Condition-based maintenance for systems with aging and cumulative damage based on proportional hazards model

Bin Liu[a1], Zhenglin Liang[b], Ajith Kumar Parlikad[b], Min Xie[a,c], Way Kuo[a]
[a]Department of Systems Engineering and Engineering Management,
City University of Hong Kong, Kowloon, Hong Kong
[b]Institute for Manufacturing, Department of Engineering,
University of Cambridge CB3 0FS, UK
[c]City University of Hong Kong, Shenzhen Research Institute, Shenzhen, China

Abstract

This paper develops a condition-based maintenance (CBM) policy for systems subject to aging and cumulative damage. The cumulative damage is modeled by a continuous degradation process. Different from previous studies which assume that the system fails when the degradation level exceeds a specific threshold, this paper argues that the degradation itself does not directly lead to system failure, but increases the failure risk of the system. Proportional hazards model (PHM) is employed to characterize the joint effect of aging and cumulative damage. CBM models are developed for two cases: one assumes that the distribution parameters of the degradation process are known in advance, while the other assumes that the parameters are unknown and need to be estimated during system operation. In the first case, an optimal maintenance policy is obtained by minimizing the long-run cost rate. For the case with unknown parameters, periodic inspection is adopted to monitor the degradation level of the system and update the distribution parameters. A case study of Asphalt Plug Joint in UK bridge system is employed to illustrate the maintenance policy.

Keywords: Condition-based maintenance; aging and degradation; proportional hazards model; unknown distribution parameters; cumulative damage

1 Introduction

[1] Corresponding author. Email address: binliu9-c@my.cityu.edu.hk. Tel: (+852)51741281

This is a peer reviewed, accepted author manuscript of the following chapter: Liu, B., Liang, Z., Parlikad, A. K., Xie, M., & Kuo, W. (2020). Condition-based maintenance for systems with aging and cumulative damage based on proportional hazards model. In Value-Based and Intelligent Asset Management: Mastering the Asset Management Transformation in Industrial Plants and Infrastructures (pp. 211-231). Springer. https://doi.org/10.1007/978-3-030-20704-5_10

With the development of sensor technologies, system condition can be monitored at a much lower expense, which prompts the application of condition-based maintenance. CBM takes advantage of the online monitoring information to make maintenance decisions. For a system subject to CBM, based on the collected condition information, maintenance actions are carried out only when “necessary” (Liu et al, 2014; Liu et al, 2016b). Compared with the traditional time-based maintenance, CBM has shown its priority in preventing unexpected failure and reducing economic losses (Zhang et al, 2014; Wu et al, 2016).

CBM is conducted based on the observation that systems usually suffer a degradation process before failure, and the degradation process can be observed by degradation indicators such as temperature, voltage and vibration. In literature, many researchers used multi-state deteriorating models to describe the degradation process and formulated the maintenance strategy as a Markov or semi-Markov decision process (Maillart, 2006; Srinivasan & Parlikad, 2013). Although Markov model is widely used in degradation modeling, one disadvantage is that the classification of system state is very arbitrary (Li et al, 2012; Chen et al, 2015; Lin et al, 2016).

Recently, more emphasis is paid to continuous degradation processes. In the framework of continuous degradation, the degradation process is usually described by a general path model or a stochastic-process-based model such as Wiener process, Gamma process and inverse Gaussian process (Ye & Xie, 2015; Ye et al, 2015; Liu et al, 2016a). Caballé et al (2015) proposed a CBM for systems with continuous degradation and external shocks. Peng & van Houtum (2016) developed a joint CBM and lot sizing policy for systems subject to continuous degradation.

An implicit assumption of the previous research is that a system fails when its degradation level exceeds a pre-specified failure threshold. However, in reality, the failure threshold is difficult to determine and usually it is a random variable depending on the environment condition and product’s characteristics. In this paper, the cumulative damage is modeled as a continuous degradation process. We argue that degradation process does not necessarily lead to system failure, but increases the likelihood of failure. Both internal aging and cumulative damage contribute to system failure. Examples of the joint effect of aging and cumulative damage on system failure can be found in systems such as high-voltage power transformers and bridge systems (Wu & Ryan, 2011; Wardhana & Hadipriono, 2003; Mo & Xie, 2016). For a new transformer, its insulation strength can withstand severe events such as transient overvoltage and lightning strikes. When a transformer ages, its internal condition degrades, which makes it more vulnerable to fluctuating environment condition and increases the risk of failure. For a bridge system, failures are usually triggered by external events such as hurricane, flood and overload. If a bridge system undergoes severe deterioration, it may hit the point where tiny external influences can lead to system failure. The degradation itself does not directly lead to system failure, but it increases the probability of failure when exposed to external events.

A convenient and prevalent way to integrate the aging and degradation effect into system failure is by a proportional hazards model (Lin & Wei, 1989). PHM incorporates a baseline hazard function which accounts for the aging effect with a link function that takes the inspection information into account to improve the prediction of failure (Pham et al, 2012). Applications of PHM can be found in various fields such as finance, manufacturing system and energy generators (Jardine & Tsang, 2013).

In literature, several studies have been conducted on maintenance policy in the PHM framework. Banjevic et al (2001) developed a control-limit maintenance policy for systems subject to periodic inspection. Ghasemi et al (2007) proposed a CBM policy for systems with imperfect information, where the condition the system cannot be directly monitored. Wu & Ryan (2010) investigated the value of condition monitoring in the PHM setting, where a continuous-time Markov chain was used to describe the system condition. Wu & Ryan (2011) further extended the model by considering Semi-Markov covariate process and continuous monitoring. Tian & Liao (2011) proposed a CBM policy for multi-component systems using PHM. Lam & Banjevic (2015) investigated the issue of inspection scheduling for CBM. In all of these previous studies, the degradation process is characterized via Markov or semi-Markov model. In addition, the distribution parameters in the PHM are assumed as known in advance.

This paper aims to develop CBM policies for systems subject to aging and cumulative damage. The system is subject to aging and extremely frequent cumulative damage (*e.g.*, traffic load to a bridge), where the extremely frequent cumulative damage is approached by a continuous degradation process. PHM is used to model the joint effect of aging and cumulative in the framework of failure rate. The effect of cumulative damage is modeled as the stochastic covariate in the PHM framework. The system is subject to periodic inspection, which is assumed to be perfect. At inspection, maintenance actions are carried out based on the observed condition information. Optimal maintenance policies are obtained by minimizing the long-run cost rate.

Specifically, two CBM models are developed by assuming respectively known distribution parameters and unknown distribution parameters. In the case where the distribution parameters are unknown, the parameters have to be estimated and updated at each inspection, and maintenance decisions are made subsequently.

This paper differs from the existing works in that: (a) It incorporates the influence of both aging and cumulative damage in modeling the failure rate. (b) It argues that degradation itself does not result in system failure, but increases the risk of failure. (c) It utilizes the observed condition information to update distribution parameters for making appropriate maintenance decisions.

The remainder of this paper is organized as follows. Section 2 presents the degradation-integrated failure model, where PHM is used to describe the impact of aging and cumulative damage. Section 3 formulates two maintenance models. One assumes known distribution parameters while the other assumes unknown distribution parameters. Application of the maintenance models to Asphalt Plug Joints in UK bridge system is presented in Section 4. Finally, concluding remarks and future research suggestions are given in Section 5.

2 Degradation-integrated failure model

This paper considers a single-unit system subject to aging and cumulative damage. The cumulative damage is modeled as a continuous degradation process. For systems such as bridges, which are subject to traffic load hours by hours, a continuous degradation process is reasonable to characterize the cumulative damage over time. In this paper, we use “cumulative damage” and “degradation process” interchangeably. In the present paper, the degradation process derives from cumulative shocks, which is an external factor. Besides the external factor, the system also suffers from internal aging factors. That is to say, the aging and the degradation are two processes. Therefore, we model the system subject to both aging and degradation process. Different from previous studies which assume that soft failure occurs when the degradation level hits a pre-specified threshold, we here consider sudden failure, which depends on both the aging and cumulative damage. For most infrastructure systems, failures usually happen due to external shocks or serious events, and degradation makes it more vulnerable when exposed to shocks. As previously described, the degradation process itself does not directly lead to system failure, but it increases the failure rate of the system. PHM is used to characterize the influences of degradation level on system failure rate. The degradation level of the system is represented as the value of covariate in the PHM framework (Lehmann, 2009). Based on PHM, the failure rate at time t is given by

$$h(t;X_t)=h_0(t)\varphi(X_t) \tag{1}$$

where $h_0(t)$ is the baseline failure rate at time t, which is a non-decreasing function of t. X_t is the degradation level at time t, and $\varphi(\cdot)$ is a positive function projecting the degradation level to the failure rate function. Let $X=\{X_t,t\geq 0\}$ be a continuous stochastic process that depicts the degradation process. Various stochastic processes can be used to describe the degradation process, among which a wide used candidate is the general path model (Lu & Meeker, 1993). Assume that $X_t=g\left(t;\theta,\alpha,\varepsilon(t)\right)$, where $g(\cdot)$ is a parametric function that characterizes the evolution of the degradation process, θ is a random variable that accounts for unit-to-unit variability, α is a random parameter that captures the initial degradation level among the components' population, $\varepsilon(t)$ is an independent and identically distributed (iid) random error term (Elwany et al, 2011). The selection of $g(\cdot)$ depends on system characteristics and can take a variety of forms such as linear, exponential or logarithmic. In this paper, for simplicity, we assume that $g(\cdot)$ is a linear function. The degradation process can be denoted as $X_t=\alpha+\theta t+\varepsilon(t)$ (Gebraeel et al, 2005; Haghighi & Bae, 2015), where the error term $\varepsilon(t)$ follows a Gaussian distribution with mean zero and variance σ^2, α and θ follow Gaussian distributions, with mean $\mu_0^{'}=\mu_0-\sigma^2/2$ and variance σ_0^2, and mean μ_1 and variance σ_1^2. Since $\varepsilon(t)$ is independent of time t, we may suppress the notation of t and denote $\varepsilon(t)$ as ε. In Eq (1), the baseline failure rate function, $h_0(t)$, accounts for the aging effect, which can be explained as the normal failure rate when no cumulative damage is imposed. The influence of cumulative damage is incorporated in the degradation level X_t. Obviously X_t follows a Gaussian distribution,

$$X_t \sim N\left(\mu_0+\mu_1 t-\sigma^2/2,\sigma_0^2+\sigma_1^2t^2+\sigma^2\right) \tag{2}$$

It is assumed that $\mu_0+\mu_1 t-\sigma^2/2 \gg \sigma_0^2+\sigma_1^2t^2+\sigma^2$, such that the probability of X_t being negative can be neglected and X_t stochastically increases with t almost surely. Given the degradation process x, the conditional reliability can be obtained as

$$R(t;x) = P(T > t \mid x_s, 0 \le s \le t) = \exp\left(-\int_0^t h_0(s)\varphi(x_s)ds\right) \tag{3}$$

where T is the time to failure and x_s is the realization of X_s at time *s*. The probability density function (pdf) is given as

$$f_T\left(t;x\right) = \lim_{\Delta t \to 0^+} \frac{P\left(t \le T < t + \Delta t \mid T > t\right)}{\Delta t} = \frac{h_0(t)\varphi(x_t)}{\exp\left(\int_0^t h_0(s)\varphi(x_s)ds\right)} \tag{4}$$

The expected lifetime of the system can be obtained as

$$E\left[T\right] = E\left[E_{|X_s, 0<s<T}\left[T\right]\right] = \int_0^\infty \int_0^\infty t \cdot f_T\left(t \mid x_s\right) f_{X_s}\left(x_s\right) dx_s dt \tag{5}$$

where f_{X_s} is the pdf of degradation level by time *s*. If the projecting function $\varphi(\cdot)$ is exponential, $h(t;X_t) = h_0(t)\exp\left(\beta X_t\right)$, where β is the coefficient, then we have $\log h(t;X_t) = \log h_0(t) + \beta X_t$, which implies that the failure rate function follows a lognormal distribution,

$$\log h(t) \sim N\left(\beta\left(\mu_0 + \mu_1 t - \sigma^2/2\right) + \log h_0(t), \beta^2\left(\sigma_0^2 + \sigma_1^2 t^2 + \sigma^2\right)\right) \tag{6}$$

The lognormal distribution fits numerous reliability data and reflects the failure due to crack propagation (Provan, 1987).

3 Maintenance model formulation

This section aims to establish maintenance models for systems with known and unknown distribution parameters respectively. The system is assumed as non-repairable; thus the inspection/replacement policy is adopted (Huynh et al, 2011). The system failure is self-announcing, but the degradation level is not evident, which can only be detected at inspection. Periodic inspection is carried out to detect the degradation level, with the cost C_i. Two maintenance actions are available upon the system: preventive replacement and corrective replacement. Preventive replacement can be an overhaul of the system, while corrective replacement refers to physical replacement of the system (Huynh et al, 2011). Both preventive replacement and corrective replacement restore the system to the "as good as new" state. At inspection, if the degradation level or the age of the system exceeds a certain threshold, preventive replacement will be implemented, with the cost C_p. If the system fails unexpectedly, corrective replacement is performed immediately, with the cost C_r. The corrective replacement cost includes the replacement cost of the system and also the cost comprising various costs with respect to failure-induced problems. Intuitively, C_r is more complex and more cost intensive ($C_p < C_r$).

Assume that the system is inspected every τ time units, where τ is a given parameter associated with the system characteristics. Given that the system functions at the *k*th inspection, the probability that the system survives through $(k+1)\tau$ is

$$P\left(T > (k+1)\tau \mid T > k\tau, x_s, k\tau \le s \le \left(k+1\right)\tau\right) = \exp\left(-\int_{k\tau}^{(k+1)\tau} h_0(s)\varphi(x_s)ds\right) \tag{7}$$

Proposition 1. Given the degradation level X_s, for $k\tau \le s \le \left(k+1\right)\tau$, the cumulative hazard rate function between two consecutive inspections increases in *k*. In addition, the inequality

$$E\left[\int_0^t h_0(s)\varphi(X_s)ds\right] \ge \int_0^t h_0(s)\varphi\left(E\left[X_s\right]\right)ds$$

holds for the cumulative hazard rate of the system.

The detailed proofs of the propositions in this paper are provided in Appendix. Proposition 1 implies that the expected cumulative failure rate is at least larger than the cumulative failure rate given at mean degradation level. Based on Proposition 1, we can readily obtain that the conditional reliability of the system surviving through the next inspection, $P\left(T > (k+1)\tau \mid T > k\tau, x_s, k\tau \le s \le \left(k+1\right)\tau\right)$, decreases with the inspection index *k*.

In the following, we will develop maintenance models for systems with known and unknown distribution parameters. The maintenance model of known parameter refers to the case that the parameters are determined or estimated by historical data or expert opinion, while no online monitoring is employed. On the other hand, the unknown parameter refers to the case that the prior distribution of the parameters are estimated with historical data or expert opinion, and the parameters are updated with monitored or inspected information. The difference between the two models lies in whether inspection information is used to update the parameters.

3.1 Maintenance model with known distribution parameters

In this section, we assume that the parameters of the failure rate function are known in advance. Since both the age and the degradation level influence the failure rate, maintenance operations are carried out based on the hazard rate at inspection, which explicitly incorporates the effects of aging and degradation. If the hazard rate at inspection exceeds a specific threshold ζ, preventive replacement is implemented. Otherwise, the system is left as it be. Long-run cost rate is used as the criterion to evaluate the maintenance policy. Our objective is to minimize the long-run cost rate by seeking an optimal ζ. According to the renewal theorem, the long-run cost rate is given by

$$\psi(\zeta)=\lim_{t\to\infty}\frac{C(t)}{t}=\frac{E\left[C(S)\right]}{E\left[S\right]}=\frac{C_pP_a+C_rP_b+C_iE\left[N_I\right]}{E\left[\min\{T,N_I\tau\}\right]} \tag{8}$$

where S is the length of a renewal cycle, P_a is the probability that a renewal cycle ends with preventive replacement, P_b is the probability that a renewal cycle ends with corrective replacement, and N_I is the number of inspections.

At time t, the probability that the failure rate of the system exceeds the threshold ζ can be obtained as

$$\begin{aligned}P\left(h(t,X_t)>\zeta\right)&=P\left(\varphi(X_t)>\frac{\zeta}{h_0(t)}\right)=P\left(X_t>\varphi^{-1}\left(\frac{\zeta}{h_0(t)}\right)\right)\\&=\Phi\left(\frac{\left(\mu_0+\mu_1t-\sigma^2/2-\varphi^{-1}\left(\zeta/h_0(t)\right)\right)}{\sqrt{\sigma_0^2+\sigma_1^2t^2+\sigma^2}}\right)\end{aligned} \tag{9}$$

Since a renewal cycle occurs either after a preventive replacement or corrective replacement, it is appealing to analyze the renewal cycle separately. The probability that preventive replacement is performed at the kth inspection is expressed as

$$\begin{aligned}P_a(k)=\exp&\left(\begin{aligned}&-\int_0^{(k-1)\tau}h_0(t)\int_0^{\ln(\zeta/h_0(t))}\varphi\left(x(t)\right)f_{X(t)}dxdt\\&-\int_{(k-1)\tau}^{k\tau}h_0(t)\int_0^{\infty}\varphi\left(x(t)\right)f_{X(t)}dxdt\end{aligned}\right)\\&\left(P\left(h(k\tau)>\zeta\right)-P\left(h((k-1)\tau)>\zeta\right)\right)\end{aligned} \tag{10}$$

The probability that failure occurs within the interval $\left((k-1)\tau,k\tau\right)$ can be obtained as

$$P_b(k)=\left(1-\exp\left(-\int_{(k-1)\tau}^{k\tau}h_0(t)\int_0^{\infty}\varphi(x(t))f_{X(t)}dxdt\right)\right)\cdot$$
$$\exp\left(-\int_0^{(k-1)\tau}h_0(t)\int_0^{\ln(\zeta/h_0(t))}\varphi(x(t))f_{X(t)}dxdt\right)\cdot \tag{11}$$
$$\Phi\left(\frac{\left(\mu_0+\mu_1 t-\sigma^2/2-\varphi^{-1}(\zeta/h_0(t))\right)}{\sqrt{\sigma_0^2+\sigma_1^2t^2+\sigma^2}}\right)$$

Detailed derivations of Eq (10) and (11) are provided in Appendix.

If preventive replacement is carried out at the *k*th inspection, the cost and length of a renewal cycle can be obtained as $C_a=C_p+kC_i$ and $S_a=k\tau$. If a failure occurs in the interval $((k-1)\tau,k\tau)$, the cost and in a renewal cycle is expressed as $C_b=C_r+(k-1)C_i$ and the expected length is calculated as $E[S_b]=k\tau-\int_{(k-1)\tau}^{k\tau}(k\tau-t)f(t)dt$. The long-run cost rate can be achieved by combining the renewal cycles ending with preventive replacement and corrective replacement. After some calculations, the long-run cost rate is given as

$$\psi(\zeta)=\frac{E[C]}{E[S]}=\frac{C_p\sum_{k=1}^{\infty}P_a(k)+C_r\sum_{k=1}^{\infty}P_b(k)+C_i\sum_{k=1}^{\infty}\left(kP_a(k)+(k-1)P_b(k)\right)}{\sum_{k=1}^{\infty}k\tau P_a(k)+\sum_{k=1}^{\infty}\left(k\tau-\int_{(k-1)\tau}^{k\tau}(k\tau-t)f(t)dt\right)P_b(k)} \tag{12}$$

The optimal maintenance threshold ζ can be obtained by minimizing Eq (12), i.e., $\zeta^*=\arg\min\{\psi(\zeta):\zeta>0\}$.

3.2 Maintenance model with unknown distribution parameters

In this section, we assume that the distribution parameters of the degradation process are unknown and have to be estimated with the inspected information. Denote X_k as the observed degradation level at the *k*th inspection. If the parameters α and θ are known, we can have the joint distribution of the observations $X_1,...,X_k$ as

$$f(X_1,...,X_k\mid\alpha,\theta)=\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^k\cdot\exp\left(-\sum_{i=1}^{k}\left(\frac{X_i-\alpha-\theta i\tau}{2\sigma^2}\right)\right) \tag{13}$$

However, the exact values of α and θ are unknown, due to the unit-to-unit variation. We assume that the prior distribution of α and θ are known, which can be obtained from the reliability characteristics of the population of the components. In accordance with previous sections, let the prior distributions of α and θ be Gaussian distributions with mean $\mu_0^{'}$ and variance σ_0^2 and mean μ_1 and variance σ_1^2.

Given the inspected information $X_1,...,X_k$, the posterior distributions of α and θ are bivariate normal distribution with parameters (Gebraeel et al , 2005)

$$\mu_\alpha=\frac{\left(\sum_{i=1}^k X_i\sigma_0^2+\mu_0^{'}\sigma^2\right)\left(\sum_{i=1}^k(i\tau)^2\sigma_1^2+\sigma^2\right)-\left(\sum_{i=1}^k i\tau\sigma_0^2\right)\left(\sum_{i=1}^k X_i i\tau\sigma_1^2+\mu_1\sigma^2\right)}{\left(k\sigma_0^2+\sigma^2\right)\left(\sum_{i=1}^k(i\tau)^2\sigma_1^2+\sigma^2\right)-\left(\sum_{i=1}^k i\tau\sigma_1^2\right)\left(\sum_{i=1}^k i\tau\sigma_0^2\right)}$$

$$\mu_\theta=\frac{\left(k\sigma_0^2+\sigma^2\right)\left(\sum_{i=1}^k X_i i\tau\sigma_1^2+\mu_1\sigma^2\right)-\left(\sum_{i=1}^k i\tau\sigma_1^2\right)\left(\sum_{i=1}^k X_i\sigma_0^2+\mu_0^{'}\sigma^2\right)}{\left(k\sigma_0^2+\sigma^2\right)\left(\sum_{i=1}^k(i\tau)^2\sigma_1^2+\sigma^2\right)-\left(\sum_{i=1}^k i\tau\sigma_1^2\right)\left(\sum_{i=1}^k i\tau\sigma_0^2\right)}$$

$$\sigma_\alpha^2=\sigma^2\sigma_0^2\frac{\sum_{i=1}^k(i\tau)^2\sigma_1^2+\sigma^2}{\left(k\sigma_0^2+\sigma^2\right)\left(\sum_{i=1}^k(i\tau)^2\sigma_1^2+\sigma^2\right)-\left(\sum_{i=1}^k i\tau\right)^2\sigma_0^2\sigma_1^2}$$

$$\sigma_\theta^2=\sigma^2\sigma_1^2\frac{k\sigma_0^2+\sigma^2}{\left(k\sigma_0^2+\sigma^2\right)\left(\sum_{i=1}^k(i\tau)^2\sigma_1^2+\sigma^2\right)-\left(\sum_{i=1}^k i\tau\right)^2\sigma_0^2\sigma_1^2}$$

$$\rho=\frac{-\sigma_0\sigma_1\sum_{i=1}^k i\tau}{\sqrt{\left(k\sigma_0^2+\sigma^2\right)\left(\sum_{i=1}^k(i\tau)^2\sigma_1^2+\sigma^2\right)}}$$

where ρ is the correlation. The above equations imply that the degradation process is nonstationary evolution process whose parameters are updated according to the observed health condition of the system. The joint pdf of $\tilde{\alpha}$ and $\tilde{\theta}$ is given as

$$f(\tilde{\alpha},\tilde{\theta})=\frac{1}{2\pi\sigma_\alpha\sigma_\theta\sqrt{1-\rho^2}}\exp\left(-\frac{z}{2\left(1-\rho^2\right)}\right) \tag{14}$$

where

$$z=\frac{\left(\tilde{\alpha}-\mu_\alpha\right)^2}{\sigma_\alpha^2}-\frac{2\rho\left(\tilde{\alpha}-\mu_\alpha\right)\left(\tilde{\theta}-\mu_\theta\right)}{\sigma_\alpha\sigma_\theta}+\frac{\left(\tilde{\theta}-\mu_\theta\right)^2}{\sigma_\theta^2}$$

Proposition 2. The correlation ρ and σ_θ^2 decrease with the inspection interval τ, while σ_α^2 increases with τ.

Proposition 2 can be used to reduce the variance of estimates by varying the inspection length. If θ exerts a dominant impact on the degradation process, which can be evaluated via the prior distribution, then inspection interval is suggested to be extended so as to reduce the uncertainty of the estimates. If the degradation process is largely influenced by α, then the inspection interval should be short so as to improve the accuracy of the estimates.

Corollary 1. Under continuous monitoring, the variances of α and θ are constant.

Corollary 1 can be achieved by letting τ approach to 0 and k approach to ∞. Corollary 1 expresses the consequence of continuous monitoring and perfect inspection, which significantly reduces the associated uncertainty. After estimating the distribution parameters at the *k*th inspection, the distribution of the degradation level at time $t+k\tau$ can be predicted, which follows a Gaussian distribution with mean (Gebraeel et al, 2005)

$$\tilde{\mu}\left(t+k\tau\right)=\mu_\alpha+\mu_\theta\left(t+k\tau\right)-\frac{\sigma^2}{2} \tag{15}$$

and variance

$$\tilde{\sigma}^2\left(t+k\tau\right)=\sigma_\alpha^2+\left(t+k\tau\right)^2\sigma_\theta^2+\sigma^2+2\rho\left(t+k\tau\right)\sigma_\alpha\sigma_\theta \tag{16}$$

Since the parameters are updated whenever an inspection is carried out, maintenance decision based on a stationary failure rate may lead to a suboptimal solution. Instead, we focus on a dynamic maintenance policy, which captures the predictive information of the degradation process. We use the “failure probability till next inspection” (FPI) as the indicator to make maintenance decisions. In this way, the maintenance procedure goes as follows: at each inspection, the distribution parameters are updated based on the inspected information, if the FPI of the system exceeds a certain threshold, preventive replacement is performed. Otherwise, the system is left unattained.

Since the maintenance decision is made one inspection after another, we focus on the expected cost till the subsequent inspection. Given that the system functions through the previous *k* inspections, and the estimates of the distribution parameters, $\tilde{\alpha}$ and $\tilde{\theta}$ are available, the FPI of the system can be denoted as

$$P\left(T<(k+1)\tau\mid T>k\tau,\tilde{\alpha},\tilde{\theta}\right)=1-\exp\left(-\int_{k\tau}^{(k+1)\tau}\int_0^\infty h_0(s)\varphi\left(x\left(s\right)\mid\tilde{\alpha},\tilde{\theta}\right)f_{x(s)|\tilde{\alpha},\tilde{\theta}}dx\left(s\right)ds\right) \tag{17}$$

If the system fails before the *k*th inspection, corrective replacement is performed at once, and a renewal cycle follows subsequently. If the FPI is larger than ω_k at the *k*th inspection, i.e.,

$\int_{k\tau}^{(k+1)\tau}\int_{0}^{\infty}h_0\left(s\right)\varphi\left(x\left(s\right)\right)f_{X(s)}\left(x\left(s\right)\right)ds>\log\left(1/\left(1-\omega_k\right)\right)$, preventive replacement is carried out. Within a renewal cycle, if the system does not fail before the *k*th inspection, then at the *k*th inspection, the expected cost within the period $\left(k\tau,\left(k+1\right)\tau\right)$ is

$$\begin{aligned}E\left[C_{pe}(k)\right]&=C_i+C_r\left(1-\exp\left(-\int_{k\tau}^{(k+1)\tau}\int_{0}^{\infty}h_0\left(s\right)\varphi\left(x\left(s\right)\right)f_{X(s)}\left(x\left(s\right)\right)ds\right)\right)\\&+C_pP\left\{\int_{k\tau}^{(k+1)\tau}\int_{0}^{\infty}h_0\left(s\right)\varphi\left(x\left(s\right)\mid\tilde{\alpha},\tilde{\theta}\right)f_{X(s)\mid\tilde{\alpha},\tilde{\theta}}ds>\log\left(1/\left(1-\omega_k\right)\right)\right\}\end{aligned}\tag{18}$$

The expected length of the period $\left(k\tau,\left(k+1\right)\tau\right)$ is

$$\begin{aligned}E\left[T_{pe}\left(k\right)\right]&=\left(\left(\int_{k\tau}^{(k+1)\tau}\left(t-k\tau\right)f_T\left(t\right)dt\right)+\tau\exp\left(-\int_{k\tau}^{(k+1)\tau}\int_{0}^{\infty}h_0(s)\varphi\left(x\left(s\right)\right)f_{X(s)}ds\right)\right)\cdot\\&P\left\{\int_{k\tau}^{(k+1)\tau}\int_{0}^{\infty}h_0\left(s\right)\varphi\left(x\left(s\right)\mid\tilde{\alpha},\tilde{\theta}\right)f_{X(s)\mid\tilde{\alpha},\tilde{\theta}}ds\le\log\left(1/\left(1-\omega_k\right)\right)\right\}\end{aligned}\tag{19}$$

Eq (19) is obtained based on the event that no preventive replacement is carried out at the *k*th inspection. The effectiveness of the maintenance policy is highly dependent on the observation data; a closed-form expression of the long-run cost rate is difficult to obtain. For simplicity, we make the period-by-period maintenance decision by comparing the FIR with the ratio of C_p and C_r. The decision π is then presented as

$$\pi=\begin{cases}1, & \text{if FPI}>C_p/C_r\\0, & \text{otherwise}\end{cases}\tag{20}$$

where 1 denotes preventive replacement and 0 implies doing nothing. Note that for safety-critical systems where a high reliability is required, more constraints are imposed on ω_k. Maintenance actions have to satisfy the reliability constraint while minimizing the maintenance cost.

4 Case study

To illustrate the practical value of the proposed approach, we apply the present model to support the maintenance decision of bridge joints in UK. Bridge joints are used to accommodate the necessary movements of bridge decks, withstand the traffic load, and protect bearing from induced moisture and chloride ion. In this example, Asphalt Plug Joint (APJ) is studied and analyzed in particular. APJ is one of the most common bridge joints due to its waterproof and noise reduction properties. It also exhibits the property of low cost and easiness to install, repair and replace. APJ is constituted by a metal plate, which spans between bridge decks to accommodate longitudinal expansion and contraction (up to 40 mm). The plate is then covered by asphaltic plug making a smooth riding surface and preventing the debris and water. Fig 1 shows the structure of a bridge and the location of APJ.

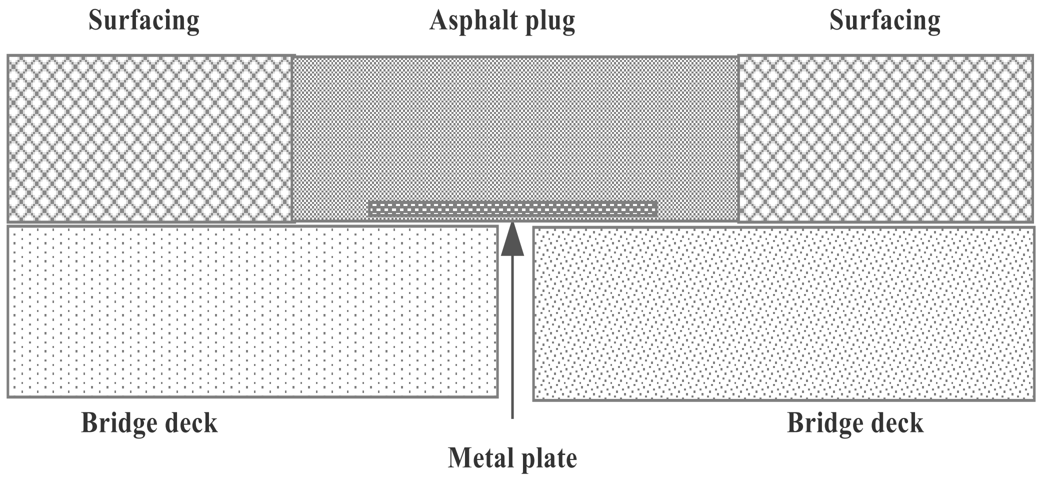

Fig 1 Sketch of asphalt plug joint

APJs have an expected lifetime of between 5 years and 15 years based on the operating environment. According to the local maintenance experts, apart from the regular aging process, the deterioration of APJ is influenced by environmental factors such as accumulated debris, corrosion and traffic load. Additionally, it is also influenced by the induced damaged from other bridge components, such as the water leakage on the underside of the deck, the performance of bearing and superstructure movement. In this example, we mimic the overall impact of the factor as a time-dependent covariate factor. When an APJ is functioning improperly, it will cause problems on the bridge deck and bearing. To mitigate the risk of APJ failure, general inspection is regulated with a two-year interval to assess the condition of APJ joints. The inspection cost is 250£. The replacement cost is 6,341£. The failure cost includes replacement cost, traffic management cost and add-on cost, which is 15,751£ in total. The local maintenance team is keen to find the optimal threshold to replacement APJ so that the operation and maintenance cost can be minimized.

4.1 CBM with known distribution parameters

According to the practical experience from the experts in UK Council, the baseline hazard rate function follows a Weibull distribution, $h_0(t) = bt^{b-1} / a^b$, where the scale parameter is set as $a = 40$ (year) and shape parameter $b = 3$. Weibull distribution was widely used in modeling the crack proposition (Bažant, 2004; Cook & Clarke, 1988). The link function is assumed as exponential, e.g., $\varphi(X_t) = \exp\left(X_t\right)$.

The parameters of the degradation process are set as $\mu_0 = 0.5$, $\mu_1 = 0.2$, $\sigma^2 = 0.01$, $\sigma_0^2 = 0.005$ and $\sigma_1^2 = 0.005$. According to Eq (6), the failure rate function follows a lognormal distribution, which is plotted in Fig 2. As can be observed, the failure rate increases rapidly after 10 years, which implies a high risk of failure and intervention actions should be implemented in time. In addition, we plot the variation of system reliability and pdf in Fig 3.

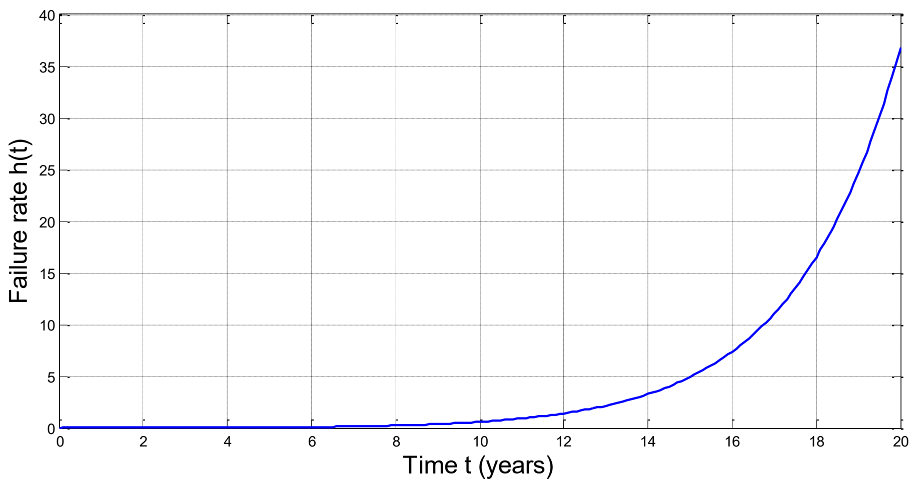

Fig 2 Plot of failure rate

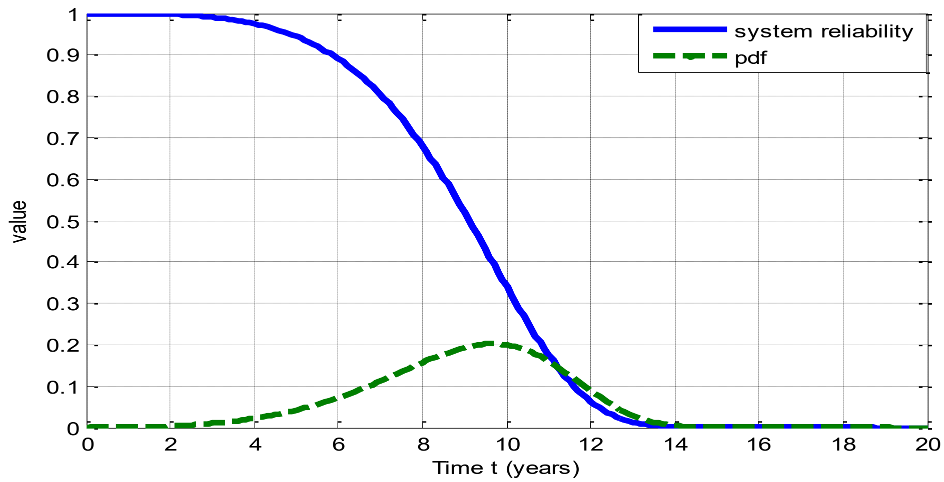

Fig 3 Plot of system reliability & pdf

Since in this model, the distribution parameters are estimated from historical data or expert opinions, estimation errors may affect the performance of the present model. Sensitivity analysis is thus conducted to investigate the influences of parameters on the lifetime distribution of the system. Fig 4 shows how the failure rate and system reliability vary with different u_α. Obviously, a larger u_α leads to a higher failure rate; system reliability function shifts to left when u_α increases. In addition, Fig 5 plots the influences of different u_θ on the failure rate and system reliability. Compared with u_α, different u_θ lead to larger differences in the failure rate and system reliability. The results imply that degradation rate exerts a significant effect on system lifetime distribution and the managers or engineers are suggested to invest more resource to accurately determine the value of degradation rate.

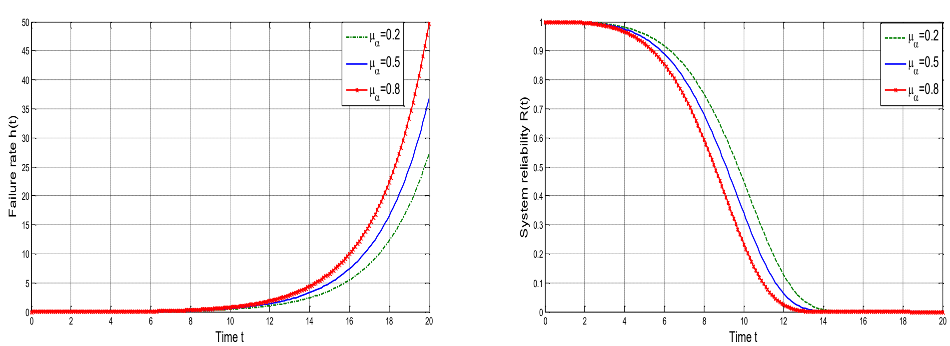

Fig 4 Sensitivity analysis on u_α (a) failure rate (b) system reliability

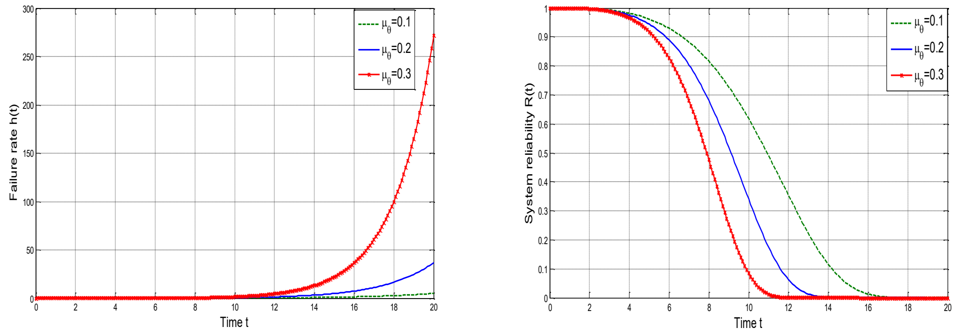

Fig 5 Sensitivity analysis on u_θ (a) failure rate (b) system reliability

In current operation, the APJs are inspected every two years, $\tau = 2$. According to Eq (12), the optimal maintenance policy is obtained as $\zeta = 0.038$, which implies that preventive replacement is carried out when the failure rate at inspection exceeds 0.038. The minimal long-run cost rate is achieved as $\psi^*(\zeta) = 1078$. Fig 6 shows how the long-run cost rate varies with respect to ζ.

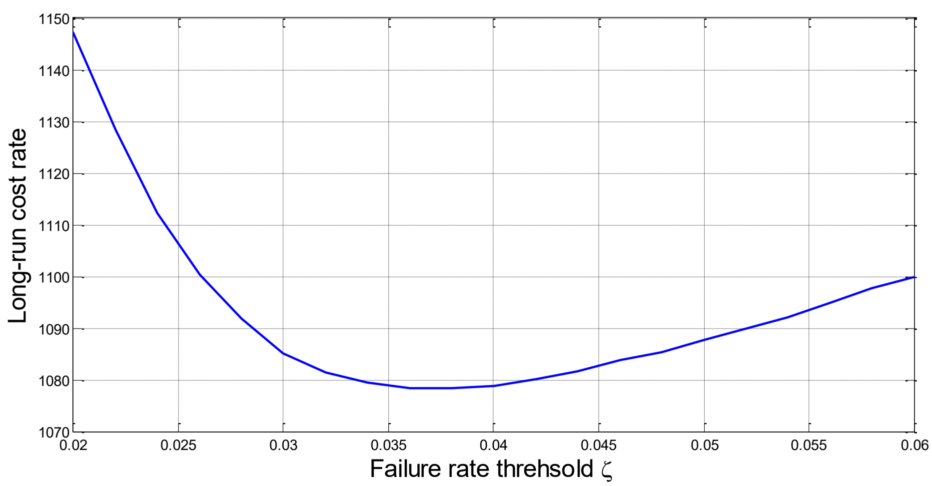

Fig 6 Long-run cost rate with respect to ζ

Based on Eq (1), we can obtain the optimal degradation threshold for preventive replacement with respect to system age, as shown in Fig 7. Obviously, the optimal degradation threshold for preventive replacement decreases with system age. The result presented in Fig 7 is useful in practice. Engineers or managers can simply make maintenance decisions by comparing the observed degradation level with the threshold, which significantly facilitates the implementation of maintenance operations.

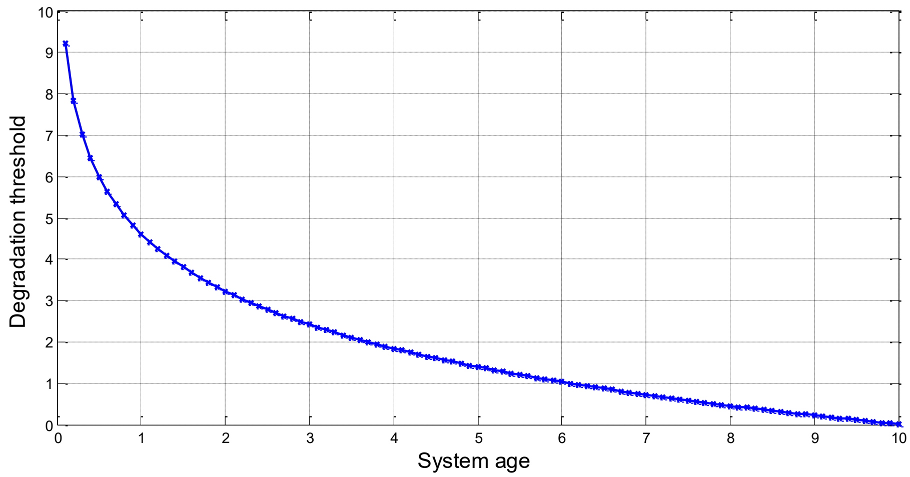

Fig 7 Degradation threshold of preventive replacement with respect to system age

4.2 CBM with unknown distribution parameters

When the parameters of the degradation process are unknown, inspection is performed to observe the system state and update the estimation of the parameters. For the bridge system, we only have eight-year inspection data, where the system is inspected every two years. For illustration purpose, we also simulate the system state data for the later eight years. Four APJs are under investigation. The parameters of the prior distribution are obtained by historical experience and expert judgment, which are given as $\mu_0 = 0.5$, $\mu_1 = 0.2$, $\sigma^2 = 0.01$, $\sigma_0^2 = 0.005$ and $\sigma_1^2 = 0.005$. The parameters are estimated according to Proposition 2 and FPI of the system is calculated based on Eq (17).

Table 1 shows the observed system state along with the estimated parameters and FPI. It can be seen that the observed system state increases with the inspection index. In addition, the FPI increases rapidly with the inspection index. This is due to the fact that the link function $\varphi(\cdot)$ is exponential, which leads to exponential increasing of the failure rate. The estimated parameters $\tilde{u}_\alpha$ and $\tilde{u}_\theta$ are close to the prior, which implies the effectiveness of the prior distribution.

Based on the proposed maintenance policy, the system is replaced preventively when the FRI is larger than $C_p / C_r = 0.4026$. With the calculated FPI, we can conclude that, if the APJs are not failed, they should be preventively replaced at the fourth inspection, so as to achieve maximal economic benefits.

Table 1 Observations, estimated parameters and FPI of the APJs

Item		Real data				Simulated data			
1	Observation	0.86	1.38	1.63	1.98	2.47	2.91	3.40	3.71
	$\tilde{u}_{\alpha}$	0.49	0.485	0.4989	0.5099	0.5016	0.491	0.4743	0.4751
	$\tilde{u}_{\theta}$	0.19	0.215	0.1971	0.189	0.1933	0.1974	0.2027	0.2025
	FPI	0.0024	0.0236	0.0855	0.2175	0.4506	0.7423	0.9476	0.9971
2	Observation	0.99	1.4	1.62	2.41	2.45	3.05	3.14	3.8
	$\tilde{u}_{\alpha}$	0.5086	0.5092	0.5315	0.4952	0.5268	0.5204	0.5546	0.5450
	$\tilde{u}_{\theta}$	0.2271	0.2238	0.1952	0.2219	0.2057	0.2038	0.1975	0.2
	FPI	0.0026	0.0249	0.0874	0.2618	0.4975	0.7931	0.9494	0.9976
3	Observation	0.84	1.32	1.71	2	2.31	2.75	3.41	3.68
	$\tilde{u}_{\alpha}$	0.4871	0.4835	0.4825	0.4955	0.5109	0.5107	0.4775	0.4719
	$\tilde{u}_{\theta}$	0.1843	0.2027	0.204	0.1943	0.1865	0.1865	0.197	0.1985
	FPI	0.0023	0.0226	0.087	0.2203	0.4331	0.706	0.9357	0.9959
4	Observation	0.81	1.22	1.85	2.09	2.21	2.85	3.5	3.85
	$\tilde{u}_{\alpha}$	0.4829	0.4815	0.4552	0.466	0.5063	0.5308	0.4756	0.4479
	$\tilde{u}_{\theta}$	0.1757	0.1823	0.2162	0.2082	0.1875	0.1779	0.1953	0.2027
	FPI	0.0023	0.0211	0.0902	0.2342	0.4348	0.6784	0.9315	0.9967

5 Conclusions

This paper investigates the condition-based maintenance policy for systems with aging and cumulative damage. The joint effect of aging and cumulative damage is described by proportional hazards model. Maintenance models are developed with consideration of known and unknown distribution parameters respectively. The results in this paper show that the degradation rate exerts a significant impact on system lifetime distribution. Engineers or managers are suggested to pay more attention to improving the accuracy of the degradation rate estimation. The proposed condition based model can be widely applied for infrastructure systems which are subject to cumulative damage and exhibit a long life cycle.

Extensions of this research can be conducted by generalizing the one-dimensional cumulative damage into multi-dimensional. Then multiple sensors should be equipped to inspect the damage (degradation) indicators. Parameter estimation of the distribution parameters could be complicated as interactions may exist among the multi-dimensional cumulative damages. The difficulty of extending to multi-dimensional degradation processes lies in the computational intensity. Approximation methods are thus appreciated in such cases. In addition, the form of link function can be explored with a variety of candidates. An exponential function is used for simplicity in this paper; in reality, various link functions can be tested if the associated data are available.

Acknowledgement

The work described in this paper was partially supported by a theme-based project grant (T32-101/15-R) of University Grants Council, and a Key Project (71532008) supported by National Natural Science Foundation of China.

References

Banjevic, D., Jardine, A. K. S., Makis, V., & Ennis, M. (2001). A control-limit policy and software for condition-based maintenance optimization. *INFOR: Information Systems and Operational Research*, 39(1), 32-50.

Bažant, Z. P. (2004). Probability distribution of energetic-statistical size effect in quasibrittle fracture. *Probabilistic engineering mechanics*, 19(4), 307-319.

Caballé, N. C., Castro, I. T., Pérez, C. J., & Lanza-Gutiérrez, J. M. (2015). A condition-based maintenance of a dependent degradation-threshold-shock model in a system with multiple degradation processes. *Reliability Engineering & System Safety*, 134, 98-109.

Chen, N., Ye, Z. S., Xiang, Y., & Zhang, L. (2015). Condition-based maintenance using the inverse Gaussian degradation model. *European Journal of Operational Research*, 243(1), 190-199.

Cook, R. F., & Clarke, D. R. (1988). Fracture stability, R-curves and strength variability. *Acta metallurgica*, 36(3), 555-562.

Elwany, A. H., Gebraeel, N. Z., & Maillart, L. M. (2011). Structured replacement policies for components with complex degradation processes and dedicated sensors. *Operations research*, 59(3), 684-695.

Gebraeel, N. Z., Lawley, M. A., Li, R., & Ryan, J. K. (2005). Residual-life distributions from component degradation signals: A Bayesian approach. *IIE Transactions*, 37(6), 543-557.

Ghasemi, A., Yacout, S., & Ouali, M. S. (2007). Optimal condition based maintenance with imperfect information and the proportional hazards model. *International Journal of Production Research*, 45(4), 989-1012.

Haghighi, F., & Bae, S. J. (2015). Reliability Estimation from Linear Degradation and Failure Time Data With Competing Risks Under a Step-Stress Accelerated Degradation Test. *IEEE Transactions on Reliability*, 64(3), 960-971.

Huynh, K. T., Barros, A., Berenguer, C., & Castro, I. T. (2011). A periodic inspection and replacement policy for systems subject to competing failure modes due to degradation and traumatic events. *Reliability Engineering & System Safety*, 96(4), 497-508.

Jardine, A. K., & Tsang, A. H. (2013). Maintenance, replacement, and reliability: theory and applications. CRC press.

Lam, J. Y. J., & Banjevic, D. (2015). A myopic policy for optimal inspection scheduling for condition based maintenance. *Reliability Engineering & System Safety*, 144, 1-11.

Lehmann, A. (2009). Joint modeling of degradation and failure time data. J*ournal of Statistical Planning and Inference*, 139(5), 1693-1706.

Li, Y. F., Zio, E., & Lin, Y. H. (2012). A multistate physics model of component degradation based on stochastic petri nets and simulation. *IEEE Transactions on Reliability*, 61(4), 921-931.

Lin, Y. H., Li, Y. F., & Zio, E. (2016). Component Importance Measures for Components With Multiple Dependent Competing Degradation Processes and Subject to Maintenance. *IEEE Transactions on Reliability*, 65(2), 547-557.

Lin, D. Y., & Wei, L. J. (1989). The robust inference for the Cox proportional hazards model. *Journal of the American statistical Association*, 84(408), 1074-1078.

Liu, B., Xie, M., & Kuo, W. (2016a). Reliability modeling and preventive maintenance of load-sharing systems with degrading components. *IIE Transactions*, 48(8), 699-709.

Liu, B., Xie, M., Xu, Z., & Kuo, W. (2016b). An imperfect maintenance policy for mission-oriented systems subject to degradation and external shocks. *Computers & Industrial Engineering*, 102, 21-32.

Liu, B., Xu, Z., Xie, M., & Kuo, W. (2014). A value-based preventive maintenance policy for multi-component system with continuously degrading components. *Reliability Engineering & System Safety*, 132, 83-89.

Lu, C. J., & Meeker, W. O. (1993). Using degradation measures to estimate a time-to-failure distribution. *Technometrics*, 35(2), 161-174.

Maillart, L. M. (2006). Maintenance policies for systems with condition monitoring and obvious failures. *IIE Transactions*, 38(6), 463-475.

Mo, H., & Xie, M. (2016). A Dynamic Approach to Performance Analysis and Reliability Improvement of Control Systems With Degraded Components. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 46(10), 1404-1414.

Peng, H., & van Houtum, G. J. (2016). Joint optimization of condition-based maintenance and production lot-sizing. *European Journal of Operational Research*, 253(1), 94-107.

Pham, H. T., Yang, B. S., & Nguyen, T. T. (2012). Machine performance degradation assessment and remaining useful life prediction using proportional hazard model and support vector machine. *Mechanical Systems and Signal Processing*, 32, 320-330.

Provan, J. W. (1987). Probabilistic approaches to the material-related reliability of fracture-sensitive structures. *In Probabilistic fracture mechanics and reliability* (pp. 1-45). Springer Netherlands.

Srinivasan, R., & Parlikad, A. K. (2013). Value of condition monitoring in infrastructure maintenance. *Computers & Industrial Engineering*, 66(2), 233-241.

Tian, Z., & Liao, H. (2011). Condition based maintenance optimization for multi-component systems using proportional hazards model. *Reliability Engineering & System Safety*, 96(5), 581-589.

Wardhana, K., & Hadipriono, F. C. (2003). Analysis of recent bridge failures in the United States. *Journal of Performance of Constructed Facilities*, 17(3), 144-150.

Wu, S., Chen, Y., Wu, Q., & Wang, Z. (2016). Linking component importance to optimisation of preventive maintenance policy. *Reliability Engineering & System Safety*, 146, 26-32.

Wu, X., & Ryan, S. M. (2010). Value of condition monitoring for optimal replacement in the proportional hazards model with continuous degradation. *IIE Transactions*, 42(8), 553-563.

Wu, X., & Ryan, S. M. (2011). Optimal replacement in the proportional hazards model with semi-markovian covariate process and continuous monitoring. *IEEE Transactions on Reliability*, 60(3), 580-589.

Ye, Z. S., Chen, N., & Shen, Y. (2015). A new class of Wiener process models for degradation analysis. *Reliability Engineering & System Safety*, 139, 58-67.

Ye, Z. S., & Xie, M. (2015). Stochastic modelling and analysis of degradation for highly reliable products. *Applied Stochastic Models in Business and Industry*, 31(1), 16-32.

Zhang, M., Ye, Z., & Xie, M. (2014). A condition-based maintenance strategy for heterogeneous populations. *Computers & Industrial Engineering*, 77, 103-114.

Appendix

1. Proof of Proposition 1

Denote $w(t)$ as the cumulative hazard rate function of the system, i.e.,

$$w(t;x_s)=\int_0^t h_0(s)\varphi(x_s)ds$$

In the following, for notational simplicity, we will suppress x_s of the $w(t;x_s)$. the derivative of $w(t)$ with respect to *t* can be obtained as

$$\frac{dw(t)}{dt}=h_0(t)\varphi(x_t)>0$$

and

$$\frac{d^2w(t)}{d^2t}=\frac{dh_0(t)}{dt}\varphi(x_t)+h_0(t)\frac{d\varphi(x_t)}{dx_t}\square\frac{dx_t}{dt}>0$$

Here we unofficially use dx_t/dt to denote the derivative of x_t. The inequality holds since $h_0(t)$ and $\varphi(x_t)$ are is non-decreasing functions in *t* and x_t, and X_t is stochastically increasing in *t*. we can conclude that $w(t)$ is a convex function in *t*. Based on the Jensen's inequality, we have

$$w(t_1)+w(t_3)>2w(t_2)\text{, for any } 0<t_1<t_2<t_3,$$

On the other hand, the cumulative hazard rate between two consecutive inspections can be rewritten as

$$\int_{k\tau}^{(k+1)\tau} h_0(s)\varphi(x_s)ds = \int_0^{(k+1)\tau} h_0(s)\varphi(x_s)ds - \int_0^{k\tau} h_0(s)\varphi(x_s)ds = w((k+1)\tau) - w(k\tau)$$

Readily we can obtain

$$\int_{k\tau}^{(k+1)\tau} h_0(s)\varphi(x_s)ds = w((k+1)\tau) - w(k\tau) > w(k\tau) - w((k-1)\tau) = \int_{(k-1)\tau}^{k\tau} h_0(s)\varphi(x_s)ds$$

On the other hand, Jensen's inequality states that $E\left[g(x)\right] > g\left(E[x]\right)$, for any convex function $g(x)$, which completes the proof.□

2. Derivation of Equation (10) and Eq (11)

Denote U_a as the event that given no failure occurs, the system is preventively replaced at the *k*th inspection, $U_a = 1\left\{h(t) \le \zeta, t \in \left[0,(k-1)\tau\right] \cap h(k\tau) > \zeta\right\}$, and V_a as the event that no failure occurs before $k\tau$, $V_a = 1\{\text{no failure occurs before } k\tau\}$. We can have

$$\begin{aligned}
&P_a\left(k\right) = P\left(V_a \cap U_a\right) = P\left(V_a \mid U_a\right)\cdot P\left(U_a\right) \\
&= P\left(V_a \mid U_a\right)\cdot P\left(h(t) \le \zeta, t \in \left[0,(k-1)\tau\right] \cap h(k\tau) > \zeta\right) \\
&= \exp\left(-\int_0^{(k-1)\tau} h_0(t)\int_0^{\ln(\zeta/h_0(t))} \varphi(x)f(x)dxdt - \int_{(k-1)\tau}^{k\tau} h_0(t)\int_0^{\infty}\varphi(x)f(x)dxdt\right)\cdot \\
&\left(P\left(h(k\tau) > \zeta\right) - P\left(h((k-1)\tau) > \zeta\right)\right) \\
&= \exp\left(-\int_0^{(k-1)\tau} h_0(t)\int_0^{\ln(\zeta/h_0(t))} \varphi(x)f(x)dxdt - \int_{(k-1)\tau}^{k\tau} h_0(t)\int_0^{\infty}\varphi(x)f(x)dxdt\right)\cdot \\
&\left(\Phi\left(\frac{\left(\mu_0 + \mu_1 k\tau - \sigma^2/2 - \varphi^{-1}\left(\zeta/h_0(k\tau)\right)\right)}{\sqrt{\sigma_0^2 + \sigma_1^2\left(k\tau\right)^2 + \sigma^2}}\right) - \Phi\left(\frac{\left(\mu_0 + \mu_1\left(k-1\right)\tau - \sigma^2/2 - \varphi^{-1}\left(\zeta/h_0(\left(k-1\right)\tau)\right)\right)}{\sqrt{\sigma_0^2 + \sigma_1^2\left(\left(k-1\right)\tau\right)^2 + \sigma^2}}\right)\right)
\end{aligned}$$

Denote U_b as the event that no preventive replacement is carried out before the *k*th inspection, $U_b = 1\left\{h(t) \le \zeta, t \in \left[0,(k-1)\tau\right]\right\}$, and V_b as the event that given no preventive replacement, failure occurs within the interval $\left(\left(k-1\right)\tau, k\tau\right)$, $V_b = 1\left\{\text{failure occurs within } \left(\left(k-1\right)\tau, k\tau\right)\right\}$. The probability that failure occurs in the period $\left(\left(k-1\right)\tau, k\tau\right)$ is given as

$$
\begin{aligned}
&P_b(k)=P\left(V_b \cap U_b\right)=P\left(V_b \mid U_b\right)\cdot P\left(U_b\right)\\
&=P\left(V_b \mid U_b\right)\cdot P\left(h(t)\le \zeta, t\in\left[0,(k-1)\tau\right]\right)\\
&=\begin{pmatrix}\exp\left(-\int_0^{(k-1)\tau} h_0(t)\int_0^{\ln\left(\zeta/h_0(t)\right)}\varphi(x)f(x)dxdt\right)\\ -\exp\left(-\int_0^{(k-1)\tau} h_0(t)\int_0^{\ln\left(\zeta/h_0(t)\right)}\varphi(x)f(x)dxdt-\int_{(k-1)\tau}^{k\tau} h_0(t)\int_0^{\infty}\varphi(x)f(x)dxdt\right)\end{pmatrix}\cdot\\
&\Phi\left(\frac{\left(\mu_0+\mu_1 t-\sigma^2/2-\varphi^{-1}\left(\zeta/h_0(t)\right)\right)}{\sqrt{\sigma_0^2+\sigma_1^2t^2+\sigma^2}}\right)\cdot\\
&=\left(1-\exp\left(-\int_{(k-1)\tau}^{k\tau} h_0(t)\int_0^{\infty}\varphi(x)f(x)dxdt\right)\right)\cdot\exp\left(-\int_0^{(k-1)\tau} h_0(t)\int_0^{\ln\left(\zeta/h_0(t)\right)}\varphi(x)f(x)dxdt\right)\cdot\\
&\Phi\left(\frac{\left(\mu_0+\mu_1 t-\sigma^2/2-\varphi^{-1}\left(\zeta/h_0(t)\right)\right)}{\sqrt{\sigma_0^2+\sigma_1^2t^2+\sigma^2}}\right)
\end{aligned}
$$

3. Proof of Proposition 2

Let

$$
\rho_2(\tau^2)=\rho^2=\frac{\left(\sum_{i=1}^{k} i\tau\right)^2\sigma_0^2\sigma_1^2}{\left(k\sigma_0^2+\sigma^2\right)\left(\sum_{i=1}^{k}\left(i\tau\right)^2\sigma_1^2+\sigma^2\right)}
$$

Then the derivative of ρ_2 with respect to τ^2 is given as

$$
\begin{aligned}
\rho_2' &=\frac{\left(\sum_{i=1}^{k} i\right)^2\sigma_0^2\sigma_1^2\left(k\sigma_0^2+\sigma^2\right)\left(\sum_{i=1}^{k}\left(i\tau\right)^2\sigma_1^2+\sigma^2\right)-\left(\sum_{i=1}^{k} i\tau\right)^2\sigma_0^2\sigma_1^2\left(k\sigma_0^2+\sigma^2\right)\left(\sum_{i=1}^{k}\left(i\right)^2\sigma_1^2\right)}{\left(k\sigma_0^2+\sigma^2\right)^2\left(\sum_{i=1}^{k}\left(i\tau\right)^2\sigma_1^2+\sigma^2\right)^2}\\
&=\frac{\left(\sum_{i=1}^{k} i\right)^2\sigma_0^2\sigma_1^2\sigma^2\left(k\sigma_0^2+\sigma^2\right)}{\left(k\sigma_0^2+\sigma^2\right)^2\left(\sum_{i=1}^{k}\left(i\tau\right)^2\sigma_1^2+\sigma^2\right)^2}>0
\end{aligned}
$$

Since $\rho=-\sqrt{\rho_2}$, it can be concluded that ρ decreases with τ .

On the other hand, σ_α^2 can be rewritten as

$$\sigma_\alpha^2 = \sigma^2 \sigma_0^2 \frac{1/\left(k\sigma_0^2 + \sigma^2\right)}{1-\rho^2}$$

which implied that σ_α^2 increases with τ .

In addition, σ_θ^2 can be rewritten as

$$\sigma_\theta^2 = \sigma^2 \sigma_1^2 \frac{k\sigma_0^2 + \sigma^2}{\left(\left(k\sigma_0^2 + \sigma^2\right)\sigma_1^2 \sum_{i=1}^{k}(i)^2 - \left(\sum_{i=1}^{k} i\right)^2 \sigma_0^2\sigma_1^2\right)\tau^2 + \left(k\sigma_0^2 + \sigma^2\right)\sigma^2}.$$

Since $k\sum_{i=1}^{k}(i)^2 > \left(\sum_{i=1}^{k} i\right)^2$ for any *k*, we can easily conclude that σ_θ^2 decreases with τ .□

4. Proof of Corollary 1

Denote $\tau k = M$, where M is a constant with bounds, $M < \infty$. Let $\sum_{i=1}^{k}(i\tau)^2 = \frac{\tau^2 k(k+1)(2k+1)}{6}$ and $\sum_{i=1}^{k} i\tau = \frac{\tau k(k+1)}{2}$, we can have

$$\frac{1}{\sigma_\alpha^2} = \frac{1}{\sigma^2\sigma_0^2}\left[\left(k\sigma_0^2 + \sigma^2\right) - \frac{3k^2(k+1)^2\sigma_0^2\sigma_1^2}{2k(k+1)(2k+1)\sigma_1^2 + 12\sigma^2\tau^{-2}}\right]$$

$$\frac{1}{\sigma_\theta^2} = \frac{1}{\sigma^2\sigma_1^2}\left\{\frac{\tau^2\sigma_1^2 k(k+1)}{2}\left[\frac{2k+1}{3} - \frac{k(k+1)\sigma_0^2}{2\left(k\sigma_0^2 + \sigma^2\right)}\right] + \sigma^2\right\}$$

When $\tau \to 0$ and $k \to \infty$, we have $\frac{1}{\sigma_\alpha^2} \to \infty$, which implies $\lim_{\tau \to 0} \sigma_\alpha^2 = 0$.

Similarly, we can obtain $\lim_{\tau \to 0} \sigma_\theta^2 = 0$, which completes the proof.□