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German Idealism and the Origins of Pure Mathematics: Riemann, Dedekind, Cantor

Ehsan Karimi Torshizi (corresponding author)
PhD in philosophy, Allameh Tabataba'i University, Tehran. Iran.
ehsankarimi44@gmail.com

Abstract

When it comes to the relation of modern mathematics and philosophy, most people tend to think of the three major schools of thought—i.e. logicism, formalism, and intuitionism—that emerged as profound researches on the foundations and nature of mathematics in the beginning of the 20th century and have shaped the dominant discourse of an autonomous discipline of analytic philosophy, generally known under the rubric of "philosophy of mathematics" since then. What has been completely disregarded by these philosophical attitudes, these foundational researches which seek to provide pure mathematics with a philosophically plausible justification by founding it on firm logico-philosophical bases, is that the genuine self-foundation of pure mathematics had been done before, namely during the 19th century, when it was developing into an entirely new and independent discipline as a concomitant of the continuous dissociation of mathematics from the physical world. This self-foundation of the 19th-century pure mathematics, however, was more akin to the German-idealist interpretations of Kant's transcendental philosophy, than the post-factum, retrospective 20th-century researches on the foundations of mathematics. This article aims to demonstrate this neglected historical fact via delving into the philosophical inclinations of the three major founders of the 19th-century pure mathematics, Riemann, Dedekind and Cantor. Consequently, pure mathematics, with respect to its idealist origins, proves to be a formalization and idealization of certain activities specific to a self-conscious transcendental subjectivity.

Keywords: pure mathematics, German idealism, logicism, formalism, intuitionism, Riemann, Dedekind, Cantor

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Introduction

In his magnum opus The Structure of Scientific Revolutions (1962), Thomas Kuhn, the major American philosopher of science, explains how scientists tend to concentrate on philosophical issues which lie at the bases of scientific theories, and scientific debates consequently become more and more philosophical in the course of a scientific revolution, or "paradigm shift", to use Kuhn's technical jargon. When a scientific discipline is faced with a crisis in its basic concepts and undergoes a revolution in its foundations, the philosophical substrata that have before been concealed behind the façade of scientific practice, come to the fore and become the prime center of attention. What scientific crises reveal, therefore, is the urgent necessity of a more radical investigation into the philosophical foundations of sciences (see Kuhn, 1996: 87 f.).

Martin Heidegger, one of the greatest philosophers of the 20th century, who is arguably among the main sources of inspiration from whom Thomas Kuhn gets much of his ideas, points out to the same idea; in the introduction of his magnum opus Sein und Zeit (1927), he writes:

> The real "movement" (Bewegung) of the sciences takes place in the revision of these basic concepts, a revision which is more or less radical and lucid with regard to itself. A science's level of development is determined by the extent to which it is capable (fähig) of a crisis in its basic concepts. In these immanent crises of the sciences, the relation of positive questioning to the matter in question becomes unstable. Today tendencies to place research on new foundations have cropped up on all sides in the various disciplines (Heidegger, 1967: 9).

The basic concepts of a scientific discipline provide us with a preliminary understanding of its thematic field within which the scientific objects are to be understood beforehand. The crisis in the basic concepts, therefore, demands an ontological investigation into the thematic field underlying all the objects of science that is more radical than any positive scientific investigation. Heidegger then proceeds to illustrate this point with some examples drawn from his contemporary scientific crises. He particularly mentions the crisis in the foundations of modern mathematics:

> The discipline which is seemingly the strictest and most securely structured, mathematics, has experienced a "crisis in its foundations" (Grundlagenkrisis). The controversy between formalism and intuitionism centers on obtaining and securing primary access to what should be the proper object of this science (ibid.).

Both philosophers thus place great emphasis on how scientific theories, at their deepest conceptual layers, are inextricably interwoven with philosophical assumptions, conceptions, and even Weltanschauungen, and both suggest a more original philosophical inquiry that must precede the positive sciences themselves. The present article, seeks to employ this profound insight in exploring the philosophical origins of the 19th-century newly emerging pure mathematics.

As mentioned above, Heidegger takes the philosophical controversy between Hilbert's formalism and Brauwer's intuitionism—one must include also Frege and Russel's logicism—as an indication of crisis par excellence in the foundations of modern mathematics at the beginning of the 20th century. What Heidegger claims here is endorsed also by Morris Kline, perhaps the most important historian of mathematics:

> By far the most profound activity of twentieth-century mathematics has been the research on the foundations. The problems thrust upon the mathematicians and others that they voluntarily assumed, concern not only the nature of mathematics but the validity of deductive mathematics (Kline, 1972: 1182).

What motivated this tendency toward the research on the foundations was the discovery of contradictions or paradoxes, notably in Cantor's set theory, an unsettling fact that disturbed mathematicians deeply at the turn of the 20th century. Needless to say, this historical view is so prevailing that is taken for granted by almost all philosophers of mathematics; they tend to begin their philosophy of modern mathematics with this "research on the foundations" by the three great schools of philosophy of mathematics, "the big three" (Shapiro, 2000: 107 ff.), at the early 20th century: logicism, formalism, and intuitionism.

It is not the whole truth, however. The research on the foundations of mathematics in the early 20th century is itself an effect of a more profound and radical development in mathematics, a "genuine" paradigm shift, occurred earlier, particularly in the second half of the 19th century. During the 19th century, "pure" mathematics, in its strictest and particular sense, began to emerge as a concomitant of continuous dissociation of mathematics from the physical world. Purely mathematical concepts that had no clear physical meaning whatsoever was invented by 19th mathematicians as proper objects of mathematical study, concepts like n-dimensional continua (manifolds), non-Euclidean geometries, complex numbers, and transfinite numbers, to name but a few, that have no clear and direct relation to our experience of physical reality. Pure mathematics defined itself over against natural sciences as an entirely independent and autonomous field of study, by introducing a new thematic field of objects all of its own. It was a genuinely new beginning that reached its pinnacle in the early 20th century; Kline's words are sufficiently clear in the articulation of the pivot of the matter:

> It was true in the nineteenth as in the two preceding centuries that the progress in mathematics brought with it larger changes barely perceptible in the year-to-year developments but vital in themselves and in their effect on future developments. The vast expansion in subject matter and the opening of new fields, as well as the extension of older ones, are of course apparent. [...] The circle within which mathematical studies appeared to be enclosed at the beginning of the 19th century was broken at all points. Mathematics exploded into a hundred branches. The flood of new results contradicted sharply the leading opinion at the end of the eighteenth century that the mine of mathematics was exhausted (Kline, 1972: 1023).

The emergent mathematics during the 19th century was not indeed what ancient Greeks thought to be the language of gods anymore; it was not a "divine thing", a necessary part of the noetic constitution of the cosmos as a twofold paradigm-copy structure that brought with it not only intelligibility but also goodness and beauty. It was neither the language of the book of nature as it was thought to be during the scientific revolution nor a mere, though indispensable, instrument of natural sciences, thoroughly overshadowed by the domination of empiricism and positivism. In order to prove its intelligibility and reliability, pure mathematics thus was in want of a philosophy; a philosophy that could provide its new thematic field of objects with a new ontology, and justify its entirely new epistemological status within the totality of human cognitions. German Idealism, from Kant to Hegel, was indeed among the most influential lines of thought both to support and to motivate this new emergent mathematics. In order to understand the origins of pure mathematics, it is inevitable to place it within this philosophical context.

In doing so, I shall constraint myself to the three thematic fields of numbers, spaces, and sets. I shall also concentrate on three well-known mathematicians, Richard Dedekind (1831-1916), Bernhard Riemann (1826-1866), and Georg Cantor (1845-1918), whose groundbreaking works on these thematic fields have subsequently given rise to totally new directions in the mathematical study. Before proceeding to the discussion of the origins of pure mathematics in detail, it is necessary first to have a conception of what German idealism is in its essence. In the next section, I attempt to provide a clear, yet succinct, account of the substance of an idealistic interpretation of Kant's Kritik der reinen Vernunft, as a backdrop for the ensuing sections on the origins of pure mathematics[1].

German Idealism: the unity of self-consciousness

One of the most central theses of Kant's first Kritik, and the entire German idealist tradition, is the correlation between consciousness and self-consciousness. As he famously claims in Transcendental Deduction:

> The I think must be able to accompany all my representations; for otherwise something would be represented in me that could not be thought at all, which is as much as to say that the representation would either be impossible or else at least would be nothing for me (B 131-2).

To become conscious of something, to know it as a possible object of experience, that thing must be grasped (begreifen) as "a" thing, as a synthetic unity instilled by the concepts (Begriff) of understanding into the manifold of sensible intuitions. The contents of my consciousness thus weave together and form a synthetic objective unity. Now, the central thesis of Transcendental Deduction implies a necessary correlation of the objective unity of consciousness with the subjective unity of consciousness, that is the unity of self-consciousness, or as Kant calls it, "synthetic unity of apperception" (A 106-107, 116). A. C. Ewing has nicely formulated this correlation; human knowledge is possible only as a synthesis that has two closely interrelated aspects:

> From the synthesis Kant passes to the systematic unity of objects on the one hand and to the transcendental unity of apperception on the other. If all knowledge requires a synthesis, on the one hand the objects known must constitute or be made into a system, and on the other hand there is a unity of consciousness presupposed in all cognition, the transcendental unity of apperception (Ewing, 1938: 78).

So to say that experience is always subject to the synthetic unity of apperception, i.e. ascribable to one identical "I", is to say that the experience is always subject to the understanding, i.e. the power of synthesizing and grasping the manifold of experience as one objective unity, "a representational unity that makes reference to an object possible" (Pippin, 2014: 147). In order to refer to an object as real, not mere seeming, a subject must have already identified itself as a unity of self-consciousness (c.f. Henrich, 1982: 135-6). It is in this sense that the transcendental unity of apperception, as Kant claims, "is the highest point to which one must affix all use of the understanding, even the whole of logic and, after it, transcendental philosophy; indeed this faculty is the understanding itself" (B 134).

Hegel radically furthers this thesis, asserting that every consciousness is self-consciousness at the bottom, and then the totality of our relations to the world (Weltbezug) is to be construed as arisen out of the inner sphere of our self-relation (Selbstbeziehung). He understands the central thesis of Kant's Transcendental Deduction in this way:

> It is one of the most profound and truest insights to be found in the Critique of Reason that the unity which constitutes the essence of

> the concept is recognized as the original synthetic unity of apperception, the unity of the "I think," or of self-consciousness.— This proposition is all that there is to the so-called transcendental deduction of the categories […] (Hegel, 2010, as cited in Pippin, 2014: 146-7)[2].

This unity of self-consciousness in its correlation with the objective unity of consciousness is thus the connecting thread running through the entire German idealist tradition, from Kant to Hegel.

The genuinely post-Kantian idealist step, however, had been already taken by Johann Gottlieb Fichte who substituted the activity of opposing (Gegensetzung) for Kant's synthesis as the basic structural constitution of consciousness (see: Henrich, 2008, 166, 174ff.). Instead of a synthetic correlation, consciousness begins with an "I" that in its absolute positing of its own existence as a self-relation (Selbsbeziehung), is at the same time absolutely opposed to a "not-I" (ibid.: 207).

The unity of consciousness is not achieved but at the pinnacle of a dialectic procedure in which oppositions are developed and reconciled in the mind. Needless to say, we are already in the heart of Hegel's dialectic philosophy. Against this background, one also may extract an idealist theory of self-consciousness as an absolute or transcendental "I".

Dedekind's Transcendental Philosophy of Elementary Arithmetic and Real Analysis

The central concept of Dedekind's mathematics is "system", in terms of which any other concept is to be defined. Let me begin with what "system" is not:

i. It is not a set, neither in its usual, intuitive sense nor in its most strict sense, i.e. in Zermelo–Fraenkel axiomatic set theory.
ii. It is not a class, i.e. a collection of sets defined by a formula whose quantifiers range only over sets. Particularly, it is not a proper class, an entity that does not belong to other entities.
iii. It is not a formal deductive system, in its Fregean sense, i.e. an abstract structure of propositions or statements used for inferring theorems from axioms according to a set of rules.

Rather, it has an essential and strong Kantian connotation. As Kantian technical jargon, "system" signifies an intellectual or mental structure, according to which a multiplicity of cognitions is thought as a genuine and perfect unity by employing an "idea" of reason to it. In Kant's own words:

> But reason cannot think this systematic unity in any other way than by giving its idea an object, which, however, cannot be given through any experience; for experience never gives an example of perfect systematic unity. Now this being of reason (ens rationis

> ratiocinatae) is [...] a mere idea [...] so as to regard all the collection of things in the world of sense as if they had their ground in this being of reason [...] and one [...] posits an idea only as a unique standpoint from which alone one can extend the unity that is so essential to reason and so salutary to the understanding (A 681/ B 709).

Dedekind's first step is to demonstrate that there is an infinite system. It is not something that can be postulated as an axiom; in order to show that the concept itself is not inconsistent, one has to introduce such a system. He surprisingly proposes his Gedankenwelt (thoughtworld) as a paradigmatic: "my thoughtworld (meine Gedankenwelt), that is the totality S of all things that can be objects of my thinking, is infinite" (Dedekind, 1932: 357). He then proceeds to mathematically prove this in a strange way; he defines φ as a transformation of S into itself, that takes any element s of S into "the thought s', that s can be an object of my thinking" (ibid.). It is easy to see, as Dedekind claims, that φ is a one-to-one, but not onto mapping, in that S' = φ(S), i.e. the image of S under φ, is a proper part of S. He proves this with recourse to a very special element of his Gedankenwelt: "because there are elements of S (e.g. my own I/ mein eigenes Ich), that are different from every such thought s' and so are not contained in S'" (ibid.). Now S is infinite by definition: "a system S is said to be infinite if it is similar (ähnlich) to a proper part of itself" or equally, "when there is a proper part of S, into which S can be similarly [= injectively] mapped"[3] (ibid.: 356).

Such systems are called "simply infinite" (einfach unendlich) that is "ordered by a mapping φ", and that special element of S which is not in φ(S) is called "basic element" (Grundelement) of S, designated by the symbol "1" (ibid.: 359).

I said before that Dedekind's Gedankenwelt is not to be considered as a set or proper class. Now, what about φ? There are some possibilities to make sense of it in terms of our philosophical logic (see McCarty, 1995: 56-58):

- φ is a syntactic operation and S a hypothetical mental system, resembling language, by having the lexical structure of a syntax algebra. On this account, φ would be both well-defined and one-to-one. The main drawback of this syntactical approach is its circularity, in that S as a syntax algebra has already what Dedekin seeks to prove.
- From a semantic viewpoint, φ consists of intensional operators embedded in its formulation, and according to whether one reads the variable s de re or de dicto, φ either is not well-defined or is not one-to-one, respectively.

Either way, φ puts forth difficulties for our elementary philosophical logic. What Dedekind suggests here is not merely a mathematical theorem and its proof; rather, the pivot of the matter is to lay a transcendental ground for the foundations of mathematics. The grounding of the inner possibility of mathematics is to be brought about via the unveiling of its transcendental roots. Yet, we need to go further in this direction.

Dedekind defines the system of natural numbers on the basis of what he calls "a simply infinite system" (einfach unendlich):

> A system N is called simply infinite if there is such a similar mapping φ of N into itself that N appears as a chain (Kette) of an element which is not contained in φ(N) (Dedekin, 1932, 359).

N is said to be ordered by φ, and that special element of N which is not in φ(N), is called "the basic element", designated by the symbol "1". Therefore, a simply infinite system N ordered by φ turns out to be the closure of the singleton system, containing "1", under φ, that is Dedekind's version of the successor function (see, McCarthy, 1995:74):

$$1, \varphi(1), \varphi(\varphi(1)), \varphi(\varphi(\varphi(1))), \ldots$$

His paradigmatic of a simply infinite system N, thus, may be constructed as a chain of his eigenes Ich, ordered by φ—the similar mapping that takes any element s of Dedekind's Gedankenwelt into the thought that s can be an object of his thinking—as follows:

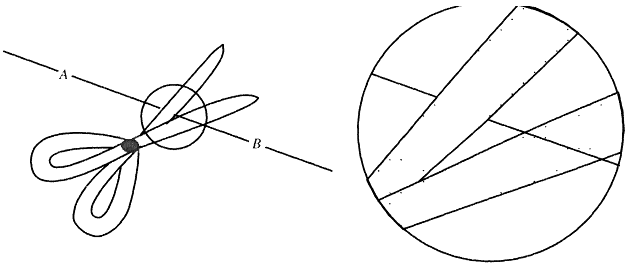
$$I, \varphi(I), \varphi(\varphi(I)), \varphi(\varphi(\varphi(I))), \ldots$$

He then defines the series of natural numbers as the residue of a procedure of abstraction, i.e. a structural reduction, from an unspecified simply infinite system:

> If one completely disregards the special nature of elements of a simply infinite system N, ordered by a mapping φ, and adheres merely to their discriminability and grasps only the relations in which they are placed with each other by the ordering map φ, then these elements are called natural numbers [...] and the basic element 1 is called the basic number of the series of numbers N. In view of this liberation of the elements from any other content (abstraction) one can rightly call the numbers a free creation (freie Schöpfung) of the human mind. (Dedekind, 1932: 360).

And lastly, since there are many such systems, he must also vindicate his definition by proving the uniqueness of N: "Every system that is similar to a simply infinite system, and so to the series of numbers N, is itself simply infinite" (Dedekind, 1932: 376). The uniqueness is proved in fact by showing that any two simply infinite systems are similar; they are isomorphic in that they have the same abstract structure. Therefore, Dedekind's criterion for being the system of natural numbers is: being obtained by the relevant abstraction from a simply infinite system (N-criterion).

There is a quite straightforward path leading from natural to rational numbers. The difficulty arises when it comes to the irrational relations (like √2, the length of the diagonal of the unite square), as the history of their discovery by Pythagoreans and the mathematical-philosophical crisis it subsequently provoked, evidently attest. Dedekind defines them as the "holes" (Lücken) between the rational numbers, so they can be recognized solely according to the ordering structure. In other words, each irrational number is defined by the system of the smaller and the system of larger numbers, and is entirely determined by this partition (Zerlegung) of the rational numbers, known as "Dedekind cut" (Dedekindscher Schnitt), as illustrated in this figure[4]:



The real numbers (that is, all rational and irrational numbers) are so constructed by analogy to a geometrical one-dimensional continuum (simply, a straight line), and irrationals are visualized as the points where this line may be cut with scissors. In Dedekind's own words, "the domain of real numbers" is defined as an ordered extension of the rational numbers satisfying the Continuity Principle:

> If the system R of all real numbers is divided into two classes U_1 and U_2 in such a way that every number α_1 of the class U_1 is smaller than every number α_2 of the class U_2, then there is one and only one number α, by which this partition is produced (Dedekind, 1932: 329).

Namely, each of its cuts has its point of division in that domain. This criterion by which Dedekind grasps "the essence of continuity" (das Wesen der Stetigkeit) is again inspired by the continuity of a straight line (Ibid: 322). Contrary to the N-criterion, however, he provides no uniqueness proof, no relevant isomorphism theorem for possibly dissimilar continua. More specifically, he explicitly speaks of two continua, the geometric continuum (straight line), and the arithmetic continuum (the domain of real numbers constructed by Dedekind cuts) (Ibid: 319-20). There is indeed an analogy between the two; real numbers can be visualized as points, i.e. distances from an origin picked up on a straight line. Dedekind himself speaks also of "arithmetically pursuing every appearance (Erscheinung) in the straight line" in order to show for the real numbers "the same completeness (Vollständigkeit) [...], the same continuity as the straight line" (Ibid: 321). This useful visual analogy, however, is never developed into a mathematically well-defined isomorphism. On the contrary, in view of Dedekind's philosophy of mathematics, the twin continua are to be considered as mathematically non-identical. They are analogous entities of an entirely different nature.

As D. C. McCarty suggests, in order to make sense of all these, one has to read them as an answer to the transcendental Kantian question: "how is mathematics possible?". Dedekind's foundational project thus constitutes a transcendental deduction from the elements of thoughts of reason to the principles of arithmetic and real analysis (McCarty, 1995: 70-71). Not only the terms but also the whole overarching conceptual framework is Kantian, i.e. articulated according to Kant's model of the transcendental mind. Here, I cannot but briefly enumerate the most important Kantian-transcendental motifs of Dedekind's philosophy of mathematics:

- Dedekind's system, as I said before, is a Kantian term, signifying an intellectual or mental structure, according to which a multiplicity of cognitions is thought as a genuine and perfect unity by employing an "idea" of reason to it.
- Dedekind's Mathematical objects are among the pure ideas of Kant.
- Dedekind's Gedankenwelt is akin to Kant's theological ideal, i.e. the idea of the totality of all beings of reason, "the absolute unity of the condition of all objects of thought in general" (A334/ B391).
- The domain of pure mathematics is the domain of pure ideas of reason.
- Included within this totality is Kant's psychological idea, transcendental "I", to which Dedekind refers as "mein eigenes Ich". So it does not refer to the contingent existence of a particular person, but the transcendental unity of pure self-consciousness.
- Dedekind's infinity proof for Gedankenwelt is modeled on Kant's way of explaining why the series of all conditions or grounds (inferential chains) for an arbitrary inevident judgment of understanding known by reason a priori is to be thought as a completed infinite totality, or an infinite regression (A 336-7/ B 393-4; A 417/ B 445). On this ground, the closure of the singleton set {Ich} under φ, may well be construed as an infinite series of conditions a priori for knowing that "Ich" is an object of thought (see McCarty, : 74-5).
- φ is both one-to-one and well-defined, since a) every Gedankending is an object of thought, i.e. internally represented, as an object is nothing but the synthetic objective unity of representations, so for every Gedankending s there is a corresponding thought φ(s); b) synthesis peculiar to pure reason is Vernunftschluss (logical consequence), i.e. they are individuated according to a purely logical notion of identity (intersubstitutability salva veritate): two objects of reason are identical (s = t) if they are logically intersubstitutable, that is indiscernible as far as the pure reason is concerned. Thus, if s = t then φ(s) = φ(t), in that they

are intersubstitutable salva veritate, logically indiscernible, as objects of pure reason; and c) with such a logical concept of identity the injectivity of φ may be similarly demonstrated, in that if φ(s) = φ(t), they are logically indiscernible, i.e. pure reason can discern at most one subject for each of them (s = t).

- So arithmetic is constructed solely on the basis of a transcendental deduction of the system of natural numbers from the transcendental unity of pure self-consciousness.
- The twin continua are non-identical; the arithmetic is among pure ideas of reason, the geometrical, which is subject to spatial and perceptual predicates, is a spatial manifold belonging to our sensational intuitions. In Kant's terms, the former belongs to Transcendental Dialectic, the latter to Transcendental Aesthetic. Since the arithmetic continuum as a pure idea has no corresponding object in our sense-experience, there exists no mathematical correspondence between the two.

In the end, it must be noted that there is a major difference between Dedekind's and Kant's philosophy of mathematics. For Kant, numbers are akin to the sense experience inasmuch as they are constructed in our inner intuition of time, whereas for Dedekind they are more logical and must be subsumed into the realm of pure ideas.

Bernhard Riemann: a Conceptual Doctrine of Space

Contrary to Dedekind whose transcendental philosophy of mathematics centers on the ideas of pure reason, Riemann begins his habilitation lecture at Güttingen entitled Über die Hypothesen welche der Geometrie zu Grunde liegen (1854), with a philosophical inquiry into the general concept (Begriff) of quantity:

> I have therefore first set myself the task of constructing the concept of a multiply extended quantity from general notions of quantity. It will be shown that a multiply extended quantity is susceptible of various metric relations, so that space constitutes only a special case of a triply extended quantity. from this, however, it is a necessary consequence that the theorems of geometry cannot be deduced from general notions of quantity, but that those properties that distinguish space from other conceivable triply extended quantities can only be deduced from experience (Riemann, 2007: 23).

Pursuing the nature and the essential properties of space as well as its relation to the multiplicity of possible geometries, Riemann embarked upon a totally new and novel project. Although he borrowed some of his main ideas from Gauss, General Investigations of Curved Surfaces–such as considering a two-dimensional curved surface as a self-sufficient space, having its own intrinsic geometry which may be different from the geometry of the encompassing three-dimensional Euclidean space, in which the curved surface is embedded–his own approach to the subject is more meticulous and comprehensive both mathematically and philosophically. The characteristics of this new approach can be described in summary fashion as follows (see, Karimi Torshizi, 2012: 38-9):

- First, from a knowing subject's perspective, Riemann intends to know what characteristics of space are essentially entwined with our experience of space, or even make it possible at all. Which certain theoretical foundations can be known about space, even anterior to perceiving it? What characteristics of space are presupposed in every experience of space, before one recognizes its dominant geometric axioms through an empirical investigation? Actually, the axioms of geometry, according to Riemann, are not necessary or self-evident truths; they are empirical datum, acquired from our empirical investigations and experience. Essential and fundamental properties of space are the properties that actualize such a fundamental experience, the very experience of space itself.
- Second, in the pursuit of this goal, Riemann uses a methodology the same as that employed by Descartes to investigate geometry, long before Riemann: the analytic methodology. Much of the impetus for this choice came from the unreliability of the classic method. The classic model of geometric proof, on the evidence of the history of geometry, occasionally misleads us about the nature of space, by representing some external, accidental properties of space as essential or intrinsic ones [Klein, Greenberg]. Therefore, by adopting the analytical approach, he tries to unearth the most fundamental characteristics of space, thereby drawing out their necessary implications. Other properties of space shall be grasped through empirical inquiries.
- Third, he also adopts a local approach to space. In other words, he uses differential geometry, which is concerned with the study of the geometry of space at the vicinity of each point instead of associating space with a unique, global geometry. This local approach prevents him from representing space already as being confined to Euclidian or Lobachevskian global geometry. This approach seems to be more consistent with the way we naturally experience our surrounding space in everyday life.

Riemann's foundational project, in the broadest perspective, may be seen as entirely orientated toward two seemingly different, yet ultimately interrelated objectives: the problem of the multiplicity of possible geometries, and reconstruction of Kant's philosophy of space.

From Riemann's standpoint, the seemingly problematic situation of non-Euclidean geometries derives from the fact that no one, before Riemann himself, distinguished topological and metric properties of space from each other (Spivak, 1979: 154-5).

Axiomatic geometry presupposes both the concept of space and the axioms that constitute geometry; the relation between these two fundamental factors, however, remains unrevealed. There is no definite and precise description of the way in which these factors are related to each other:

> As is well known, geometry presupposes the concept of space, as well as assuming the basic principles for constructions in space. It gives only nominal definitions of these things, while their essential specifications appear in the form of axioms. The relationship between these presuppositions is left in the dark; we do not see whether, or to what extent, any connection between them is necessary, or a priori whether any connection between them is even possible (Riemann, 2007: 23).

By advancing the theory of manifold, Riemann proceeds beyond the theoretical framework of the traditional debates on the Euclidean vs. non-Euclidean geometries. He begins with the notion of a continuum that is devoid of any geometrical structure; this continuum contains no metric factors, including straight lines, geodesics, and the measure of angles. However, two fundamental properties can, prima facie, be attributed to this non-geometric continuum; one can attribute to it either a finite or an infinite number of dimensions, and one can attribute to it a kind of continuity in that for every two points of the continuum it is possible to non-metrically discern whether or not they are infinitesimally close to each other.

These properties are called topological features of space. A topological manifold is a space with these properties. Therefore, a finite-dimensional continuous manifold is an n-fold extended continuum or an n-fold extended continuous quantity. These topological properties are already attached to the concept from which a manifold emerges. Thus, what Riemann means by a manifold is, according to the terminology of modern mathematics, a topological manifold. On the contrary, metric relations of space, i.e. those relations according to which one can define geodesics, the distance between two points, and the measure of angles, are externally attached to the space. They may, say, result from the binding forces externally act on space. They come from somewhere outside the manifold (Ibid.: 33). However, these external, accidental properties, metric relations, determine exactly the dominant geometry of space. Any variation in these metric relations results in a variation in the dominant geometry of space. To recognize the dominant geometry of space, one has to determine its governing metric relations:

> The question of the validity of the hypotheses of geometry in the infinitely small is connected with the question of the basis for the metric relations of space. In connection with this question, which may indeed still be ranked as part of the study of space, the above remark is applicable, that in a discrete manifold the principle of

> metric relations is already contained in the concept of the manifold, but in a continuous one it must come from something else. Therefore, either the reality underlying space must form a discrete manifold, or the basis for the metric relations must be sought outside it, in binding forces acting upon it. (Riemann, 2007: 33).

A manifold is capable of different metric relations, and thereby different geometries. Therefore, our three-dimensional physical space is a particular case of a three-dimensional manifold; it is a three-fold extended continuum onto which a special geometrical structure, whether Euclidean or non-Euclidean, is superposed. One has to undertake an empirical investigation to learn about the metric relations reigning over the physical space. Therefore, the task of identifying the geometry of physical space, according to Riemann, has to be delegated to physicists: "This leads us into the domain of another science, of physic, into which the object of this work does not allow us to go to-day" (Ibid.).

What geometry dominates on physical space cannot be determined a priori, merely on the basis of its topological properties. By advancing the theory of manifolds, Riemann can draw a sharp distinction between topological and metric properties of space. He demonstrates that there is no necessary and a priori relation between these two kinds of properties. Therefore, the problem of whether or not the geometry of space is Euclidean cannot be settled a priori; it is a matter for natural scientists and has to be resolved by empirical inquiries. This consequence is evidently inconsistent with this Kantian assumption that the geometry of physical reality is to be assumed Euclidian a priori. However, Riemann's entire approach may be taken as a reconstruction of Kant's transcendental theory of space, though its articulation lies beyond the scope of this article. For the purposes of this study, it seems sufficient to emphasize that Riemann's project may plausibly and arguably embedded whithin a Kantian transcendental framework, though its further relation to the German idealistic interpretation of Kant remains to be more closely and comprehensively investigated.

Georg Cantor: the ideational character of Sets

"Set-theory" is a name for a fundamental field of modern mathematics; yet it is not so much an independent discipline as it is a foundation for the entire mathematics, an underlying background in terms of which the whole ontology of modern mathematics is to be formulated. But it must be noted that this concept, in regard to its origin, has philosophical connotations; Georg Cantor, as the true founder of modern set-theory (Mengenlehre), had a very peculiar philosophical conception of a set (Menge). It is the first description of what Cantor understands by set or what he thought a set is to be (1880):

> I designate a manifold (an Inbegriff, a set) of elements that belongs to any conceptual sphere (Begriffssphäre) as well-defined, if on the basis of its definition and as a result of the logical principle of the excluded middle, it must be thought of as internally determined, whether any object belonging to the same conceptual sphere belongs also to the thought manifold or not, and also whether two objects belonging to the set, despite the formal differences in the way they are given, are equal to each other or not (Cantor, 1932: 150).

In this first characterization, sets are prima facie considered as mere logical modifications of conceptual spheres, as purely conceptual (Begrifflich) entities. One may thus make sets subordinate to the concepts, as Frege does, inasmuch as for any concept (Begriff)—according to Frege, a Begriff is not but a special kind of function—there always exists the set of those objects which satisfy this concept. Sets are thus conceived as Begriffsumfänge (scops of concepts) and essentially inseparable from the logic of concepts. as Frege puts it: "The scope of a concept does not consist of the objects that fall under the concept, like a forest of trees, but it has its hold on the concept itself and only on this. The concept thus has logical priority over its scope" (as cited in Steiner, 1980: 1045). It is also possible to conceive sets as themselves concepts of a sort: inclusive concepts. sets, in this sense, may be considered to be second-order concepts that are brought about as a result of the combination or synthesization of first-order concepts, intuitions, or thoughts.

But it must be noted that a set is defined by Cantor as an Inbegriff. This term has a very peculiar, somehow philosophical sense, the sense in which it has been used since the 18th century. Inbegriff primarily signifies an "aggregation" that exists in the mind, or in the thinking subject. In a more particular sense, it denotes "something that has been synthesized" as an indefinable abstract. This synthesis, needless to say, amounts to a unity of a sort: an abstract synthetic unity. It is in this sense that Boltzano had defined a set in terms of Inbegriff, even before Cantor (1851): "an Inbegriff that we subordinate to such a concept in which the arrangement of its parts is indifferent (gleichgültig) (in which nothing essential to us changes if only this [arrangement] alone changes), I call [such an Inbegriff] a set" (Bolzano, 1851: 4).

In the second description (1883), this essential characteristic of a set concept is more explicitly emphasized. The second definition thus reads as follows:

> Namely, by a "manifold" or a "set" I generally understand every plurality that can be thought of as a unity, i.e. every Inbegriff of certain elements which can be connected to a whole by a law (Cantor, 1932: 204).

So set is already much more than a mere aggregate; it captures the moment of an abstract unity, from the outset. It is not a substantial, metaphysically real unity; it is relative to plurality and totality, based on a law (Gesetz). In order to be able to form a set, a plurality is to be brought together according to a law, and a set is thus obtained by this law-based unification of a multiplicity in the mind. One may construe the law here as a purely logical one, or as a condition (Bedingung) in its Fregean sense—i.e. functions. But it is more likely that by law Cantor has something in mind like what Dedekind ascribes to a system as an Inbegriff, that is a "point of view": "It very often happens that different things a, b, c [...] for some reason are understood from a common point of view (Gesichtspunkt), are put together in the mind, and one then says that they form a system S [...] Such a system S (or an Inbegriff, a manifold, a totality) is also a thing as an object of our thinking (Dedekind, 1932: 344-5). From this perspective, sets are to be conceived as abstract pure unification or synthesization proceeded from a mental aspect or consideration; it is an originally ideational unity, a unification that bounds the multiple elements into a whole by rational ideation. What guarantees the inner unity of a set, what determines a set as such is a mental aspect.

Now the third description (1895) reads as follows:

> By a set, we understand every combination M of certain well-differentiated objects m of our intuition or our thinking (which are called the elements of M) into a whole (Cantor, 1932: 282).

A set is characterized here as a combination (Zusammenfassung) of intuited or thought entities. So the third description's emphasis is again on the mental character of sets; a set has not any external reality, it is not a metaphysical category, but it is a modification of what is given to us by intuition or arises from our intellect's spontaneity.

To sum up, according to Cantor, sets may be essentially characterized in this ways:

As ideational entities (Inbegriff), sets are mental aspects or mental points of view (Gesichtspunkte) according to which a multiplicity of mental entities (intuitions, concepts, or thoughts), however disparate, can be taken into consideration as a unity.

In any case, sets essentially are concomitants of ideational acts of transcendental subjectivity, that bring together a plurality of contents of consciousness according to mental ideation.

Conclusion

Now, let us return to what we began with; every science qua a regional ontology—qua what is concerned with a specific region of being—is inevitably grounded on a somehow philosophical (in its most general sense) pre-understanding, a pre-ontological interpretation of its thematic field of objects. More technically formulated, every self-founding of science is possible only on the basis of a preceding projection of the ontological constitution of its thematic field of objects. Modern mathematics, as mentioned by Heidegger himself, is not to be excluded; the genuine self-founding of pure mathematics, however, has taken place at the second half of the 19th century, and not to be sought after in the so-called "research on the foundations" and has been acknowledged as the "crisis in foundations" of mathematics at the early 20th century. What the standard philosophy of mathematics occupies itself with, what is generally known as the three basic schools of 20th century philosophy of mathematics—logicism, formalism, and intuitionism—is not so much a genuine self-founding of pure mathematics as it is a post factum, retrospective attempt to reconstruct what has been already founded. In the projection of its thematic field, in its (pre-) ontological interpretation of its possible objects, and in the justification of its entirely new epistemological status within the totality of human cognitions, 19th-century pure mathematics was heavily reliant on the transcendental philosophy of Kant, more particularly on German idealistic interpretation of the central core of Kritik der reinen Vernunft: the transcendental unity of pure self-consciousness and the correlation between the world-consciousness with the self-consciousness, in terms of which the fundamental concepts of the new developing pure mathematics, n-dimensional spaces, non-Euclidean geometries, complex and real numbers, sets, systems and so on, have been construed.

Why are such considerations significant at all? One reason is that they essentially change our understanding of what pure mathematics is. Contemporary mathematicians, for the most part, conceive of pure mathematics as the science of formal systems, and knowing how to skillfully manipulate the formal language of such systems (Cf. Dieck, 2013: 136-7). Such conceptions are mostly reliant on the post-factum retrospective reconstructions of pure mathematics in the early 20th century and totally abstain from regarding pure mathematics in its genuine origination. its true origin taken into account, pure mathematics is not so much a mere system of logico-mathematical propositions or the science of formal systems or a formal language and the like, as it is a purely formal activity, an ideation, of a transcendental self-conscious subjectivity, and its philosophy thus bears close affinity with German idealism, rather than logicism, formalism, or intuitionism.

Notes

1 For my understanding I am extremely indebted to two authority figures in German idealism: Dieter Henrich and Robert Pippin.

2 Georg Wilhelm Fredrich Hegel, *The Science of Logic*. Translated and edited by George di Giovanni. Cambridge: Cambridge University Press, 2010.

3 In other words, there is a one-to-one but not onto transformation of *S* into itself.

4 The figure is extracted from (Paugh, : 2002: 12)

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