Competitive Location Problems: Balanced Facility Location and the One-Round Manhattan Voronoi Game

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Abstract

We study competitive location problems in a continuous setting, in which facilities have to be placed in a rectangular domain R of normalized dimensions of 1 and $\rho \geq 1$, and distances are measured according to the Manhattan metric. We show that the family of *balanced* facility configurations (in which the Voronoi cells of individual facilities are equalized with respect to a number of geometric properties) is considerably richer in this metric than for Euclidean distances. Our main result considers the *One-Round Voronoi Game* with Manhattan distances, in which first player White and then player Black each place n points in R; each player scores the area for which one of its facilities is closer than the facilities of the opponent. We give a tight characterization: White has a winning strategy if and only if $\rho \geq n$; for all other cases, we present a winning strategy for Black.

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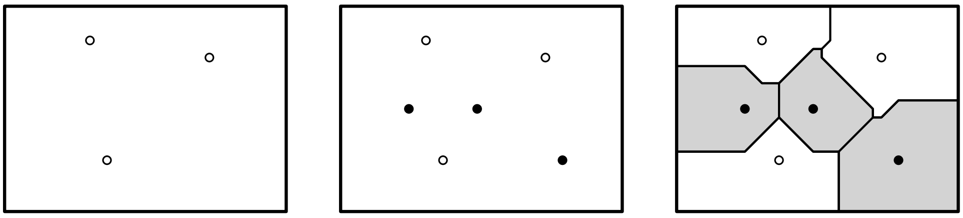
1 Introduction

Problems of optimal location are arguably among the most important in a wide range of areas, such as economics, engineering, and biology, as well as in mathematics and computer science. In recent years, they have gained a tremendous amount of importance through clustering problems in artificial intelligence. In all scenarios, the task is to choose a set of positions from a given domain, such that some optimality criteria with respect to the resulting distances to a set of demand points are satisfied; in a geometric setting, Euclidean or Manhattan distances are natural choices. Another challenge of facility location problems is that they often happen in a *competitive* setting, in which two or more players contend for the best locations. This change to competitive, multi-player versions can have a serious impact on the algorithmic difficulty of optimization problems: for example, the classic Travelling Salesman Problem is NP-hard, while the competitive two-player variant is even PSPACE-complete [10].

2 Competitive Location Problems



(a) White places 3 points. **(b)** Black places 3 points. **(c)** The dominated areas.

Figure 1 Example of a one-round Manhattan Voronoi game.

In this paper, we consider problems of facility location under Manhattan distances; while frequently studied in location theory and applications (e.g., see [15,16,19]), they have received limited attention in a setting in which facilities compete for customers. We study a natural scenario in which facilities have to be chosen in a rectangle R of normalized dimensions with height 1 and width $\rho \geq 1$. A facility dominates the set of points for which it is strictly closer than any other facility, i.e., the respective (open) Voronoi cell, subject to the applicable metric. While for Euclidean distances a bisector (the set of points that are of equal distance from two facilities) is the boundary of the open Voronoi cells, so its area is zero, Manhattan bisectors may have positive area, as shown in Figure 2. As we show below, accounting for fairness and local optimality, we consider *balanced* configurations for which the respective Voronoi cells are equalized.

Exploiting the geometric nature of Voronoi cells, we completely resolve a classic problem of competitive location theory for the previously open case of Manhattan distances. In the *One-Round Voronoi Game*, first player *White* and then player *Black* each place n points in R. Each player scores the area consisting of the points that are closer to one of their facilities than to any one of the opponent's; see Figure 1 for an example. The goal for each player is to obtain the higher score. Owing to the different nature of the Manhattan metric, both players may dominate strictly less than $\rho/2$, the remaining area belonging to *neutral zones*.

1.1 Related Work

Problems of location are crucial in economics, optimization, and geometry; see the classic book of Drezner [8] with over 1200 citations, or the more recent book by Laporte et al. [17]. Many applications arise from multi-dimensional data sets with heterogeneous dimensions, so the Manhattan metric (which compares coordinate distances separately) is a compelling choice. The ensuing problems have also received algorithmic attention. Fekete et al. [12] provide several algorithmic results, including an NP-hardness proof for the k-median problem of minimizing the average distance. Based on finding an optimal location for an additional facility in a convex region with n existing facilities, Averbakh et al. [3] derive exact algorithms for a variety of conditional facility location problems.

An important scenario for competitive facility location is the *Voronoi game*, first introduced by Ahn et al. [1], in which two players take turns placing one facility a time. In the end, each player scores the total area of all of their Voronoi regions. (For an overview of work on Voronoi diagrams, see the surveys by Aurenhammer and Klein [2].) As Teramoto et al. [18] showed, the problem is PSPACE-complete, even in a discrete graph setting.

Special attention has been paid to the *One-Round Voronoi Game*, in which each player places their n facilities at once. Cheong et al. [7] showed that for Euclidean distances in the plane, White can always win for a one-dimensional region, while Black has a winning strategy if the region is a square and n is sufficiently large. Fekete and Meijer [11] refined this by showing that in a rectangle of dimensions $1 \times \rho$ with $\rho \geq 1$, Black has a winning strategy for $n \geq 3$ and $\rho < n/\sqrt{2}$, and for $n = 2$ and $\rho < 2/\sqrt{3}$; White wins in all other cases. In this paper, we give a complementary characterization for the case of Manhattan distances; because of the different geometry, this requires several additional tools.

There is a considerable amount of other work on variants of the Voronoi game. Bandyapadhyay et al. [4] consider the one-round game in trees, providing a polynomial-time algorithm for the second player. As Fekete and Meijer [11] have shown, the problem is NP-hard for polygons with holes, corresponding to a planar graph with cycles. For a spectrum of other variants and results, see [5, 9, 13, 14].

1.2 Main Results

Our main results are twofold.

- We show that for location problems with Manhattan distances in the plane, the properties of *fairness* and *local optimality* lead to a geometric condition called *balancedness*. While the analogue concept for Euclidean distances in a rectangle implies grid configurations [11], we demonstrate that there are balanced configurations of much greater variety.
- We give a full characterization of the One-Round Manhattan Voronoi Game in a rectangle R with aspect ratio $\rho \geq 1$. We show that White has a winning strategy if and only if $\rho \geq n$; for all other cases, Black has a winning strategy.

2 Preliminaries

Let P denote a finite set of points in a rectangle R. For two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$, we define $\Delta_x(p_1, p_2) := |x_1 - x_2|$ and $\Delta_y(p_1, p_2) := |y_1 - y_2|$. Then their Manhattan distance is given by $d_M(p_1, p_2) := \Delta_x(p_1, p_2) + \Delta_y(p_1, p_2)$.

Defining $D(p_1, p_2) := \{p \in R \mid d_M(p, p_1) < d_M(p, p_2)\}$ as a set of points that are closer to p_1 than to p_2, the *Voronoi cell* of p in P is

$$V^P(p) := \bigcap_{q \in P \setminus \{p\}} D(p, q).$$

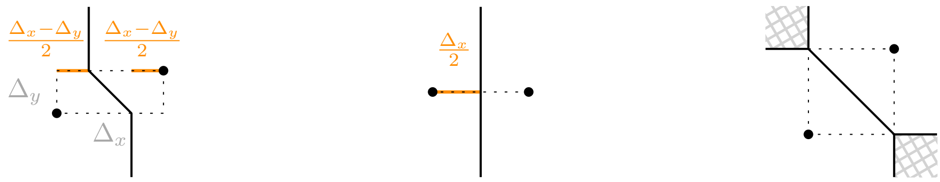
The *Manhattan Voronoi diagram* $\mathcal{V}(P)$ is the complement of the union of all Voronoi cells of P. In contrast to the Euclidean case, for which the Voronoi diagram has measure zero and every Voronoi cell is convex:

- the Manhattan Voronoi diagram may contain *neutral zones* of positive measure, and
- Manhattan Voronoi cells need not be convex, but they are star-shaped.

Both of these properties can easily be observed when analyzing the bisectors. The *bisector* of p_1 and p_2 is the set of all points that have equal distance to p_1 and p_2, i.e.,

$$\mathcal{B}(p_1, p_2) := \{q \in R \mid d_M(q, p_1) = d_M(q, p_2)\}.$$

There are three types of bisectors, as shown in Figure 2. Typically, a bisector consists of three one-dimensional parts, namely two (vertical or horizontal) segments that are connected by a segment of slope ± 1; see Figure 2(a). If $\Delta_x(p_1, p_2) = 0$ or $\Delta_y(p_1, p_2) = 0$, then the diagonal segment shrinks to a point and the bisector consists of a (vertical or horizontal) segment; see Figure 2(b). However, when $\Delta_x(p_1, p_2) = \Delta_y(p_1, p_2)$, then the bisector $\mathcal{B}(p_1, p_2)$ contains two regions; see Figure 2(c). We call a bisector of this type *degenerate*. Further, a non-degenerate bisector is *vertical (horizontal)* if it contains vertical (horizontal) segments.



(a) General vertical bisector. **(b)** Case: $\Delta_y(p_1, p_2) = 0$. **(c)** Degenerate bisector.

Figure 2 Illustration of the three types of bisectors.

For $p = (x_p, y_p) \in P$, both the vertical line $\ell_v(p)$ and the horizontal line $\ell_h(p)$ through p split the Voronoi cell $V^P(p)$ into two pieces, which we call *half cells*. We denote the set of all half cells of P obtained by vertical lines by $\mathcal{H}^{|}$ and those obtained by horizontal lines by $\mathcal{H}^{-}$. Furthermore, we define $\mathcal{H} := \mathcal{H}^{|} \cup \mathcal{H}^{-}$ as the set of all half cells of P. Applying both $\ell_v(p)$ and $\ell_h(p)$ to p yields a subdivision into four quadrants, which we denote by $Q_i(p)$, $i \in \{1, \ldots, 4\}$; see Figure 3(a). Moreover, $C_i(p) := V^P(p) \cap Q_i(p)$ is called the ith *quarter cell* of p. We also consider the eight regions of every $p \in P$ obtained by cutting R along the lines $\ell_v(p)$, $\ell_h(p)$, and the two diagonal lines of slope ± 1 through p. We refer to each such (open) region as an *octant* of p denoted by $O_i(p)$ for $i \in \{1, \ldots, 8\}$, see Figure 3(b); a closed octant is denoted by $\overline{O}_i(p)$. The area of a subset S of R is denoted by $\mathrm{area}(S)$.

For a point $p \in P$, we call the four horizontal and vertical rays rooted at p, contained with $V^P(p)$, the four *arms* of $V^P(p)$ (or of p). Two arms are *neighbouring* if they appear consecutively in the cyclic order; otherwise they are *opposite*. Moreover, we say an arm is a *boundary arm* if its end point touches the boundary of R; otherwise it is *inner*. For later reference, we note the following.

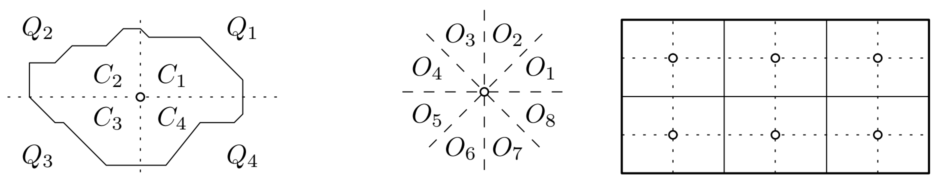
▶ **Observation 1.** *The following properties hold:*

(i) *If the bisector $\mathcal{B}(p, q)$ is non-degenerate and vertical (horizontal), then it does not intersect both the left and right (top and bottom) half cells of p.*

(ii) *For every i and every $q_1, q_2 \in O_i(p)$, the bisectors $\mathcal{B}(p, q_1)$ and $\mathcal{B}(p, q_2)$ have the same type (vertical/horizontal).*

(iii) *A Voronoi cell is contained in the axis-aligned rectangle spanned by its arms.*

Proof. Properties (i) and (ii) follow immediately from the shape of the bisectors. Property (ii) implies that a Manhattan Voronoi cell consists of four (x- and y-) monotone paths connecting the tips of its arms. Consequently, each Voronoi cell is contained in the axis-aligned rectangle spanned by its arms and, thus, Property (iii) holds. ◀



(a) The quadrants and quarter cells. **(b)** The octants. **(c)** A 2×3 grid.

Figure 3 Illustration of crucial definitions.

3 Balanced Point Sets

In a competitive setting for facility location, it is a natural *fairness property* to allocate the same amount of influence to each facility. A second *local optimality property* arises from choosing an efficient location for a facility within its individual Voronoi cell. Combining both properties, we say a point set P in a rectangle R is *balanced* if the following two conditions are satisfied:

Fairness: for all $p_1, p_2 \in P$, $V^P(p_1)$ and $V^P(p_2)$ have the same area.

Local optimality: for all $p \in P$, p minimizes the average distance to the points in $V^P(p)$.

For Manhattan distances, there is a simple geometric characterization for the local optimality depending on the area of the half and quarter cells; see Figure 3(a).

▶ **Lemma 2.** *A point p minimizes the average Manhattan distance to the points in $V^P(p)$ if and only if either of the following properties holds:*

(i) *p is a* Manhattan median *of $V^P(p)$: all four half cells of $V^P(p)$ have the same area.*

(ii) *p satisfies the* quarter-cell property*: diagonally opposite quarter cells of $V^P(p)$ have the same area.*

Proof. For an illustration we refer to Figure 3(a). Let a_i denote the area of the quarter cell $C_i(p)$. First we consider a point $p = (x_p, y_p)$ that minimizes the average Manhattan distance to all points in $V^P(p)$. Suppose the area of the top half cell exceeds the area of the bottom half cell, i.e., $a_1 + a_2 > a_3 + a_4$. Then replacing p by $p' = (x_p, y_p + \varepsilon)$ for an appropriately small $\varepsilon > 0$ reduces the average y-distance and leaves the average x-distance unchanged. This contradicts the optimality of p. Similarly, we can exclude $a_1 + a_2 < a_3 + a_4$, so $a_1 + a_2 = a_3 + a_4$, making p a y-median of $V^P(p)$. Analogously, we conclude that $a_1 + a_4 = a_2 + a_3$, making p an x-median of $V^P(p)$. This shows that all half cells of $V^P(p)$ have the same area. Moreover, note that the half-cell condition uniquely defines both x_p and y_p, so it is both necessary and sufficient.

We now show that (i) is equivalent to (ii). By adding the equations

$$a_1 + a_2 = a_3 + a_4$$
$$a_1 + a_4 = a_2 + a_3$$

it follows that $2a_1 + a_2 + a_4 = 2a_3 + a_2 + a_4 \iff a_1 = a_3$; by subtracting the equations, we get $a_2 - a_4 = a_4 - a_2 \iff a_2 = a_4$. Hence, the quarter-cell property is fulfilled.

Conversely, $a_1 = a_3$ and $a_2 = a_4$ imply $a_1 + a_2 = a_3 + a_4 = a_1 + a_4 = a_2 + a_3$. ◀

Lemma 2 immediately implies the following characterization.

▶ **Corollary 3.** *A point set P in a rectangle R is balanced if and only if all half cells of P have the same area.*

A simple family of balanced sets arise from regular, $a \times b$ grids; see Figure 3(c). In contrast to the Euclidean case, there exist a large variety of other balanced sets: Figure 4 depicts balanced point sets for which *no* cell is a rectangle.

▶ **Lemma 4.** *The configurations $\mathcal{R}_{2,\rho}, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5$, depicted in Figure 4, are balanced. Moreover, $\mathcal{R}_{2,\rho}, \rho \in [1, 3/2]$, and $\mathcal{R}_3$ are the only balanced non-grid point sets with two and three points, respectively.*

6 Competitive Location Problems

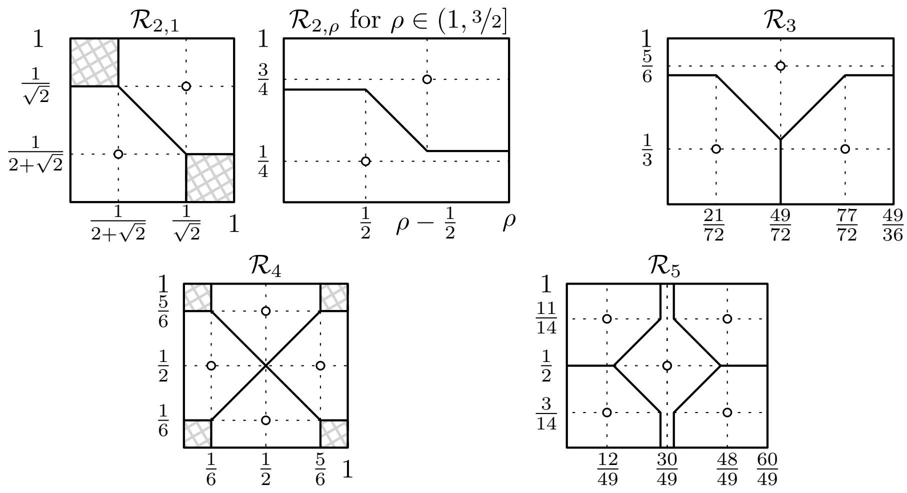


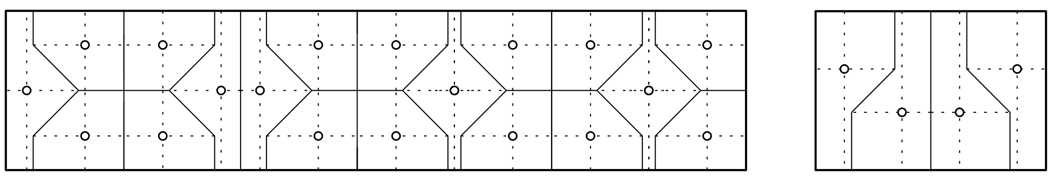
Figure 4 Non-grid examples of balanced point sets of cardinality 2, 3, 4, and 5.

Simple calculations show that the configurations are balanced. In order to prove the uniqueness, we make use of Lemma 2. While the analysis for $n = 2$ can be easily conducted manually, for $n = 3$, the relative point positions lead to about 20 cases of structurally different Voronoi diagrams, which were checked using MATLAB®. We present the details in Appendix A.

Observe that $\mathcal{R}_{2,\rho}, \mathcal{R}_3, \mathcal{R}_4$, and $\mathcal{R}_5$ are *atomic*, i.e., they cannot be decomposed into subconfigurations whose union of Voronoi cells is a rectangle. We show how they serve as building blocks to induce large families of balanced configurations.

▶ **Theorem 5.** *For every n, $n \neq 7$, there exists a rectangle R and a set P of n points such that P is balanced and no Voronoi cell is a rectangle.*

Proof. For every $n = 3k+5\ell$ with $k, \ell \in \{0, 1, \dots\}$, we construct a configuration by combining k blocks of $\mathcal{R}_3$ and ℓ blocks of $\mathcal{R}_5$, as shown in Figure 5(a). This yields configurations with $n = 3k$ for $k \geq 1$, $n = 3k+2 = 3(k-1)+5$ for $k \geq 1$, or $n = 3k+1 = 3(k-3)+10$ for $k \geq 3$, so we obtain configurations for all $n \geq 8$ and $n = 3, 5, 6$. Configurations with $n = 2k$ $(k \in \mathbb{N})$ points are obtained by combining k blocks of $\mathcal{R}_2$ as shown in Figure 5(b); alternatively, recall the configurations in Figure 4. ◀



(a) Combining k blocks of $\mathcal{R}_3$ and l blocks of $\mathcal{R}_5$ for a configuration with $n = 3k + 5l$ points in a rectangle R with $\rho(R) = 1/49(36k + 60l)$.

(b) Combining k blocks of $\mathcal{R}_2$ for $n = 2k$ points.

Figure 5 Illustration of the proof of Theorem 5.

While none of the configurations in Theorem 5 contains a rectangular Voronoi cell, they contain many immediate repetitions of the same atomic components. In fact, there are arbitrarily large *non-repetitive* balanced configurations without directly adjacent congruent atomic subconfigurations.

▶ **Theorem 6.** *There is a injection between the family of 0-1 strings and a family of non-repetitive balanced configurations without any rectangular Voronoi cells.*

Proof. For a given 0-1 string $\mathcal{S}$ of length s, we use s pairs of blocks $\mathcal{R}_3$ and its reflected version $\mathcal{R}'_3$ to build a sequence of $2s$ blocks. As shown in Figure 6, we insert a block $\mathcal{R}_5$ after the ith pair if $\mathcal{S}$ has a 1 in position i; otherwise the block sequence remains. ◀

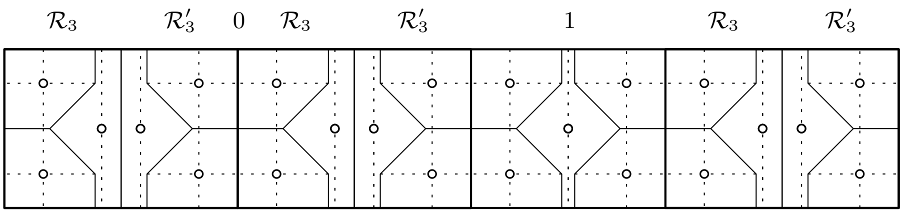


Figure 6 Illustration of the proof of Theorem 6. The configuration represents the string 01.

4 The Manhattan Voronoi Game

An instance of the One-Round Manhattan Voronoi Game consists of a rectangle R and the number n of points to be played by each player. Without loss of generality, R has height 1 and width $\rho \geq 1$. Player White chooses a set W of n white points in R, followed by the player Black selecting a set B of n black points, with $W \cap B = \varnothing$. Each player scores the area consisting of the points that are closer to one of their facilities than to any one of the opponent's. Hence, if two points of one player share a degenerate bisector, the possible neutral regions are assigned to this player. Therefore, by replacing each degenerate bisector between points of one player by a (w.l.o.g. horizontal) non-degenerate bisector, each player scores the area of its (horizontally enlarged) Manhattan Voronoi cells. With slight abuse of notation, we denote the resulting (horizontally enlarged) Voronoi cells of colored point sets by $V^{W\cup B}(p)$ similar as before. The player with the higher score wins, or the game ends in a tie.

For an instance (R, n) and a set W of white points, a set B of n black points is a *winning set* for Black if Black wins the game by playing B; likewise, B is a *tie set* if the game ends in a tie. A black point b is a *winning point* if its cell area area$(V^{W\cup B}(b))$ exceeds $1/2n \cdot \text{area}(R)$. A white point set W is *unbeatable* if it does not admit a winning set for Black, and W is a *winning set* if there exists neither a tie nor a winning set for Black. If Black or White can always identify a winning set, we say they have a *winning strategy*.

Despite the possible existence of degenerate bisectors for Manhattan distances, we show that Black has a winning strategy if and only if Black has a winning point. We make use of the following two lemmas.

▶ **Lemma 7.** *Consider a rectangle R with a set W of white points. Then for every $\varepsilon > 0$ and every half cell H of W, Black can place a point b such that the area of $V^{W\cup\{b\}}(b) \cap H$ is at least $(area(H) - \varepsilon)$.*

Proof. Without loss of generality, we consider the left half cell H of some $w \in W$ as in Figure 7. By placing b slightly to the left of w, the bisector $\mathcal{B}(b, w)$ is a vertical segment between b and w. Therefore $V^{W\cup\{b\}}(b)$ contains all points of H to the left of $\mathcal{B}(b, w)$. The area difference of $V^{W\cup\{b\}}(b) \cap H$ and H is upper bounded by the product of the total length of the top and bottom arm of H and (half) the distance of b and w. Consequently, by placing b close enough, the difference drops below any fixed $\varepsilon > 0$. ◀

In fact, White must play a balanced set; otherwise Black can win.

▶ **Lemma 8.** *Let W be a set of n white points in a rectangle R. If any half cell of W has an area different from $1/2n \cdot$ area(R), then Black has a winning strategy.*

Proof. If not all half cells of W have the same area, then there exists a half cell H with $\text{area}(H) > 1/2n \cdot \text{area}(R)$. We assume w.l.o.g. that H is a half cell of $\mathcal{H}^|$; otherwise we consider $\mathcal{H}^-$. Denoting the n largest half cells of $\mathcal{H}^|$ by $H_1, \ldots, H_n$, it follows that there exists $\delta > 0$ such that $\sum_{i=1}^{n} H_i = 1/2 \cdot \text{area}(R) + \delta$.

By Lemma 7, Black can place a point b_i to capture the area of H_i up to any $\varepsilon > 0$. More precisely, by choosing $\varepsilon < \delta/n$, Lemma 7 guarantees that there exists a placement of n black points $b_1, \ldots, b_n$ such that

$$\sum_{i=1}^{n} \text{area}(V^{W \cup B}(b_i) \cap H_i) \geq \sum_{i=1}^{n} (\text{area}(H_i) - \varepsilon) = \frac{1}{2}\text{area}(R) + \delta - n\varepsilon > \frac{1}{2}\text{area}(R).$$

Consequently, Black has a winning strategy by placing these points. ◀

These insights enable us to prove the main result of this section.

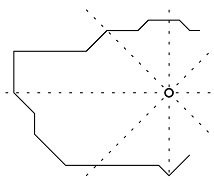
▶ **Theorem 9.** *Black has a winning strategy for a set W of n white points in a rectangle R if and only if Black has a winning point.*

Proof. If Black wins and their score exceeds $1/2 \cdot \text{area}(R)$ then, by the pigeonhole principle, a black cell area exceeds $1/2n \cdot \text{area}(R)$, confirming a winning point. Otherwise Black has a winning set that scores at most half the area of R.

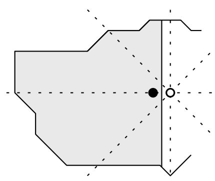
▷ Claim 10. Let B be a winning set for Black such that Black scores at most $1/2n \cdot \text{area}(R)$. Then there is also a black winning set B' such that Black's score exceeds $1/2n \cdot \text{area}(R)$.

Proof. If Black wins with B, then Black's score exceeds White's score. Moreover, since Black scores at most $1/2n \cdot \text{area}(R)$, there exist neutral zones and degenerate bisectors between black and white points.

Consider a white point w and a black point b with $\Delta_x(w, b) = \Delta_y(w, b)$. Black can avoid this degeneracy by choosing a slightly perturbed location. By moving on either side of their diagonal line, Black can win either of the neutral regions up to any $\varepsilon > 0$. If the neutral regions are of different size, then Black can ensure a net gain. If the areas are the same, Black has a net loss of $\varepsilon > 0$. However, since Black wins, it can allow for some net loss $\varepsilon > 0$. The argument applies even if b is part of more than one degeneracy by consideration of the sum of the losses and gains in the resulting cells. Therefore, b can avoid degeneracy by an arbitrarily small net loss. The repeated application for all its points shows that Black has a winning set without forcing neutral regions. Consequently, Black's score exceeds $1/2n \cdot \text{area}(R)$. ◁



(a) A left half cell H.



(b) A point capturing H up to any $\varepsilon > 0$.

Figure 7 Illustration of the proof of Lemma 7.

Now suppose that there exists a winning point b, i.e., $\mathrm{area}(V^{W\cup\{b\}}(b)) = {}^1\!/\!{}_{2n}\cdot\mathrm{area}(R)+\delta$ for some $\delta > 0$. If $n = 1$, Black clearly wins with b. If $n \geq 2$, Black places $n-1$ further black points: consider $w_i \in W$. By Lemma 8, we may assume that each half cell of W has area ${}^1\!/\!{}_{2n}\cdot\mathrm{area}(R)$. By Observation 1(i), w_i has a half cell H_i that is disjoint from $V^{W\cup\{b\}}(b)$. By Lemma 7, Black can place a point b_i to capture the area of H_i up to every $\varepsilon > 0$. Choosing $\varepsilon < {}^\delta\!/\!{}_{n-1}$ and placing one black point for $n-1$ distinct white points with Lemma 7, Black achieves a score of $\sum_{p\in B}\mathrm{area}(V^{W\cup B}(p)) = ({}^1\!/\!{}_{2n}\cdot\mathrm{area}(R)+\delta)+(n-1)\,({}^1\!/\!{}_{2n}\cdot\mathrm{area}(R)-\varepsilon) > {}^1\!/\!{}_2\cdot\mathrm{area}(R)$. Consequently, Black has a winning strategy. ◀

5 Properties of Unbeatable Winning Sets

In this section, we identify necessary properties of *unbeatable* white sets, for which the game ends in a tie or White wins. We call a cell a *bridge* if it has two opposite boundary arms.

▶ **Theorem 11.** *If W is an unbeatable white point set in a rectangle R, then it fulfils the following properties:*

(P1) *The area of every half cell of W is ${}^1\!/\!{}_{2n}\cdot\mathrm{area}(R)$.*

(P2) *The arms of a non-bridge cell are equally long; the opposite boundary arms of a bridge cell are of equal length and, if $|W| > 1$, they are shortest among all arms.*

Proof. Because W is unbeatable, property (P1) follows immediately from Lemma 8. Moreover, in case $|W| = 1$, (P1) implies that opposite arms of the unique (bridge) cell have equal length, i.e., (P2) holds for $|W| = 1$.

It remains to prove property (P2) for $|W| \geq 2$. By Theorem 9, it suffices to identify a black winning point if (P2) is violated. We start with an observation.

▷ Claim 12. Let P be a point set containing $p = (0,0)$ and let P' be obtained from P by adding $p' = (\delta,\delta)$ where $\delta > 0$ such that p' lies within $V^P(p)$. Restricted to $Q := Q_1(p')$, the cell $V^{P'}(p')$ contains all points that are obtained when the boundary of $V^P(p)\cap Q$ is shifted upwards (rightwards) by δ (if it does not intersect the boundary of R).

Proof. To prove this claim, it suffices to consider the individual bisectors of p' and any other point $q \in P'$. Note that all points shaping the cell of p' in quadrant Q are contained in an octant $O_i(p')$ with $i \in \{1,2,3,8\}$. We show that vertical bisectors move rightwards and horizontal bisectors move upwards.

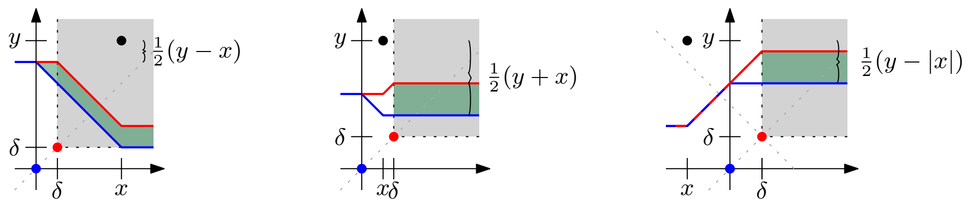
For a point $q = (x,y)$ in $\overline{O}_2(p')$, the part of the bisector $\mathcal{B}(q,p')\cap Q$ can be obtained from $\mathcal{B}(q,p)\cap Q$ by shifting it upwards by an amount of δ; see also Figure 8(a). In particular, the initial height of the diagonal segment remains unchanged because its vertical distance to q is ${}^1\!/\!{}_2(\Delta_y(q,p')-\Delta_x(q,p')) = {}^1\!/\!{}_2((y-\delta)-(x-\delta)) = {}^1\!/\!{}_2(y-x) = {}^1\!/\!{}_2(\Delta_y(q,p)-\Delta_x(q,p))$. Note also that this holds for degenerate bisectors, because only their diagonal segment is contained in Q.

For a point $q = (x,y)$ in $\overline{O}_3(p')\cap O_2(p)$, the vertical distance of q to the horizontal segment of $\mathcal{B}(q,p)$ within Q is ${}^1\!/\!{}_2(y-x)+x = {}^1\!/\!{}_2(y+x)$ while vertical distance of q to the horizontal segment of $\mathcal{B}(q,p')$ is ${}^1\!/\!{}_2((y-\delta)-(\delta-x)) = {}^1\!/\!{}_2(y+x)-\delta$; see also Figure 8(b).

For a point $q = (x,y)$ in $\overline{O}_3(p')\cap\overline{O}_3(p)$, the vertical distance of q to the horizontal segment of the bisector within Q is ${}^1\!/\!{}_2((y-\delta)-(|x|+\delta)) = {}^1\!/\!{}_2(y-|x|)-\delta$ for p' and ${}^1\!/\!{}_2(y-|x|)$ for p; see also Figure 8(c).

Note that for $q \in O_2(p')\cup O_3(p')$, the bisector $\mathcal{B}(q,p)$ is horizontal. Consequently, all shifted segments are horizontal or diagonal. Shifting them rightwards yields a region contained in $V^{P'}(p')$. By symmetry, all (vertical) bisectors of points within $O_1(p')\cup O_8(p')$ are shifted rightwards. This implies the claim. ◁

10 Competitive Location Problems



(a) $q \in \overline{O}_2(p')$ **(b)** $q \in \overline{O}_3(p') \cap O_2(p)$ **(c)** $q \in \overline{O}_3(p') \cap \overline{O}_3(p)$

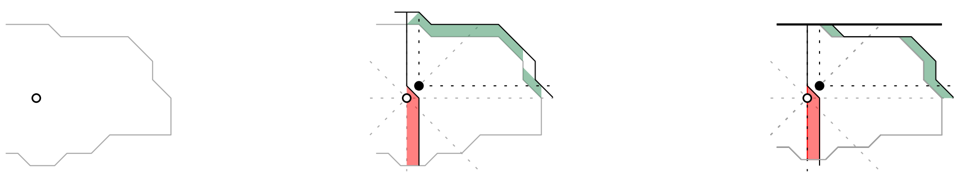
Figure 8 Illustration of Claim 12. If $q = (x, y)$ lies in $\overline{O}_2(p') \cup \overline{O}_3(p')$, the part of the bisector $\mathcal{B}(q, p')$ within the first quadrant Q of p' coincides with $\mathcal{B}(q, p) \cap Q$ shifted upwards by δ.

We use our insight of Claim 12 to show property (P2) in two steps.

▷ Claim 13. Let $w \in W$ be a point such that an arm A_1 of $V^W(w)$ is shorter than a neighbouring arm A_2 and the arm A_3 opposite to A_1 is inner. Then Black has a winning point.

Proof. Without loss of generality, we consider the case that A_1 is the bottom arm of $V^W(w)$, A_2 its right arm, and $w = (0, 0)$; see Figure 9(b). We denote the length of A_i by $|A_i|$. Now we consider Black placing a point b within $V^W(w)$ at (δ, δ) for some $\delta > 0$. To ensure that the cell of b contains almost all of the right half cell of $V^W(w)$, we infinitesimally perturb b rightwards; for ease of notation in the following analysis, we omit the corresponding infinitesimal terms and assume that the bisector of b and w is vertical. We compare the area of $V(b) := V^{W+b}(b)$ with the right half cell H of w. In particular, we show that there exists $\delta > 0$ such that the area of $V(b)$ exceeds the area of H. Because $\text{area}(H) = {}^1\!/_{2n} \cdot \text{area}(R)$ by (P1), b is a winning point.

Clearly, all points in H to the right of the (vertical) bisector of b and w are closer to b. Consequently, when compared to H, the loss of $V(b)$ is upper bounded by $\delta|A_1| + {}^1\!/_2\delta^2$; see also Figure 9(b). By Claim 12 and the fact that A_3 is inner, $V(b) \cap Q_1(p)$ gains at least $\delta(|A_2| - \delta)$ when compared to $H \cap Q_1(p)$. When additionally guaranteeing $\delta < {}^2\!/_3(|A_2| - |A_1|)$, the gain exceeds the loss and thus b is a winning point. ◁



(a) The right half cell H. **(b)** Case: The bottom arm A_1 is shorter than the right arm A_2 and the top arm A_3 is inner. **(c)** Case: The bottom arm A_1 is shorter than the top arm A_3 and the right arm A_2 is inner.

Figure 9 Illustration of Claim 13 and Claim 14: the gain and loss of $V(b)$ compared to H.

For a cell with two neighbouring inner arms, Claim 13 implies that all its arms have equal length. Consequently, it only remains to prove (P2) for bridges. With arguments similar to those proving Claim 13, we obtain the following result. For an illustration, see Figure 9(c).

▷ Claim 14. If there exists a point $w \in W$ such that two opposite arms of $V^W(w)$ have different lengths and a third arm is inner, then Black has a winning point.

Proof. Without loss of generality, we consider the case that A_1 is the bottom arm, A_1 is shorter than the top arm A_3, and the right arm A_2 is inner. Analogously to the proof of Claim 13, Black places a point b at (δ, δ) for some $\delta > 0$ and chooses the vertical bisector with w. As above, when compared to the right half cell H of w, the loss of $V^{W+b}(b)$ is upper bounded by $\delta|A_1| + 1/2 \cdot \delta^2$. By Claim 12 and the fact that A_2 is inner, the gain is lower bounded by $\delta(|A_3| - \delta)$. Guaranteeing $\delta < 2/3(|A_3| - |A_1|)$, the gain exceeds the loss. Thus, in case $|A_3| > |A_1|$, Black has a winning point. ◁

If $|W| > 1$, every cell has at least one inner arm. Therefore Claim 14 yields that opposite boundary arms of a bridge cell have equal length. Moreover, Claim 13 implies that the remaining arms are not shorter. This proves (P2) for bridges. ◀

We now show that unbeatable white sets are grids; in some cases they are even *square grids*, i.e., every cell is a square.

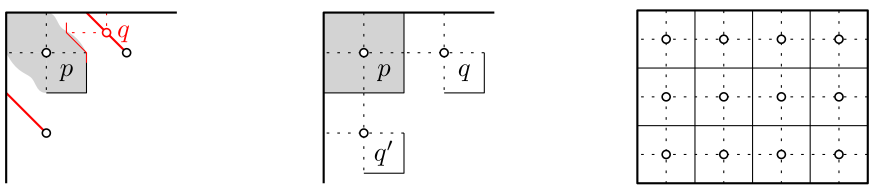
▶ **Lemma 15.** *Let P be a set of n points in a $(1 \times \rho)$ rectangle R with $\rho \geq 1$ fulfilling properties (P1) and (P2). Then P is a grid. More precisely, if $\rho \geq n$, then P is a $1 \times n$ grid; otherwise, P is a square grid.*

Proof. We distinguish two cases.

Case: $\rho \geq n$. By (P1), every half cell has area $\frac{1}{2n}\text{area}(R) = \frac{1}{2n}\rho \geq \frac{1}{2}$. Since the height of every half cell is bounded by 1, every left and right arm has a length of at least $1/2$. Then, property (P2) implies that each top and bottom arm has length $1/2$, i.e., every $p \in P$ is placed on the horizontal centre line of R. Finally, again by (P1), the points must be evenly spread. Hence, P is a $1 \times n$ grid.

Case: $\rho < n$. We consider the point p whose cell $V^P(p)$ contains the top left corner of R and denote its quarter cells by C_i. Then, C_2 is a rectangle. Moreover, $V^P(p)$ is not a bridge; otherwise its left half cell has area $\geq \frac{1}{2} > \frac{1}{2n}\rho = \frac{1}{2n}\text{area}(R)$. Therefore, by (P2), all arms of $V^P(p)$ have the same length; we denote this length by d. Together with the fact that C_2 and C_4 have the same area by (P1) and Lemma 2, it follows that C_2 and C_4 are squares of side length d.

We consider the right boundary of C_4. Since the right arm of $V^P(p)$ has length d (and the boundary continues vertically below), some point q has distance $2d$ to p and lies in $Q_1(p)$. The set of all these possible point locations forms a segment, which is highlighted in red in Figure 10(a). Consequently, the left arm of q has length d. By (P2), the top arm of q must also have length d. Hence, q lies at the grid location illustrated in Figure 10(b). Moreover, it follows that q is the unique point whose cell shares part of the boundary with C_1; otherwise the top arm of q does not have length d.



(a) Points $q \in Q_1(p) \cup Q_3(p)$ with $l_1(p, q) = 2d$. **(b)** Unique neighbouring cell generators. **(c)** Final square grid.

Figure 10 Illustration of the proof of Lemma 15.

By symmetry, a point q' lies at a distance $2d$ below p and distance d to the boundary. Thus, every quarter cell of $V^P(p)$ is a square of size d; thus, the arms of all cells have length at least d. Moreover, the top left quarter cells of $V^P(q)$ and $V^P(q')$ are squares, so their bottom right quadrants must be as well. Using this argument iteratively along the boundary implies that boundary cells are squares. Applying it to the remaining rectangular hole shows that P is a square grid. ◀

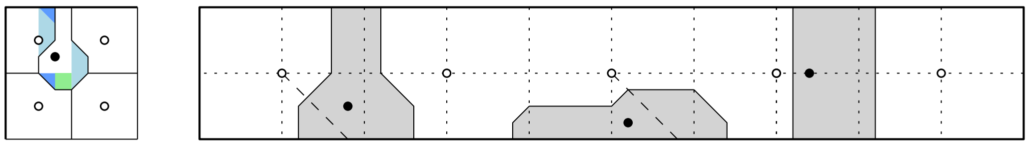
We now come to our main result.

▶ **Theorem 16.** *White has a winning strategy for placing n points in a $(1 \times \rho)$ rectangle with $\rho \geq 1$ if and only if $\rho \geq n$; otherwise Black has a winning strategy. Moreover, if $\rho \geq n$, the unique winning strategy for White is to place a $1 \times n$ grid.*

Proof. First we show that Black has a winning strategy if $\rho < n$. Suppose that Black cannot win. Note that $\rho < n$ implies $n \geq 2$. Consequently, by Theorem 11 and Lemma 15, the white point set W is a square $a \times b$ grid with $a, b \geq 2$, and thus the four cells in the top left corner induce a 2×2 grid. By Theorem 9, it suffices to identify a winning point for Black. Thus, we show that

▷ Claim 17. Black has a winning point in a square 2×2 grid.

Proof. suppose the arms of all cells have length d. Then a black point p is a winning point if its cell has an area exceeding $2d^2$. With p placed at a distance $^{3d}/_2$ from the top and left boundary as depicted in Figure 11(a), the cell of p has an area of $2d^2 + {}^{d^2}/_4$. ◁



(a) A black winning point in a 2×2-grid. **(b)** Every black cell has an area $\leq {}^1/_{2n} \cdot \text{area}(R)$. Moreover, only $n-1$ locations result in cells of that size.

Figure 11 Illustration of the proof of Theorem 16.

Secondly, we consider the case $\rho \geq n$ and show that White has a winning strategy. Theorem 11 and Lemma 15 imply that White must place its points in a $1 \times n$ grid; otherwise Black can win. We show that Black has no option to beat this placement; i.e., if $\rho \geq n$, then

▷ Claim 18. Black has no winning point and cannot force a tie in a $1 \times n$ grid.

Proof. By symmetry, there essentially exist two different placements of a black point b with respect to a closest white point w_b. Without loss of generality, we assume that w_b is to the left and not below b. Let x and y denote the horizontal and vertical distance of b to w_b, respectively. For a unified presentation, we add half of potential neutral zones in case $x = y$ to the area of the black cell. As a consequence, Black loses if its cells have an area of less than ${}^1/_2 \cdot \text{area}(R)$.

If $x > y$, the cell of b evaluates to an area of (at most) ${}^1/_{2n} \cdot \text{area}(R) - y^2$. In particular, it is maximized for $y = 0$, i.e., when b is placed on the horizontal centre line of R and if there exist white points to the left and right of b. In this case the cell area is exactly ${}^1/_{2n} \cdot \text{area}(R)$.

If $x \leq y$, the cell area of b has an area of (at most) ${}^1/_{2n} \cdot \text{area}(R) - y(w' - h') - {}^1/_4(3y^2 + x^2)$, where $w' := {}^w/_{2n}$ and $h' := {}^h/_2$ denote the dimensions of the grid cells. Note that $w' \geq h'$ because $\rho \geq n$. Consequently, the cell area is maximized for $x = 0, y = 0$. However, this placement coincides with the location of a white point and is thus forbidden. Therefore every valid placement results in a cell area strictly smaller than $1/2n \cdot \mathrm{area}(R)$. Consequently, Black has no winning point.

Note that the cell area is indeed strictly smaller than the above mentioned maximum values if the black point does not have white points on both sides. Therefore the (unique) best placement of a black point is on the centre line between two white points, as illustrated by the rightmost black point in Figure 11(b). However, there exist only $n-1$ distinct positions of this type; all other placements result in strictly smaller cells. Consequently, Black cannot force a tie and so loses. ◁

This completes the proof of the theorem. ◀

6 Open Problems

There are various directions for future work.

We demonstrated that there is a spectrum of balanced configurations, based on identifying a number of small *atomic* (i.e., non-decomposable) configurations that can be concatenated in a strip-like fashion. Are there further atomic configurations? Is it possible to combine them into more intricate two-dimensional patterns rather than just putting together identical strip-based configurations? Beyond that, the biggest challenge is clearly to provide a full characterization of balanced configurations, with further generalizations to other metrics and dimensions.

As our main result, we presented a full characterization of the *One-Round Voronoi Game* with Manhattan distances. Just as for the previously studied Euclidean metric, this still leaves the multi-round variant as a wide open (and, most probably, quite difficult) problem. Further interesting problems arise from considering higher-dimensional variants.

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A Appendix: Details for Lemma 4 of Section 3

In this section we present the remaining details for Lemma 4.

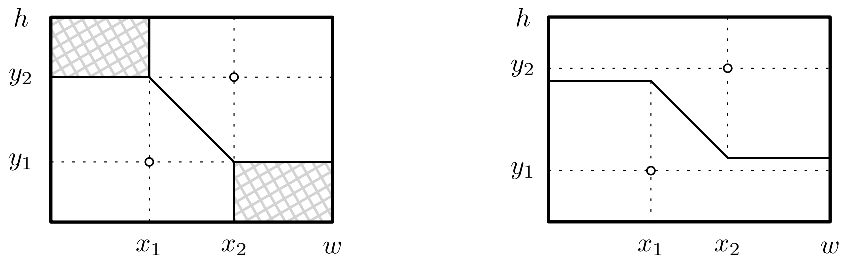
▶ **Lemma 4.** *The configurations $\mathcal{R}_{2,\rho}, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5$, depicted in Figure 4, are balanced. Moreover, $\mathcal{R}_{2,\rho}, \rho \in [1, 3/2]$, and $\mathcal{R}_3$ are the only balanced non-grid point sets with two and three points, respectively.*

In particular, we prove uniqueness of the balanced configurations for $n = 2$ and $n = 3$. Claim 19 shows the uniqueness for $n = 2$; Claim 20 shows the uniqueness for $n = 3$.

A.1 Balanced sets with two points

▷ Claim 19. For every $(1 \times \rho)$ rectangle with $1 \le \rho \le 3/2$, there exists (up to reflection) a unique point set P with $|P| = 2$ such that P is not a grid and fulfils (P1). The resulting configuration is $\mathcal{R}_{2,\rho}$ as illustrated in Figure 4.

Moreover, if $\rho > 3/2$, there exists no such point set.



(a) Case: $|x_1 - x_2| = |y_1 - y_2|$ **(b)** Case: $|x_1 - x_2| < |y_1 - y_2|$

Figure 12 Illustration of the proof of Claim 19.

Proof. Let P be a point set in an $(h \times w)$ rectangle R consisting of two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ which is different from a grid and fulfils (P1). Without loss of generality, we assume that p_2 lies in the top right quadrant of p_1. We distinguish two cases depending on whether or not the vertical and horizontal distances of p_1 and p_2 are equal.

Firstly, we consider the case that $\Delta_x(p_1, p_2) = \Delta_y(p_1, p_2) =: d$. By (P1) and Lemma 2, diagonally opposite quarter cells have the same area. Consequently, the second and fourth rectangular quarter cells of p_1 and p_2 imply that $dx_1 = dy_1 \iff x_1 = y_1$, and $d(w - x_2) = d(h - y_2) \iff w - x_2 = h - y_2$. Hence $w = x_1 + d + (w - x_2) = y_1 + d + (h - y_2) = h$ so it follows that $\rho = 1$. This yields $\mathcal{R}_{2,1}$ as shown in Figure 4.

Secondly, we consider the case that $\Delta_x(p_1, p_2) \neq \Delta_y(p_1, p_2)$. Without loss of generality, we assume that $\Delta_x(p_1, p_2) < \Delta_y(p_1, p_2)$ as illustrated in Figure 12(b). Thus, $\mathcal{B}(p_1, p_2)$ is horizontal. By (P1), the bottom half cell of p_1 and the top half cell of p_2 have an area of $1/4wh$ each. Because their width is w, it follows that $y_1 = 1/4h$ and $y_2 = 3/4h$. By symmetry of the bisector, it follows that the height of the left half cell of p_1 equals the height of the right half cell of p_2. Because the areas of these half cells are equal, their respective widths must also agree, i.e., $x_2 = w - x_1$. Moreover, the left half cell of p_1 has an area of

$$x_1(3/4h - 1/2(1/2h - (x_2 - x_1)) = x_1/2(h + x_2 - x_1) = x_1/2(h + w - 2x_1) \overset{!}{=} wh/4$$
$$\iff x_1 \in \{h/2, w/2\}\,.$$

If $x_1 = {}^w\!/2$, P is a grid. For $x_1 = {}^h\!/2$, we obtain the configuration $\mathcal{R}_{2,\rho}$ depicted in Figure 4. Note that it is necessary that ${}^h\!/2 < w - {}^h\!/2 \iff h < w$ and $(y_2 - y_1) \geq (x_2 - x_1) \iff w \leq {}^3\!/2h$. This completes the proof of the claim. ◀

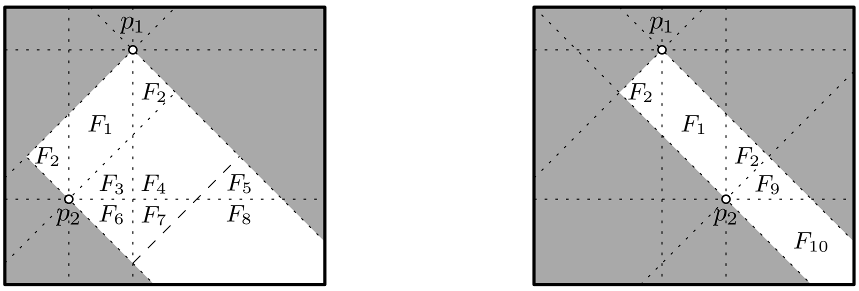
A.2 Balanced sets with three points

▷ Claim 20. Let P be a point set in a rectangle R such that $|P| = 3$, P is not a grid, and P satisfies (P1). Then $\rho(R) = {}^{49}\!/36$ and (R, P) is the configuration $\mathcal{R}_3$.

Proof. We denote the height and width of R by h and w respectively, and distinguish four cases depending on the number of degenerate bisectors of the points in P. We start with the case of no degenerate bisector.

No degenerate bisectors Firstly, let us assume that P contains no degenerate bisector. Therefore every corner of R is contained in one of three cells. Consequently, the cell of one point $p_1 = (x_1, y_1)$ contains two corners of R; without loss of generality, we assume that p_1 contains the two top corners. Then, the top half cell of p_1 has width w and an area of ${}^1\!/6 \cdot hw$ by property (P1). Consequently, $y_1 = {}^5\!/6h$. Moreover, the other two points lie in $O_6(p_1) \cup O_7(p_1)$; otherwise the cell of p_1 does not contain both top corners.

Furthermore, at least one other point $p_2 = (x_2, y_2)$ contains a corner of R in its cell, without loss of generality the bottom left corner of R. This implies that the third point p_3 lies in $\bigcup_{i \in \{1,2,3,8\}} O_i(p_2)$. We distinguish the cases $x_1 > x_2$ and $x_1 \leq x_2$ which are illustrated in Figures 13(a) and 13(b) respectively. Moreover, the octants of p_1 and p_2 as well as the so-called partition line completing the diamond around the rightmost breakpoint of the bisector between $\mathcal{B}(p_1, p_2)$ (the dashed line in Figure 13(a); see Averbakh et al. [3] for more details) subdivide the possible locations of the third point into regions which are illustrated in Figure 13. As a result, for every position of p_3 within a region, the resulting Voronoi diagram is structurally identical; see also Figure 14. Note also that all regions with the same label result in fully symmetric configurations.

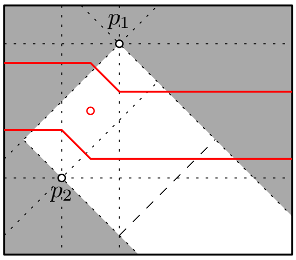


(a) Configurations where $x_1 > x_2$. **(b)** Configurations where $x_1 \leq x_2$.

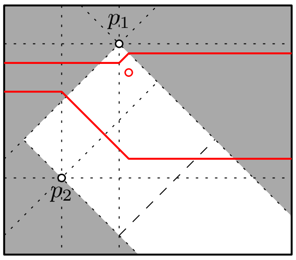
Figure 13 Illustration of the cases in the proof of Claim 20.

For each configuration, we can describe the areas of the quarter cells dependent on the three point coordinates. Using (P2) and Lemma 2, we obtain a number of equations. Carrying out the involved calculations by hand is rather tedious, so we made use of MATLAB®. It turns out that there exists a solution if and only if $\rho = {}^{49}\!/36$ and that $\mathcal{R}_3$ is the unique solution.

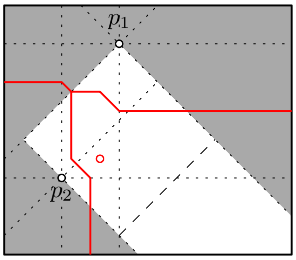
In the following, we present all combinatorially different Voronoi diagrams of three points containing one, two, and three degenerate bisectors, respectively. Exploiting (P2) and Lemma 2, we then obtain a system of equations for each diagram dependent on the points' coordinates. Using MATLAB®, we guarantee that none of them supports a balanced set.



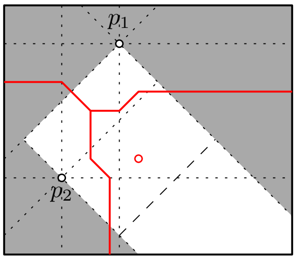
(a) p_3 in F_1



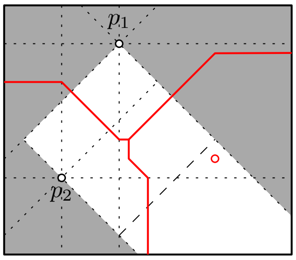
(b) p_3 in F_2



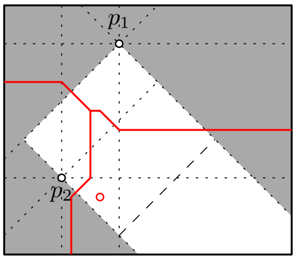
(c) p_3 in F_3



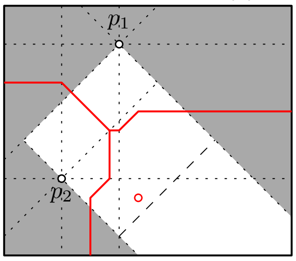
(d) p_3 in F_4



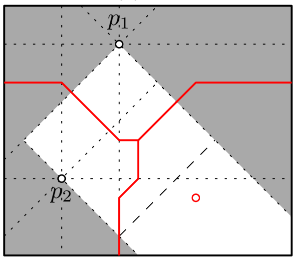
(e) p_3 in F_5



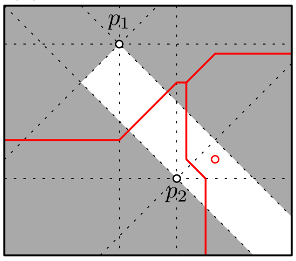
(f) p_3 in F_6



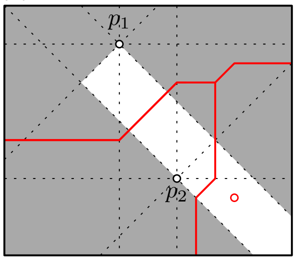
(g) p_3 in F_7



(h) p_3 in F_8



(i) p_3 in F_9

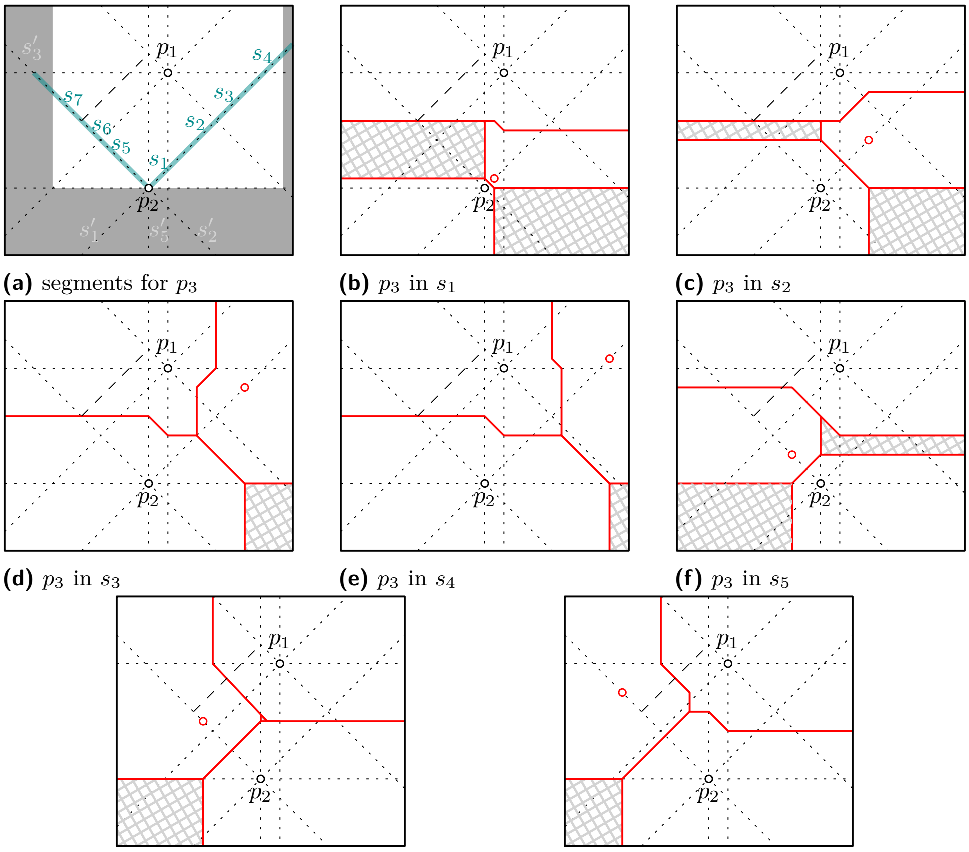


(j) p_3 in F_{10}

Figure 14 The ten combinatorially different Voronoi diagrams with no degenerate bisector dependent on the relative position of the third point p_3.

18 Competitive Location Problems

One degenerate bisector Secondly, we consider the case that $P = \{p_1, p_2, p_3\}$ contains one degenerate bisector. We may assume without loss of generality that p_1 has no degenerate bisector and that $\Delta_y(p_1, p_2)$ exceeds all of $\Delta_x(p_1, p_2)$, $\Delta_x(p_1, p_3)$, and $\Delta_y(p_1, p_3)$; otherwise we exchange the labels of p_2 and p_3 or rotate the configuration. Furthermore, we may assume that p_2 lies below p_1 and does not lie to the right of p_1 as depicted in Figure 15(a).



(a) segments for p_3 **(b)** p_3 in s_1 **(c)** p_3 in s_2

(d) p_3 in s_3 **(e)** p_3 in s_4 **(f)** p_3 in s_5

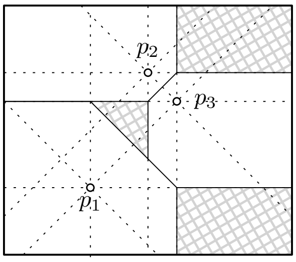
(g) p_3 in s_6 **(h)** p_3 in s_7

Figure 15 The seven Voronoi diagrams with one degenerate bisector.

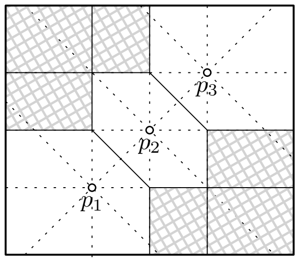
By assumption, p_2 and p_3 share a degenerate bisector. Therefore p_3 lies on one of the diagonal lines through p_2. Moreover, p_3 lies above p_2 and not too far to the left and right of p_1, because of our assumption that $\Delta_y(p_1, p_2) > \Delta_x(p_1, p_3), \Delta_y(p_1, p_3)$. As before, the octants of p_1 and p_2 as well as the partition line induced by the leftmost breakpoint of the bisector $\mathcal{B}(p_1, p_2)$ (represented by the dashed segment in Figure 15(a) that completes the diamond about the leftmost breakpoint of $\mathcal{B}(p_1, p_2)$) subdivide the location of p_3 into seven segments which are illustrated in Figure 15(a). Placing p_3 on different segments results in combinatorially different Voronoi diagrams which are depicted in Figure 15.

Two degenerate bisectors Without loss of generality, we may assume that p_3 has a degenerate bisector with both p_1 and p_2. Since $\mathcal{B}(p_1, p_2)$ is non-degenerate, p_1 and p_2 lie on different diagonals and we may assume without loss of generality that p_3 is the rightmost point of P and $\Delta_x(p_1, p_3) \geq \Delta_x(p_1, p_2)$. Consequently, we obtain a configuration as depicted in Figure 16(a).

Three degenerate bisectors If all three bisectors are degenerate, then the three points lie on a common diagonal line: consider two points on a line of slope $+1$. Since the diagonals



(a) Two degenerate bisectors.



(b) Three degenerate bisectors.

Figure 16 Voronoi diagrams of three points containing two degenerate bisectors.

through the two points of slope -1 are parallel, the third point must lie on the diagonal of slope $+1$; see Figure 16(b). Without loss of generality (allowing for reflection and rotation and renaming the points) we may assume the illustrated labelling.

In showing that the resulting systems of equations have no solutions, we prove that $\mathcal{R}_3$ is the unique balanced non-grid configuration with three points. ◀