Comment on "Bejan's flow visualization of buoyancy-driven flow of a hydromagnetic Casson fluid from an isothermal wavy surface" Physics of Fluids, 33(9), p.093113 (2021]

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This recent paper [1] claims to study the "thermofluidic transport of non-Newtonian Casson fluid from a wavy surface". In fact, the model studied is neither due to Casson [2,3] nor is it non-Newtonian: it is simply a Newtonian fluid with a rescaled viscosity.

The Casson model [2] is a simple model for a yield stress fluid and is classically defined (see e.g the standard "DPL" textbook (p243) of Bird et al [3]) in a shear flow as:

$$\sqrt{\tau_{xy}} = \sqrt{\tau_0} + \sqrt{\mu_0}\sqrt{du/dy} \quad \text{for} \quad \tau_{xy} > \tau_0 \qquad (1)$$

$$du/dy = 0 \quad \text{for} \quad \tau_{xy} < \tau_0$$

where τ_{xy} is the shear stress, τ_0 is the yield stress, μ_0 is a viscosity and $du/dy \geq 0$, is the shear rate. The model states that above the yield stress there is deformation and below it there is no deformation (and the stress is indeterminate). The similarity with the very familiar Bingham model for viscoplastic (yield stress) fluids is clear.

In marked contrast, Kumar and Mondal [1] define their "Casson" fluid (their Eqn 2) as:

$$\tau_{ij} = 2\left(\mu_B + \tau_y/\sqrt{2\pi}\right)d_{ij}, \quad \text{for} \quad \delta > \delta_C, \qquad (2)$$
$$\tau_{ij} = 2\left(\mu_B + \tau_y/\sqrt{2\pi_c}\right)d_{ij}, \quad \text{for} \quad \delta < \delta_C$$

where d_{ij} and τ_{ij} signify the "component of the rate of deformation tensor" and the "shear stress tensor", respectively, τ_y is the yield "strength", $\delta = d_{ij}d_{ij}$ represents the "component of deformation product", δ_C "symbolizes a critical value of this material" and μ_B is the plastic viscosity. The symbols π and π_c are nowhere defined in the manuscript (neither is the rate of deformation tensor but at least this is a well known quantity). From the references in [1], where the same model is also used we may deduce that $\pi = d_{ij}d_{ij}$, with π_c the critical value of π that is needed to make Eqn 2 continuous, i.e. $\delta = \pi$.

The above model is evidently a bi-viscous Bingham fluid. This is one of many forms of regularization which are routinely applied to the standard Bingham fluid model; see [4]. The Bingham fluid is recovered by taking $\delta_C \to 0$. Eqn 2 is not a Casson fluid.

A far bigger concern than the name of this model is that when this "Casson fluid" finds its way into the governing equations, namely Eqns 4 & 5 from Kumar and Mondal [1], the stress associated with this fluid is treated as if the fluid simply has a constant viscosity, e.g. we have:

$$\ldots. = -\frac{\partial p}{\partial x} - \mu\left[1 + \frac{1}{\gamma}\right]\nabla^2 u \ldots. \qquad (3)$$

in the x-momentum equation, where $\gamma = \mu_B\sqrt{2\delta_C}/\tau_y$. The second term on the right hand side of Eqn 3 represents the divergence of the stress (viz $\boldsymbol{\nabla}.\boldsymbol{\tau}$) which only simplifies to the above expression for the case of a fluid of constant viscosity, i.e. you cannot take the viscosity outside of the divergence operator. In other words the above expression is only valid for the low shear range $\delta < \delta_C$. We conclude that the Equations presented in Kumar and Mondal [1] simply represent a Newtonian fluid with a rescaled viscosity. The results represent neither a true Casson fluid nor any real non-Newtonian effects.

Our concern is not only with this paper [1]. Approximately 100 others in various scientific journals also mistakenly use the same "Casson fluid" model, citing each other in a spiralling frenzy (this is the only example in *Physics of Fluids* of which we are aware). In our view there are three issues here. First, misnaming detracts from the many papers that study the Casson fluid model, which is a legitimate constitutive model of wide application. Second, we learn the product rule of differentiation in our first calculus classes. The above mathematical error is obvious and should not have been repeated unquestioned in this and other similar papers. Thirdly, the assertion that the model is non-Newtonian suggests a departure from linearity; a degree of novelty and difficulty in dealing with the constitutive law and any subsequent analysis. This novelty is simply not present in these "rogue Casson" papers, which mostly repeat analysis that has been already performed for Newtonian fluids.

We have chosen not to cite the other similarly erroneous rogue Casson papers, as to do so would perpetuate the above issues. The origin of the above mistakes appears to be [5], or similar papers from these same authors in the same time period. They erroneously cite [6] as reference for the Casson model. However, in [6] Eqn (2) is properly and clearly identified as a bi-viscous (Bingham) model and is only compared with results from a Casson fluid. Also in [5] we find the above-mentioned missuse of the product rule where only the low shear branch of Eqn (2) is used, i.e. the shear stresses are treated as in Eqn (3) leading to a Newtonian fluid. Slightly worrying is that the subject of [5] is boundary layer flow, where usually the high shear regions are found near the wall and low shear away from the wall. Thus, the low shear branch of Eqn (2) is physically incorrect to use at all in this context.

Lastly, moving away from the rogue Casson papers, let us comment that a number of authors have correctly used and analysed the bi-viscous Bingham. The first use for unsteady shear flows that we know of was [7], as a numerical simplification for 1D flows. The model is still used as a simple regularization in many CFD codes. In [8] the authors use the model analytically to solve different Stokes problems. The same approach has also been frequently used to approximate thin-film and lubrication flows with Bingham fluids. In analytical studies the bi-viscous Bingham model is typically used to derive an analytical solution, from which the Bingham limit of the solution is resolved. This is however a singular perturbation method that does not always work. In fact the bi-viscous approach is rarely necessary: boundary layer flows [9], Stokes-type problems [10] and start-up/stopping flows [11] can all be dealt with using the exact Bingham model, as can many others. In the same vein, many of these problems could be dealt with using the true Casson fluid model.

References

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