Optimal Deterministic Clock Auctions and Beyond

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Abstract

We design and analyze deterministic and randomized clock auctions for single-parameter domains with downward-closed feasibility constraints, aiming to maximize the social welfare. Clock auctions have been shown to satisfy a list of compelling incentive properties making them a very practical solution for real-world applications, partly because they require very little reasoning from the participating bidders. However, the first results regarding the worst-case performance of deterministic clock auctions from a welfare maximization perspective indicated that they face obstacles even for a seemingly very simple family of instances, leading to a logarithmic inapproximability result; this inapproximability result is information-theoretic and holds even if the auction has unbounded computational power. In this paper we propose a deterministic clock auction that achieves a logarithmic approximation for *any* downward-closed set system, using black box access to a solver for the underlying optimization problem. This proves that our clock auction is optimal and that the aforementioned family of instances exactly captures the information limitations of deterministic clock auctions. We then move beyond deterministic auctions and design randomized clock auctions that achieve improved approximation guarantees for a generalization of this family of instances, suggesting that the earlier indications regarding the performance of clock auctions may have been overly pessimistic.

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1 Introduction

In this paper, we revisit the well-studied family of single-parameter mechanism design problems where a service provider needs to decide which subset of n customers to serve. Each customer i has a value v_i for receiving the service but, due to scarcity, it is infeasible to serve all the customers. Depending on the type of service, these limitations are captured by a set $\mathcal{F}$ which contains all the subsets of customers that can be simultaneously served. For example, if the service provider is an airline company and the customers wish to board one of its flights, the scarcity is due to the limited number of seats in the plane, and $\mathcal{F}$ contains any subset of customers that can fit in the plane. Another well-studied example arises in combinatorial auctions with single-minded bidders, where there is a set M of m heterogeneous and indivisible items and each customer i needs a specific bundle $B_i \subseteq M$ of

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these items. In this case, two customers i and j can be served simultaneously only if the bundles of items that they each need are disjoint, i.e., $B_i \cap B_j = \emptyset$.

Our goal throughout the paper is to identify and serve a feasible subset $A \in \mathcal{F}$ of customers, aiming to maximize the social welfare, i.e., the total value of the served customers, $\sum_{i \in A} v_i$. Depending on the structure of the feasibility constraint $\mathcal{F}$, this setting captures many classic optimization problems, like the knapsack problem or the maximum weight independent set problem. However, in addition to the computational obstacles that arise in solving these classic problems, the service provider is also facing an information limitation: the value v_i of each customer i is private and unknown to the provider. The standard solution to this problem is to design an auction aiming to elicit the values of the customers and identify a subset with high social welfare.

Most of the proposed auctions for solving instances of these single-parameter mechanism design problems take the form of *sealed-bid auctions*: each bidder is asked to truthfully report their private value to the auctioneer, who then uses this information to determine the set of winners, who are served, along with the price that each of them should pay for the service. These auctions need to carefully determine the outcome so that they not only optimize the social welfare, but also incentivize true reports from the bidders. However, the classic VCG sealed-bid auction [47, 20, 34] cannot be used when the underlying problem is computationally intractable, and sealed-bid auctions, in general, can often be rather impractical. For example, they require that the bidders reveal all their private information, suffering a privacy cost, while there is very limited transparency or oversight regarding how the auctioneer actually uses this information to derive the final outcome. Also, even if, in theory, the bidders cannot benefit by misreporting their values, in practice bidders have been observed to misreport anyway, possibly because they cannot easily verify these incentive guarantees [38].

To address these shortcomings, Milgrom and Segal [43, 44] recently proposed the class of (deferred-acceptance) *clock auctions* as a very practical alternative to sealed-bid auctions. Rather than asking the bidders to trust the auctioneer with all their private information, clock auctions run over a sequence of rounds and offer a (personalized) price to each bidder that weakly increases over time. In each round, the bidder can either accept the latest price offered to her and remain in the auction, or reject it and permanently drop out of the auction. The bidders that remain active when the auction terminates are served, at the cost of the most recent price that they accepted. Every clock auction is guaranteed to satisfy a unique combination of very appealing properties, like *obvious strategyproofness*, *unconditional winner privacy*, and *credibility*. We defer a detailed discussion regarding the important benefits of clock auctions to Section 1.3.

The multiple benefits of clock auctions over sealed-bid ones provided renewed motivation for computer scientists to revisit classic single-parameter mechanism design problems and evaluate the performance of clock auctions from a worst-case approximation standpoint. Note that these benefits come at the cost of additional information limitations: unlike sealed-bid auctions, clock auctions can learn about the values of the bidders only by offering them a carefully chosen sequence of prices, and each time a price is increased there is a risk that the corresponding bidder will drop out. In fact, initial results on the performance of clock auctions by Dütting et al. [24] showed that no deterministic clock auction, *even with unbounded computational power*, can achieve a $O(\log^{1-\epsilon} n)$ approximation for any constant $\epsilon > 0$ for a seemingly very simple family of instances where $\mathcal{F}$ contains only two disjoint maximal sets. This suggested that things could be even worse for more complicated feasibility constraints, like combinatorial auctions with single-minded bidders, for which no known deterministic clock auction came close to this logarithmic approximation guarantee.

1.1 Our Results

Our main result in this paper is a deterministic clock auction that guarantees an approximation of $4 \log n$ not only for combinatorial auctions with single-minded bidders but for the much wider family of instances induced by *downward-closed feasibility constraints* (that is, if $F \in \mathcal{F}$ then any subset of F is also in $\mathcal{F}$). This shows that the lower bound construction of Dütting et al. [24] exactly captures the information limitations of deterministic clock auctions, and it paints a much more optimistic picture regarding the performance of clock auctions. Motivated by this positive result, we then move on to also consider the performance of *randomized* clock auctions and prove that combining our deterministic auction with one based on randomized sampling achieves better than logarithmic approximation for an interesting family of instances that generalizes the ones used by Dütting et al. [24] to prove the logarithmic lower bound.

Our deterministic *water filling clock auction* (WFCA) uses black-box access to an algorithm that returns a feasible subset of active bidders (approximately) maximizing the revenue, given their current clock prices. In each round, the WFCA uses this black box to decide which subset of the active bidders' prices it should raise, until the remaining active bidders are all feasible. This auction is inspired by the combinatorial auctions proposed in the elegant work of Babaioff et al. [10], which were shown to achieve a $O(\log \overline{v})$ approximation for single-minded bidders and $O(\log^2 \overline{v})$ for single-value multi-minded bidders[1], where $\overline{v} = \max_{i,j}\{v_i/v_j\}$ is the ratio between the highest and the lowest bidder value. On the negative, they also provide a construction (see Proposition C.1) showing that even for the case of single-minded bidders, the approximation factor of their auction is $\Omega(n)$ and $\Omega(m)$, where m is the number of items. The WFCA outperforms these results by simultaneously achieving an approximation of $O(\log n)$ and $O(\log \overline{v})$ for *any* downward-closed feasibility constraint (Theorem 3 and Theorem 7, respectively), as well as $O(\log m)$ for all combinatorial auctions with m items and single-value multi-minded bidders (Corollary 4). Since no deterministic clock auction can achieve a bound of $O(\log^{1-\epsilon} n)$, $O(\log^{1-\epsilon} \overline{v})$, or $O(\log^{1-\epsilon} m)$ for a constant $\epsilon > 0$ [24], our auction is essentially optimal with respect to all of these parameters.

In light of this positive result, which resolves the information limitations of deterministic clock auctions from a welfare maximization standpoint, a fundamental open question is whether better approximation guarantees can be achieved using randomization. Note that, unlike the case for deterministic clock auctions, the best known inapproximability results for *randomized* clock auctions are small constants (e.g., see Section 4.3). In this paper, we make a first step in that direction with a positive result that combines the deterministic WFCA with a clock auction based on randomized sampling to achieve a $O(\sqrt{\log n})$ approximation for a family of instances that contains the ones used for the logarithmic lower bound construction in [24]. Specifically, the approximation guarantee holds for any set system whose maximal feasible sets are disjoint, while the worst-case instances for deterministic clock auctions comprise just two disjoint feasible sets, one of which is a singleton. In fact, this class of instances is also known to pose important obstacles in both prophet inequality and secretary problems [45]. Thus, our positive result provides an important separation between randomized and deterministic clock auctions, along with a natural approach for leveraging randomness in clock auctions. We conclude by generalizing this result to show that our randomized auction achieves a $O(\min\{\log n, \sqrt{\log k}\})$ approximation for any downward-closed feasibility constraint $\mathcal{F}$, where k is the number of maximal feasible sets in $\mathcal{F}$.

[1] In a combinatorial auction with single-value multi-minded bidders, each bidder i is interested in multiple different bundles of items, rather than just one, and she has the same value v_i for receiving any one of them.

1.2 Related Work

After Milgrom and Segal [44] introduced the class of (deferred-acceptance) clock auctions and noted their many desirable practical properties, subsequent work studied their performance in a variety of settings. Most relevant to our paper is the work of Dütting et al. [24] who initiated the analysis of clock auctions using worst-case analysis and aiming to approximate the optimal social welfare. In that work, the authors provided a computationally efficient clock auction achieving a $O(\log m)$-approximation for a knapsack auction setting (where m is the number of items) as well as a $O(\sqrt{m \log m})$-approximate auction for a combinatorial auction setting with single-minded bidders.

Clock auctions are contained in the more general class of obviously strategyproof (OSP) mechanisms proposed by Li [41] (discussed in greater detail in Section 1.3). This more general class of mechanisms has also been studied in the context of combinatorial auctions with single-minded bidders. Even for this more general class of auctions the family of instances with feasibility constraints consisting of disjoint maximal sets proves a challenging obstacle. Ferraioli et al. [30] demonstrated that no deterministic OSP mechanism can achieve a $O(\log^{0.5-\epsilon} n)$-approximation for constant $\epsilon > 0$ in such an instance. Moreover, De Keijzer et al. [21] demonstrate that no OSP mechanism can obtain better than a 2-approximation to the welfare in such an instance even when bidders can have only two possible values.

While we focus on "forward auctions" where the auctioneer is selling a service to a set of bidders with independent values, deferred acceptance clock auctions have recently also been studied in other auction contexts. For example, their performance has been examined in procurement auction settings [27, 39, 37, 15, 13], double auction settings [25, 42], multilateral settings [16], as well as single-item auction settings where bidders have interdependent values [32]. Clock auctions have also been generalized to apply beyond single-parameter binary allocation problems to accommodate multiple levels of service while retaining the same desirable properties [31].

In this work we also study randomized clock auctions and examine how randomization can be used to overcome the challenging logarithmic lower bound. De Keijzer et al. [21] considered the design of universally OSP randomized mechanisms in the context of combinatorial auctions with single-minded bidders and describe a mechanism which achieves a $\min\left\{|D|, \frac{V_{|D|}}{V_1} + 1\right\}$-approximation where D is the known domain of possible values of all bidders and $V_{|D|}$ and V_1 are the largest and smallest values in this domain, respectively. The way in which we utilize randomization differs from [21] and instead takes a "random sampling" approach which has seen application in many problems in mechanism design, such as revenue maximization (see, e.g., [33, 2, 29, 12]), combinatorial auction settings (see, e.g., [23, 22, 6, 5, 26]), and budget-feasible mechanisms (see, e.g., [3, 4, 40, 11, 14]).

As discussed above, the feasibility constraint of disjoint maximal sets poses a major obstacle in obtaining good deterministic guarantees for both clock auctions [24] and OSP mechanisms more broadly [30]. Notably, this class of instances also proves quite challenging in online mechanism design settings involving secretary problems and prophet inequalities where an auction designer faces a sequence of bidders arriving one at a time and must make an irrevocable decision in each round to either serve or permanently reject the current bidder. An instance with disjoint maximal sets is used to provide a $\Omega(\log n / \log\log n)$ lower bound for these settings and this lower bound persists even when the bidders have values drawn from known i.i.d. distributions over the set $\{0, 1\}$ and the mechanism designer can choose the order in which to approach the bidders [8, 18, 45].

1.3 Clock Auction Properties

In this section, we provide a more detailed discussion regarding the multiple benefits of clock auctions over sealed-bid ones. In a sealed-bid auction, the bidders are asked to report their values directly to the auctioneer in the form of bids. These bids are then used by the auctioneer to select the subset of bidders receiving the service and to determine the prices charged to the bidders. There are many strategyproof sealed-bid mechanisms proposed in the literature for a variety of single-parameter settings. For example, the well-known second-price auction for allocating a single good is known to be optimal and strategyproof. On the other hand, it has been well-established empirically (see, e.g., [38]) that, although the second-price auction is strategyproof, bidders are far less likely to accurately report their values to the auction than they are to follow the dominant strategy implied by this value in the equivalent ascending clock (Japanese) auction.

The notion of *obvious strategyproofness* was proposed by Li [41] in an attempt to provide theoretical reasoning for this empirical reality. Obvious strategyproofness (OSP) is a strict strengthening of strategyproofness and whether or not a mechanism is OSP is dependent on its implementation. For example, while the allocation and payments of the second-price auction and Japanese auctions are identical, the Japanese auction is OSP and the second-price auction is not. An auction $\mathcal{M}$ is OSP if every bidder has an *obviously dominant strategy*. Such a strategy exists if in the extensive-form game tree of $\mathcal{M}$ whenever a bidder is asked to take an action the best-case utility from deviating from her obviously dominant strategy *over all possible beliefs about the other bidders* is weakly less than her worst-case utility from following the strategy over all possible beliefs about the other bidders. Li [41] provides theoretical and empirical justification demonstrating that bidders can identify their obviously dominant strategy in OSP mechanisms. Further, he demonstrates that clock auctions admit an obviously dominant strategy for all bidders where they remain in the auction while the clock price is less than or equal to their value and exit when the clock price first exceeds their value. Notably, Li shows that the family of OSP mechanisms in single-parameter auction settings are essentially equivalent to the family of clock auctions where each agent faces either an ascending or descending price. As OSP is such a strong incentive property, it also guarantees weak group-strategyproofness, i.e., no coalition of bidders can simultaneously deviate from their dominant truthful strategy and all benefit. This strong guarantee is not satisfied by many standard sealed-bid auctions.

Clock auctions also offer an appealing level of transparency to the bidders. In a sealed-bid auction, the bidders implicitly trust that the auctioneer is following the stated rules of the auction. The bidders must then believe that the auctioneer will not charge them more than they should be charged despite the fact that they have reported their true values to the auctioneer. As this trust is, in practice, unlikely, some real-world applications have begun using the first-price sealed-bid auction (where bidders can easily verify they are being charged the correct amount) rather than the second-price sealed bid auction (where a bidder cannot verify that she has not been overcharged). Aiming to address the potential strategic considerations of the revenue maximizing auctioneer, Akbarpour and Li [1] defined the notion of a credible auction, wherein it is optimal for the auctioneer to follow the prescribed auction protocol, and demonstrated that in the case of selling a single item that the ascending price clock auction (i.e., the Japanese auction) is the unique auction form that guarantees strategyproofness and credibility. In more complicated settings, the truthfulness of, e.g., the VCG auction requires that the auctioneer does not make any computational errors, so even if a bidder trusts that the auctioneer is not strategic they still need to believe that the auctioneer will accurately compute the winning set and prices. This makes clock auctions even more appealing for such settings since the computation of the clock prices does not affect the obvious strategyproofness of clock auctions in any way.

Since clock auctions terminate when the remaining active bidders form a feasible set and clocks stop increasing, any winning bidder never reveals their true value for the service to the auctioneer (or any other participants). This is in contrast to sealed-bid auctions where all bidders report their value to the auctioneer. This *unconditional winner privacy* (UWP) [44] (based on the notion of unconditional privacy in computer science defined by Brandt and Sandholm [17]) states that an auction should require the winning bidders to reveal the minimum amount of information about their value possible to prove that they should be winning. Milgrom and Segal [44] demonstrate that clock auctions are the unique family of auctions which guarantee UWP which makes clock auctions appealing in high-stakes settings where winning bidders could be concerned about revealing their true values to the auctioneer.

Finally, in many practical applications it is unappealing or infeasible to use highly complicated mechanisms since, for example, the participants may not fully understand the functionality of the mechanism or the optimal mechanism may not be readily implementable. To this end, there is a growing line of recent work in algorithmic mechanism design regarding the study of "simple" mechanisms (see, e.g., [35, 9, 46]). In this work, the algorithmic underpinnings of the mechanisms are generally made quite straightforward to allow bidders to understand their functionality. In contrast, clock auctions offer similarly simple interactions to the bidders, the ascending clock prices, while maintaining deep algorithmic richness, the computation of the price trajectories. As clock auctions retain obvious strategyproofness regardless of how the prices are computed "behind the scenes", bidders face a simple interface even if the auctioneer uses complex techniques to calculate the prices.

We note that randomized clock auctions are merely distributions over deterministic clock auctions which all have each of the aforementioned properties *ex-post*. Thus, randomized clock auctions are *universally* OSP, so the bidders do not have to reason about the possible randomization within the auction as they would have to do if the guarantee was only in expectation.

2 Preliminaries

We consider a single-parameter setting where an auctioneer faces a set N of n bidders each competing to receive some service. Each bidder $i \in N$ has a private value v_i which indicates her willingness to pay for being served. The service is limited and there is a publicly known feasibility constraint $\mathcal{F} \subseteq 2^N$ which comprises the subsets of bidders which can be feasibly served simultaneously. We assume that $\mathcal{F}$ is downward-closed, i.e., if $F \in \mathcal{F}$ then all subsets of F are in $\mathcal{F}$. The auctioneer aims to serve a subset of bidders $F \in \mathcal{F}$ of high social welfare $v(F) = \sum_{i \in F} v_i$. Although $\mathcal{F}$ is public, the values of the bidders are private so the auctioneer must elicit these values from the bidders. The bidders, however, are self-interested and may seek to misrepresent their value in order to reach a more preferable outcome for themselves (e.g., being served). To prevent such manipulations the auctioneer may charge payments to the bidders.

We aim to design clock auctions for this problem. A deterministic clock auction is a multi-round auction in which each bidder faces a personal "clock" corresponding to the price that she would be required to pay if the auction terminated. At the beginning of the auction, the clocks are initialized at some arbitrarily small price, and each of these prices then weakly increases over time, giving rise to an ascending auction. At the outset of the auction the set of active bidders A contains all the bidders, i.e., $A = N$, and the clock price $p_{i,0}$ of each bidder i is set to 0. In each round t, the clock auction broadcasts a price $p_{i,t} \geq p_{i,t-1}$ to each bidder $i \in A$ at which point every bidder chooses to remain in the auction at her new price or permanently exit the auction and pay 0 (in which case i is removed from A). The auction can terminate when it is feasible to serve the remaining active bidders, i.e., $A \in \mathcal{F}$, at which point these bidders are served and charged their most recent clock price. The clock price trajectory for each bidder is computed using only public information, i.e., the feasibility constraint, the history of prices, and the points at which bidders exited the auction.

Any clock auction is guaranteed to satisfy numerous desirable properties that make them a particularly appealing practical solution (see Section 1.3 for more details). We also study randomized clock auctions which are probability distributions over deterministic clock auctions. Notably, this means that randomized clock auctions satisfy these properties *ex-post* (rather than in expectation).

We measure the performance of a randomized clock auction $\mathcal{M}$ as the worst-case ratio of its expected social welfare over the social welfare of the optimal set $\mathrm{OPT} = \mathrm{argmax}_{F\in\mathcal{F}} \sum_{i\in F} v_i$. For some instance I and realization of the random decisions within an auction $\mathcal{M}$, let $\mathcal{M}(I)$ denote the set of bidders served by $\mathcal{M}$ on I and $\mathrm{OPT}(I)$ denote the optimal set of bidders in I. We say that $\mathcal{M}$ obtains an α-approximation for a class of instances $\mathcal{I}$ if

$$\alpha \geq \sup_{I\in\mathcal{I}} \frac{v(\mathrm{OPT}(I))}{\mathbb{E}\left[v(\mathcal{M}(I))\right]},$$

where the expectation is taken over the randomization of the auction, if it uses any.

The revenue achieved by an auction is equal to $\sum_{i\in N} p_i$. Let $P = \{p = x \cdot \epsilon : x \in \mathbb{N}\}$ denote the set of possible prices that the auction will consider. Central to our analysis is the quantity $r^* = \max_{p\in P, F\in\mathcal{F}} \{p \cdot |i \in F : v_i \geq p|\}$, the maximum possible revenue an auction could obtain by offering one of the prices in P to all bidders. The uniform price p^* yielding revenue r^* is an important threshold for the clock prices of the bidders in our auctions.

3 The Deterministic Water-Filling Clock Auction

Our deterministic water-filling clock auction (WFCA) proceeds as follows: all bidders are initialized to a price of 0. At each round of the auction we find the feasible subset of active bidders with the highest revenue (i.e., the highest total sum of clock prices) and let the bidders in that set be "conditional winners" in the round. Among the bidders which are not conditional winners (the "conditional losers"), we select the bidder(s) with the lowest price and increase the corresponding price(s) by some small constant ϵ. We continue this process until the set of bidders that remain active is feasible.

Mechanism 1 The deterministic *water-filling clock auction* (WFCA)

let $t \leftarrow 0$, $A \leftarrow N$, and $p_i \leftarrow 0$ for all $i \in N$
while $A \notin \mathcal{F}$ **do**
    $t \leftarrow t + 1$
    let $W \leftarrow \arg\max_{S\in\mathcal{F}: S\subseteq A} \left\{\sum_{i\in S} p_i\right\}$ be the latest set of conditional winners
    let $\ell \leftarrow \min_{i\in A\setminus W}\{p_i\}$ be the lowest price among conditional losers
    foreach *bidder i with $p_i = \ell$* **do**
        update $p_i \leftarrow p_i + \epsilon$
        if *i rejects updated price* **then**
            let $A \leftarrow A \setminus \{i\}$
return A

3.1 Welfare Approximation for General Downward-Closed Constraints

We now proceed to analyze the performance of the WFCA and prove its optimality with respect to social welfare approximation among deterministic clock auctions. To clearly refer to the values of different variables in this auction depending on the round t, we use $p_{i,t}$ to denote the value of the price offered to agent i at the beginning of round t, i.e., before the update, and ℓ_t to denote the minimum such price among the conditional losers of this round. We use A_t to denote the active bidders and W_t to denote the conditional winners of round t.

Recall that $r^* = \max_{p\in P, S\in\mathcal{F}} \{p \cdot |i \in S : v_i \geq p|\}$ is the maximum revenue achievable by offering a uniform price to all bidders and we let p^* denote the price which yields r^*. Let $N^* = \{i \in N | v_i \geq p^*\}$ denote the set of bidders with value above p^* and $S^* = \text{argmax}_{S\in\mathcal{F}; S\subseteq N^*}\{|S|\}$ denote the largest feasible subset of bidders with value above p^*. Our main result for this section shows that the welfare obtained by the WFCA is always at least half of $r^* = p^* \cdot |S^*|$.

Note that, if we were given the welfare maximizing feasible set, it is rather well-known that there exists a fixed price such that, if this price is offered to everyone in this set, the resulting revenue will be a logarithmic approximation of the total welfare of that set. However, in our setting we do not know which set is the optimal one, and there could be multiple feasible sets that overlap in highly non-trivial ways. The main result of this paper shows that the WFCA nevertheless always extracts at least half of the optimal uniform-price revenue, which then implies a logarithmic approximation of the optimal social welfare.

▶ **Theorem 1.** *The WFCA obtains welfare at least $r^*/2$ in any downward-closed set system.*

Proof. This theorem is clearly satisfied if S^* is the final winning set of the WFCA, since $v(S^*) \geq r^*$, so we henceforth assume that S^* is not the winning set and let $\tilde{t}$ be the first round at which some subset of the bidders in S^* rejects the prices offered to them. Since $v_i \geq p^*$ for all $i \in S^*$, it must be that the lowest price among conditional losers in this round is $\ell_{\tilde{t}} \geq p^*$ and only bidders $i \in W_{\tilde{t}}$ could still be facing prices less than p^*. To prove Theorem 1, we consider two possible cases, depending on the size of the overlap between $W_{\tilde{t}}$ and S^*.

Case 1. $|W_{\tilde{t}} \cap S^*| \leq |S^*|/2$:

This case is the easier one of the two. Note that, by definition of $\tilde{t}$, at the beginning of round $\tilde{t}$ all the bids in S^* are still active and every bid in $S^* \setminus W_{\tilde{t}}$ has accepted a price of at least p^*; the latter is true because at the beginning of round $\tilde{t}$ all bidders in this set are among the conditional losers, and the minimum price among them is at least p^*. From this we can deduce that $\sum_{i\in W_{\tilde{t}}} p_{i,\tilde{t}}$, i.e., the revenue of the set $W_{\tilde{t}}$ at the beginning of round $\tilde{t}$ is at least $p^* \cdot |S^* \setminus W_{\tilde{t}}|$, otherwise the set $S^* \setminus W_{\tilde{t}}$ would provide higher revenue and $W_{\tilde{t}}$ would not be the conditional winner; a contradiction.

We therefore know that the revenue of $W_{\tilde{t}}$ at the beginning of round $\tilde{t}$ is at least $p^* \cdot |S^* \setminus W_{\tilde{t}}|$ and we also know that $|W_{\tilde{t}} \cap S^*| \leq |S^*|/2$ (since we are considering Case 1), i.e., $|S^* \setminus W_{\tilde{t}}| \geq |S^*|/2$. This implies that the revenue at the beginning of $\tilde{t}$ is at least $p^* \cdot |S^*|/2 = r^*/2$. To conclude the proof for this case, we observe that the set of conditional winners of the WFCA satisfy revenue monotonicity over t, i.e., the revenue of W_t at the beginning of round t is at least as high as the revenue of W_{t-1} at the beginning of round $t-1$. This is true because the prices of the bidders in W_{t-1} are the same in $t-1$ and t (because W_{t-1} were the winners in round $t-1$ and their prices were not updated) and W_t were the conditional winners in t so their revenue in t is at least as high as that of W_{t-1}.

Therefore, the revenue at the end of the auction, extracted from the eventual winners is at least $r^*/2$ and the social welfare obtained by the mechanism in Case 1 is at least that much as well (since each agent's value is at least as high as the price they accepted).

Case 2. $|W_{\tilde{t}} \cap S^*| > |S^*|/2$:

For this case, we show that for every $t \geq \tilde{t}$ there exists a feasible set that satisfies a carefully chosen invariant. For each round t of the auction, let $L_t = \{i \in S^* : \; p_{i,t} < p^*\}$ be the subset of S^* whose prices all remain less than p^*, and given some set of active bidders $F \subseteq A_t$ let $\overline{\text{Rev}}(F, t) = \sum_{i \in F: \; p_{i,t} \geq p^*} p_{i,t}$ denote the sum of the clock prices over the bidders in F whose price is at least p^* at round t. Our invariant shows that as long as we are in Case 2, i.e., $|W_{\tilde{t}} \cap S^*| > |S^*|/2$, for every round $t \geq \tilde{t}$ there exists some feasible subset F of active bidders such that the sum of $p^* \cdot |F \cap L_t|$ (which is a lower bound for the total value among the bidders in $F \cap L_t$) and $\overline{\text{Rev}}(F, t)$ (which is a lower bound for the total value of the bidders in F that have price at least p^*) is at least $r^*/2$. Note that this invariant intentionally disregards the potential contributions from any bidders in $F \setminus L_t$ whose price remains less than p^*, which significantly simplifies the proof below.

▶ **Lemma 2.** *If $|W_{\tilde{t}} \cap S^*| > |S^*|/2$, then in every round $t \geq \tilde{t}$ of the WFCA there is a feasible set of active bidders $F \subseteq A_t$ such that*

$$p^* \cdot |F \cap L_t| + \overline{Rev}(F, t) \geq r^*/2. \tag{1}$$

Proof. Our proof for this lemma is constructive: for each round t we define a feasible set $F_t \subseteq A_t$ and prove it satisfies Inequality (1). Specifically, for any round t where the lowest price among the conditional losers is $\ell_t \geq p^*$, we let F_t be the set of conditional winners of that round, i.e., $F_t = W_t$. On the other hand, for any round t where $\ell_t < p^*$, we show that the same set that satisfied the inequality in round $t-1$ also satisfies it in round t, i.e., $F_t = F_{t-1}$. A crucial implication of the way we choose these sets is that, as we show later on, every active bidder i with $p_{i,t} < p^*$ must belong to $F_{t-1} \cap F_t$ (see property (3)), which allows our inductive argument to go through.

Formally, let $T^* = \{t \geq \tilde{t} : \; \ell_t \geq p^*\}$ be the subset of rounds $t \geq \tilde{t}$ where $\ell_t \geq p^*$. Also, for each $t \geq \tilde{t}$ let $f(t) = \max\{t' \leq t : \; t' \in T^*\}$ be the most recent round that was in T^* (so $f(t) = t$ for $t \in T^*$) and let $F_t = W_{f(t)}$ be the winning set in round $f(t)$. To prove this lemma, we show that in each round $t \geq \tilde{t}$, the set F_t satisfies Inequality (1). This means that during rounds when $t \notin T^*$, the same set F_{t-1} that satisfied the invariant in $t-1$ still satisfies the invariant in t, even if it is not the set of conditional winners in t. However, when $t \in T^*$, it could be that the price update at the end of round $t-1$ raised some prices to p^* or above, potentially affecting the left hand side of (1) for set F_{t-1}. In this case we prove that choosing F_t to be equal to the set of conditional winners, W_t, ensures that the invariant remains true even if some of the price updates at the end of round $t-1$ reached or exceeded p^*.

First, for the round $t = \tilde{t}$, we have $f(\tilde{t}) = \tilde{t}$, since $\tilde{t} \in T^*$, and thus $F_{\tilde{t}} = W_{\tilde{t}}$. To verify that Inequality (1) is satisfied by the set $W_{\tilde{t}}$ note that, by definition of $\tilde{t}$, all the bidders in $W_{\tilde{t}} \cap S^*$ remain active up to this round, and there are at least $|S^*|/2$ of them. Let q_1 be the number of these bidders that still have a price less than p^* at round $\tilde{t}$ and q_2 be the number of remaining bidders who have price at least p^*, so that $q_1 + q_2 \geq |S^*|/2$. Each of the q_1 bidders belongs to the set $W_{\tilde{t}} \cap L_{\tilde{t}}$ so $p^* \cdot |W_{\tilde{t}} \cap L_{\tilde{t}}| = q_1 p^*$. Also, each of the remaining q_2 bidders contributes at least p^* to the revenue $\overline{\text{Rev}}(W_{\tilde{t}}, \tilde{t})$, so the left hand side of the inequality is at least $(q_1 + q_2)p^* \geq p^* \cdot |S^*|/2 = r^*/2$.

For the inductive step, we now show that for every round $t > \tilde{t}$, if Inequality (1) holds for set F_{t-1} at round $t-1$ then it also holds for set F_t at round t. Specifically, it suffices to show that for any round $t > \tilde{t}$ we have

$$p^* \cdot |F_t \cap L_t| + \overline{\text{Rev}}(F_t, t) \geq p^* \cdot |F_{t-1} \cap L_{t-1}| + \overline{\text{Rev}}(F_{t-1}, t-1). \tag{2}$$

If Inequality (1) holds for F_{t-1} at round $t-1$, then the right hand side of Inequality (2) is at least $r^*/2$ and this inequality would imply that the left hand side is at least $r^*/2$ as well. As a result F_t would satisfy the invariant in round t, which would conclude the proof of the lemma.

◇ We first prove that Inequality (2) holds for all $t > \tilde{t}$ such that $t \notin T^*$. Note that for all $t \notin T^*$ we have $f(t) = f(t-1)$, i.e., the most recent round in T^* for t and $t-1$ is the same. Therefore $F_t = F_{t-1}$ and Inequality (2) reduces to

$$p^* \cdot |F_t \cap L_t| + \overline{\text{Rev}}(F_t, t) \geq p^* \cdot |F_t \cap L_{t-1}| + \overline{\text{Rev}}(F_t, t-1).$$

It is not too hard to see that this actually holds with equality, since neither of the summands on the right hand side were affected by the price updates that took place in round $t-1$. If $t-1 \in T^*$, then $F_t = W_{t-1}$ and no prices within F_t were updated and neither were L_{t-1} or $\overline{\text{Rev}}(F_t, t-1)$. On the other hand, if $t-1 \notin T^*$, then $\ell_{t-1} < p^*$, so no price was raised to p^* during round $t-1$, implying, once again, that neither L_{t-1} nor $\overline{\text{Rev}}(F_t, t-1)$ were affected by these price updates.

◇ We now prove that Inequality (2) also holds for all $t \in T^*$. In order to do so we first make the crucial observation that for any bidder i with price $p_{i,t} < p^*$ at round $t > \tilde{t}$ it must be that $i \in F_{t-1} \cap F_t$. To verify this, note that if $i \notin F_t$, i.e., $i \notin W_{f(t)}$, then $p_{i,f(t)} \geq p^*$ because $f(t) \in T^*$ and the lowest price among the losers in round $f(t)$ is $\ell_{f(t)} \geq p^*$. But, since bidder i's price weakly increases over time, this would imply that $p_{i,t} \geq p^*$, leading to a contradiction. The same argument also shows that $i \in F_{t-1}$, which implies the following important property of the WFCA.

$$\text{Any bidder } i \text{ with } p_{i,t} < p^* \text{ at some round } t > \tilde{t} \text{ satisfies } i \in F_{t-1} \cap F_t. \tag{3}$$

Now, the fact that $F_t = W_t$ was the set of conditional winners for round t means that the sum of its bidders' prices at that round was at least as high as that of any other feasible subset of active bidders. Specifically, this means that $\sum_{i \in F_t} p_{i,t} \geq \sum_{i \in F_{t-1}} p_{i,t}$. As we showed above in (3), for every bidder i with $p_{i,t} < p^*$ we know that $i \in F_{t-1} \cap F_t$ so the prices of these bidders contribute to the revenue of both F_{t-1} and F_t equally in round t. We can therefore deduce that the revenue of F_t at round t is at least as high as that of F_{t-1} at round t, even if we restrict our attention to bids with price at least p^*. Thus, we have shown that $\overline{\text{Rev}}(F_t, t) \geq \overline{\text{Rev}}(F_{t-1}, t)$.

If round $t-1$ was also in T^*, this would essentially conclude the proof, because F_{t-1} would be W_{t-1}, the set of conditional winners in $t-1$, implying that its bidders' prices did not change from $t-1$ to t. As a result, we would get $\overline{\text{Rev}}(F_{t-1}, t-1) = \overline{\text{Rev}}(F_{t-1}, t) \leq \overline{\text{Rev}}(F_t, t)$. Also, we would get $F_{t-1} \cap L_{t-1} = F_{t-1} \cap L_t$ and this would also be equal to $F_t \cap L_t$, because $L_t \subseteq F_t$ from (3). These observations directly imply that Inequality (2) would be satisfied.

If, on the other hand, round $t-1$ was not in T^*, the prices in F_{t-1} could differ between $t-1$ and t. However, since $t-1 \notin T^*$, we know that $\ell_{t-1} < p^*$, so no price above p^* would be affected. What could affect the argument above is that the price of some bidders in L_{t-1} could be updated from $p^* - \epsilon$ to p^*, so they would not be in L_t. The argument still goes through, however, because all these bidders would accept the updated prices (because we know that every bidder $i \in L_{t-1}$ has value $v_i \geq p^*$) and they would each contribute a price of p^* toward $\overline{\text{Rev}}(F_{t-1}, t)$, while they were not contributing anything toward $\overline{\text{Rev}}(F_{t-1}, t-1)$ (since their price was below p^*). As a result, using the facts that $L_t \subseteq F_t$, from (3), and the fact that $\overline{\text{Rev}}(F_t, t) \geq \overline{\text{Rev}}(F_{t-1}, t)$, we get

$$\begin{aligned}
\overline{\text{Rev}}(F_{t-1}, t) - \overline{\text{Rev}}(F_{t-1}, t-1) &\geq p^* \cdot (|F_{t-1} \cap L_{t-1}| - |F_{t-1} \cap L_t|) \Rightarrow \\
\overline{\text{Rev}}(F_t, t) - \overline{\text{Rev}}(F_{t-1}, t-1) &\geq p^* \cdot (|F_{t-1} \cap L_{t-1}| - |F_t \cap L_t|) \Rightarrow \\
p^* \cdot |F_t \cap L_t| + \overline{\text{Rev}}(F_t, t) &\geq p^* \cdot |F_{t-1} \cap L_{t-1}| + \overline{\text{Rev}}(F_{t-1}, t-1).
\end{aligned}$$ ◀

Using Lemma 2 we can now also conclude the proof of Theorem 1 for Case 2, by observing that the set A_τ of bidders accepted when the auction terminates, at some round $\tau \geq \tilde{t}$, will also satisfy Inequality (1), implying that $p^* \cdot |A_\tau \cap L_\tau| + \overline{\text{Rev}}(A_\tau, \tau) \geq r^*/2$. This, however, directly implies that the social welfare of the WFCA in this case would be at least as high as the left hand side since each bidder in $A_\tau \cap L_\tau$ has value at least p^*, by definition of L_τ, and the social welfare of the remaining bidders (even if we restrict our attention to the ones that have price at least p^*) is at least $\overline{\text{Rev}}(A_\tau, \tau)$ since their total value is at least as high as the sum of the prices that they have accepted. We therefore conclude that the social welfare is at least $r^*/2$ in this case as well. ◀

Leveraging the result of Theorem 1 we can now use standard techniques to demonstrate that the WFCA obtains asymptotically optimal bounds with respect to several parameters of interest. We first consider the parameter n, i.e., the number of bidders, and argue that r^* is a $O(\log n)$ approximation of the welfare in OPT. We note that, as we have discretized the set of prices P that the WFCA considers, that p^* is a multiple of ϵ and we may lose an additive $n\epsilon$ against the optimal revenue. However, as ϵ goes to zero, the following theorems and corollaries can disregard this loss. For example, if we let $\epsilon = 1/n^2$ we ensure that the revenue lost by choosing p^* to be a multiple of ϵ is no more than $1/n$.

▶ **Theorem 3.** *The WFCA obtains a $4 \log n$-approximation to the optimal social welfare in any downward-closed set system.*

Proof. Fix a feasible set S of bidders and index them in non-increasing order of their value using indices from 1 to $|S|$. Then, there exists an index $i \in \{1, \dots, |S|\}$ for which we have $i \cdot v_i \geq v(S)/(2 \log |S|)$. To verify this fact, assume that for all indices i we have $v_i < v(S)/(2i \log |S|)$. Summing over all i we then obtain that $v(S) < v(S) \cdot \sum_{i=1}^{|S|} \frac{1}{i}/(2 \log |S|) \leq v(S)$, a contradiction. As a result, if we were to offer a uniform price of $p = v_i$ to all bidders in S, then i of them would accept this price, leading to a revenue of $i \cdot p = i \cdot v_i \geq v(S)/(2 \log |S|)$.

Since $|\text{OPT}| \leq n$, we therefore conclude that there exists a uniform price p, which when offered to all bidders in OPT would yield revenue which is a $2 \log n$-approximation to the welfare in OPT. By Theorem 1 we know that the WFCA obtains welfare which is a 2-approximation to r^* which is the best possible revenue attainable by offering a uniform price to all bidders. But then, the WFCA must obtain welfare which is a 2-approximation to the revenue obtained from bidders in OPT when offering them price p, so the welfare obtained by the WFCA is a $4 \log n$-approximation to the optimal welfare. ◀

In fact, the proof of Theorem 3 actually shows that we obtain a $4 \log o$-approximation where $o = |\text{OPT}|$ is the size of the optimal set. We may then also obtain another approximation guarantee for the special class of downward-closed feasibility constraints corresponding to combinatorial auctions with single-minded or multi-minded bidders, which have received a lot of attention in the past. In these auctions, each bidder is served by receiving some (non-empty) subset of a set M of items and the feasibility constraints are implied by the fact that each item can be allocated to at most one bidder. Therefore, the number of items $m = |M|$ is an upper bound to the size of the optimal set as well (since every bidder needs to obtain at least one item to receive positive value), i.e., $o \leq m$. We then directly obtain the following logarithmic approximation with respect to m as a corollary of the proof of Theorem 3, matching the known lower bound from [24] for deterministic clock auctions.

▶ **Corollary 4.** *The WFCA obtains a $O(\log m)$-approximation to the optimal social welfare in any combinatorial auction setting with single-value multi-minded bidders where m is the number of items.*

Using Black-Box Access to Approximation Algorithms

In order to focus on the information limitations of clock auctions, we have so far assumed that the auction is equipped with a black box that can identify the revenue maximizing feasible set in every round. However, computing this set can be computationally intractable in some cases, depending on the feasibility constraint $\mathcal{F}$, so it is important to note that the WFCA can be adjusted to also work with black-box access to *approximation* algorithms. In each round t, when the WFCA needs to determine the set of conditional winners W_t given the latest prices, it could invoke that black box. Then, the only required adjustment would be to ensure revenue monotonicity over t: the auction would just need to check that the revenue of the returned set W_t is at least as high as that of W_{t-1} and, if not, set $W_t \leftarrow W_{t-1}$ instead. In doing so the proof of Theorem 1 can be easily adapted to show that the WFCA obtains welfare at least $r^*/(2\alpha)$ where $\alpha > 1$ is the approximation factor of the algorithm used as a black box. We can then conclude the following theorem.

▶ **Theorem 5.** *The WFCA obtains welfare at least $r^*/(2\alpha)$ in any downward-closed set system where $\alpha > 1$ is the approximation ratio of the black box algorithm used in each round to select the set with the highest sum of clock prices.*

As a consequence of Theorem 5, and using the same argument as above, we conclude that the WFCA achieves a $O(\alpha \log o)$-approximation to the social welfare in any downward-closed setting.

▶ **Corollary 6.** *The WFCA obtains a $4\alpha \log o$-approximation to the optimal social welfare for any downward-closed feasibility constraint $\mathcal{F}$ where $\alpha > 1$ is the approximation ratio of the black box algorithm used in each round to select the set with the highest sum of clock prices.*

3.2 Comparison to Ascending Wrapper Auction of Babaioff et al. [10]

The WFCA is inspired by the *iterative ascending wrapper auction* proposed by Babaioff et al. [10]. They prove that their auction achieves a $2 \log \overline{v} + 1$ approximation of the optimal social welfare for the special case of combinatorial auctions with single-minded bidders, where $\overline{v} = \max_i v_i$ is the highest value among all bidders when the smallest value is normalized to 1 (or, equivalently, $\overline{v}$ is the ratio between the largest and smallest bidder value).

Both the WFCA auction and the "wrapper" auction use black-box access to an algorithm to determine the set of conditional winners and then raise the prices of (some) conditional losers. The main difference between the two is that the wrapper auction raises the prices of *all* the conditional losers, whereas the WFCA only raises the prices of the conditional losers with the lowest price. Another difference is that the wrapper auction *doubles* the prices of the losers, whereas the WFCA raises them by a fixed constant, but this is not an important difference (in fact, our bounds would be affected by no more than a factor 2 if we were to apply this change to the WFCA as well).

The biggest difference between the two auctions comes in the analysis. The analysis of the wrapper auction heavily depends on the fact that this auction completes within at most $2\log\overline{v}+1$ rounds. However, this fact holds only if *every* loser's price is increased and it applies only to the special case of feasibility constraints induced by combinatorial auctions with single-minded bidders (where the constraints are "pairwise": two bidders are either always compatible with each other or always incompatible, depending on whether their bundles overlap). Therefore, this analysis cannot be extended to general downward-closed feasibility constraints. For example, even for the case of single-parameter combinatorial auctions with double-minded bidders, which is also considered in their paper, the analysis does not follow (see Proposition 3.5 in [10]). More importantly, even for the case of combinatorial auctions with single-minded bidders the approximation factor of the wrapper auction with respect to parameters n and m is $\Omega(n)$ and $\Omega(m)$ (see Proposition C.1 in [10]). For a more complete comparison of the performance of the WFCA and the wrapper auction, our next result shows that, even if we parameterize our bounds with respect to $\overline{v}$, the WFCA matches the asymptotic approximation guarantee of the wrapper auction. We defer the proof of this theorem to Appendix A.

▶ **Theorem 7.** *The WFCA obtains a* $4(\log\overline{v}+1)$*-approximation to the optimal social welfare in any downward-closed set system.*

Finally, note that Babaioff et al. [10] also provide an auction that achieves a $O(\log^2\overline{v})$ approximation for the case of single-value multi-minded bidders. Since this class of instances is downward-closed, all our positive results apply to it as well, so the WFCA guarantees a $O(\log n)$, $O(\log\overline{v})$, and $O(\log m)$ approximation for the single-value multi-minded case.

4 Randomized Clock Auction for Disjoint Maximal Sets

The logarithmic lower bound for deterministic clock auctions is shown using a simple class of instances involving just two disjoint maximal feasible sets. It is easy to observe that the use of randomization can readily overcome the logarithmic barrier for this class of instances: we could just choose one of the two maximal feasible sets uniformly at random and let its bidders be the winners, which would yield a 2-approximation because the set of winners is optimal with probability at least $1/2$. Choosing a maximal feasible set uniformly at random, however, would not yield a good approximation when the number of these sets is large.

Another natural way to use randomization in clock auctions would be to partition the bidders into two groups uniformly at random and then "sample" the values from one group by raising the clocks of its bidders until every one of them drops out. This way, we observe the values of the bidders in the "sample group" and we can use them as a guide in order to estimate the value of each feasible set. This approach of using some random subset of the bidders to gather information about the instance and then use this information on the non-sampled bidders has been used widely in mechanism design (see, e.g., [23, 14]) and is the main idea behind the Sampling Auction, presented below as Mechanism 2.

The Sampling Auction is a randomized clock auction that uses the sampled bidder values to compute a threshold $\hat{q}$ and then chooses a maximal feasible set of bidders R uniformly at random among the ones whose sampled value is at least $\hat{q}$. The winners of this auction are the non-sampled bidders of R, who are all served for a price of zero.

Mechanism 2 The Sampling Auction: A randomized clock auction

Input : The set $\mathcal{S}$ of all maximal sets in $\mathcal{F}$
let $T \leftarrow \emptyset$
for *each bidder* $i \in N$ **do**
  with probability $1/2$ increase the clock of i until she rejects and let $T \leftarrow T \cup \{i\}$
let $S_{\text{sample}} \leftarrow S \cap T$ for all $S \in \mathcal{S}$
let $\hat{q} \leftarrow \frac{\max_{S \in \mathcal{S}}\{v(S_{\text{sample}})\}}{4}$
let $\mathcal{R}$ be the collection of all sets $S \in \mathcal{S}$ with $v(S_{\text{sample}}) \geq \hat{q}$
let R be a set chosen uniformly at random from $\mathcal{R}$
return $R \setminus T$

Unfortunately, it is not too hard to see that the Sampling Auction can perform very poorly even in simple instances. Take, for example, an instance where all bidders except one have a small value (e.g., 1) and the remaining bidder has a very high value (e.g., n^3). If the high value bidder is in the sample, this means that she already dropped out and we lost her value. If, on the other hand, she is not in the sample, then the sampled values do not provide any useful guidance toward choosing the optimal feasible set, which would need to include the high-value bidder.

4.1 The Hedging Auction

In this section we present the Hedging Auction, a randomized clock auction which "hedges" between the WFCA and the Sampling Auction by flipping a fair coin to decide which one of these two clock auctions to run. We later show that this auction achieves a $O(\sqrt{\log n})$ approximation guarantee for the class of instances with disjoint maximal feasible sets and thus overcomes the logarithmic barrier for a class of instances that generalizes the lower bound construction[2]. Beyond this class of instances, we show that this auction achieves a $O(\min\{\log n, \sqrt{\log k}\})$ approximation guarantee for any downward-closed feasibility constraint $\mathcal{F}$, where k is the number of maximal feasible sets in $\mathcal{F}$ (note that for disjoint maximal feasible sets we have $k \leq n$).

The success of the Hedging Auction is due to the "complemenarity" of the WFCA and the Sampling Auction. As we already showed in the previous section, the social welfare guaranteed by the WFCA is at least half of the optimal uniform-price revenue. Therefore, the auction performs well whenever this revenue is a good approximation of the optimal social welfare. Our analysis shows that whenever this is not the case, i.e., when there is no uniform price whose revenue is a good approximation of the optimal social welfare, then the Random Sampling auction performs better. Roughly speaking, if the optimal uniform-price revenue is a bad approximation of the optimal social welfare, this means that the welfare is not concentrated within a small number of bidders. But, if most of the welfare is distributed among multiple bidders, then the sampling phase of the Sampling Auction is very likely to provide useful guidance, leading to a set of winners with high welfare.

Mechanism 3 The Hedging Auction

Input : The feasibility constraint $\mathcal{F}$, and set $\mathcal{S}$ of all maximal sets in $\mathcal{F}$
With probability $1/2$: Run the WFCA (Mechanism 1) on $\mathcal{F}$
With the remaining $1/2$ probability: Run the Sampling Auction (Mechanism 2) on $\mathcal{F}$

[2] Also, note that this improved bound matches the $\Omega(\sqrt{\log n})$ lower bound recently shown by Ferraioli et al. [30] for arbitrary deterministic OSP mechanisms (even beyond clock auctions) in this setting.

4.2 Analysis of the Hedging Auction

In order to analyze the performance of the Hedging Auction, we begin by stating a useful lemma regarding the sum of the values of bidders in the two disjoint subsets created by partitioning a set uniformly at random when we have upper bounds on the values of the bidders. Similar arguments regarding the concentration of the values of two subsets created by a random partition have been used in the context of general combinatorial auctions, see, e.g., [23], and budget-feasible procurement auctions, see, e.g., [14], but these results required weaker concentration guarantees and, thus, could use weaker bounds on the values of the bidders. We defer the proof of this lemma to Appendix A.

▶ **Lemma 8.** *Consider a set S indexed in non-increasing order by bidder value with $v_i \leq \frac{\alpha v(S)}{i}$ for some $0 < \alpha < 1$. When bidders are partitioned into two sets S_1 and S_2 uniformly at random we have that $\sum_{i \in S_1} v_i \geq \frac{v(S)}{4}$ and $\sum_{i \in S_2} v_i \geq \frac{v(S)}{4}$ with probability at least $1 - 2 \cdot e^{\frac{-1}{14\alpha^2}}$.*

From the proof of Lemma 8 we also obtain the following corollary.

▶ **Corollary 9.** *Consider a set S indexed in non-increasing order by bidder value with $v_i \leq \frac{\alpha v(S)}{i}$. If we remove each bidder from the set independently with probability $1/2$ then the remaining value in the set will be less than $\frac{v(S)}{4}$ with probability at most $e^{\frac{-1}{14\alpha^2}}$.*

With these in hand, we are ready to show that the Hedging Auction achieves an $O(\sqrt{\log n})$-approximation to the social welfare for any disjoint maximal feasible sets.

▶ **Theorem 10.** *The Hedging Auction obtains a $O(\sqrt{\log n})$-approximation to the optimal social welfare in any instance where the feasibility constraint comprises disjoint maximal sets.*

Proof. If the optimal revenue obtainable in hindsight from a set is a $64\sqrt{\log n}$-approximation to the social welfare, then because the water-filling auction obtains a 2-approximation to the optimal revenue, we obtain the desired bound. So suppose not. This enforces constraints on the values of bidders in any set S with $v(S) \geq v(\mathrm{OPT})/16$. More precisely, if we index the bidders in S in non-increasing order of value, then it must be that $v_i \leq \frac{1}{64\sqrt{\log n}} \cdot \frac{v(\mathrm{OPT})}{i} \leq \frac{1}{4\sqrt{\log n}} \cdot \frac{v(S)}{i}$. Consider that these constraints on bidder values must, in particular, apply to the optimal set. Then by Lemma 8 we have that with probability at least $1 - 2 \cdot e^{\log n} = 1 - \frac{2}{n}$ the optimal set must have sample value at least $\frac{v(\mathrm{OPT})}{4}$ and unsampled value at least $\frac{v(\mathrm{OPT})}{4}$. Assume that this event happens. Since, by definition, no set can have sample value more than $v(\mathrm{OPT})$, if we set the threshold $\hat{q}$ at $1/4$ times the largest sample welfare, then the optimal set will pass the first phase of the sampling auction and retain a constant fraction of its welfare. Moreover, we know that $\hat{q} \geq \frac{v(\mathrm{OPT})}{16}$. Therefore any set S' which could possibly exceed $\hat{q}$ must have total welfare at least $\frac{v(\mathrm{OPT})}{16}$. But then by Corollary 9, when we sample each bidder in S' independently with probability $1/2$, the remaining unsampled value in S' will be less than $v(S')/4$ with probability at most $e^{\frac{-16 \log n}{14}} \leq \frac{1}{2n}$.

Since there are at most n maximal sets in our set system and since we have that any set capable of passing the threshold phase with threshold $\hat{q}$ retains less than a quarter of its welfare with probability no more than $1/(2n)$, by a union bound, we know that with probability at least $1/2$, all sets capable of passing the threshold retain at least $1/4$ of their welfare. Thus, with probability at least $1/2$, when the optimal revenue in hindsight is not a $64\sqrt{\log n}$-approximation (and therefore the water-filling auction possibly fails to give a $O(\sqrt{\log n})$-approximation), selecting a set uniformly at random from among those with sample welfare above $\hat{q}$ yields a constant approximation. Finally, we may conclude that the

Hedging Auction which runs either the water-filling auction or the sampling auction each with probability 1/2 must yield a $O(\sqrt{\log n})$-approximation as desired. ◀

In fact, using the same approach we can prove the following theorem which generalizes the bound of Theorem 10 to general downward-closed feasibility constraints, parameterized by the number of (not necessarily disjoint) maximal feasible sets, k. This, for example, shows that the Hedging Auction achieves an approximation factor better than $O(\log n)$ for any feasibility constraint for which $k = n^{o(\log n)}$ (rather than just $k \leq n$, as in the case of disjoint maximal sets). Moreover, when $k = \text{poly}(n)$ the auction runs in polynomial time since we may directly compute the revenue of each maximal feasible set in the WFCA and the sample welfare of each maximal feasible set in the Sampling Auction.

▶ **Theorem 11.** *The Hedging Auction obtains a $O(\min\{\log n, \sqrt{\log k}\})$-approximation to the optimal social welfare for any downward-closed feasibility constraint $\mathcal{F}$, where k is the number of maximal feasible sets in $\mathcal{F}$.*

Proof. First observe that the Hedging Auction runs the WFCA with probability 1/2 and, thus, always obtains no worse than a $O(\log n)$ approximation. The remainder of the proof follows essentially the exact same structure as the proof of Theorem 10. If the optimal revenue in hindsight from any set is a $64\sqrt{\log k}$-approximation to the social welfare, then because the water-filling auction obtains a 2-approximation to this revenue, we obtain the desired bound. If not, then, as above, we obtain bounds in terms of k on the values of the bidders in any set S with $v(S) \geq v(\text{OPT})/16$. Following the same logic as above, we have that when bidders in S are indexed in non-increasing order of value, then $v_i \leq \frac{1}{4\sqrt{\log k}} \cdot \frac{v(S)}{i}$. We then similarly have that the optimal set must have sample value $\frac{v(\text{OPT})}{4}$ and unsampled value at least $\frac{v(\text{OPT})}{4}$ with probability at least $1 - \frac{2}{k}$ by Lemma 8. If this event happens then since no set can have sample value more than $v(\text{OPT})$, we know OPT will pass the sampling threshold $\hat{q}$ and that $\hat{q} \geq \frac{v(\text{OPT})}{16}$. Moreover, any set S' which could possibly have sample value above $\hat{q}$ must have welfare at least $\frac{v(\text{OPT})}{16}$ which means that S' will have remaining unsampled value less than $v(S')/4$ with probability at most $\frac{1}{2k}$ by Corollary 9. Finally, since there are only k maximal sets in our set system, by a union bound, we know that with probability at least 1/2 all sets capable of passing the threshold retains at least 1/4 of their welfare. Thus, with probability at least 1/2, when the optimal revenue in hindsight is not a $64\sqrt{\log k}$-approximation (and therefore the water-filling auction possibly fails to give a $O(\sqrt{\log k})$-approximation), selecting a maximal set uniformly at random from among those with sample welfare above $\hat{q}$ yields a constant approximation. Finally, we may conclude that the Hedging Auction which runs either the water-filling auction or the sampling auction each with probability 1/2 must yield a $O(\sqrt{\log k})$-approximation as desired. ◀

4.3 Approximation Lower Bounds for Randomized Clock Auctions

In this section, we present information theoretic lower bounds on the performance of any randomized clock auction for two instances where the feasibility constraint corresponds to disjoint maximal sets. Given that the existing lower bounds for deterministic clock auctions which consists of two disjoint maximal sets can easily be overcome by randomly selecting one of the two maximal sets (thereby obtaining a 2-approximation without any need to increase clock prices of the bidders), one may hope that a randomized clock auction may be able to obtain, e.g., a $1 + o(1)$-approximation on such instances. Our first lower bound demonstrates that this is not the case and that no randomized clock auction can achieve better than a

11/10-approximation to the social welfare even when the set system comprises only two maximal sets, one with a single bidder and the other with two bidders. As a consequence, this lower bound demonstrates that a randomized clock auction cannot achieve arbitrarily good approximation guarantees even when the feasibility constraint can be represented by matchings in a bipartite graph where the bidders are the edges (since this instance corresponds to the path graph of three edges).

▶ **Theorem 12.** *No randomized clock auction (even with unbounded computational power) can achieve better than a $11/10 - \epsilon$-approximation for $\epsilon > 0$ even when the feasibility constraint comprises exactly two maximal sets, one containing a single bidder and the other containing two bidders.*

Proof. We proceed via Yao's lemma. We construct three instances, $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$, as follows. In each instance, there are two disjoint maximal sets S_1 and S_2, where S_1 contains one bidder i and S_2 contains two bidders j and k. In $\mathcal{I}_1$, $v_i = 1$ and $v_j = v_k = 1/3$. In $\mathcal{I}_2$, $v_i = v_j = 1$ and $v_k = 1/3$. Finally, in $\mathcal{I}_3$ $v_i = v_k = 1$ and $v_j = 1/3$. Consider an equiprobable distribution over these three instances. Observe that the expected optimal welfare on this distribution is $\frac{1}{3} + \frac{2}{3} \cdot \frac{4}{3} = \frac{11}{9}$. Consider a deterministic clock auction $\mathcal{M}$ which achieves an $11/10 - \epsilon$-approximation for some $\epsilon > 0$ on this distribution of instances. It must be that $\mathcal{M}$ obtains expected welfare strictly more than $\frac{10}{9}$. If $\mathcal{M}$ were to accept S_1 without increasing the clock price of at least one bidder in S_2 above $1/3$ then $\mathcal{M}$ would obtain expected welfare 1. On the other hand, if $\mathcal{M}$ were to accept S_2 without first increasing the clock price of at least one bidder in S_2 above $1/3$ then $\mathcal{M}$ would obtain expected welfare $\frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{4}{3} = \frac{10}{9}$. Thus, $\mathcal{M}$ must raise the clock price of one of the two bidders in S_2 above $1/3$. Without loss of generality (since instances $\mathcal{I}_2$ and $\mathcal{I}_3$ are equiprobable) suppose $\mathcal{M}$ raises the clock of j above $1/3$ first. Then, in instances $\mathcal{I}_1$ and $\mathcal{I}_3$ bidder j would reject this offer and $\mathcal{M}$ would obtain welfare no more 1. In instance $\mathcal{I}_2$ bidder j would accept and $\mathcal{M}$ would obtain welfare no more than $\frac{4}{3}$. Thus, $\mathcal{M}$ obtains expected welfare no more than $\frac{1}{3} \cdot \frac{4}{3} + \frac{2}{3} = \frac{10}{9}$. But then, $\mathcal{M}$ cannot raise the price of either bidder above $1/3$ first if it wants to obtain expected welfare strictly more than $\frac{10}{9}$, a contradiction. Since no deterministic mechanism can achieve a $\frac{11}{10} - \epsilon$-approximation for any $\epsilon > 0$ when facing this distribution of instances, by Yao's lemma no randomized mechanism can achieve a $\frac{11}{10} - \epsilon$-approximation for any $\epsilon > 0$ against an adversarial instance. ◀

In Appendix A.3, we show that when we instead have two disjoint maximal sets with one bidder and many bidders, respectively, we may obtain a stronger lower bound on the performance of any randomized clock auction. We note, however, that the inapproximability result we show is still far away from the simple upper bound of 2. Thus, we view tightening the gap between the lower and upper bound even for this simple family of instances as an interesting open question.

5 Conclusions and Open Questions

In this work, we provide an asymptotically optimal deterministic clock auction for arbitrary downward-closed feasibility constraints. Throughout the paper we focus on the information-theoretic limitations of clock auctions and set aside the computational ones, but it is worth noting that our WFCA gives rise to asymptotically optimal *polynomial time* clock auctions for several feasibility constraints of interest. For example, if the feasibility constraint is a knapsack constraint (e.g., when selling seats on a plane or in a theater), we can solve the underlying optimization problem within a small constant factor approximation. If the feasibility constraint corresponds to weighted interval scheduling (e.g., when scheduling jobs that each need a machine during a specific time interval), we can exactly solve the underlying optimization problem using dynamic programming. There are also several natural generalizations of weighted interval scheduling that admit constant factor approximation algorithms and correspond to important real-world problems. For example, the weighted group interval scheduling problem captures settings where the bidders are *multi-minded* over intervals, i.e., there are multiple different intervals that each one bidder could use for scheduling her job [28]. Alternatively, the bidders' requests could be multi-dimensional (e.g., requiring rectangle pieces of land) with the constraints implied by the geometric overlaps of these requests (e.g., [7, 19]). Since all these families of instances include the lower bound construction of [24], the logarithmic approximation achieved by the WFCA, when paired with the corresponding (constant factor approximation) algorithms, is optimal.

Moving beyond deterministic clock auctions, we also provide a way of using randomization to achieve improved approximation guarantees for interesting families of instances. We believe that the most exciting direction for future research in this line of work is the study of the power and limitations of randomized clock auctions. The best known lower bounds for randomized clock auctions are small constants so, if we set aside the computational constraints, it could even be possible to design a randomized clock auction that achieves a constant factor approximation for arbitrary downward-closed feasibility constraints. Whether such a constant factor approximation is possible or not is, arguably, the most important open question. Alternatively, rather than general downward-closed feasibility constraints, one can also focus on specific feasibility constraints of interest, aiming to design optimal randomized clock auctions for them. Given the highly appealing incentive properties of clock auctions, achieving small constant approximations can yield very practical solutions.

Another interesting direction is to study the best approximation that can be achieved by (deterministic) clock auctions using alternative parameters. While the WFCA gives optimal bounds with respect to the parameters n, m, and $\overline{v}$, there are other parameters that have been used in the literature, for which we do not know what the optimal bound is. For example, when the feasibility constraint corresponds to the intersection of p matroids, one may be able to obtain guarantees in terms of p. Note that the existing construction of [24] implies a $\Omega(\log^{1-\epsilon} p)$ lower bound with respect to this parameter, but we do not know whether this is tight or not. Similarly, in a combinatorial auction with single-minded bidders with bundle size at most d (i.e., each bidder wants at most d items), the same construction implies only a $\Omega(\log^{1-\epsilon} d)$ lower bound. Determining if the WFCA or another clock auction can obtain bounds with respect to these parameters is an interesting open question.

Finally, every clock auction is obviously strategyproof (OSP), but there could also exist other OSP mechanisms that do not take the form of a clock auction. Recently, Ferraioli et al. [30] proved a lower bound showing that the approximation factor of any deterministic OSP mechanism for downward-closed feasibility constraints is $\Omega(\sqrt{\log n})$. To the best of our knowledge, the $O(\log n)$ approximation of the WFCA is the best known bound achieved by a deterministic OSP mechanism. It would be interesting to see whether there exist (non-clock-auction) deterministic OSP mechanisms that can overcome the logarithmic approximation, or whether the WFCA is the optimal OSP mechanism in this setting.

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A Omitted Proofs

A.1 Proof of Theorem 7

Proof. We begin by partitioning the bidders by value into $\lceil \log \overline{v} \rceil$ buckets where bucket B_1 contains all bidders with value between 1 and 2 (inclusive) and bucket B_k contains all bidders with value in the range $(2^{k-1}, 2^k]$ for all $k \in \{2, \dots, \lceil \log \overline{v} \rceil\}$. Consider the bucket containing the bidders in the optimal set OPT of highest total value, that is, the bucket B_{k^*} with

$$k^* = \text{argmax}_{k \in [\lceil \log \overline{v} \rceil]} \left\{ \sum_{i \in \text{OPT} \cap B_k} v_i \right\}.$$

We then have that $v(\text{OPT}) = \sum_{k \in \lceil \log \overline{v} \rceil} \sum_{i \in O \cap B_k} v_i \leq (\log \overline{v} + 1) \cdot v(\text{OPT} \cap B_{k^*})$. On the other hand, the welfare obtained by serving the bidders in $\text{OPT} \cap B_{k^*}$ is at most a factor 2 greater than the revenue earned by offering each such bidder a price equal to 2^{k^*-1}. But the water-filling auction obtains welfare at least half the optimal revenue achievable in hindsight, so it then obtains a $4(\log \overline{v} + 1)$-approximation to the social welfare as well. ◀

A.2 Proof of Lemma 8

Proof. Let $\mathcal{E}_1$ denote the event that S_1 has total value at least $v(S)/4$ and $\mathcal{E}_2$ denote the event that S_2 has total value at least $v(S)/4$. We then seek to lower bound $Pr[\mathcal{E}_1 \bigcap \mathcal{E}_2]$. Consider the complement of this event, $\bar{\mathcal{E}}_1 \bigcup \bar{\mathcal{E}}_2$. By an application of a union bound $Pr[\bar{\mathcal{E}}_1 \bigcup \bar{\mathcal{E}}_2] \leq Pr[\bar{\mathcal{E}}_1] + Pr[\bar{\mathcal{E}}_2]$. By coupling the event that S_1 is some subset $T \subseteq S$ with the event that S_1 is exactly the subset $S \setminus T$ we observe that $Pr[\bar{\mathcal{E}}_1] + Pr[\bar{\mathcal{E}}_2] = 2 \cdot Pr[\bar{\mathcal{E}}_2]$. On the other hand, $\mathcal{E}_2$ only fails to occur if set S_1 has total value strictly greater than $\frac{3v(S)}{4}$ (since $v(S_1) + v(S_2) = v(S)$). Since bidders are partitioned uniformly at random into S_1 and S_2 we have that $\mathbb{E}[v(S_1)] = \frac{S}{2}$. We can then upper bound the probability of $\bar{\mathcal{E}}_2$ by an application of the Hoeffding bound. Let X_i be a random variable denoting the contribution of item i to S_1. In other words, $X_i = v_i$ if bidder i is in S_1 and 0 otherwise. We then have that $v(S_1) = \sum_{i=1}^{|S|} X_i$. By applying a Hoeffding bound [36] we obtain that

$$\begin{aligned} Pr[\bar{\mathcal{E}}_2] = Pr\left[v(S_1) - \mathbb{E}[v(S_1)] \geq \frac{v(S)}{4}\right] &\leq \exp\left(\frac{-2(v(S)/4)^2}{\sum_{j=1}^{|S|} v_i^2}\right) \\ &\leq \exp\left(\frac{-v(S)^2/8}{\sum_{j=1}^{|S|}\left(\frac{\alpha v(S)}{j}\right)^2}\right) \\ &\leq \exp\left(\frac{-v(S)^2/8}{\alpha^2 v(S)^2 \pi^2/6}\right) \\ &\leq e^{-\frac{1}{14\alpha^2}} \end{aligned}$$

where the second inequality comes from our defined upper bound on each v_i and the third inequality comes from the convergence of the sum of the squared reciprocals of the natural numbers. With this in hand, we may conclude that $Pr[\mathcal{E}_1 \bigcap \mathcal{E}_2] \geq 1 - 2 \cdot e^{\frac{-1}{14\alpha^2}}$ as desired. ◀

A.3 Stronger lower bound for two disjoint maximal sets

▶ **Theorem 13.** *No randomized auction (even with unbounded computational power) can obtain better than a $1.144 - \epsilon$ approximation for constant $\epsilon > 0$ in a combinatorial auction setting with $n+1$ single-minded bidders even when the feasibility constraint comprises exactly two maximal sets, one containing a single bidder and the other containing n bidders.*

Proof. We proceed again via Yao's lemma but instead use a family of $n+1$ instances $\{\mathcal{I}_i\}_{i\in[n+1]}$. In each instance, there are two disjoint maximal sets S_1 and S_2 where S_1 contains a single bidder and S_2 contains n bidders. In all instances the single bidder in S_1 has value 1. In instance $\mathcal{I}_i$ the i-th bidder in S_2 has value $\frac{17}{20}$ and the remaining bidders each have value $\frac{3}{5n}$. In instance $\mathcal{I}_n$ all of the bidders in S_2 have value $\frac{3}{5n}$. Consider a distribution over these instances where $\mathcal{I}_i$ occurs with probability $\frac{3}{5n}$ for all $i \in [n]$ and $\mathcal{I}_n$ occurs with the remaining probability $\frac{2}{5}$. The expected optimal welfare is then $\frac{3}{5} \cdot \left(\frac{17}{20} + \frac{3(n-1)}{5n}\right) + \frac{2}{5} = \frac{127}{100} - \frac{9}{25n}$. Consider designing a deterministic clock auction for this distribution of instances. The only form for a deterministic auction is to raise the price of some (possibly adaptive) number of bidders in S_2 and to choose based on their responses to either accept S_1 or S_2. Note that an auction which accepts the bidders of S_2 after having only approached bidders in S_2 which reject the increased clock price is weakly worse than one which accepts the bidders in S_2 without raising any clock prices. This is because the conditional probability that one of the bidders in S_2 has high value after some number of bidders have rejected the higher price only decreases and the auction has lost value from the bidders who rejected their price. Thus, an optimal auction for this setting can only take on three forms: (i) accepting S_1 without raising the price of any bidders in S_2, (ii) accepting S_2 without raising the price of any bidders in S_2, or (iii) raising the prices of the bidders in S_2 one by one and accepting S_2 if a bidder accepts the increased price and enough value in S_2 remains (otherwise the auction accepts S_1). An auction which always accepts S_1 obtains welfare exactly 1 and fails to give the desired bound. An auction which always accepts S_2 without raising the price of any of the bidders in S_2 obtains expected welfare $\frac{3}{5} \cdot \left(\frac{17}{20} + \frac{3(n-1)}{5n}\right) + \frac{2}{5} \cdot \frac{3}{5} = \frac{111}{100} - \frac{9}{25n}$. But then for large enough n the auction always accepting S_2 fails to obtain a guarantee of $\frac{127}{111} - \epsilon \geq 1.144 - \epsilon$.

Assume that a mechanism $\mathcal{M}$ obtains the desired approximation of $1.144 - \epsilon$. By the above arguments, it must raise the prices of the bidders in S_2 until some bidder accepts. Since $\mathcal{M}$ is deterministic, it must have a fixed order in which it raises the clocks of the bidders in S_2 assuming all bidders reject. Fix this particular order over the bidders. Observe that, if fewer than $\frac{n}{4}$ bidders in S_2 remain active before $\mathcal{M}$ reaches a bidder who accepts the increased clock price then $\mathcal{M}$ can maximize its welfare by accepting S_1. On the other hand, since if S_2 contains a high value bidder each bidder is equally likely to have high value, we can then calculate the expected welfare that $\mathcal{M}$ obtains if a high value bidder exists. If such a bidder exists, $\mathcal{M}$ obtains value 1 if it is in the final $\frac{n}{4}$ bidders and $\frac{17}{20} + \frac{3}{5n} \cdot (n-k)$ if it is the k-th bidder. If no high value bidder is contained in S_2 (i.e., the instance is $\mathcal{I}_n$) then $\mathcal{M}$ obtains welfare 1. We then find that $\mathcal{M}$ obtains expected welfare equal to

$$1 \cdot \frac{2}{5} + 1 \cdot \frac{3}{20} + \sum_{k=1}^{3n/4} \frac{3}{5n} \cdot \left(\frac{29}{20} - \frac{3k}{5n} \right) = \frac{481}{400} - \frac{9(3n/4+1)(3n/8)}{25n^2} = \frac{881}{800} - \frac{27}{200n}.$$

But then for large enough n, $\mathcal{M}$ fails to achieve an approximation of $\frac{1016}{800} - \epsilon \geq 1.144 - \epsilon$ if it raises the prices of bidders in S_2 until one bidder accepts the higher price, a contradiction. Since no deterministic mechanism can achieve a $1.144-\epsilon$-approximation for any $\epsilon > 0$ when facing this distribution of instances, by Yao's lemma no randomized mechanism can achieve a $1.144-\epsilon$-approximation for any $\epsilon > 0$ against an adversarial instance. ◀