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THE DYNAMICAL INVERSE PROBLEM FOR A LAMÉ TYPE SYSTEM (THE BC METHOD)

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In the paper, for a Lamé type system the inverse problem of recovering the fast and slow wave velocities from the boundary dynamical data (the response operator) is solved. The velocities are determined in a near-boundary domain, the depth of determination being proportional to the observation time. The BC-method, which is an approach to inverse problems based on their connections with boundary control theory, is used. Bibliography: 20 titles.

1. Introduction

About the paper. In [3], it was solved the inverse problem of restoring the velocities of fast (p-) and slow (s-) waves in a model Lamé type system, using dynamical boundary data. The approach was based on the division of boundary controls into two classes. The controls from these classes initiate only p-waves or only s-waves, respectively. Such an approach cannot be applied to the full Lamé system (with the variable density and lower terms), because the division of controls is impossible in it.

Later on, in [10] it was suggested an approach that does not use the division of controls. This approach, as such, is more perspective for the work with the full system, but the corresponding inverse problem has not been solved as yet.

In the present paper, for a *Lamé type system* we suggest another approach based on the ideas of paper [8], which also does not use the division of controls. We are pinning our hopes for a progress in solving the problem for the full Lamé system on this paper.

Similarly to the previous ones, the new approach is a version of the boundary control method (the BC method), making use of controllability properties of dynamical systems for solving the inverse problems. For the Lamé system, these properties were established in [9].

The main result. We consider a dynamical Lamé type system in which there are wave modes of two types (p-waves and s-waves) and the velocities of the modes c_p and c_s depend on the point, but $c_p > c_s$ everywhere. It is assumed that the density in the domain is constant ($\rho = 1$).

The main result of the paper is the recovery of the velocities c_p and c_s in a near-boundary (regular) domain from the response operator, and the depth of determination is proportional to the observation time.

2. Geometry

2.1. Metrics. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth[1] boundary Γ. In $\overline{\Omega}$, smooth functions (velocities) $c_\alpha = c_\alpha(x)$ ($\alpha = p, s$) such that $0 < c_s < c_p$ are given. In $\overline{\Omega}$, they determine conformally Euclidean metrics

$$ds_\alpha^2 := \frac{|dx|^2}{c_\alpha^2}, \tag{2.1}$$

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[1] Throughout the paper, relative to surfaces, functions, fields, and so on, *smooth* means C^∞-smooth.

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where $|dx|$ is a Euclidean length element in $\mathbb{R}^3$. By $\tau_\alpha(x,y)$ we denote distances in these metrics. The quantities $T^*_\alpha := \max_{\Omega} \tau_\alpha(\,\cdot\,,\Gamma)$ are called *filling times*.

For a subset $A \subset \overline{\Omega}$ we define its metric neighborhoods

$$\Omega^r_\alpha[A] := \left\{x \in \overline{\Omega} \;\middle|\; \tau_\alpha(x,A) < r\right\}, \qquad r > 0,$$

and by $\Omega^r_\alpha := \Omega^r_\alpha[\Gamma]$ we denote the neighborhoods of the boundary (the near-boundary layers of thickness r). From the relation for velocities it follows that $\tau_p(x,y) < \tau_s(x,y)$ and $\Omega^r_s[A] \subset \Omega^r_p[A]$ for any $x,y \in \overline{\Omega}$ $(x \neq y)$, $A \subset \overline{\Omega}$, and $r > 0$. The term "filling times" is motivated by the relations $T^*_\alpha = \inf\{r > 0 \mid \Omega^r_\alpha = \overline{\Omega}\}$.

For $A \subset \overline{\Omega}$ we define the equidistant surfaces

$$\Gamma^r_\alpha[A] := \left\{x \in \overline{\Omega} \;\middle|\; \tau_\alpha(x,A) = r\right\}, \qquad r > 0,$$

and denote by $\Gamma^r_\alpha := \Gamma^r_\alpha[\Gamma]$ the equidistant curves of the boundary.

2.2. The regular domain. With a point $x \in \overline{\Omega}$ we associate the sets $\gamma_\alpha(x) := \{\gamma \in \Gamma \mid \tau_\alpha(x,\gamma) = \tau_\alpha(x,\Gamma)\}$ of nearest points of the boundary. As is known, for $r > 0$ small enough, for any $x \in \Omega^r_\alpha$ the sets $\gamma_\alpha(x)$ consist of a single point, and the system of semigeodesic (ray) coordinates with the base Γ is regular in Ω^r_α. Let T^{reg}_α be the least upper bounds of those r for which such regularity holds. The near-boundary layers $\Omega^{T^{\mathrm{reg}}_\alpha}$ are called regular domains of the respective metrics.

We define $T^{\mathrm{reg}} := \min\{T^{\mathrm{reg}}_p, T^{\mathrm{reg}}_s\}$ and the common regular domain $\Omega^{T^{\mathrm{reg}}} := \Omega^{T^{\mathrm{reg}}}_p$. All further considerations will be conducted in this common regular domain.

2.3. The influence domains. In the sequel, the variable $t \geq 0$ plays the role of time. We fix $T > 0$ and denote by

$$Q^T := \Omega \times (0,T), \qquad \Sigma^T := \Gamma \times [0,T]$$

the space-time cylinder and its lateral area.

For a point $(x_0,t_0) \in \overline{Q^T} = \overline{\Omega} \times [0,T]$ we define the influence cones

$$K^T_\alpha[(x_0,t_0)] \;:=\; \left\{(x,t) \in \overline{Q^T} \mid \tau_\alpha(x,x_0) \leq t - t_0\right\}.$$

For $B \subset \overline{Q^T}$, the subdomain

$$K^T_\alpha[B] \;:=\; \overline{\bigcup_{(x_0,t_0)\in B} K^T_\alpha[(x_0,t_0)]}$$

is called the *influence domain* of the set B.

From the definitions it is seen that the cross-section $t = \xi$ of the influence domain $K^T_\alpha[\Sigma^T]$ coincides with the ξ-neighborhood of the respective metric Γ in Ω:

$$\left\{x \in \overline{\Omega} \mid (x,\xi) \in K^T_\alpha[\Sigma^T]\right\} = \overline{\Omega^\xi_\alpha}, \qquad 0 < \xi \leq T. \tag{2.2}$$

2.4. Functions and fields. We consider the following sets of real number and vector ($\mathbb{R}^3$-valued) functions. These latter are called *fields*.

The space $\mathcal{H}$. The space of fields

$$\mathcal{H} := L_2(\Omega;\mathbb{R}^3)$$

with scalar product

$$(y,v)_{\mathcal{H}} := \int_\Omega y(x)\cdot v(x)\,dx,$$

where "$\cdot$" is the standard scalar product in $\mathbb{R}^3$, plays the main role. For measurable $A \subset \Omega$, we define the subspace

$$\mathcal{H}[A] \ := \left\{ y \in \mathcal{H} \;\middle|\; \operatorname{supp} y \subset \overline{A} \right\}.$$

In the space $\mathcal{H}$ we distinguish the subspaces

(1) of *solenoidal* fields

$$\mathcal{J} := \{ y \in \mathcal{H} \mid \operatorname{div} y = 0 \quad \text{in } \Omega \} \tag{2.3}$$

(the operation div is meant in the sense of distributions); the set of smooth fields $\mathcal{J} \cap C^\infty(\overline{\Omega}; \mathbb{R}^3)$ is dense in $\mathcal{J}$;

(2) *of potential fields*

$$\mathcal{G} := \{ h \in \mathcal{H} \mid h = \nabla \varphi,\ \varphi \in W_2^1(\Omega),\ \varphi|_\Gamma = 0 \}; \tag{2.4}$$

the set of smooth fields $\mathcal{G} \cap C^\infty(\overline{\Omega}; \mathbb{R}^3)$ is dense in $\mathcal{G}$. The subspaces of $\mathcal{J}$ and $\mathcal{G}$ consisting of fields localized in A are denoted by $\mathcal{J}[A]$ and $\mathcal{G}[A]$.

The following relation (Weyl decomposition) is valid:

$$\mathcal{H} = \mathcal{J} \oplus \mathcal{G} \tag{2.5}$$

(for example, see [11, 15, 16]).

The space $\mathcal{F}^T$. Define the space $\mathcal{F}^T := L_2(\Sigma^T; \mathbb{R}^3)$ with scalar product

$$(f, g)_{\mathcal{F}^T} := \int\limits_{\Sigma^T} f(\gamma, t) \cdot g(\gamma, t)\, d\Gamma\, dt,$$

where $d\Gamma$ is the Euclidean area element on Γ. The class of smooth fields

$$\mathcal{M}^T := \left\{ f \in C^\infty(\Sigma^T; \mathbb{R}^3) \mid \operatorname{supp} f \subset \Gamma \times (0, T] \right\}$$

is dense in $\mathcal{F}^T$. Note that the fields from $\mathcal{M}^T$ vanish near $t = 0$.

With a subset $B \subset \Sigma^T$ we associate the subspace

$$\mathcal{F}^T[B] := \left\{ f \in \mathcal{F}^T \mid \operatorname{supp} f \subset \overline{B} \right\}.$$

It involves the dense set of smooth fields $\mathcal{M}^T[B] := \mathcal{M}^T \cap \mathcal{F}^T[B]$.

A vector $a \in \mathbb{R}^3$ at a point of the boundary is decomposed into the sum

$$a = a_\nu + a_\theta = a^\nu\, \nu + a_\theta, \tag{2.6}$$

where ν is the Euclidean external unit normal to Γ, $a^\nu = a \cdot \nu$; a_ν and a_θ are the normal and tangential components. This decomposition will also be written as

$$a = \begin{pmatrix} a^\nu \\ a_\theta \end{pmatrix}. \tag{2.7}$$

Consider the scalar and vector spaces

$$\mathcal{F}_p^T := L_2(\Sigma^T), \quad \mathcal{F}_s^T := \left\{ f \in \mathcal{F}^T \mid (\nu \cdot f)|_\Gamma = 0 \right\}.$$

Their subspaces $\mathcal{F}_\alpha^T[B]$ $(\alpha = p, s)$ consist of elements with supports in $\overline{B}$; denote

$$\mathcal{M}_\alpha^T[B] := \mathcal{M}^T \cap \mathcal{F}_\alpha^T[B], \tag{2.8}$$

which are smooth functions and fields vanishing near $t = 0$.

In accordance with (2.7) we write

$$\mathcal{F}^T = \begin{pmatrix} \mathcal{F}_p^T \\ \mathcal{F}_s^T \end{pmatrix}.$$

3. A Lamé type system

3.1. An initial boundary value problem. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ. Fix $T \in (0, \infty)$, and denote $\varkappa := c_p^2$ and $\mu := c_s^2$.

Consider the initial boundary value problem

$$u_{tt} = \nabla \varkappa \operatorname{div} u - \operatorname{rot} \mu \operatorname{rot} u \qquad \text{in } Q^T, \tag{3.1}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \qquad \text{in } \overline{\Omega}, \tag{3.2}$$

$$u = f \qquad \text{on } \Sigma^T, \tag{3.3}$$

with smooth variable coefficients $\mu = \mu(x) > 0$, $\varkappa = \varkappa(x) > 0$ in $\overline{\Omega}$. Note that $\varkappa = \lambda + 2\mu$ (λ and μ are standard Lamé coefficients). We call this system the *a Lamé type system* and denote it by the symbol α^T. Equation (3.1) is obtained from the full [2] Lamé equation, which describes the wave propagation in an elastic medium, by retaining higher terms (with respect to the order of differentiation); moreover, we assume that the density $\rho = 1$ in the domain (see [3]). Note that the main properties of the full system (the regularity of solutions, controllability) [9] remain valid for a Lamé type system [10].

An $\mathbb{R}^3$-valued function $f = f(\gamma, t)$ is called a (Dirichlet) *boundary control*. It describes the displacements of the points of the boundary initiating a wave process in Ω. A solution $u = u^f(x, t)$ (*wave*) is an $\mathbb{R}^3$-valued function describing the displacements of the points of a medium in Ω. For controls of the class $\mathcal{M}^T$, problem (3.1)–(3.3) has a unique classical smooth solution u^f.

The map $f \mapsto u^f$ from $\mathcal{F}^T$ to $L_2((0,T);\, L_2(\Omega; \mathbb{R}^3))$ is continuous (see [9]). Consequently, it can be extended from $\mathcal{M}^T$ to controls in $\mathcal{F}^T$ by continuity. By a (generalized) solution of problem (3.1)–(3.3) for controls of this class we mean the image of f under this extension.

3.2. Finiteness of the influence domain. The functions

$$c_p = \sqrt{\varkappa}, \qquad c_s = \sqrt{\mu}$$

($0 < c_s < c_p$) have the meaning of the velocities of longitudinal (fast) and transverse (slow) waves. The velocities determine two conformally Euclidean metrics (2.1). Each of them specifies its own distances, neighborhoods, geodesics, influence domains, and so on (see Sec. 2).

The equation of the Lamé type is hyperbolic and has two families of characteristics $\chi_\alpha(x,t) =$ const in Q^T determined by known equations $\left(\frac{\partial \chi_\alpha}{\partial t}\right)^2 - c_\alpha^2 \, |\nabla \chi_\alpha|^2 = 0$ ($\alpha = p, s$.) Since problem (3.1)–(3.3) is hyperbolic, we have the relation

$$\operatorname{supp} u^f \;\subset\; K_p^T[\operatorname{supp} f]\,, \tag{3.4}$$

which is referred to as the finiteness principle of the influence domain. It shows that the waves in a Lamé type system propagate with velocity not exceeding the velocity of the fast mode c_p.

Let $f \in \mathcal{F}^T[\Sigma^T]$, i.e., the control f acts from Γ. In view of (2.2), relation (3.4) implies that

$$\operatorname{supp} u^f(\,\cdot\,, t) \;\subset\; \overline{\Omega_p^t}, \qquad t > 0\,. \tag{3.5}$$

[2] The full Lamé equation in coordinate-free form is $\rho u_{tt} = \nabla(\lambda + 2\mu) \operatorname{div} u - \operatorname{rot} \mu \operatorname{rot} u + 2\,\{(\nabla\mu, \nabla)u - \operatorname{div} u \, \nabla\mu + \nabla\mu \times \operatorname{rot} u\}$ (see [14]).

3.3. The system α^T. Henceforth we regard problem (3.1)–(3.3) as a dynamical system and endow it with attributes of control theory – spaces and operators.

The space of controls $\mathcal{F}^T$ is called the *external space* of the system α^T. A solution u^f is interpreted as a *trajectory* of the system, and $u^f(\,\cdot\,,t)$ is its *state* at the time t. The space $\mathcal{H}$ is said to be internal. By the property of L_2-regularity of solutions (see the end of Sec. 3.1), all the waves $u^f(\,\cdot\,,t)$ are its elements.

By (3.5), the relation $f \in \mathcal{F}^T[\Sigma^T]$ implies that $u^f(\,\cdot\,,t) \in \mathcal{H}[\Omega_p^t]$ for all $0 < t \le T$, i.e., the trajectory u^f of the system α^T does not leave the subspace $\mathcal{H}[\Omega_p^T]$.

3.4. The response operator. On the fields of the class $\mathbf{H}^2(\Omega)$ (henceforth $\mathbf{H}^k(\dots)$ are vector Sobolev classes), we introduce the operator

$$L := \nabla \varkappa \operatorname{div} - \operatorname{rot} \mu \operatorname{rot},$$

which determines the evolution of the system α^T. Integrating by parts, for smooth u and v we get the relation (Green's formula)

$$\begin{aligned}(Lu,v)_{\mathcal{H}} - (u,Lv)_{\mathcal{H}} &= \int\limits_\Gamma \left[\begin{pmatrix}\varkappa \operatorname{div} u\\ \mu \operatorname{rot} u \times \nu\end{pmatrix}\cdot\begin{pmatrix}v^\nu\\ v_\theta\end{pmatrix} - \begin{pmatrix}u^\nu\\ u_\theta\end{pmatrix}\cdot\begin{pmatrix}\varkappa \operatorname{div} v\\ \mu \operatorname{rot} v \times \nu\end{pmatrix}\right] d\Gamma\\ &= (Nu, Dv)_{L_2(\Gamma;\,\mathbb{R}^3)} - (Du, Nv)_{L_2(\Gamma;\,\mathbb{R}^3)}\,;\end{aligned}$$

we have used the agreement about the representation (2.6)–(2.7) and have denoted

$$Du := \begin{pmatrix}u^\nu\\ u_\theta\end{pmatrix}, \qquad Nu := \begin{pmatrix}\varkappa \operatorname{div} u\\ \mu \operatorname{rot} u \times \nu\end{pmatrix} \quad \text{on } \Gamma. \tag{3.6}$$

The "input–output" correspondence in the dynamical system α^T is described by the response operator $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $\operatorname{Dom} R^T = \mathcal{M}^T$:

$$R^T f := N u^f \qquad \text{on } \Sigma^T, \tag{3.7}$$

where N is the (Neumann) operator defined by the second formula in (3.6). The response operator is well defined in view of the remark at the end of Sec. 3.1. Its action on a control vector $f = \begin{pmatrix}f^\nu\\ f_\theta\end{pmatrix}$, in accordance with the definition (3.7) and the agreement on the representation (2.6)–(2.7), can be written in the form [3]:

$$R^T f := \begin{pmatrix}\varkappa \operatorname{div} u^f\\ \mu \operatorname{rot} u^f \times \nu\end{pmatrix} \quad \text{on } \Sigma^T. \tag{3.8}$$

Note that R^T can be extracted from the measurements on the boundary Γ upon interaction with it of the waves generated by the controls f (see [14]). The response operator is adequate to the information that is available to the external observer who studies the dynamical system using its mapping "input–output."

3.5. The inverse problem. The statement of the dynamical inverse problem is as follows. Using the response operator R^{2T} given for fixed $T > 0$, it is required to find the velocities of waves: c_p in the domain Ω_p^T and c_s in the domain Ω_s^T. This statement is adequate to the finiteness property of the data influence domain (see [3, 5, 8]). The problem will be solved under the additional assumption $T < T^{\mathrm{reg}}$, i.e., in the regular domain.

3.6. Controllability. In the system α^T, the set of states (waves)

$$\mathcal{U}[\Sigma^T] := \{u^f(\,\cdot\,,T) \mid f \in \mathcal{M}^T[\Sigma^T]\}$$

is said to be *reachable* (from the boundary Γ in time $t=T$). By (3.5), we have the embedding

$$\mathcal{U}[\Sigma^T] \subset \mathcal{H}[\Omega_p^T], \qquad T>0\,. \tag{3.9}$$

Properties of reachable sets and the character of embeddings of the type (3.9) are central questions of boundary control theory. We mention here a result of this sort established in [9] for the full Lamé equation[3] with the use of the fundamental theorem on the uniqueness of the extension of the solution across a noncharacteristic surface [13].

Let X_s^T be an (orthogonal) projection in $\mathcal{H}$ onto $\mathcal{H}[\Omega_s^T]$. Its action is reduced to cutting off the vector fields to the subdomain Ω_s^T:

$$X_s^T\,y = \begin{cases} y & \text{in } \Omega_s^T, \\ 0 & \text{in } \Omega\backslash\Omega_s^T. \end{cases}$$

The relation

$$\overline{X_s^T\,\mathcal{U}[\Sigma^T]} = \mathcal{H}[\Omega_s^T]\,, \quad T>0, \tag{3.10}$$

is valid (the closure is taken in the metric of $\mathcal{H}$).

From (3.10) it follows that any vector field $y \in L_2\left(\Omega_s^T;\mathbb{R}^3\right)$ localized in the subdomain captured by the slow mode can be approximated (with any precision) by the wave $u^f(\,\cdot\,,T)$ for an appropriate choice of the control $f \in \mathcal{M}^T[\Sigma^T]$. In control theory this property is interpreted as approximate boundary controllability of the system α^T in the domain Ω_s^T.

To the final instant of time $t=T$, the waves initiated by the controls $f \in \mathcal{F}^T[\Sigma^T]$ fill the domain Ω_p^T containing the subdomain Ω_s^T. Roughly speaking, relation (3.10) means that the shape of the wave $u^f(\,\cdot\,,T)$ in Ω_s^T may be arbitrary. At the same time, this is certainly not the case in the subdomain $\Omega_p^T\backslash\Omega_s^T$ (see [9,10]).

3.7. The subsystem α_p^T. In a Lamé type system, two subsystems, *acoustic* and *Maxwell*, stand out naturally.

Consider the scalar initial boundary value problem

$$\varphi_{tt} = c_p^2\,\Delta\varphi \qquad \text{in } Q^T, \tag{3.11}$$

$$\varphi|_{t=0} = \varphi_t|_{t=0} = 0 \qquad \text{in } \overline{\Omega}\,, \tag{3.12}$$

$$\varphi = a \qquad \text{on } \Sigma^T\,, \tag{3.13}$$

where $c_p = \sqrt{\varkappa}$. For controls of the class $\mathcal{M}_p^T$ (see (2.8)), it has a unique classical smooth solution $\varphi = \varphi^a(x,t)$. The correspondence $a \mapsto \varphi^a$ is continuous from $\mathcal{F}^T$ to $L_2\left((0,T);L_2(\Omega)\right)$, which enables one to define solutions for $a \in \mathcal{F}_p^T$ (see [5]).

The corresponding dynamical system is said to be *acoustic* and is denoted by α_p^T. Its external and internal spaces are $\mathcal{F}_p^T$ and $L_2(\Omega)$. Since the influence domain for the wave equation (3.11) is finite, we have the relation

$$\operatorname{supp}\varphi^a \,\subset\, K_p^T[\operatorname{supp}a]$$

and its simple consequence

$$\operatorname{supp}\varphi^a(\,\cdot\,,t) \,\subset\, \overline{\Omega_p^t}, \qquad t>0\,.$$

[3]The result is also valid for a Lamé type system (see [3,10]).

The acoustic system is approximately controllable from the boundary. We define the reachable sets

$$\Phi[\Sigma^T] := \left\{\varphi^a(\,\cdot\,,T)\,\middle|\, a\in\mathcal{M}_p^T[\Sigma^T]\right\}\,.$$

With the help of the Holmgren–John–Tataru theorem [4–6], the relation

$$\overline{\Phi[\Sigma^T]} \;=\; L_2[\Omega_p^T] \tag{3.14}$$

can be proved (the closure in $L_2(\Omega)$), which is valid for all $T>0$. We give a consequence of the property (3.14), which will be used in the sequel. Denote

$$\nabla\Phi[\Sigma^T] := \left\{\nabla\varphi^a(\,\cdot\,,T)\,\middle|\, a\in\mathcal{M}_p^T[\Sigma^T]\right\}\,.$$

Let $T<T^{\mathrm{reg}}$; the following relation is valid (see [10]):

$$\overline{\nabla\Phi[\Sigma^T]} \;=\; \left\{\nabla q\,\middle|\, q\in W_2^1(\Omega),\ \operatorname{supp} q\subset\overline{\Omega_p^T},\ q|_\Gamma=0\right\} \overset{(2.4)}{=} \mathcal{G}[\Omega_p^T] \tag{3.15}$$

(the closure in $\mathcal{H}$). It means the completeness of the gradients of waves in the space of potential fields localized in Ω_p^T.

3.8. The subsystem α_s^T. Consider the vector initial boundary value problem

$$\psi_{tt} = -c_s^2\operatorname{rot}\operatorname{rot}\psi \qquad \text{in } Q^T, \tag{3.16}$$

$$\psi|_{t=0} = \psi_t|_{t=0} = 0 \qquad \text{in } \overline{\Omega}\,, \tag{3.17}$$

$$\psi\times\nu = b \qquad \text{on } \Sigma^T\,, \tag{3.18}$$

where $c_s=\sqrt{\mu}<c_p$, $\times$ is the vector product in $\mathbb{R}^3$. For controls $b\in\mathcal{M}_s^T$ (see (2.8)), it has a unique classical smooth solution $\psi=\psi^b(x,t)$. Note that the mapping $b\mapsto\psi^b$, defined on the smooth class $\mathcal{M}_s^T$, is not continuous from $\mathcal{F}_s^T$ to $L^2\left((0,T);L_2(\Omega;\mathbb{R}^3)\right)$ (see [12]). But this complication is of a technical nature, and in the sequel we shall be able to get by with smooth controls and solutions.

The corresponding dynamical system is called *Maxwell* and is denoted by α_s^T. Its external space is $\mathcal{F}_s^T$. It is convenient to regard the space $\mathcal{H}$ as an internal one, but the following is essential.

The quantity $\operatorname{div}\psi^b$ is the integral of the movement of the system α_s^T, and, by the initial conditions (3.17), we have $\operatorname{div}\psi^b(\,\cdot\,,t)=0$ for all $t\geq 0$. For this reason, the waves are solenoidal fields and the trajectory of the system lies in the subspace $\mathcal{J}$ (see (2.3)).

Equation (3.16) is derived from the full system of Maxwell equations by removing one of the components (the magnetic field). Since the influence domain for Maxwell equations is finite, we have the relation

$$\operatorname{supp}\psi^b \;\subset\; K_s^T[\operatorname{supp} b]$$

and its consequence

$$\operatorname{supp}\psi^b(\,\cdot\,,t) \;\subset\; \overline{\Omega_s^t}, \qquad t>0.$$

The system α_s^T is approximately controllable from the boundary in the following sense. We define the reachable sets

$$\Psi[\Sigma^T] := \left\{\psi^b(\,\cdot\,,T)\,\middle|\, b\in\mathcal{M}_s^T[\Sigma^T]\right\}$$

and introduce the subspace

$$\mathcal{J}[\Omega_s^T] := \{y\in\mathcal{J}\,|\,\operatorname{supp} y\subset\overline{\Omega_s^T}\}\,.$$

Using the uniqueness of the extension of the solution of Maxwell equations across a noncharacteristic surface [13], one can obtain the relation

$$\overline{\Psi[\Sigma^T]} \;=\; \mathcal{J}[\Omega_s^T] \tag{3.19}$$

(the closure in $\mathcal{H}$), which is valid for all $T > 0$ (see [6, Theorem 3]).

Now we give a consequence of the property (3.19), which will be used in the next section. Denote

$$\operatorname{rot}\Psi[\Sigma^T] := \left\{\operatorname{rot}\psi^b(\,\cdot\,,T) \;\middle|\; b \in \mathcal{M}_s^T[\Sigma^T]\right\}.$$

The following relations are valid:

$$\overline{\operatorname{rot}\Psi[\Sigma^T]} \;=\; \left\{\operatorname{rot} y \;\middle|\; y \in W_2^1(\Omega;\mathbb{R}^3),\ \operatorname{supp} y \subset \overline{\Omega_s^T}\right\} \;=\; \mathcal{J}[\Omega_s^T]\,. \tag{3.20}$$

The first relation in (3.20) is derived with the help of (3.19) (see [10]), and the second one is a consequence of the density of the rotors of smooth fields in $\mathcal{J}[\Omega_s^T]$ (see [16]).

Thus we have (see (3.19) and (3.20)):

$$\overline{\Psi[\Sigma^T]} \;=\; \overline{\operatorname{rot}\Psi[\Sigma^T]} \;=\; \mathcal{J}[\Omega_s^T]\,. \tag{3.21}$$

3.9. The relationship between the trajectories. In system (3.1)–(3.3), we choose a control $f \in \mathcal{M}^T$ and set

$$a_{tt} := \left[\varkappa \operatorname{div} u^f\right]\big|_{\Sigma^T} \in \mathcal{M}_p^T\,, \quad b_{tt} := \left[\mu\,(\operatorname{rot} u^f)_\theta \times \nu\right]\big|_{\Sigma^T} \in \mathcal{M}_s^T\,.$$

As shown in [3], the following representation is valid:

$$u^f \;=\; \nabla\varphi^a \;+\; \operatorname{rot}\psi^b \qquad \text{in } Q^T\,; \tag{3.22}$$

it relates the trajectories of the system α^T and its subsystems α_p^T and α_s^T. It means that the waves in a Lamé type system split into potential and solenoidal components.

On the other hand, for arbitrary $a \in \mathcal{M}_p^T$ and $b \in \mathcal{M}_s^T$ the fields $\nabla\varphi^a = u^{f'}$ and $\operatorname{rot}\psi^b = u^{f''}$ are trajectories of the system α^T that correspond to the controls

$$f' = \begin{pmatrix} \nu\cdot\nabla\varphi^a \\ (\nabla\varphi^a)_\theta \end{pmatrix}, \quad f'' = \begin{pmatrix} \nu\cdot\operatorname{rot}\psi^b \\ (\operatorname{rot}\psi^b)_\theta \end{pmatrix},$$

and thus $\nabla\varphi^a + \operatorname{rot}\psi^b = u^{f'} + u^{f''} = u^{f'+f''}$. These relations and (3.22) imply that the following representation in algebraic sum form is valid:

$$\mathcal{U}[\Sigma^T] \;=\; \nabla\Phi[\Sigma^T] \;+\; \operatorname{rot}\Psi[\Sigma^T]\,.$$

Using (3.15) and (3.21) and passing to the closures, one can easily obtain

$$\overline{\mathcal{U}[\Sigma^T]} \;=\; \mathcal{G}[\Omega_p^T] \;+\; \mathcal{J}[\Omega_s^T]\,. \tag{3.23}$$

Note that the terms in this sum have a nonzero intersection.

4. The acoustic subsystem

In Sec. 4, we consider the objects (velocity, eikonal, geodesics, normals, divergences, wave fronts) that are related only to the fast metric

$$ds_p^2 = \frac{|dx|^2}{c_p^2} \tag{4.1}$$

and, simplifying the notation, we omit the subscript "p" in all quantities. Thus, we shall denote the fast velocity by $c := c_p$, the distance between the points x and y of the domain by $\tau(x, y) := \tau_p(x, y)$, the eikonal by $\tau(x) := \tau_p(x, \Gamma)$, the equidistant curves of the boundary by $\Gamma^r := \Gamma_p^r$, and so on. Note that dynamically the eikonal $\tau(x)$ at a point x is equal to the time of traveling the fast waves from the boundary Γ to this point, and its level surfaces Γ^τ correspond to wave front sets.

4.1. Semigeodesic coordinates. We fix $T : 0 < T < T^{\mathrm{reg}}$. To each point x of the regular domain $\Omega^T := \Omega_p^T$ there corresponds a unique point $\gamma(x)$ of the boundary nearest to x: $\tau(x, \gamma(x)) = \tau(x)$. The pair $(\gamma(x), \tau(x)) =: i(x)$ is called the semigeodesic coordinates of the point x with the base Γ, and the set

$$\Theta^T := i(\Omega^T) \tag{4.2}$$

is said to be the pattern of the subdomain Ω^T.

Agreement 4.1 (about the notation)**.** (1) *The point of the regular domain with geodesic coordinates (γ, τ) is denoted by $x(\gamma, \tau)$;*

(2) *if φ is a scalar or vector-valued function on Ω^T, then we denote by the same symbol φ the function $\varphi \circ i^{-1}$ defined on Θ^T (so that $\varphi(\gamma, \tau) := \varphi(x(\gamma, \tau))$); if ψ is given on Θ^T, then by the same symbol ψ we denote the function $\psi \circ i$ on Ω^T (so that $\psi(x) := \psi(\gamma(x), \tau(x))$);*

(3) *the writing $\varphi(x) = \psi(\gamma, \tau)$ implies two relations: $\varphi(x(\gamma, \tau)) = \psi(\gamma, \tau)$ on Θ^T and $\varphi(x) = \psi(\gamma(x), \tau(x))$ in Ω^T.*

Take $x \in \Omega^T$ and choose local coordinates $\widetilde{\gamma}^1$, $\widetilde{\gamma}^2$ in a neighborhood $\sigma \subset \Gamma$ of the point $\gamma(x)$. The functions $\widetilde{\gamma}^\alpha(\cdot) := \widetilde{\gamma}^\alpha(\gamma(\cdot))$, $\alpha = 1, 2$; $\tau = \tau(\cdot)$ form a system of semigeodesic coordinates on the following set (tube) containing x:

$$B_\sigma^T := \{x' \in \Omega^T \,|\, \gamma(x') \in \sigma, \quad 0 \le \tau(x') < T\}. \tag{4.3}$$

In a system of semigeodesic coordinates, the Euclidean elements of length and volume have the known form[4]

$$|dx|^2 = g_{\alpha\beta} d\gamma^\alpha d\gamma^\beta + c^2 d\tau^2; \; dx = cJ d\gamma^1 d\gamma^2 d\tau = c d\Gamma^\tau d\tau = c\frac{J}{J_0} d\Gamma d\tau, \tag{4.4}$$

where $J(\gamma, \tau) := (\det\{g_{\alpha\beta}\})^{\frac{1}{2}}$, $J_0(\gamma, \tau) := J(\gamma, 0)$, $d\Gamma^\tau$ and $d\Gamma$ are Euclidean surface elements on Γ^τ and Γ. The length element of the fast metric in the semigeodesic coordinates has the form

$$ds^2 = h_{\alpha\beta} d\gamma^\alpha d\gamma^\beta + d\tau^2\,; \tag{4.5}$$

comparing (4.4) and (4.5) and taking (4.1) into account, we obtain

$$h_{\alpha\beta} = \frac{1}{c^2} g_{\alpha\beta}\,. \tag{4.6}$$

[4]Henceforth summation is implied over the repeating indices α, $\beta = 1, 2$.

4.2. Recovery of the velocity from the tensor h. Here we prepare one of the fragments of the procedure solving the inverse problem. Let $T < T^{\rm reg}$. By this assumption, the pattern (4.2) of the subdomain Ω^T is $\Theta^T = \Gamma \times [0, T)$. The mapping $i : \Omega^T \to \Theta^T$ induces on the pattern two metrics (two tensors) g and h such that i^{-1} is the isometry (Θ^T, g) on Ω^T with the Euclidean metric and the isometry (Θ^T, h) on Ω^T with the fast metric. By (4.1), the metrics g and h are conformally equivalent: $h = c^{-2}g$. Assume that we know the velocity $c = c(\gamma, \tau)$ on Θ^T. The following statement is valid (see [8]).

Theorem 4.1. *The velocity $c = c(\gamma, \tau)$ on the pattern $\Theta^T = \Gamma \times [0, T)$ uniquely determines the velocity $c(x)$ in Ω^T.*

Proof. The velocity $c = c(\gamma, \tau)$ on the pattern Θ^T uniquely determines the tensor h on Θ^T, which enables one to find the Euclidean metric: $g = c^2 h$.

The tensor g determines the correspondence $i^{-1} \colon \Theta^T \to \Omega^T$. Indeed, let x^1, x^2, x^3 be Carlesian coordinates in Ω^T. Since they are harmonic, we have

$$\Delta_g x^k = 0 \quad \text{on} \quad \Theta^T \tag{4.7}$$

(Δ_g is the Laplacian in the g metric). Since x^k and $\frac{\partial x^k}{\partial \tau}$ are known on Γ, the elliptic equation (4.7) determines the functions $x^k = x^k(\gamma, \tau)$ on Θ^T uniquely. The correspondence i^{-1} is the mapping $(\gamma, \tau) \to x(\gamma, \tau) = \{x^1(\gamma, \tau), x^2(\gamma, \tau), x^3(\gamma, \tau)\}$.

The velocity in Ω^T is restored in accordance with the formula

$$c(x) = \left[\sum_{k=1}^{3} \left(\frac{\partial x^k(\gamma, \tau)}{\partial \tau} \right)^2 \right]^{1/2}, \quad x \in \Omega^T,$$

(see Agreement 4.1). The theorem is proved. $\square$

4.3. Representation of fields. In the regular domain, the eikonal is smooth; it determines the field of Euclidean normals to the surfaces Γ^τ :

$$\nu(x) := \frac{\nabla \tau(x)}{|\nabla \tau(x)|}, \qquad x \in \Omega^T, \;\; 0 < T < T^{\rm reg}.$$

Note that $\nu|_\Gamma$ is the internal normal to the boundary.

Any vector field y in Ω^T can be represented in the form

$$y = y_\theta + y_\nu \,,$$

where $y_\nu := (y \cdot \nu)\nu$ and $y_\theta := y - y_\nu$ are the longitudinal and transverse components of y.

Let $r = r(x)$ be the radius vector of the point $x = x(\gamma, \tau)$; γ^1, γ^2, τ are semigeodesic coordinates in the tube B^T_σ (see (4.3)) containing x; for $\alpha = 1, 2$ we denote

$$r_\alpha := \frac{\partial r}{\partial \gamma^\alpha}, \qquad r_0 := \frac{\partial r}{\partial \tau}\,;$$

the vectors r_1 and r_2 are tangent and the vector r_0 is normal to the surface Γ^τ. The field y in the tube can be represented in the form

$$y = y^\alpha r_\alpha + y^0 r_0 = y_\theta + y^0 r_0 \,.$$

Agreement 4.2. *We shall use the matrix representation, identifying $y = y^0 r_0 + y_\theta$ with the column* $\begin{pmatrix} y^0 \\ y_\theta \end{pmatrix}$.

We say that a field v is longitudinal if $v = v^0 r_0$ (i.e., $v_\theta = 0$). We recall known relations for the Euclidean metric tensor:

$$g_{\alpha\beta} = r_\alpha \cdot r_\beta; \qquad g_{00} = r_0 \cdot r_0 = c^2 \,. \tag{4.8}$$

4.4. Parallel translation. Below we shall use the parallel translation in the metric (4.1). Let B^T_σ be a tube covered by the system γ^1, γ^2, τ of semigeodesic coordinates; assume that $v(x) = v^0(x) r_0(x)$ be the vector at a point $x \in B^T_\sigma$ orthogonal to the surface $\Gamma^{\tau(x)}$. Denote $v^0 := v^0(x)$; the vector

$$[v(x)]^\wedge := v^0 r_0(\gamma(x))$$

is the result of parallel translation of the initial vector $v(x)$ from the point $x \in \Gamma^{\tau(x)}$ to the point $\gamma(x) \in \Gamma$ along the geodesic of the fast metric; obviously, it is orthogonal to Γ.

The fast and Euclidean metrics are conformally equivalent; the scalar product in the fast metric is invariant relative to the parallel translation. From the above it follows that for any two longitudinal vectors u and v, the relation

$$\frac{1}{c^2(x)} u(x) \cdot v(x) = \frac{1}{c^2(\gamma(x))} [u(x)]^\wedge \cdot [v(x)]^\wedge \tag{4.9}$$

is valid.

4.5. The mapping π. Let $T < T^{\mathrm{reg}}$, and let v be a longitudinal field in Ω^T. We associate with it a field on the pattern Θ^T, i.e., a function of (γ, τ) the values of which are vectors orthogonal to the surface Γ, in accordance with the rule

$$(\pi v)(\gamma, \tau) := [v(x(\gamma, \tau))]^\wedge, \qquad (\gamma, \tau) \in \Theta^T.$$

The following properties of the mapping π are easy consequences of the definition:

(1) let φ be a scalar function in Ω^T. We denote by the same symbol the operation of multiplication of fields by φ. The relation

$$\pi\varphi = \varphi\pi \tag{4.10}$$

is valid (here Agreement 4.1 is implied).

(2) Denote $c_0(\gamma, \tau) := c(\gamma, 0)$. As is easily seen from (4.9), the mapping $v \to \frac{c}{c_0}\pi v$ is a pointwise isometry in the sense of the Euclidean norm:

$$\left| \left(\frac{c}{c_0} \pi v \right) (\gamma, \tau) \right| = |v(x(\gamma, \tau))|, \qquad (\gamma, \tau) \in \Theta^T. \tag{4.11}$$

(3) Let $\frac{D}{d\tau}$ be the covariant derivative (in the fast metric). On smooth fields we have the relation

$$\frac{\partial}{\partial \tau} \pi = \pi \frac{D}{d\tau}. \tag{4.12}$$

4.6. The operator Π^T. The field $\nu := \frac{\nabla\tau}{|\nabla\tau|}$ is defined in Ω^T ($T < T^{\rm reg}$). It determines the decomposition

$$\mathcal{H}^T := \mathcal{H}[\Omega^T] = \mathscr{L}_\theta^T \oplus \mathscr{L}_\nu^T\,,$$

in which

$$\mathscr{L}_\theta^T := \{w \in L_2(\Omega^T;\mathbb{R}^3) \mid w\cdot\nu = 0 \quad \text{in} \quad \Omega^T\}, \tag{4.13}$$

$$\mathscr{L}_\nu^T := \{v \in L_2(\Omega^T;\mathbb{R}^3) \mid v\times\nu = 0 \quad \text{in} \quad \Omega^T\} \tag{4.14}$$

are subspaces of transverse and longitudinal (with respect to ν) fields.

On the pattern $\Theta^T = \Gamma\times[0,T)$ we consider the Hilbert space of fields normal to Γ:

$$\mathcal{F}_\nu^T := \{f \in L_2(\Sigma^T;\mathbb{R}^3) \,|\, f\times\nu_0 = 0 \qquad \text{on } \Theta^T\} \tag{4.15}$$

(with measure $d\Gamma d\tau$), where $\nu_0(\gamma,\tau) := \nu(\gamma,0)$. Recall the notation

$$J_0(\gamma,\tau) := J(\gamma,0), \quad c_0(\gamma,\tau) := c(\gamma,0)$$

and define a function $\kappa = \kappa(\gamma,\tau)$ on Θ^T:

$$\kappa := \frac{c}{c_0}\sqrt{c\frac{J}{J_0}}. \tag{4.16}$$

Introduce the operator $\Pi^T : \mathscr{L}_\nu^T \to \mathcal{F}_\nu^T$,

$$\Pi^T v := \kappa\pi v. \tag{4.17}$$

Lemma 4.1. *The operator Π^T possesses the following properties*:

(1) Π^T *is unitary*;
(2) *for bounded scalar functions χ, the relation $\Pi^T\chi = \chi\Pi^T$ holds*;
(3) *the operator Π^T retains smoothness*:

$$\Pi^T[\mathscr{L}_\nu^T \cap C^\infty(\overline{\Omega^T})] = \mathcal{F}_\nu^T \cap C^\infty(\overline{\Theta^T}).$$

Proof. All the functions that occur in the right-hand side of the definition of κ are smooth and positive on Θ^T. For arbitrary $u, v \in \mathscr{L}_\nu^T$ we have

$$\begin{aligned}(u,v)_{\mathscr{L}_\nu^T} &= \int\limits_{\Omega^T} u\cdot v\,dx \overset{(4.4)}{=} \int\limits_{\Theta^T} u(x(\gamma,\tau))\cdot v(x(\gamma,\tau))\left(c\frac{J}{J_0}\right)(\gamma,\tau)\,d\Gamma\,d\tau\\ &\overset{(4.9)}{=} \int\limits_{\Theta^T}\left(\frac{c}{c_0}\pi u\right)(\gamma,\tau)\cdot\left(\frac{c}{c_0}\pi v\right)(\gamma,\tau)\left(c\frac{J}{J_0}\right)(\gamma,\tau)\,d\Gamma\,d\tau\\ &\overset{(4.17)}{=} \left(\Pi^T u, \Pi^T v\right)_{\mathcal{F}_\nu^T},\end{aligned}$$

i.e., Π^T is an isometry. It is easy to see that Ran $\Pi^T = \mathcal{F}_\nu^T$. Property (2) follows from the definitions and (4.10); property (3) is an easy consequence of the fact that the mapping i is a diffeomorphism. The lemma is proved. $\square$

4.7. Projection in the space of potential fields. In the space of potential fields (2.4)

$$\mathcal{G} = \{h \in \mathcal{H} \mid h = \nabla\varphi,\ \varphi \in W_2^1(\Omega),\ \varphi|_\Gamma = 0\},$$

we separate out a chain of subspaces

$$\mathcal{G}^\xi := \left\{h \in \mathcal{G} \,\middle|\, \operatorname{supp} h \subset \overline{\Omega^\xi}\right\}, \qquad 0 \le \xi \le T\,;$$

we mention some properties of their elements (see [2]).

Proposition 4.1. *Let $T < T^{\rm reg}$, and let $\xi \in (0,T)$ be fixed.*

(1) *The trace* $h|_{\Gamma^\xi - 0}$ *of a field* $h \in \mathcal{G}^\xi$ *smooth in* Ω^ξ *is a field normal to* Γ^ξ.
(2) *Any smooth normal field on* Γ^ξ *is the trace of a field from* $\mathcal{G}^\xi$ *smooth in* Ω^ξ.

Denote by Q^ξ the projection in $\mathcal{G}^T$ onto $\mathcal{G}^\xi$ ($T < T^{\rm reg}$). One can show that the family $\{Q^\xi\}$ is continuous:

$$s\text{-}\lim_{\tau\to\xi} Q^\tau = Q^\xi, \quad 0 \le \xi \le T; \quad Q^0 = \mathbb{O}_{\mathcal{G}^T}; \quad Q^T = \mathbb{I}_{\mathcal{G}^T}.$$

Let $T < T^{\rm reg}$. We describe a representation for $Q^\xi\colon \mathcal{G}^T \to \mathcal{G}^\xi$. Choose a smooth field $h = h_\theta + h_\nu = \nabla\varphi = (\nabla\varphi)_\theta + \frac{\partial\varphi}{\partial\nu}\nu \in \mathcal{G}^T$, fix $\xi \in (0,T)$, and consider the problem

$$\Delta r = 0 \qquad \text{in } \Omega^\xi, \tag{4.18}$$

$$(\nabla r)_\theta = h_\theta = (\nabla\varphi)_\theta \qquad \text{in } \Gamma^\xi, \qquad \int\limits_{\Gamma^\xi} \frac{\partial r}{\partial\nu}\, d\Gamma^\xi = 0\,, \tag{4.19}$$

$$r = 0 \qquad \text{on } \Gamma. \tag{4.20}$$

The first condition in (4.19) is equivalent to the relation

$$r = \varphi + \text{const} \qquad \text{on } \Gamma^\xi; \tag{4.21}$$

The second condition enables one to find the constant in (4.21) uniquely. This implies that problem (4.18)–(4.20) is solvable in a unique way; its solution $r = r^\xi(x)$ is a function smooth in Ω^ξ.

Lemma 4.2. *For any smooth field* $h \in \mathcal{G}^T$*, the following representation holds*:

$$Q^\xi\, h = \begin{cases} h - \nabla r^\xi & \text{in } \Omega^\xi, \\ 0 & \text{in } \Omega^T\backslash\Omega^\xi, \end{cases} \tag{4.22}$$

here r^ξ *is the solution of problem* (4.18)–(4.20).

Proof. Let

$$h^\xi = \begin{cases} h - \nabla r^\xi & \text{in } \Omega^\xi, \\ 0 & \text{in } \Omega^T\backslash\Omega^\xi. \end{cases}$$

Denote $h^\xi_\perp := h - h^\xi$, so that

$$h = h^\xi + h^\xi_\perp. \tag{4.23}$$

We mention the following properties of h^ξ:

(1) $h^\xi = h - \nabla r^\xi = \nabla\varphi - \nabla r^\xi = \nabla(\varphi - r^\xi)$ in Ω^ξ;
(2) $h^\xi_\theta\big|_{\Gamma^\xi - 0} = (h - \nabla r^\xi)_\theta\big|_{\Gamma^\xi-0} = h_\theta\big|_{\Gamma^\xi-0} - (\nabla r^\xi)_\theta\big|_{\Gamma^\xi-0} \overset{(4.19)}{=} 0$;
(3) $h^\xi_\theta\big|_\Gamma = (h - \nabla r^\xi)_\theta\big|_\Gamma = 0$.

Properties (1)–(3) imply the inclusion $h^\xi \in \mathcal{G}^\xi$.

Next, for any $w \in \mathcal{G}^\xi \cap C^\infty(\overline{\Omega^\xi}; \mathbb{R}^3)$ that can be represented in the form $w = \nabla\psi\,:\psi\big|_\Gamma = 0$, $\psi\big|_{\Gamma^\xi} = 0$, we have

$$\begin{aligned}(h^\xi_\perp, w)_{\mathcal{H}} &= \int\limits_{\Omega^\xi} \nabla r^\xi \cdot w\, dx = \int\limits_{\Omega^\xi} \nabla r^\xi \cdot \nabla\psi\, dx \\ &= \int\limits_\Gamma (\nabla r^\xi)^\nu \psi\, d\Gamma + \int\limits_{\Gamma^\xi} (\nabla r^\xi)^\nu\psi\, d\Gamma^\xi - \int\limits_{\Omega^\xi} \Delta r^\xi \psi\, dx \overset{(4.18)-(4.19)}{=} 0.\end{aligned}$$

Thus, $(h^\xi_\perp, w)_{\mathcal{H}} = 0$ and, since the smooth w's in $\mathcal{G}^\xi$ are dense, we get $h^\xi_\perp \in \mathcal{G}^T \ominus \mathcal{G}^\xi$, i.e., in (4.23) all terms are orthogonal. The lemma is proved. □

We note an important fact: generally speaking, the field $Q^\xi h$ is discontinuous on Γ^ξ, and the discontinuity $Q^\xi h\big|_{\Gamma^\xi-0}$ is a field *normal* to Γ^ξ:

$$Q^\xi h\big|_{\Gamma^\xi-0} \overset{(4.22)}{=} (h_\theta + h_\nu - (\nabla r^\xi)_\theta - (\nabla r^\xi)_\nu)\big|_{\Gamma^\xi} \overset{(4.19)}{=} (h_\nu - (\nabla r^\xi)_\nu)\big|_{\Gamma^\xi}. \tag{4.24}$$

4.8. The Calderón operator. We fix $\xi : 0 < \xi \le T < T^{\mathrm{reg}}$ and introduce the operator $\Lambda^\xi\colon L_2(\Gamma^\xi) \to L_2(\Gamma^\xi)$, $\operatorname{Dom}\Lambda^\xi = C^\infty(\Gamma^\xi)$, acting by the rule

$$\Lambda^\xi g = \frac{\partial q}{\partial \nu}\bigg|_{\Gamma^\xi-0},$$

where q is a solution of the problem

$$\begin{aligned} \Delta q &= 0 \quad \text{in } \Omega^\xi, \\ q &= g \quad \text{on } \Gamma^\xi, \\ q &= 0 \quad \text{on } \Gamma. \end{aligned}$$

This is the known Calderón operator. We cite some of its properties (see [8]):

(1) there is an estimate for the norm: $\|(\Lambda^\xi)^{-1}\| \le C\xi$, $0 < \xi \le T$; moreover, we define
$$\Lambda^0 := 0. \tag{4.25}$$

(2) The operator Λ^ξ is self-adjoint:
$$\int_{\Gamma^\xi} \Lambda^\xi \varphi\, \psi\, d\Gamma^\xi = \int_{\Gamma^\xi} \varphi \Lambda^\xi \psi\, d\Gamma^\xi, \tag{4.26}$$
and for $\xi > 0$ it is positive,
$$(\Lambda^\xi g, g)_{L_2(\Gamma^\xi)} > 0, \quad g \ne 0,$$
and thus injective.

(3) The operator Λ^ξ retains smoothness: $\Lambda^\xi C^\infty(\Gamma^\xi) = C^\infty(\Gamma^\xi)$, $\xi > 0$.

(4) The following estimate is valid:
$$\|(\Lambda^\xi)^{-1} g\|_{H^1(\Gamma^\xi)} \le C\xi \|g\|_{H^1(\Gamma^\xi)}, \; 0 \le \xi \le T \tag{4.27}$$
($H^1(\dots) := W_2^1(\dots)$ is the Sobolev class).

(5) Λ^ξ is an elliptic pseudodifferential operator of the first order with the *principal symbol* $|k|_g$ (see [18]), where
$$|k|_g := (g^{\alpha\beta}(\gamma^1, \gamma^2, \xi) k_\alpha k_\beta)^{1/2}; \tag{4.28}$$
here k_1 and k_2 are variables dual to the variables γ^1 and γ^2; $\gamma = (\gamma^1, \gamma^2) \in \Gamma$.

4.9. The operator Λ. We fix $T < T^{\mathrm{reg}}$ and mention the representation

$$\overline{\Omega^T} = \bigcup_{0 \le \xi \le T} \Gamma^\xi.$$

In the space of scalar functions $L_2(\Omega^T)$ we define an operator Λ, $\operatorname{Dom}\Lambda = C^\infty(\overline{\Omega^T})$, acting *in layers* (in accordance with the representation) by the rule

$$(\Lambda\varphi)\big|_{\Gamma^\xi} := \Lambda^\xi[\varphi\big|_{\Gamma^\xi}], \quad 0 \le \xi \le T.$$

We cite some of its properties:

(1) the operator Λ is unbounded and injective; it is not local, i.e., it does not preserve the support of a function in Ω^T. At the same time, the inclusion $\operatorname{supp}\varphi \in \Omega^{\xi''}\backslash\Omega^{\xi'}$ implies the inclusion $\operatorname{supp}\Lambda\varphi \in \Omega^{\xi''}\backslash\Omega^{\xi'}$ $(0 \leq \xi' \leq \xi'' \leq T)$;

(2) using the smooth character of the dependence of Λ^ξ on ξ and the property (3) in Sec. 4.8, one can prove that

$$\Lambda C^\infty(\overline{\Omega^T}) \subset C^\infty(\overline{\Omega^T});$$

(3) by (4.25), for smooth φ we have

$$(\Lambda\varphi)\big|_\Gamma = 0\,.$$

Lemma 4.3.

$$\Lambda^* = c^{-1}\Lambda\, c\,. \tag{4.29}$$

Proof. For any smooth φ and ψ, we have

$$\begin{aligned}(\Lambda\varphi,\psi)_{L_2(\Omega^T)} &= \int\limits_{\Omega^T} \Lambda\varphi\,\psi\,dx = \int\limits_0^T d\tau \int\limits_{\Gamma^\tau} c\Lambda^\tau\varphi\,\psi\,d\Gamma^\tau \\ &\overset{(4.26)}{=} \int\limits_0^T d\tau \int\limits_{\Gamma^\tau} \varphi\,\Lambda^\tau(c\psi)\,d\Gamma^\tau = \int\limits_0^T d\tau \int\limits_{\Gamma^\tau} c\varphi\frac{1}{c}\,\Lambda^\tau(c\psi)\,d\Gamma^\tau \\ &= \int\limits_{\Omega^T} \varphi\,\frac{1}{c}\Lambda c\,\psi\,dx = (\varphi, \frac{1}{c}\Lambda c\,\psi)_{L_2(\Omega^T)} = (\varphi, \Lambda^*\psi)_{L_2(\Omega^T)}.\end{aligned}$$

The lemma is proved. □

4.10. $\mathcal{N}^T$-transformation. In the description of the operators introduced below, we use semigeodesic coordinates (assume that $T < T^{\mathrm{reg}}$). Recall that $\mathcal{L}_\theta^T$ is the space of transverse vector fields (4.13). We express the gradient and divergence in semigeodesic coordinates:

$$(\nabla\varphi)(x) = \left[\left(g^{\alpha\beta}\frac{\partial\varphi}{\partial\gamma^\beta}\right)r_\alpha + \left(g^{00}\frac{\partial\varphi}{\partial\tau}\right)r_0\right](\gamma,\tau); \tag{4.30}$$

$$(\operatorname{div} y)(x) = \left[\frac{1}{cJ}\frac{\partial}{\partial\gamma^\alpha}(cJy^\alpha) + \frac{1}{cJ}\frac{\partial}{\partial\tau}(cJy^0)\right](\gamma,\tau), \tag{4.31}$$

where $\{g^{\alpha\beta}\}$ is the matrix inverse to $\{g_{\alpha\beta}\}$, $g^{00} = \frac{1}{c^2}$; $y = y^\alpha r_\alpha + y^0 r_0$. We define

- the *transverse gradient* $\nabla_\theta\colon L_2(\Omega^T) \to \mathcal{L}_\theta^T$ acting on the functions smooth in $\overline{\Omega^T}$ by the rule

$$(\nabla_\theta\varphi)(x) = \left[\left(g^{\alpha\beta}\frac{\partial\varphi}{\partial\gamma^\beta}\right)r_\alpha\right](\gamma,\tau);$$

- the *transverse divergence* $\operatorname{div}_\theta\colon \mathcal{L}_\theta^T \to L_2(\Omega^T)$ that acts on the smooth transverse fields $v = v^\alpha r_\alpha$ by the rule

$$(\operatorname{div}_\theta v)(x) = \left[\frac{1}{cJ}\frac{\partial}{\partial\gamma^\alpha}(cJv^\alpha)\right](\gamma,\tau).$$

We emphasize their layerwise character: the relations $\varphi\big|_{\Gamma^\xi} = 0$, $v\big|_{\Gamma^\xi} = 0$ imply $(\nabla_\theta\varphi)\big|_{\Gamma^\xi} = 0$ and $(\operatorname{div}_\theta v)\big|_{\Gamma^\xi} = 0$. The following relation is valid:

$$(\nabla_\theta\varphi, v)_{\mathcal{L}_\theta^T} = -\,(\varphi, \operatorname{div}_\theta v)_{L_2(\Omega^T)}\,, \tag{4.32}$$

which is easily derived by integration by parts in layers.

We restrict the transverse divergence to the set of smooth potential fields projected onto the subspace $\mathscr{L}^T_\theta$; for this operator we keep the same notation $\operatorname{div}_\theta$.

Also we consider the operator $\operatorname{div}_\theta^{-1}\colon L_2(\Omega^T) \to \mathcal{L}^T_\theta$, which acts (in layers) on functions smooth in $\overline{\Omega^T}$ and is connected with the transverse divergence:

$$\operatorname{div}_\theta \circ \operatorname{div}_\theta^{-1} = \mathrm{Id},$$

where Id is the identity operator in $L_2(\Omega^T)$.

The family of projections $\{Q^\xi\}$ introduced in Sec. 4.7 defines an operator $\mathcal{N}^T\colon \mathcal{G}^T \to \mathscr{L}^T_\nu$; $\operatorname{Dom} \mathcal{N}^T = \mathcal{G}^T \cap C^\infty(\overline{\Omega^T}; \mathbb{R}^3)$ in accordance with the rule

$$\mathcal{N}^T h = Q^\xi h\big|_{\Gamma^\xi - 0} \quad \text{on } \Gamma^\xi,\ 0 < \xi \le T < T^{\mathrm{reg}}.$$

Thus, the image $\mathcal{N}^T h$ is composed of the discontinuities that arise on the surfaces Γ^ξ when projecting h to $\mathcal{G}^\xi$.

By (4.24), for $h = h_\theta + h_\nu$ we have a layerwise representation

$$\mathcal{N}^T h = h_\nu - (\nabla r^\xi)_\nu = h_\nu - \left(\frac{\partial r^\xi}{\partial \nu}\bigg|_{\Gamma^\xi}\right)\nu \quad \text{on } \Gamma^\xi, \qquad 0 < \xi \le T < T^{\mathrm{reg}}. \tag{4.33}$$

With the family of problems (4.18)–(4.20) we associate the operator $\nabla_\theta^{-1} : \mathscr{L}^T_\theta \to L_2(\Omega^T)$ acting on the transverse components h_θ of smooth field $h \in \mathcal{G}^T$ by the rule

$$\nabla_\theta^{-1} h_\theta := r^\xi \quad \text{on } \Gamma^\xi,\ 0 < \xi \le T;$$

as is easily seen, $\nabla_\theta \nabla_\theta^{-1} = \mathrm{id}_\theta$ (id_θ is the identity operator in $\mathscr{L}^T_\theta$). Using the operator Λ, we can write the representation (4.33) in the form

$$\mathcal{N}^T h = h_\nu - \left(\Lambda \nabla_\theta^{-1} h_\theta\right)\nu\,. \tag{4.34}$$

Proposition 4.2. *The transformation $\mathcal{N}^T$ is an isometry of $\mathcal{G}^T$ onto $\mathscr{L}^T_\nu$.*

This fact was established in [2].

Recall the Weyl decomposition

$$\mathcal{H}^T = \mathcal{J}^T \oplus \mathcal{G}^T; \tag{4.35}$$

let $\mathcal{P}^T_\mathcal{G}$ is the projection in $\mathcal{H}^T = \mathcal{H}[\Omega^T]$ onto $\mathcal{G}^T$.

Lemma 4.4. *The adjoint operator $(\mathcal{N}^T)^*$ is well defined on smooth longitudinal fields $v \in \mathcal{L}^T_\nu$ and admits the representation*

$$(\mathcal{N}^T)^* v = \mathcal{P}^T_\mathcal{G}\left(v + \operatorname{div}_\theta^{-1}\left[c^{-1}\Lambda c\, v^\nu\right]\right), \tag{4.36}$$

where $v^\nu = v \cdot \nu$.

Proof. For smooth $h = h_\theta + h_\nu \in \mathcal{G}^T$ and $v \in \mathscr{L}^T_\nu$, we have

$$\begin{aligned}
(\mathcal{N}^T h, v)_{\mathscr{L}^T_\nu} &\overset{(4.34)}{=} (h_\nu - (\Lambda\nabla_\theta^{-1} h_\theta)\nu, v)_{\mathscr{L}^T_\nu} = (h_\nu, v)_{\mathscr{L}^T_\nu} - (h_\theta, (\nabla^*_\theta)^{-1}\Lambda^*\, v^\nu)_{\mathscr{L}^T_\theta} \\
&\overset{(4.32),(4.29)}{=} (h_\nu, v)_{\mathscr{L}^T_\nu} + (h_\theta, \operatorname{div}_\theta{}^{-1}[c^{-1}\Lambda c\, v^\nu])_{\mathscr{L}^T_\theta} \\
&= (h_\theta + h_\nu, v + \operatorname{div}_\theta{}^{-1}[c^{-1}\Lambda c\, v^\nu])_{\mathcal{H}^T} = \left(h, \mathcal{P}^T_\mathcal{G}\left(v + \operatorname{div}_\theta{}^{-1}[c^{-1}\Lambda c\, v^\nu]\right)\right)_{\mathcal{G}^T}.
\end{aligned}$$

The lemma is proved. $\square$

4.11. The operator $\nabla\varkappa\,\mathrm{div}$. We return to the acoustic subsystem α_p^T and apply the operator ∇ to both sides of relations (3.11)–(3.13). Denoting $h := \nabla\varphi$, we obtain the system

$$h_{tt} = \nabla c^{\,2}\,\mathrm{div}\,h \qquad \text{in } \ Q^T, \tag{4.37}$$

$$h|_{t=0} = h_t|_{t=0} = 0 \qquad \text{in } \ \overline{\Omega}\,, \tag{4.38}$$

$$h = f \qquad \text{on } \ \Sigma^T, \tag{4.39}$$

where $f\big|_{\Sigma^T} = \nabla\varphi^a\big|_{\Sigma^T} = \begin{pmatrix}\nabla\varphi^a\cdot\nu\\ (\nabla\varphi^a)_\theta\end{pmatrix} = \begin{pmatrix}\frac{\partial\varphi^a}{\partial\nu}\\ \nabla_\theta a\end{pmatrix}$.

On the fields of the class $\mathbf{H}^2(\Omega)$ we introduce the operator

$$\mathcal{L} := \nabla c^{\,2}\ \mathrm{div}, \tag{4.40}$$

determining the evolution of system (4.37)–(4.39).

Lemma 4.5. *In the subdomain Ω^T covered by the system of pseudogeodesic coordinates, for a smooth field $y = y^0 r_0 + y_\theta$ we have the representation*[5]

$$\mathcal{L}y = \begin{pmatrix}(\mathcal{L}y)^0\\ (\mathcal{L}y)_\theta\end{pmatrix} = \begin{pmatrix}\frac{1}{c^2}\frac{\partial}{\partial\tau}\frac{c}{J}\left[\frac{\partial}{\partial\tau}cJy^0 + cJ\,\mathrm{div}_\theta\,y_\theta\right]\\[2ex] \nabla_\theta\frac{c}{J}\left[\frac{\partial}{\partial\tau}cJy^0 + cJ\,\mathrm{div}_\theta\,y_\theta\right]\end{pmatrix}. \tag{4.41}$$

Proof. Using the expressions (4.30) and (4.31) of the gradient and the divergence in the semi-geodesic coordinates and taking into account the fact that $g^{00} = \frac{1}{c^2}$, we obtain

$$\nabla\varphi = g^{00}\frac{\partial\varphi}{\partial\tau}r_0 + \nabla_\theta\varphi; \qquad c^2\,\mathrm{div}\,y = c^2\left(\frac{1}{cJ}\frac{\partial}{\partial\tau}cJy^0 + \mathrm{div}_\theta\,y_\theta\right);$$

$$\mathcal{L}y \overset{(4.40)}{=} \nabla c^2\,\mathrm{div}\,y = \frac{1}{c^2}\frac{\partial(\frac{c}{J}\frac{\partial}{\partial\tau}cJy^0 + c^2\,\mathrm{div}_\theta\,y_\theta)}{\partial\tau}r_0 + \nabla_\theta\left(\frac{c}{J}\frac{\partial}{\partial\tau}cJy^0 + c^2\,\mathrm{div}_\theta\,y_\theta\right).$$

The lemma is proved. □

4.12. The operator $\mathcal{N}^T(\nabla\varkappa\,\mathrm{div})(\mathcal{N}^T)^*$. By (4.34) and (4.36), for any smooth $h \in \mathcal{G}^T$ and $v \in \mathscr{L}_\nu^T$ we get

$$\mathcal{N}^T h = \mathcal{N}^T\begin{pmatrix}h^0\\ h_\theta\end{pmatrix} = \begin{pmatrix}1 & -\frac{1}{c}\,\Lambda\,\nabla_\theta^{-1}\\ 0 & 0\end{pmatrix}\begin{pmatrix}h^0\\ h_\theta\end{pmatrix}, \tag{4.42}$$

$$(\mathcal{N}^T)^* v = (\mathcal{N}^T)^*\begin{pmatrix}v^0\\ 0\end{pmatrix} = \mathcal{P}_{\mathcal{G}}^T\begin{pmatrix}v^0\\ {\mathrm{div}_\theta}^{-1}[c^{-1}\Lambda c^2\,v^0]\end{pmatrix} \tag{4.43}$$

(we used the fact that $\nu = \frac{r_0}{|r_0|} = \frac{1}{c}r_0$ and $v^\nu = v^0|r_0| = cv^0$).

Let w be an arbitrary smooth field in $\overline{\Omega^T}$. By the Weyl decomposition (4.35), we get

$$w = \mathcal{P}_{\mathcal{G}}^T w + \mathcal{P}_{\mathcal{J}}^T w, \tag{4.44}$$

where $\mathcal{P}_{\mathcal{G}}^T$ is the projection in $\mathcal{H}^T$ onto $\mathcal{G}^T$ and $\mathcal{P}_{\mathcal{J}}^T$ is the projection in $\mathcal{H}^T$ onto $\mathcal{J}^T$; we note that the projection $\mathcal{P}_{\mathcal{G}}^T$ maintains the smoothness:

$$\mathcal{P}_{\mathcal{G}}^T C^\infty(\overline{\Omega^T};\mathbb{R}^3) \subset \mathcal{G}^T \cap C^\infty(\overline{\Omega^T};\mathbb{R}^3)$$

(see [11]). On smooth fields, we have

$$\mathcal{L}\mathcal{P}_{\mathcal{G}}^T = \mathcal{L}. \tag{4.45}$$

Indeed,

$$\mathcal{L}\mathcal{P}_{\mathcal{G}}^T w \overset{(4.40)}{=} \nabla c^2\,\mathrm{div}\,\mathcal{P}_{\mathcal{G}}^T w \overset{(4.44)}{=} \nabla c^2\,\mathrm{div}(w - \mathcal{P}_{\mathcal{J}}^T w) \overset{(2.3)}{=} \mathcal{L}w.$$

[5] Henceforth, we use Agreement 4.2 about the matrix notation.

Lemma 4.6. *On smooth fields* $v=\begin{pmatrix} v^0\\ 0\end{pmatrix}\in\mathscr{L}_\nu^T$ *the following representation is valid*:

$$\mathcal{N}^T\mathcal{L}(\mathcal{N}^T)^*v^0r_0=\left(\frac{1}{c^2}\frac{\partial}{\partial\tau}-\frac{1}{c}\Lambda\right)\frac{c}{J}\left(\frac{\partial}{\partial\tau}cJ+J\Lambda c^2\right)v^0r_0. \tag{4.46}$$

Proof.

$$\begin{aligned}\mathcal{L}(\mathcal{N}^T)^*\begin{pmatrix} v^0\\ 0\end{pmatrix} &\overset{(4.43)}{=}\mathcal{L}\mathcal{P}_{\mathcal{G}}^T\begin{pmatrix} v^0\\ \operatorname{div}_\theta{}^{-1}[c^{-1}\Lambda c^2\,v^0]\end{pmatrix}\overset{(4.45)}{=}\mathcal{L}\begin{pmatrix} v^0\\ \operatorname{div}_\theta{}^{-1}[c^{-1}\Lambda c^2\,v^0]\end{pmatrix}\\
&\overset{(4.41)}{=}\begin{pmatrix}\frac{1}{c^2}\frac{\partial}{\partial\tau}\frac{c}{J}\left[\frac{\partial}{\partial\tau}cJv^0+cJ\operatorname{div}_\theta\operatorname{div}_\theta{}^{-1}[c^{-1}\Lambda c^2\,v^0]\right]\\ \\ \nabla_\theta\frac{c}{J}\left[\frac{\partial}{\partial\tau}cJv^0+cJ\operatorname{div}_\theta\operatorname{div}_\theta{}^{-1}[c^{-1}\Lambda c^2\,v^0]\right]\end{pmatrix}\\
&=\begin{pmatrix}\frac{1}{c^2}\frac{\partial}{\partial\tau}\frac{c}{J}\left[\frac{\partial}{\partial\tau}cJ+J\Lambda c^2\right]v^0\\ \\ \nabla_\theta\frac{c}{J}\left[\frac{\partial}{\partial\tau}cJ+J\Lambda c^2\right]v^0\end{pmatrix};\end{aligned}$$

$$\begin{aligned}\mathcal{N}^T\mathcal{L}(\mathcal{N}^T)^*\begin{pmatrix} v^0\\ 0\end{pmatrix}&\overset{(4.42)}{=}\begin{pmatrix}1 & -\frac{1}{c}\Lambda\nabla_\theta^{-1}\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{c^2}\frac{\partial}{\partial\tau}\frac{c}{J}\left[\frac{\partial}{\partial\tau}cJ+J\Lambda c^2\right]v^0\\ \\ \nabla_\theta\frac{c}{J}\left[\frac{\partial}{\partial\tau}cJ+J\Lambda c^2\right]v^0\end{pmatrix}\\
&=\begin{pmatrix}\left(\frac{1}{c^2}\frac{\partial}{\partial\tau}-\frac{1}{c}\Lambda\right)\frac{c}{J}\left(\frac{\partial}{\partial\tau}cJ+J\Lambda c^2\right)v^0\\ 0\end{pmatrix}.\end{aligned}$$

The lemma is proved. □

4.13. Images. The operators Π^T and $\mathcal{N}^T$ are unitary; their composition

$$\mathcal{I}^T=\Pi^T\mathcal{N}^T \tag{4.47}$$

is a unitary operator from $\mathcal{G}^T$ onto $\mathcal{F}_\nu^T$ ($0<T<T^{\rm reg}$). We call $\mathcal{I}^T$ the *image operator*; $\widetilde{h}=\mathcal{I}^Th$ is called the *image* of the field h; the image is the field on the pattern Θ^T normal to Γ. The operator $\mathcal{I}^T$ will play an important role in the inverse problem.

Let

$$\mathcal{T}:=\{g\in L_2(\Gamma,\mathbb{R}^3)\,|\,g\times\nu=0\}$$

be the space of normal fields on Γ (ν is an external normal to Γ). The space $\mathcal{F}_\nu^T$ (4.15) can be regarded as a space of $\mathcal{T}$-valued functions of the variable $\tau\in[0,T]$:

$$\mathcal{F}_\nu^T=L_2([0,T];\mathcal{T}); \tag{4.48}$$

A family of projections that are cut-off functions acts in it:

$$(X^\xi f)(\tau):=\begin{cases} f(\tau), & 0\le\tau\le\xi;\\ 0, & \xi<\tau\le T\end{cases}$$

($0\le\xi\le T$).

Lemma 4.7. *The following relation is valid*:

$$\mathcal{I}^TQ^\xi\;=\;X^\xi\mathcal{I}^T. \tag{4.49}$$

Proof. In the space of longitudinal fields $\mathscr{L}_\nu^T$, we separate out an expanding family of subspaces

$$\mathscr{L}_\nu^\xi:=\{v\in\mathscr{L}_\nu^T\mid\operatorname{supp}v\subset\overline{\Omega^\xi}\},\quad 0\le\xi\le T<T^{\rm reg};$$

by Y^ξ we denote the projection in $\mathscr{L}_\nu^T$ onto $\mathscr{L}_\nu^\xi$; its action is reduced to cutting off a field to the subdomain Ω^ξ. One can show [2] that

$$\mathcal{N}^T Q^\xi = Y^\xi \mathcal{N}^T, \quad 0 \le \xi \le T < T^{\mathrm{reg}}.$$

Now, relation (4.49) is a consequence of this relation and the definitions of the operators $\mathcal{I}^T = \Pi^T \mathcal{N}^T$ and Π^T (see (4.17)). The lemma is proved. □

In addition, we note that by property (3) of Lemma 4.1, the correspondence "field–image" preserves smoothness:

$$\mathcal{I}^T[\mathcal{G}^T \cap C^\infty(\overline{\Omega^T})] = \mathcal{F}_\nu^T \cap C^\infty(\overline{\Theta^T});$$

moreover, the relation

$$(\mathcal{I}^T h)|_{\tau=0} = \kappa_0 h_\nu|_\Gamma \tag{4.50}$$

with $\kappa_0 := \kappa|_\Gamma = \sqrt{c_0}$ holds; it is a consequence of the relation[6]

$$(\mathcal{N}^T h)|_{\Gamma^\xi} = \{h_\nu - \left(\Lambda \nabla_\theta^{-1} h_\theta\right) \nu\}|_{\Gamma^\xi} \xrightarrow{\xi \to 0} h_\nu|_\Gamma$$

and the definition of the operator Π^T.

We define an operator $\mathcal{L}^T \colon \mathcal{G}^T \to \mathcal{G}^T$, $\operatorname{Dom} \mathcal{L}^T = \mathcal{G}^T \cap C^\infty(\overline{\Omega^T})$, which acts on smooth potential fields h by the rule

$$\mathcal{L}^T h \;:=\; \mathcal{L} h \;=\; \nabla c^2 \operatorname{div} h.$$

The transformation $\mathcal{I}^T$ induces in $\mathcal{F}_\nu^T$ the operator

$$\widetilde{\mathcal{L}}^T := (\mathcal{I}^T)\mathcal{L}^T(\mathcal{I}^T)^* \tag{4.51}$$

with the domain of definition $\operatorname{Dom} \widetilde{\mathcal{L}}^T = \mathcal{F}_\nu^T \cap C^\infty(\overline{\Theta^T})$. For the inverse problem, of importance is the representation of $\widetilde{\mathcal{L}}^T$ that is described below.

We say that an operator $S \colon \mathcal{F}_\nu^T \to \mathcal{F}_\nu^T$ is a layer operator if it is determined by a family of operators $S(\tau) \colon \mathcal{T} \to \mathcal{T} (0 \le \tau \le T)$ and acts in accordance with the rule[7]

$$(Sf)(\tau) = S(\tau) f(\tau), \;\; \tau \in [0, T].$$

Next, let $\sigma \subset \Gamma$ be a neighborhood covered by local coordinates γ^1, γ^2; let $\widetilde{r}_0$ be a base field in $\sigma \times [0, T] \subset \overline{\Theta^T}$ that does not depend on τ and is defined by the relation

$$\widetilde{r}_0(\gamma, \tau) = r_0(\gamma, 0);$$

a field $f \in \mathcal{F}_\nu^T$ is representable on $\sigma \times [0, T]$ in the form $f = f^0 \widetilde{r}_0$.

Theorem 4.2. *For $0 < T < T^{\mathrm{reg}}$, for a smooth normal field $f = f^0 \widetilde{r}_0$ on $\sigma \times [0, T]$ the following representation is valid*:

$$\widetilde{\mathcal{L}}^T f \;=\; \begin{pmatrix} \left(\frac{\partial^2}{\partial \tau^2} - \widetilde{\Lambda}^2\right) f^0 \\ 0 \end{pmatrix} + \widetilde{S} f, \tag{4.52}$$

where $\widetilde{\Lambda} := \pi \sqrt{\dfrac{J}{c}} \Lambda c \sqrt{\dfrac{c}{J}} \pi^{-1}$ and $\widetilde{S}$ is a layer pseudodifferential operator on the pattern Θ^T of order not exceeding 1.

Before proving the theorem, we state several lemmas. We recall that an operator K in $L_2(\Omega^T)$ is said to be a layer operator if it acts by the rule

$$(K\varphi)\big|_{\Gamma^\xi} \;=\; K(\xi)[\varphi\big|_{\Gamma^\xi}], \;\; 0 < \xi \le T,$$

where the $K(\xi)$ are operators in $L_2(\Gamma^\xi)$.

---

[6] It follows from property (3): see Sec. 4.9.

[7] Here, in view of the representation (4.48), f is meant as a $\mathcal{T}$-valued function of τ.

Lemma 4.8. *For a function χ smooth in $\overline{\Omega^T}$, the following representation is valid:*

$$\frac{\partial}{\partial\tau}\Lambda\chi - \Lambda\chi\frac{\partial}{\partial\tau} \;=\; K, \tag{4.53}$$

where K is a layer operator such that all of the $K(\xi)$ are pseudodifferential operators of order 1.

Omitting the proof, we note that the representation (4.53) is justified with the help of standard results of elliptic theory [17]. We also clarify that the operator K proves to be pseudodifferential, because the Calderón operators determining Λ are the same: each Λ^ξ is a pseudodifferential operator of order 1 (for example, see [18]).

Lemma 4.9. *For any smooth field $v = \begin{pmatrix} v^0 \\ 0 \end{pmatrix} \in \mathscr{L}_\nu^T$,*

$$\mathcal{N}^T\mathcal{L}^T(\mathcal{N}^T)^*v = \begin{pmatrix} \left(\dfrac{\partial^2}{\partial\tau^2} - \bar\Lambda^2\right)v^0 \\ 0 \end{pmatrix} + Sv, \tag{4.54}$$

where $\bar\Lambda := \Lambda^* c = \frac{1}{c}\Lambda c^2$ and S is a layer pseudodifferential operator of order not exceeding 1.

Proof. In the calculations given below, by $\sim$ we denote the passages with removing the operators of lesser order.

$$\begin{aligned}
\mathcal{N}^T\mathcal{L}^T(\mathcal{N}^T)^*v^0 r_0 &\overset{(4.46)}{=} \left(\frac{1}{c^2}\frac{\partial}{\partial\tau} - \frac{1}{c}\Lambda\right)\frac{c}{J}\left(\frac{\partial}{\partial\tau}cJ + J\Lambda c^2\right)v^0 r_0 \\
&= \left(\frac{1}{c^2}\frac{\partial}{\partial\tau}\frac{c}{J}\frac{\partial}{\partial\tau}cJ + \frac{1}{c^2}\frac{\partial}{\partial\tau}c\Lambda c^2 - \frac{1}{c}\Lambda\frac{c}{J}\frac{\partial}{\partial\tau}cJ - \frac{1}{c}\Lambda c\Lambda c^2\right)v^0 r_0 \\
&\sim \left(\frac{1}{c^2}\frac{\partial}{\partial\tau}c^2\frac{\partial}{\partial\tau} + \frac{1}{c}\frac{\partial}{\partial\tau}\Lambda c^2 - \frac{1}{c}\Lambda c^2\frac{\partial}{\partial\tau} - \frac{1}{c}\Lambda c\Lambda c^2\right)v^0 r_0 \\
&\sim \left(\frac{\partial^2}{\partial\tau^2} + \frac{1}{c}\left[\frac{\partial}{\partial\tau}\Lambda c^2 - \Lambda c^2\frac{\partial}{\partial\tau}\right] - \frac{1}{c}\Lambda c\Lambda c^2\right)v^0 r_0 \\
&\overset{(4.53)}{\sim} \left(\frac{\partial^2}{\partial\tau^2} - \frac{1}{c}\Lambda c\Lambda c^2\right)v^0 r_0.
\end{aligned}$$

Denoting $\bar\Lambda := \frac{1}{c}\Lambda c^2$ and recalling that $\Lambda^* = \frac{1}{c}\Lambda c$, we obtain (4.54). The lemma is proved. $\square$

Now we are ready to complete the proof of Theorem 4.2. By definition (4.17), the operator $\Pi^T : \mathscr{L}_\nu^T \to \mathcal{F}_\nu^T$ acts as follows:

$$\Pi^T v = \kappa\pi v, \tag{4.55}$$

where, by (4.16),

$$\kappa = \frac{c}{c_0}\sqrt{c\frac{J}{J_0}}\,. \tag{4.56}$$

Using the fact that Π^T is unitary, we get

$$\widetilde{\mathcal{L}}^T \overset{(4.51)}{=} (\mathcal{I}^T)\mathcal{L}^T(\mathcal{I}^T)^* \overset{(4.47)}{=} \Pi^T\mathcal{N}^T\mathcal{L}^T(\mathcal{N}^T)^*(\Pi^T)^* \overset{(4.55)}{=} \kappa\pi\mathcal{N}^T\mathcal{L}^T(\mathcal{N}^T)^*\pi^{-1}\kappa^{-1}.$$

It remains to use Lemma 4.9. We have

$$\widetilde{\mathcal{L}}^T = \kappa\pi\left(\frac{\partial^2}{\partial\tau^2} - \bar\Lambda^2\right)\pi^{-1}\kappa^{-1} + (\mathcal{I}^T)S(\mathcal{I}^T)^*. \tag{4.57}$$

Since $\frac{D}{\partial\tau} = \frac{\partial}{\partial\tau}$ on the longitudinal fields, relation (4.12) takes the form $\frac{\partial}{\partial\tau}\pi = \pi\frac{\partial}{\partial\tau}$. Therefore

$$\kappa\pi\frac{\partial^2}{\partial\tau^2}\pi^{-1}\kappa^{-1} = \kappa\frac{\partial^2}{\partial\tau^2}\kappa^{-1} \sim \frac{\partial^2}{\partial\tau^2}. \tag{4.58}$$

Next we denote

$$\widetilde{\Lambda}^2 := \kappa\pi\bar{\Lambda}^2\pi^{-1}\kappa^{-1} \overset{(4.10)}{=} \pi\kappa\bar{\Lambda}^2\kappa^{-1}\pi^{-1}, \tag{4.59}$$

where

$$\widetilde{\Lambda} := \pi\kappa\bar{\Lambda}\kappa^{-1}\pi^{-1} \overset{(4.56)}{=} \pi\sqrt{cJ}\,\bar{\Lambda}\,\frac{1}{\sqrt{cJ}}\,\pi^{-1} \overset{\bar{\Lambda}=\frac{1}{c}\Lambda c^2}{=} \pi\sqrt{\frac{J}{c}}\,\Lambda c\sqrt{\frac{c}{J}}\,\pi^{-1}.$$

Considering (4.58) and (4.59) and denoting $\widetilde{S} := (\mathcal{I}^T)S(\mathcal{I}^T)^*$ in (4.57), we obtain (4.52) ($f = f^0\widetilde{r}_0$):

$$\widetilde{L}^T f = \left(\frac{\partial^2}{\partial\tau^2} - \widetilde{\Lambda}^2\right) f^0\,\widetilde{r}_0 + \widetilde{S}f. \tag{4.60}$$

It is easy to see that $\widetilde{S}$ is a layer pseudodifferential operator on the pattern of order not exceeding 1. Theorem 4.2 is proved.

Note that Λ is a layer operator in which each Λ^ξ, by (4.28), is a pseudodifferential operator of the first order with principal symbol $|k|_g$. Using this fact, as well as properties of principal symbols under composition of operators and multiplication of them by functions, we conclude that the result of Theorem 4.2 admits an invariant statement in terms of pseudodifferential operators.

Theorem 4.3. *The following representation is valid*:

$$\widetilde{L}^T = \frac{\partial^2}{\partial\tau^2} + H, \tag{4.61}$$

where H is a layer operator such that each

$$H(\tau)\colon \mathcal{T} \to \mathcal{T}, \quad 0 < \tau \le T,$$

is a pseudodifferential operator of the second order with principal symbol

$$\operatorname{Symb}_{H(\tau)}(\gamma, k_1, k_2) = -c^2(\gamma,\tau)|k|_g^2\,\mathrm{Id}_\gamma = -|k|_h^2\,\mathrm{Id}_\gamma; \tag{4.62}$$

here k_1 and k_2 are variables dual to γ^1 and γ^2; Id *is the identity operator on the cotangent space $T^*_\gamma\Gamma$; $|k|_h := (h^{\alpha\beta}(\gamma^1,\gamma^2,\tau)k_\alpha k_\beta)^{1/2}$.*

5. Dynamics

5.1. Forward problem. The control operator. We fix an arbitrary $T > 0$ and consider the problem (4.37)–(4.38) with control h_ν on Σ^T (as shown in [3], h_θ is uniquely determined by h_ν and the response operator R^T on Σ^T):

$$h_{tt} - \mathcal{L}h = 0 \qquad \text{in } Q^T, \tag{5.1}$$

$$h|_{t=0} = h_t|_{t=0} = 0 \qquad \text{in } \overline{\Omega}, \tag{5.2}$$

$$h_\nu = f \qquad \text{on } \Sigma^T; \tag{5.3}$$

here, $\mathcal{L} := \nabla c^2\operatorname{div}$, ν is an external normal, $h_\nu = (h\cdot\nu)\nu$, $f \in \mathcal{F}^T_\nu \subset \mathcal{F}^T$ is a control. Its solution $h = h^f(x,t)$ can be regarded as a $\mathcal{G}^T$-valued function dependent on time.

In the dynamical system described by problem (5.1)–(5.3), the correspondence "input-state" is realized by the control operator

$$\mathcal{W}^T\colon \mathcal{F}^T_\nu \to \mathcal{G}^T, \quad \operatorname{Dom}\mathcal{W}^T = \mathcal{F}^T_\nu \cap \mathcal{M}^T,$$
$$\mathcal{W}^T f = h^f(\cdot\,, T). \tag{5.4}$$

It admits closure, and for $T < T^{*}$[8] it is injective: $\operatorname{Ker}\mathcal{W}^T = \{0\}$ (see [9]).

[8] We recall that T^* is the time of filling Ω with waves traveling from the boundary: see Sec. 2.1.

5.2. Controllability. The set

$$G^T := \operatorname{Ran} \mathcal{W}^T \subset \mathcal{G}^T$$

is said to be *reachable* to the instant of time T.

Proposition 5.1. *For $T < T^{\mathrm{reg}}$, the relation*

$$\overline{G^T} = \mathcal{G}^T \tag{5.5}$$

is valid (closure in the metric of $\mathcal{H}$).

It is derived similarly to relation (3.15).

From (5.5) it follows that any potential field in the subdomain Ω^T can be approximated by waves $h^f(\cdot, T)$ in L_2-norm. In control theory, this property is referred to as an *approximate controllability* of system (5.1)–(5.3).

In the external space $\mathcal{F}_\nu^T$, we consider a family of subspaces

$$\mathcal{F}_\nu^{T,\xi} := \{f \in \mathcal{F}_\nu^T \mid f(\cdot, t) = 0,\, 0 \le t < T - \xi\}, \quad 0 \le \xi \le T, \tag{5.6}$$

formed by *delayed controls* ($\mathcal{F}_\nu^{T,0} = \{0\}, \mathcal{F}_\nu^{T,T} = \mathcal{F}_\nu^T$). The delay of a control leads to the delay of the wave: since $\operatorname{supp} h^f(\cdot, \xi) \subset \overline{\Omega^\xi}$ $(0 < \xi \le T)$ and system (5.1)–(5.3) is stationary (the independence of $\mathcal{L}$ from time), for $f \in \mathcal{F}_\nu^{T,\xi}$ we have the inclusion $\operatorname{supp} h^f(\cdot, T) \subset \overline{\Omega^\xi}$, i.e., $h^f(\cdot, T) \subset \mathcal{G}^\xi$.

Consider an extending family of reachable sets

$$G^\xi := \mathcal{W}^T \mathcal{F}_\nu^{T,\xi} \subset \mathcal{G}^\xi.$$

The projections P^ξ in $\overline{G^T}$ on $\overline{G^\xi}$ are called wave projections; the complementary projections are

$$P_\perp^\xi := \mathbb{I}_{G^T} - P^\xi. \tag{5.7}$$

The fact that the system is stationary and relation (5.5) imply the relation

$$\overline{G^\xi} = \mathcal{G}^\xi, \tag{5.8}$$

which, in turn, yields

$$P^\xi = Q^\xi, \quad P_\perp^\xi = Q_\perp^\xi \qquad (0 \le \xi \le T < T^{\mathrm{reg}}) \tag{5.9}$$

(Q^ξ is the projection in $\mathcal{G}^T$ onto $\mathcal{G}^\xi$: see Sec. 4.7).

Of course, both Q^ξ and P^ξ are determined by the behavior of the velocity c, but their coincidence is a consequence of the controllability of the system.

5.3. Discontinuities in the forward problem. Considerations in this and the next sections concern a known property of hyperbolic systems: discontinuous controls generate discontinuous waves. The description of discontinuities of waves is the subject matter of geometric optics; relevant formulas play a key role in the BC method.

We fix $T : 0 < T < T^{\mathrm{reg}}$ and $\xi \in (0, T)$; denote

$$\theta^j(t) := \begin{cases} 0, & t < 0; \\ \frac{t^j}{j!} & t \ge 0 \end{cases}$$

($j = 0, 1, \ldots; \theta^0(t)$ is the Heaviside function). We set

$$\theta_s^j(t) := \theta^j(t - s), \quad s, t \in \mathbb{R}.$$

We take a normal field $a \in \mathcal{T} \cap C^\infty(\Gamma, \mathbb{R}^3)$ and consider system (5.1)–(5.3):

$$h_{tt} - \mathcal{L}h = 0 \qquad \text{in } Q^T, \tag{5.10}$$

$$h|_{t=0} = h_t|_{t=0} = 0 \qquad \text{in } \overline{\Omega}, \tag{5.11}$$

$$h_\nu = \theta^0_{T-\xi} a \qquad \text{on } \Sigma^T, \tag{5.12}$$

with a control of a special form:

$$f = f(\gamma, t) = \theta^0(t - (T - \xi))a(\gamma).$$

It is a delayed control: $\theta^0_{T-\xi} a \in \mathcal{F}^{T,\xi}_\nu$, and it is discontinuous for $t = T - \xi$. Since the velocity of wave propagation is finite, we have

$$\operatorname{supp} h^{\theta^0_{T-\xi}a} \subset \{(x,t) \in \overline{Q^T} \,|\, t \geq \tau(x) + (T - \xi)\};$$

the characteristic surface that confines the support

$$\chi^{T,\xi} := \{(x,t) \in \overline{Q^T} \,|\, t = \tau(x) + (T - \xi)\}$$

proves to be the discontinuity surface of the solution

$$h^{\theta^0_{T-\xi}a}(x, \tau(x) + T - \xi + 0) = A(x)[a(\gamma(x))]^\vee, \tag{5.13}$$

where $A := \left(\frac{c_0 J_0}{cJ}\right)^{1/2}$ is the amplitude factor, $[a(\gamma(x))]^\vee$ is the result of parallel translation (in the fast metric) of the vector a from the point $\gamma(x) \in \Gamma$ to the point $x \in \Omega^T$ along the geodesic $l_{\gamma(x)}$ (see [1]).

The description of discontinuities and the derivation of formulas of the type (5.13) are substantially simplified in passing to images [8]. It can be shown that the operator $\mathcal{I}^T$ can be extended by g^T to the set of solutions of problem (5.1)–(5.3); for this operator we keep the same notation $\mathcal{I}^T$. In accordance with the representation (4.48), we regard the image of the wave $\widetilde{h} = \mathcal{I}^T h$ as a $\mathcal{T}$-valued function of the variable $\tau \in [0,T]$ dependent on time as a parameter; by the representation $\mathcal{F}^T_\nu = L_2((0,T);\mathcal{T})$, the controls are $\mathcal{T}$-valued functions of time $t \in [0,T]$. Applying the operator $\mathcal{I}^T$ in problem (5.10)–(5.12), taking into account relation (4.50) and the representation (4.61), we obtain the system

$$\widetilde{h}_{tt} - \widetilde{h}_{\tau\tau} - H(\tau)\widetilde{h} = 0 \qquad (\tau, t) \in (0,T) \times (0,T), \tag{5.14}$$

$$\widetilde{h}|_{t=0} = \widetilde{h}_t|_{t=0} = 0 \qquad \tau \in (0,T), \tag{5.15}$$

$$\widetilde{h}_\nu|_{\tau=0} = \theta^0_{T-\xi}\kappa_0 a. \tag{5.16}$$

Acting in accordance with the ray method [1, 19], we seek a solution of the system in the form “anzatz + residual”:

$$\widetilde{h}(\tau,t) = \sum_{j=0}^{N} \theta^j_{T-\xi}(t-\tau)A_j(\tau) + d_{N+1}(\tau,t). \tag{5.17}$$

The substitution of (5.17) in (5.14) leads to known transport equations for $\mathcal{T}$-valued “amplitudes”:

$$2\frac{\partial A_j}{\partial \tau} - \left[\frac{\partial^2}{\partial \tau^2} + H(\tau)\right] A_{j-1} = 0, \qquad j = 0, 1, \dots;$$

$A_{-1} := 0$. Successively solving them with regard to the conditions $A_0(0) = \kappa_0 a$ (see (5.16)), $A_j(0) = 0$, $j = 1, 2, \dots$, we find

$$A_0(\tau) = \kappa_0 a, \quad A_1(\tau) = \frac{1}{2}\int_0^\tau [H(s)\kappa_0 a]\, ds \dots.$$

Restricting ourselves to the case $N = 1$, we obtain the representation

$$\widetilde{h}^{\theta^0_{T-\xi}a}(\tau,t) = \theta^0_{T-\xi}(t-\tau)\kappa_0 a + \theta^1_{T-\xi}(t-\tau)\frac{1}{2}\int\limits_0^\tau [H(s)\kappa_0 a]\,ds + d_2(\tau,t), \tag{5.18}$$

and for the residual we have the estimate

$$|d_2(\tau,t)| \le C\theta^2_{T-\xi}(t-\tau), \quad (\tau,t) \in [0,T]\times[0,T], \tag{5.19}$$

which can be derived in the same way as in the case of the wave equation [19,20]. From (5.18), we can get the formula of geometric optics

$$\widetilde{h}^{\theta^0_{T-\xi}a}(\xi-0,T) \;=\; \kappa_0 a; \tag{5.20}$$

$$2\,\frac{d}{d\xi}\left[\frac{\widetilde{h}^{\theta^0_{T-\xi}a}(\tau,T)-\kappa_0 a}{\xi-\tau}\Bigg|_{\tau=\xi-0}\right] \;=\; H(\xi)\kappa_0 a; \tag{5.21}$$

formula (5.20) is in essence another form of relation (5.13).

Turning back to the initial definition of the image, the solution $\widetilde{h}^{\theta^0_{T-\xi}a}$ can be interpreted as a wave traveling across the pattern Θ^T; to the instant of time t ($t \ge T-\xi$), it covers part of the pattern

$$\operatorname{supp}\widetilde{h}^{\theta^0_{T-\xi}a}(\cdot,t) \subset \Gamma\times[0,t-(T-\xi)];$$

the representation (5.18) describes the shape of the wave in a neighborhood of its leading front set $\Gamma\times\{\tau = t-(T-\xi)\}$.

5.4. The dual system. The system

$$w_{tt} - \mathcal{L}w = 0 \qquad \text{in } Q^T, \tag{5.22}$$

$$w|_{t=T} = 0,\;\; w_t|_{t=T} = y \qquad \text{in } \overline{\Omega}, \tag{5.23}$$

$$w_\nu = 0 \qquad \text{on } \Sigma^T. \tag{5.24}$$

is said to be *dual* to system (5.1)–(5.3); its solution $w = w^y(x,t)$ possesses the following properties:

(1) let $y \in \mathcal{G}^T \cap C_0^\infty(\overline{\Omega^T};\mathbb{R}^3)$; in this case, the problem has a unique classical solution $w^y \in C^\infty(\overline{Q^T};\mathbb{R}^3)$;
(2) for $y \in \mathcal{G}^T$, a solution $w^y \in C([0,T];\mathcal{G}^T)$ is well defined, and the map $y \to w^y$ is continuous in the respective norms;
(3) the hyperbolicity of Eq. (5.22) on potential fields leads to the known property of finiteness of the influence domain: the solution w^y on the set $\{(x,t)\in Q^T \mid \tau(x) < t\}$ is determined by the values $y|_{\Omega^T}$ (does not depend on the behavior of y in $\Omega\setminus\Omega^T$).

Lemma 5.1. *If f and y are such that the solutions h^f and w^y are smooth in $\overline{Q^T}$, then the duality relation*

$$(h^f(\,\cdot\,,T),y)_{\mathcal{G}} = (f,\varkappa\operatorname{div}w^y\,\nu)_{\mathcal{F}^T_\nu}. \tag{5.25}$$

holds.

Proof. Integrating by part in the identity, we have

$$0 = \int\limits_{Q^T}\left[h^f_{tt}(x,t) - \nabla(\varkappa\operatorname{div}h^f)\right]\cdot w^y\,dx\,dt$$

$$= \int\limits_{\Omega} \Big\{ \Big[h^f_t(x,t) \cdot w^y(x,t) - h^f(x,t) \cdot w^y_t(x,t) \Big] \Big|_{t=0}^{t=T} + \int\limits_0^T h^f(x,t) \cdot w^y_{tt}(x,t)\, dt \Big\} dx$$

$$- \int\limits_0^T \int\limits_{\Gamma} \Big[(\varkappa \operatorname{div} h^f)(w^y \cdot \nu) - (h^f \cdot \nu)(\varkappa \operatorname{div} w^y) \Big] (\gamma, t)\, d\Gamma\, dt$$

$$- \int\limits_0^T \int\limits_{\Omega} \Big[h^f \cdot \nabla(\varkappa \operatorname{div} w^y) \Big] (x,t)\, dx\, dt$$

$$= \int\limits_0^T \int\limits_{\Gamma} [(\underbrace{h^f}_{f} \cdot \nu) \varkappa \operatorname{div} w^y](\gamma, t)\, d\Gamma\, dt - \int\limits_{\Omega} h^f(x,T) \cdot \underbrace{w^y_t(x,T)}_{y(x)}\, dx.$$

Thus,

$$\int\limits_{\Omega} h^f(x,T) \cdot y(x)\, dx = \int\limits_{\Sigma^T} (f \cdot \varkappa \operatorname{div} w^y\, \nu)(\gamma, t)\, d\Gamma\, dt.$$

The lemma is proved. □

The map $\mathcal{O} \colon y \to \varkappa \operatorname{div} w^y\, \nu|_{\Sigma^T}$ is well defined on smooth $y \in \mathcal{G}^T$; the closability of the map $f \to h^f$, property (2) of the solution of system (5.22)–(5.24), and relation (5.25) enable one to extend it to a continuous map from $\mathcal{G}^T$ to $\mathcal{F}^T_\nu$. Denote $\mathcal{O}^T := \mathcal{O}|_{\mathcal{G}^T}$. The following result is derived from the same duality relation.

Proposition 5.2. *The following relation is valid*:

$$\mathcal{O}^T = \left(\mathcal{W}^T\right)^*.$$

The operator $\mathcal{O}^T$ is called the *observation* operator.

5.5. The response operator $\mathcal{R}^T$. The correspondence "input-output" in system (5.1)–(5.3) is realized by the *response operator* $\mathcal{R}^T \colon \mathcal{F}^T_\nu \to \mathcal{F}^T_\nu$, $\operatorname{Dom} \mathcal{R}^T = \mathcal{F}^T_\nu \cap \mathcal{M}^T$,

$$\mathcal{R}^T f := \begin{pmatrix} \varkappa \operatorname{div} h^f \\ 0 \end{pmatrix}. \tag{5.26}$$

It is simply connected with the response operator of the system of the Lamé type (see (3.8)): for any $f \in \mathcal{F}^T_\nu \cap \mathcal{M}^T$, $\mathcal{R}^T f = R^T(h^f|_{\Sigma^T})$ on Σ^T (see the remark at the beginning of Sec. 5.1). The response operator is unbounded.

Consider system (5.1)–(5.3) with doubled final instant of time $2T$; let $\mathcal{R}^{2T}$ be the corresponding response operator. Since the velocity of wave propagation is finite, this operator depends on $\varkappa = c^2$ *locally*: $\mathcal{R}^{2T}$ is determined by the values of $\varkappa$ in Ω^T and does not depend on its behavior in $\Omega \backslash \Omega^T$.

Below, the operator $\mathcal{R}^{2T}$ will play the role of data of the inverse problem.

5.6. Discontinuities in the dual system. Let $y \in \mathcal{G}^T$ be a smooth field; we choose $\xi \in (0,T)$, $T < T^{\mathrm{reg}}$, and consider a system of the form (5.22)–(5.24):

$$\begin{aligned} & w_{tt} - \mathcal{L} w = 0 && \text{in } Q^T, \\ & w|_{t=T} = 0, \;\; w_t|_{t=T} = P^\xi_\perp y && \text{in } \overline{\Omega}, \\ & w_\nu = 0 && \text{on } \Sigma^T \end{aligned}$$

(the projection $P^\xi_\perp$ is defined by formula (5.7)). The action of the projection leads to the appearance of a discontinuity in the Cauchy data on the equidistant curve Γ^ξ. Discontinuous data initiate a discontinuous wave $w^{P^\xi_\perp y}$. The discontinuity of the wave propagates (in the reverse time) along the space-time rays forming the characteristic $\mathcal{X}^{T,\xi}$, and for $t=T-\xi$ it interacts with the boundary. As a result, the trace observable on Γ

$$\left[\varkappa\,\mathrm{div}\,w^{P^\xi_\perp y}\,\nu\right]\Big|_{\Sigma^T}=\mathcal{O}^T P^\xi_\perp y$$

proves to be discontinuous for $t=T-\xi$; our nearest aim is to describe this discontinuity.

We recall that the operator $\mathcal{O}^T:\mathcal{G}^T\to\mathcal{F}^T_\nu$ is defined by the relation

$$(\mathcal{W}^T f, y)_{\mathcal{G}^T}=(f,\mathcal{O}^T y)_{\mathcal{F}^T_\nu}. \tag{5.27}$$

Here it is convenient to us to regard $\mathcal{O}^T P^\xi_\perp y$ as a $\mathcal{T}$-valued function of time $t\in[0,T]$; the product $\Big((\mathcal{O}^T P^\xi_\perp y)(t),a\Big)_{\mathcal{T}}$ is defined for $a\in\mathcal{T}$ and is square-summable with respect to t.

Proposition 5.3. *The following inclusion holds*:

$$\mathrm{supp}\,\mathcal{O}^T P^\xi_\perp y\subset[0,T-\xi]. \tag{5.28}$$

Indeed, for delayed controls $f\in\mathcal{F}^{T,\xi}_\nu$ we have

$$(f,\mathcal{O}^T P^\xi_\perp y)_{\mathcal{F}^T_\nu}\;=\;(\mathcal{W}^T f, P^\xi_\perp y)_{\mathcal{G}^T}\;=\;0$$

(because $\mathcal{W}^T\mathcal{F}^{T,\xi}_\nu\subset\mathcal{G}^\xi$), which is equivalent to (5.28).

Lemma 5.2. *For $y\in\mathcal{G}^T\cap C^\infty(\overline{\Omega^T};\mathbb{R}^3)$ and $a\in\mathcal{T}\cap C^\infty(\Gamma,\mathbb{R}^3)$ the following relation is valid*:

$$\lim_{\delta\to+0}\frac{1}{\delta}\int\limits_{T-\xi-\delta}^{T-\xi}\Big((\mathcal{O}^T P^\xi_\perp y)(t),a\Big)_{\mathcal{T}}\,dt\;=\;(\kappa_0\widetilde{y}(\xi),a)_{\mathcal{T}}; \tag{5.29}$$

here, $\xi\in(0,T)$ and $\widetilde{y}=\mathcal{I}^T y$ is the image of the field y.

Proof. Take a small $\delta>0$; consider a control $\theta^0_{T-\xi-\delta}a\in\mathcal{F}^T_\nu:\mathrm{supp}\,\theta^0_{T-\xi-\delta}a\subset[T-\xi-\delta,T]$. By the location of supports (5.28), we have

$$(\mathcal{O}^T P^\xi_\perp y,\theta^0_{T-\xi-\delta}a)_{\mathcal{F}^T_\nu}=\int\limits_{T-\xi-\delta}^{T-\xi}\Big((\mathcal{O}^T P^\xi_\perp y)(t),a\Big)_{\mathcal{T}}\,dt\,. \tag{5.30}$$

Let $X^\xi_\perp:=\mathbb{I}-X^\xi_\perp$ is the projection in $\mathcal{F}^T_\nu$ cutting the elements to the interval $[T-\xi,T]$; by (4.49), the relation $\mathcal{I}^T Q^\xi\;=\;X^\xi\mathcal{I}^T$ holds. Hence, using (5.9) ($P^\xi=Q^\xi$), we derive that

$$\mathcal{I}^T P^\xi_\perp\;=\;X^\xi_\perp\mathcal{I}^T. \tag{5.31}$$

By (5.18) and (5.19) for the image $\widetilde{h}^{\theta^0_{T-\xi-\delta}a}$ we have the representation

$$\widetilde{h}^{\theta^0_{T-\xi-\delta}a}(\tau,T)=\theta^0(\xi+\delta-\tau)\kappa_0 a+d_1(\tau,T) \tag{5.32}$$

with the estimate

$$|d_1(\tau,T)|\le C\theta^1(\xi+\delta-\tau). \tag{5.33}$$

Next, the following relations are valid:

$$
\begin{aligned}
(\mathcal{O}^T P_\perp^\xi y, \theta^0_{T-\xi-\delta} a)_{\mathcal{F}^T_\nu} &\overset{(5.27)}{=} (P_\perp^\xi y, \mathcal{W}^T[\theta^0_{T-\xi-\delta} a])_{\mathcal{G}^T} \\
&= (\mathcal{I}^T P_\perp^\xi y, \mathcal{I}^T \mathcal{W}^T[\theta^0_{T-\xi-\delta} a])_{\mathcal{F}^T_\nu} \overset{(5.31)}{=} (X_\perp^\xi \widetilde{y}, \widetilde{h}^{\theta^0_{T-\xi-\delta} a}(\cdot, T))_{\mathcal{F}^T_\nu} \\
&\overset{(5.32)}{=} \int\limits_\xi^{\xi+\delta} (\kappa_0\, a + d_1(\tau, T), \widetilde{y}(\tau))_{\mathcal{T}} \overset{(5.33)}{=} \delta(\kappa_0\, \widetilde{y}(\xi), a)_{\mathcal{T}} + o(\delta)
\end{aligned}
\tag{5.34}
$$

(we have used the unitary property of the operator $\mathcal{I}^T$). Combining (5.30) and (5.34), we get (5.29). The lemma is proved. □

In view of property (5.28), the result obtained can be interpreted as a description of the discontinuity of the function $(\mathcal{O}^T P_\perp^\xi y)(t)$ for $t = T - \xi$. We agree to write (5.29) in the form

$$
(\mathcal{O}^T P_\perp^\xi y)(T - \xi - 0) \;=\; (\kappa_0\, \mathcal{I}^T y)(\xi), \quad 0 < \xi < T, \tag{5.35}
$$

interpreting the limit in the sense of the lemma. By arguments of a dynamical nature, given at the beginning of the section, relation (5.35) represents the image of a field as a collection of discontinuities passed through the medium filling Ω^T and detected on the boundary Γ. We call (5.35) the *amplitude formula* [5].

5.7. Connecting operator. The operator $\mathcal{C}^T \colon \mathcal{F}^T_\nu \to \mathcal{F}^T_\nu$,

$$
C^T := \big(\mathcal{W}^T\big)^* \mathcal{W}^T,
$$

is called the *connecting operator* of system (5.1)–(5.3). This name is explained by the fact that for $f,\, g \in \mathcal{F}^T_\nu$ we have

$$
(\mathcal{C}^T f, g)_{\mathcal{F}^T_\nu} = (\mathcal{W}^T f, \mathcal{W}^T g)_{\mathcal{G}^T} = (h^f(\cdot, T), h^g(\cdot, T))_{\mathcal{G}^T}, \tag{5.36}
$$

i.e., $\mathcal{C}^T$ connects the scalar products of the external and internal spaces of the dynamical system. This is a continuous operator nonnegative in $\mathcal{F}^T_\nu$.

It is important that the connecting operator can be calculated from the response operator. Consider the operator of odd extension $\mathcal{S}^T \colon \mathcal{F}^T_\nu \to \mathcal{F}^{2T}_\nu$,

$$
(\mathcal{S}^T f)(\cdot, t) \;:=\; \begin{cases} f(\cdot, t), & 0 \le t < T; \\ -f(\cdot, 2T - t), & T \le t \le 2T; \end{cases}
$$

and the operator of integration $\mathcal{J}^{2T} \colon \mathcal{F}^{2T}_\nu \to \mathcal{F}^{2T}_\nu$,

$$
(\mathcal{J}^{2T} f)(\cdot, t) := \int\limits_0^t f(\cdot, s)\, ds\,, \quad 0 \le t \le 2T.
$$

Denote $M^T_\nu := \mathcal{F}^T_\nu \cap \mathcal{M}^T$ and $\mathcal{M}^{T,0}_\nu := \{f \in M^T_\nu \mid S^T f \in M^{2T}_\nu\}$. We mention the inclusion $\mathcal{S}^T \mathcal{M}^{T,0}_\nu \subset \operatorname{Dom} R^{2T}$ and the relation

$$
\Big(\big(\mathcal{S}^T\big)^* f\Big)(\cdot, t) = f(\cdot, t) - f(\cdot, 2T - t), \quad 0 \le t \le 2T.
$$

Lemma 5.3. *For fields of the class $\mathcal{M}^{T,0}_\nu$ the following representation is valid:*

$$
\mathcal{C}^T \;=\; \frac{1}{2}(\mathcal{S}^T)^* \mathcal{J}^{2T} \mathcal{R}^{2T} \mathcal{S}^T. \tag{5.37}
$$

The proof is quite similar to that given in [9]. As seen from (5.37), to find $\mathcal{C}^T$ it is sufficient to have the values of $\mathcal{R}^{2T}$ only on $\mathcal{S}^T\mathcal{M}_\nu^{T,0}$.

The operator $\mathcal{C}^T$ enables one to find the images of waves, using the so-called wave bases. In the subspace $\mathcal{F}_\nu^{T,\xi} \subset \mathcal{F}_\nu^T$ we choose a complete system of controls $\{f_j^\xi\}$: $\overline{\mathrm{Lin}}\{f_j^\xi\} = \mathcal{F}_\nu^{T,\xi}$ [9]; $(\mathcal{C}^T f_i^\xi, f_j^\xi)_{\mathcal{F}_\nu^T} = \delta_{ij}$.

By property (5.8) and in view of (5.36), the corresponding system of waves forms an orthonormal basis in the subspace $\mathcal{G}^\xi$.

Take $f \in \mathcal{M}_\nu^T$; for the wave $h^f(\cdot, T) = \mathcal{W}^T f$ we have the representation

$$\begin{aligned} P_\perp^\xi \mathcal{W}^T f &= \mathcal{W}^T f - \sum_j (\mathcal{W}^T f, \mathcal{W}^T f_j^\xi)_{\mathcal{G}^T} \mathcal{W}^T f_j^\xi \\ &\overset{(5.36)}{=} \mathcal{W}^T f - \sum_j (\mathcal{C}^T f, f_j^\xi)_{\mathcal{F}_\nu^T} \mathcal{W}^T f_j^\xi. \end{aligned} \tag{5.38}$$

Next, in the external space $\mathcal{F}_\nu^T$ we take an L_2-orthonormal basic $\{g_k\}$. We have the relations

$$\begin{aligned} (\mathcal{O}^T P_\perp^\xi \mathcal{W}^T f, g_k)_{\mathcal{F}_\nu^T} &= (P_\perp^\xi \mathcal{W}^T f, \mathcal{W}^T g_k)_{\mathcal{G}^T} \\ &\overset{(5.38)}{=} (\mathcal{W}^T f, \mathcal{W}^T g_k)_{\mathcal{G}^T} - \sum_j (\mathcal{C}^T f, f_j^\xi)_{\mathcal{F}_\nu^T} (\mathcal{W}^T f_j^\xi, \mathcal{W}^T g_k)_{\mathcal{G}^T} \\ &\overset{(5.36)}{=} (\mathcal{C}^T f, g_k)_{\mathcal{F}_\nu^T} - \sum_j (\mathcal{C}^T f, f_j^\xi)_{\mathcal{F}_\nu^T} (\mathcal{C}^T f_j^\xi, g_k)_{\mathcal{F}_\nu^T}. \end{aligned}$$

They yield the relation

$$\begin{aligned} \mathcal{O}^T P_\perp^\xi \mathcal{W}^T f &= \sum_k (\mathcal{O}^T P_\perp^\xi \mathcal{W}^T f, g_k)_{\mathcal{F}_\nu^T} g_k \\ &= \sum_k \left\{ (\mathcal{C}^T f, g_k)_{\mathcal{F}_\nu^T} - \sum_j (\mathcal{C}^T f, f_j^\xi)_{\mathcal{F}_\nu^T} (\mathcal{C}^T f_j^\xi, g_k)_{\mathcal{F}_\nu^T} \right\} g_k. \end{aligned} \tag{5.39}$$

In the amplitude formula (5.35), we set $y = h = \mathcal{W}^T f$:

$$(\mathcal{O}^T P_\perp^\xi h)(T - \xi - 0) = (\mathcal{O}^T P_\perp^\xi \mathcal{W}^T f)(T - \xi - 0) = (\kappa_0\, \mathcal{I}^T \mathcal{W}^T f)(\xi) = \kappa_0 \widetilde{h}^f(\xi, T).$$

Calculating the left-hand side of the relation

$$\kappa_0{}^{-1} (\mathcal{O}^T P_\perp^\xi \mathcal{W}^T f)(T - \xi - 0) = \widetilde{h}^f(\xi, T) \tag{5.40}$$

with the help of the representation (5.39), we restore the image of the wave h^f.

5.8. Recovery of the velocities. Assume that we possess the following data concerning the Lamé type system (3.1)–(3.3); its response operator R^{2T} is given for fixed $T > 0$ and the functions $c_\alpha|_\Gamma$, $\frac{\partial c_\alpha}{\partial \nu}\big|_\Gamma$ $(\alpha = p, s)$ are known. The *inverse problem* consists of recovery of the velocities c_s in Ω_s^T and c_p in Ω_p^T from these data. We give our main result.

Theorem 5.1. *For any positive $T < T^{\mathrm{reg}}$, the data of the inverse problem determine the velocities $c_\alpha|_{\Omega_\alpha^T}$ $(\alpha = p, s)$ in a unique way.*

[9] Lin is a linear span.

Proof. To prove the theorem, it suffices to sum up the previous considerations. Using them, we present a general plan of solving the inverse problem:

(1) the response operator R^{2T} determines the response operator $\mathcal{R}^{2T}$ of system (5.1)–(5.3): $\mathcal{R}^{2T} f = R^{2T}(h^f\big|_{\Sigma^T})$ for any $f \in \mathcal{F}^{2T}_\nu \cap \mathcal{M}^{2T}$.

(2) Using $\mathcal{R}^{2T}$, we find the connecting operator $\mathcal{C}^T$ of system (5.1)–(5.3) (Lemma 5.3).

(3) Making use of chosen $a \in \mathcal{T} \cap C^\infty(\Gamma, \mathbb{R}^3)$ and $\xi \in (0, T)$, we find

$$\kappa_0 \widetilde{h}^{\theta^0_{T-\xi} a}(\tau, T) = (\mathcal{O}^T P^\tau_\perp \mathcal{W}^T[\theta^0_{T-\xi} a])(T - \tau - 0)$$

in accordance with (5.40).

(4) By (5.20), we have $\kappa_0^2 = |a|^{-1} |\kappa_0 \widetilde{h}^{\theta^0_{T-\xi} a}(\xi - 0, T)|$. Thus, the function $\kappa_0 = \kappa_0(\gamma)$ is determined.

(5) By (5.21), we find $H(\xi)\kappa_0\, a$; changing a and ξ, we restore the family of operators $H(\xi)$, $0 < \xi < T$.

(6) In accordance with (4.62), the operators $H(\xi)$ determine the matrix $\{h^{\alpha\beta}(\gamma, \xi)\}$ and a function $c_p(\gamma, \xi)$ on the pattern Θ^T, from which the fast velocity $c_p(x)$ is uniquely determined in the subdomain Ω^T_p (Theorem 4.1).

The recovery of the slow velocity c_s in Ω^T_s from the response operator $\mathcal{R}^{2T}_m$ of the Maxwell subsystem (3.16)–(3.18), which is defined by the relation $\mathcal{R}^{2T}_m f = R^{2T}(\psi^f\big|_{\Sigma^T})$ on the controls of the class $\mathcal{M}^{2T}$ that are tangent to Γ, is conducted in [8]. Note that in this case, the main space is $\mathcal{H}^T$ and its subspace J^T of solenoidal fields.

Steps (2)–(5) of our proof are in essence steps (i)–(vi) of paper [8]. Using the family of operators $H(\xi)$, $0 < \xi < T$, corresponding to the Maxwell subsystem, we first determine the tensor $\{h_{\alpha\beta}\}$ of the slow metric and only after that the velocity $c_s(\gamma, \xi)$ on Θ^T. To this end we consider the so-called Jamabé problem the solution of which is reduced to the solution of a certain elliptic equation. For this equation to have a unique solution, it is necessary to know also the values of c_s and $\frac{\partial c_s}{\partial \tau}$ on Γ (Theorem 1.1 in [8]). Using $c_s(\gamma, \xi)$ on Θ^T, we restore the slow velocity $c_s(x)$ in Ω^T_s (Theorem 4.1). The theorem is proved. □

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