eta_i} \mathbb{E}_{\eta_0} L(\eta) &= \frac{(\eta_{0,i} - \eta_i)}{n^{-1}g^2(\eta_i)} + \frac{(\eta_{0,i} - \eta_i)^2 g'(\eta_i)}{n^{-1}g^3(\eta_i)} + \frac{g'(\eta_i)}{g^3(\eta_i)} (g^2(\eta_{0,i}) - g^2(\eta_i)), \\
\frac{\partial^2}{\partial \eta_i \partial \eta_j} \mathbb{E}_{\eta_0} L(\eta) &= 0, \\
(D_0^2)_{i,j} &= -\frac{\partial^2}{\partial \eta_i \partial \eta_j} \mathbb{E}_{\eta_0} L(\eta_0) = 0, \\
\frac{\partial^2}{\partial \eta_i^2} \mathbb{E}_{\eta_0} L(\eta) &= -\frac{1}{n^{-1}g^2(\eta_i)} - \frac{2(\eta_{0,i} - \eta_i)g'(\eta_i)}{n^{-1}g^3(\eta_i)} \\
&\quad - \frac{2(\eta_{0,i} - \eta_i)g'(\eta_i)}{n^{-1}g^3(\eta_i)} + \frac{(\eta_{0,i} - \eta_i)^2 g''(\eta_i)}{n^{-1}g^3(\eta_i)} - \frac{3(\eta_{0,i} - \eta_i)^2 [g'(\eta_i)]^2}{n^{-1}g^4(\eta_i)} \\
&\quad + \frac{g''(\eta_i)}{g^3(\eta_i)} (g^2(\eta_{0,i}) - g^2(\eta_i)) - \frac{g'(\eta_i)}{g^3(\eta_i)} (2g(\eta_i)g'(\eta_i)) \\
&\quad - \frac{3[g'(\eta_i)]^2}{g^4(\eta_i)} (g^2(\eta_{0,i}) - g^2(\eta_i)) \\
&= -\frac{1}{n^{-1}g^2(\eta_i)} - g'(\eta_i) \frac{4(\eta_{0,i} - \eta_i)}{n^{-1}g^3(\eta_i)} \\
&\quad + g''(\eta_i) \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^3(\eta_i)} \\
&\quad - [g'(\eta_i)]^2 \frac{3(\eta_{0,i} - \eta_i)^2 + 2n^{-1}g^2(\eta_i) + 3n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^4(\eta_i)}, \\
(D_0^2)_{i,i} &= -\frac{\partial^2}{\partial \eta_i^2} \mathbb{E}_{\eta_0} L(\eta_0) = \frac{1}{n^{-1}g^2(\eta_{0,i})} + 2\left[\frac{g'(\eta_{0,i})}{g(\eta_{0,i})}\right]^2 = (\Sigma_0^2)_{i,i}.
\end{aligned}$$

C.3 Assumption ED_2 : Proofs

C.3.1 Derivation of $\nabla^2 \zeta(\eta)$

Recall,

$$\begin{aligned}
(\nabla^2 \zeta(\eta))_{i,i} &= -\frac{2g'(\eta_i)(Y_i - \eta_{0,i})}{n^{-1}g^3(\eta_i)} + \frac{g''(\eta_i)}{n^{-1}g^3(\eta_i)} [(Y_i - \eta_i)^2 - (\eta_{0,i} - \eta_i)^2 - n^{-1}g^2(\eta_{0,i})] \\
&\quad + \frac{g'(\eta_i)}{n^{-1}g^3(\eta_i)} [-2(Y_i - \eta_i) + 2(\eta_{0,i} - \eta_i)] \\
&\quad - \frac{3[g'(\eta_i)]^2}{n^{-1}g^4(\eta_i)} [(Y_i - \eta_i)^2 - (\eta_{0,i} - \eta_i)^2 - n^{-1}g^2(\eta_{0,i})].
\end{aligned}$$

Note,

$$\begin{aligned}
Y_i - \eta_i &= Y_i - \eta_i + \eta_{0,i} - \eta_{0,i} \\
&= (Y_i - \eta_{0,i}) + (\eta_{0,i} - \eta_i), \\
(Y_i - \eta_i)^2 &= [(Y_i - \eta_{0,i}) + (\eta_{0,i} - \eta_i)]^2 \\
&= (Y_i - \eta_{0,i})^2 + (\eta_{0,i} - \eta_i)^2 + 2(Y_i - \eta_{0,i})(\eta_{0,i} - \eta_i).
\end{aligned}$$

Hence,

$$\begin{aligned}
(\nabla^2 \zeta(\eta))_{i,i} &= -\frac{2g'(\eta_i)(Y_i - \eta_{0,i})}{n^{-1}g^3(\eta_i)} + \frac{g''(\eta_i)}{n^{-1}g^3(\eta_i)}[(Y_i - \eta_{0,i})^2 + 2(\eta_{0,i} - \eta_i)(Y_i - \eta_{0,i}) - n^{-1}g^2(\eta_{0,i})] \\
&\quad + \frac{g'(\eta_i)}{n^{-1}g^3(\eta_i)}[-2(Y_i - \eta_{0,i})] \\
&\quad - \frac{3[g'(\eta_i)]^2}{n^{-1}g^4(\eta_i)}[(Y_i - \eta_{0,i})^2 + 2(\eta_{0,i} - \eta_i)(Y_i - \eta_{0,i}) - n^{-1}g^2(\eta_{0,i})] \\
&= a_i X_i^2 + b_i X_i + c_i,
\end{aligned}$$

where

$$\begin{aligned}
X_i &= \frac{Y_i - \eta_{0,i}}{n^{-1/2}g(\eta_{0,i})} \sim N(0, 1), \\
a_i &= n^{-1}g^2(\eta_{0,i})\left[\frac{g''(\eta_i)}{n^{-1}g^3(\eta_i)} - \frac{3[g'(\eta_i)]^2}{n^{-1}g^4(\eta_i)}\right] \\
&= g^2(\eta_{0,i})\left[\frac{g''(\eta_i)}{g^3(\eta_i)} - \frac{3[g'(\eta_i)]^2}{g^4(\eta_i)}\right], \\
b_i &= n^{-1/2}g(\eta_{0,i})\left[-\frac{4g'(\eta_i)}{n^{-1}g^3(\eta_i)} + 2(\eta_{0,i} - \eta_i)\frac{a_i}{n^{-1}g^2(\eta_{0,i})}\right], \\
c_i &= -a_i.
\end{aligned}$$

C.4 Assumption L_0 : Proofs

Proof: Rough Bound using Corollary 5.7 Observe, for any functions $f(x)$ and $g(x)$,

$$[f(x) + g(x)]^2 \leq [\max_x |f(x)| + \max_x |g(x)|]^2.$$

Consequently, using the above inequality and noting that the $D_0(\eta)$ are diagonal matrices,

$$\begin{aligned}
\|D_0^{-1}D_0^2(\eta)D_0^{-1} - I_p\|^2 &= \max_{1 \leq i \leq p} \left((D_0^2)_{i,i}^{-1}(D_0^2(\eta))_{i,i} - 1 \right)^2 \\
&= \max_{1 \leq i \leq p} \left(\left(\frac{1 + 2n^{-1}[g'(\eta_{0,i})]^2}{n^{-1}g^2(\eta_{0,i})} \right)^{-1} \left[-\frac{1}{n^{-1}g^2(\eta_i)} - g'(\eta_i) \frac{4(\eta_{0,i} - \eta_i)}{n^{-1}g^3(\eta_i)} \right. \right. \\
&\quad \left. \left. + g''(\eta_i) \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^3(\eta_i)} \right. \right. \\
&\quad \left. \left. - [g'(\eta_i)]^2 \frac{3(\eta_{0,i} - \eta_i)^2 + 2n^{-1}g^2(\eta_i) + 3n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{n^{-1}g^4(\eta_i)} \right] - 1 \right)^2 \\
&= \max_{1 \leq i \leq p} \left(-\frac{1}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - \frac{4(\eta_{0,i} - \eta_i)}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} \frac{g'(\eta_i)}{g(\eta_i)} \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} \right. \\
&\quad \left. + \frac{g''(\eta_i)}{g(\eta_i)} \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} \frac{(\eta_{0,i} - \eta_i)^2 + n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} \right. \\
&\quad \left. - \frac{[g'(\eta_i)]^2}{g^2(\eta_i)} \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} \frac{3(\eta_{0,i} - \eta_i)^2 + 3n^{-1}(g^2(\eta_{0,i}) - g^2(\eta_i))}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} \right. \\
&\quad \left. - 2(D_0^{-2})_{i,i} \frac{[g'(\eta_i)]^2}{g^2(\eta_i)} - 1 \right)^2 \\
&\leq \max_{1 \leq i \leq p} \left[\frac{m_{r,1,i}^2}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} + \frac{4m_{r,2,i}m_{r,1,i}^2r(D_0^{-1})_{i,i}}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} \right. \\
&\quad \left. + m_{r,2,i}^2m_{r,1,i}^2 \frac{3r^2(D_0^{-2})_{i,i} + 3n^{-1}m_{r,4,i}}{1 + 2n^{-1}[g'(\eta_{0,i})]^2} + 2(D_0^{-2})_{i,i}m_{r,2,i}^2 + 1 \right]^2 := \delta(r),
\end{aligned}$$

where the terms $(m_{r,k,i})$ are defined in Corollary 5.7.

C.5 Proof of $\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - \log\left(\frac{g^2(\eta_{0,i})}{g^2(\eta_i)}\right) - 1 \geq 0$ for all η_i

Let $y := \frac{g^2(\eta_{0,i})}{g^2(\eta_i)}$, then

$$y - \ln y - 1 \geq 0 \iff y \geq \ln y + 1 \iff e^y \geq ey, \quad (62)$$

for $y \geq 0$. Define $a(y) = e^y$ and $b(y) = ey$.

Case 1: When $y = 1$, $a(1) = b(1)$. Furthermore,

$$a'(y) = e^y > b'(y) = e,$$

for $y > 1$, hence $a(y)$ will grow faster than $b(y)$ (for $y > 1$). Hence, (62) is indeed true for $y \geq 1$.

Case 2: When $y = 0$, $a(0) > b(0)$. Furthermore, when $0 < y < 1$, $a'(y), b'(y) > 0$ and $a'(y) < b'(y)$, hence both $a(y)$ and $b(y)$ are increasing, and $b(y)$ is growing faster than $a(y)$, (which could lead to them intersecting, after which $b(y) > a(y)$). Hence, (62) will not hold if $b(y)$ intersects $a(y)$ inside the interval $(0, 1)$. Fortunately, while $b(y)$ does

intersect $a(y)$, it does so at $y = 1$ which is outside the interval $(0, 1)$. Hence, $a(y) > b(y)$ for $0 \leq y < 1$.

Specifically, $a(y) > b(y)$ for $y \neq 1$ and $a(y) = b(y)$ for $y = 1$, therefore (62) is indeed true for $y \geq 0$.

C.6 Proof for unique maximum of $\mathbb{E}_{\eta_0} L(\eta)$

Recall,

$$\begin{aligned}\mathbb{E}_{\eta_0} L(\eta) &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \frac{n}{2} \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{g^2(\eta_i)} - \sum_{i=1}^p \log(g(\eta_i)) - \frac{1}{2} \sum_{i=1}^p \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} \\ &= \frac{p}{2} \log\left(\frac{n}{2\pi}\right) - \frac{n}{2} \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{g^2(\eta_i)} - p \log(g(\eta_{0,i})) + \sum_{i=1}^p \log\left[\frac{g(\eta_{0,i})}{g(\eta_i)}\right] - \frac{1}{2} \sum_{i=1}^p \frac{g^2(\eta_{0,i})}{g^2(\eta_i)} \\ &= \frac{p}{2} \log\left(\frac{n}{2\pi g^2(\eta_{0,i})}\right) - \frac{n}{2} \sum_{i=1}^p \frac{(\eta_{0,i} - \eta_i)^2}{g^2(\eta_i)} - \frac{1}{2} \sum_{i=1}^p \left(\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - \log\left[\frac{g^2(\eta_{0,i})}{g^2(\eta_i)}\right]\right), \\ \max_{\eta} \mathbb{E}_{\eta_0} L(\eta) &\leq \frac{p}{2} \log\left(\frac{n}{2\pi g^2(\eta_{0,i})}\right) - \frac{n}{2} \sum_{i=1}^p \min_{\eta_i} \left[\frac{(\eta_{0,i} - \eta_i)^2}{g^2(\eta_i)}\right] - \frac{1}{2} \sum_{i=1}^p \min_{\eta_i} \left(\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - \log\left[\frac{g^2(\eta_{0,i})}{g^2(\eta_i)}\right]\right).\end{aligned}$$

Note,

$$\min_{\eta_i} \frac{(\eta_{0,i} - \eta_i)^2}{g^2(\eta_i)} = \frac{(\eta_{0,i} - \eta_i)^2}{g^2(\eta_i)} \Big|_{\eta_i = \eta_{0,i}} = 0.$$

Furthermore, using C.5,

$$\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - \log\left[\frac{g^2(\eta_{0,i})}{g^2(\eta_i)}\right] \geq 1 \implies \min_{\eta_i} \left(\frac{g^2(\eta_{0,i})}{g^2(\eta_i)} - \log\left[\frac{g^2(\eta_{0,i})}{g^2(\eta_i)}\right]\right) = 1,$$

when $g(\eta_i) = g(\eta_{0,i})$. Therefore, η_0 will be the unique maximum of $\mathbb{E}_{\eta_0} L(\eta)$ iff $g(\cdot)$ is unique at every $\eta_{0,i}$.

C.7 Proof for Proposition 5.1

Recall,

$$\begin{aligned}\tilde{\eta}_0 &= \eta_0 + D_0^{-2} \nabla L(\eta_0), \\ (D_0^{-2})_{i,i} &= \frac{n^{-1} g^2(\eta_{0,i})}{1 + 2n^{-1} [g'(\eta_{0,i})]^2}, \\ (\nabla L(\eta_0))_i &= \frac{g'(\eta_{0,i})}{n^{-1} g^3(\eta_{0,i})} (Y_i - \eta_{0,i})^2 + \frac{1}{n^{-1} g^2(\eta_{0,i})} (Y_i - \eta_{0,i}) - \frac{g'(\eta_{0,i})}{g(\eta_{0,i})}.\end{aligned}$$

Thus,

$$\begin{aligned}(\eta - \tilde{\eta}_0)_i &= \eta_i - \eta_{0,i} - (D_0^{-2})_{i,i} \left[\frac{g'(\eta_{0,i})}{n^{-1} g^3(\eta_{0,i})} (Y_i - \eta_{0,i})^2 + \frac{1}{n^{-1} g^2(\eta_{0,i})} (Y_i - \eta_{0,i}) - \frac{g'(\eta_{0,i})}{g(\eta_{0,i})} \right] \\ &= \eta_i - \eta_{0,i} - [A_i (Y_i - \eta_{0,i})^2 + B_i (Y_i - \eta_{0,i}) + C_i],\end{aligned}$$

where

$$A_i = \frac{g'(\eta_{0,i})}{g(\eta_{0,i})(1 + 2n^{-1}[g'(\eta_{0,i})]^2)}, \quad B_i = \frac{1}{1 + 2n^{-1}[g'(\eta_{0,i})]^2}, \quad \text{and} \quad C_i = -A_i n^{-1} g^2(\eta_{0,i}).$$

Therefore, applying Theorem 2 to the direct case, and the indirect case i.e. $\eta_i := k_i \mu_i$

$$\begin{aligned} \mathbb{E}((\eta - \tilde{\eta}_0)_i | Y_i) = 0 &\implies \mathbb{E}(\eta_i | Y_i) = \tilde{\eta}_{0,i} = \eta_{0,i} + [A_i(Y_i - \eta_{0,i})^2 + B_i(Y_i - \eta_{0,i}) + C_i], \\ &\implies \mathbb{E}(\mu_i | Y_i) = \mu_{0,i} + k_i^{-1}[A_i(Y_i - k_i \mu_{0,i})^2 + B_i(Y_i - k_i \mu_{0,i}) + C_i], \\ \mathbb{V}ar((D_0)_{i,i} \eta_i | Y_i) = 1 &\implies \mathbb{V}ar(\eta_i | Y_i) = (D_0)_{i,i}^{-2}, \\ &\implies \mathbb{V}ar(\mu_i | Y_i) = k_i^{-2} (D_0^{-2})_{i,i}. \end{aligned}$$

For the direct case, let $X_i := A_i(Y_i - \eta_{0,i})^2 + B_i(Y_i - \eta_{0,i}) + C_i$ and assume $Y_i \sim N(\eta_i, n^{-1}g^2(\eta_i))$, then using C.8 ,

$$\begin{aligned} \mathbb{E}_{Y_i|\eta_{0,i}}(X_i) &= A_i \mathbb{E}_{Y_i|\eta_{0,i}}(Y_i - \eta_{0,i})^2 + B_i \mathbb{E}_{Y_i|\eta_{0,i}}(Y_i - \eta_{0,i}) + C_i \\ &= A_i n^{-1} g^2(\eta_{0,i}) + C_i = 0, \\ \mathbb{V}ar_{Y_i|\eta_{0,i}}(X_i) &= A_i^2 \mathbb{V}ar_{Y_i|\eta_{0,i}}(Y_i - \eta_{0,i})^2 + B_i^2 \mathbb{V}ar_{Y_i|\eta_{0,i}}(Y_i - \eta_{0,i}) \\ &= A_i^2 2n^{-2} g^4(\eta_{0,i}) + B_i^2 n^{-1} g^2(\eta_{0,i}). \end{aligned}$$

C.8 Proof for $\mathbb{V}ar(Y_i - \eta_{0,i})^2$

Recall $\mathbb{E}(X^p) = \sigma^p(p-1)!! \mathbb{I}\{p \text{ is even}\}$ when $X \sim N(0, \sigma^2)$. Hence, let $X := Y_i - \eta_{0,i}$, then $X \sim N(0, \sigma^2)$, where $\sigma^2 = n^{-1}g^2(\eta_{0,i})$. Consequently,

$$\begin{aligned} \mathbb{E}(Y_i - \eta_{0,i}) &= \mathbb{E}(X) = 0, \\ \mathbb{E}(Y_i - \eta_{0,i})^2 &= \mathbb{E}(X^2) = \sigma^2, \\ \mathbb{E}(Y_i - \eta_{0,i})^4 &= \mathbb{E}(X^4) = 3!!\sigma^4 = 3\sigma^4, \\ \mathbb{V}ar(Y_i - \eta_{0,i})^2 &= \mathbb{V}ar(X^2) = \mathbb{E}(X^4) - [\mathbb{E}(X^2)]^2 = 2\sigma^4. \end{aligned}$$

C.9 Proof for Corollary 5.5

For the posterior distribution, where we set $\eta := \eta|Y$ for brevity,

$$\begin{aligned} \mathbb{E}||\eta - \eta_0||^2 &= \mathbb{E} \sum_{i=1}^p (\eta_i - \eta_{0,i})^2 = \sum_{i=1}^p \mathbb{V}ar(\eta_i) + (\mathbb{E}\eta_i - \eta_{0,i})^2 = \sum_{i=1}^p \mathbb{V}ar(\eta_i) + X_i^2, \\ \mathbb{E}_{\eta_0} \mathbb{E}||\eta - \eta_0||^2 &= \sum_{i=1}^p \mathbb{V}ar(\eta_i) + \mathbb{E}_{\eta_0}(X_i^2) = \sum_{i=1}^p \mathbb{V}ar(\eta_i) + \mathbb{V}ar_{Y|\eta_0}(X_i) + [\mathbb{E}_{\eta_0}(X_i)]^2 \\ &= \sum_{i=1}^p (D_0^{-2})_{i,i} + A_i^2 2n^{-2} g^4(\eta_{0,i}) + B_i^2 n^{-1} g^2(\eta_{0,i}), \end{aligned}$$

where $X_i^2 = (\mathbb{E}\eta_i - \eta_{0,i})^2$.

For the indirect case, i.e. when $\eta_i = k_i \mu_i$

$$\begin{aligned}\mathbb{E}||\mu - \mu_0||^2 &= \mathbb{E} \sum_{i=1}^p (\mu_i - \mu_{0,i})^2 = \sum_{i=1}^p \mathbb{V}ar(\mu_i) + (\mathbb{E}\mu_i - \mu_{0,i})^2 = \sum_{i=1}^p \mathbb{V}ar(\mu_i) + \tilde{X}_i^2, \\ \mathbb{E}_{\mu_0} \mathbb{E}||\mu - \mu_0||^2 &= \sum_{i=1}^p \mathbb{V}ar(\mu_i) + \mathbb{E}_{\mu_0}(\tilde{X}_i^2) = \sum_{i=1}^p \mathbb{V}ar(\mu_i) + \mathbb{V}ar_{\mu_0}(\tilde{X}_i) + [\mathbb{E}_{\mu_0}(\tilde{X}_i)]^2 \\ &= \sum_{i=1}^p k_i^{-2} (D_0^{-2})_{i,i} + k_i^{-2} [A_i^2 2n^{-2} g^4(k_i \mu_{0,i}) + B_i^2 n^{-1} g^2(k_i \mu_{0,i})],\end{aligned}$$

where $\tilde{X}_i^2 = (\mathbb{E}\mu_i - \mu_{0,i})^2$.

We proved the above statements using the following lemma:

Lemma C.1. *If $\{X_i\}_{i=1}^p$ are random variables and $(c_i)_{i=1}^p \in \mathbb{R}^p$, then*

$$\begin{aligned}\mathbb{E} \sum_{i=1}^p (X_i - c_i)^2 &= \sum_{i=1}^p \mathbb{E}X_i^2 + c_i^2 - 2c_i \mathbb{E}X_i \\ &= \sum_{i=1}^p \mathbb{V}ar(X_i) + (\mathbb{E}X_i)^2 + c_i^2 - 2c_i \mathbb{E}X_i \\ &= \sum_{i=1}^p \mathbb{V}ar(X_i) + (\mathbb{E}X_i - c_i)^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}\mathbb{E} \sum_{i=1}^p X_i^2 &= \sum_{i=1}^p \mathbb{E}([X_i - \mathbb{E}X_i] + \mathbb{E}X_i)^2 \\ &= \sum_{i=1}^p \mathbb{E}(X_i - \mathbb{E}X_i)^2 + [\mathbb{E}X_i]^2 + 2\mathbb{E}(X_i - \mathbb{E}X_i)\mathbb{E}X_i \\ &= \sum_{i=1}^p \mathbb{V}ar(X_i) + [\mathbb{E}X_i]^2.\end{aligned}$$

C.10 Proof for $\mathbb{V}ar(aX + bX^2)$

Note, for $X \sim N(0, \sigma^2)$, $\mathbb{E}(X^p) = \sigma^p(p-1)!!\mathbb{I}\{p \text{ is even}\}$. Consequently,

$$\begin{aligned}\mathbb{V}ar(aX + bX^2) &= \mathbb{E}(aX + bX^2)^2 - [\mathbb{E}(aX + bX^2)]^2 = \mathbb{E}(a^2X^2 + b^2X^4 + 2abX^3) - [b\mathbb{E}(X^2)]^2 \\ &= a^2\sigma^2 + 3b^2\sigma^4 - [b\sigma^2]^2 = a^2\sigma^2 + 2b^2\sigma^4,\end{aligned}$$

where $a, b \in \mathbb{R}$.

C.11 Proof of $\mathbb{E}[\exp(AX^2 + BX + C)]$, where $X \sim N(0, \sigma^2)$

Given that $X \sim N(0, \sigma^2)$,

$$\begin{aligned}\mathbb{E}[\exp(AX^2 + BX + C)] &= \int_x \frac{1}{\sqrt{2\pi\sigma^2}} \exp(Ax^2 + Bx + C) \exp(-\frac{x^2}{2\sigma^2}) dx \\ &= \exp(C) \int_x \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2}[-2A + \frac{1}{\sigma^2}]x^2 - 2Bx) dx.\end{aligned}$$

Let $\tilde{A} = (-2A + \frac{1}{\sigma^2}) = \frac{1-2A\sigma^2}{\sigma^2}$, and $\tilde{B} = B\tilde{A}^{-1}$, then

$$(-2A + \frac{1}{\sigma^2})x^2 - 2Bx = \tilde{A}(x^2 - 2\tilde{B}x) = \tilde{A}[(x - \tilde{B})^2 - \tilde{B}^2] = \tilde{A}(x - \tilde{B})^2 - \tilde{A}\tilde{B}^2.$$

Hence,

$$\begin{aligned}\mathbb{E}[\exp(AX^2 + BX + C)] &= \exp(C) \int_x \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2}[-2A + \frac{1}{\sigma^2}]x^2 - 2Bx) dx \\ &= \exp(C) \int_x \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2}[\tilde{A}(x - \tilde{B})^2 - \tilde{A}\tilde{B}^2]) dx \\ &= \exp(C + \frac{1}{2}\tilde{A}\tilde{B}^2) \int_x \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2}\tilde{A}(x - \tilde{B})^2) dx \\ &= \frac{\tilde{\sigma}}{\sigma} \exp(C + \frac{1}{2}\tilde{A}\tilde{B}^2) \int_x \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} \exp(-\frac{(x - \tilde{B})^2}{2\tilde{\sigma}^2}) dx,\end{aligned}$$

where $\tilde{\sigma}^2 = \tilde{A}^{-1}$.

Thus, if \tilde{B} and $\tilde{\sigma}$ are well-defined,

$$\begin{aligned}\mathbb{E}[\exp(AX^2 + BX + C)] &= \frac{(\frac{1-2A\sigma^2}{\sigma^2})^{-1/2}}{\sigma} \exp(C + \frac{1}{2}B^2(\frac{1-2A\sigma^2}{\sigma^2})^{-1}) \\ &= \frac{1}{\sqrt{1-2A\sigma^2}} \exp(C + \frac{B^2\sigma^2}{2(1-2A\sigma^2)}).\end{aligned}$$

C.12 Proof for Lemma 5.11

Proof. The derivatives of g can be expressed using g itself, i.e.

$$\begin{aligned}g' &= \frac{a}{2g} \iff a = (g^2)' = 2gg', \\ g'' &= -\frac{ag'}{2g^2} = -\frac{a^2}{4g^3}.\end{aligned}$$

□

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