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On the topology design of large wireless sensor networks for distributed consensus with low power consumption

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Abstract

Sensor-based structural health monitoring systems are commonly used to provide real-time information and detect damage in complex structures. In particular, wireless structural health monitoring systems are of low cost but, since wireless sensors are powered with batteries, a low power consumption is critical. A common approach for wireless structural health monitoring is to use a distributed computation strategy, which is usually based on consensus algorithms. Power consumption in such wireless consensus networks depends on the number of connections of the network. If sensors are randomly connected, there is no control on the power consumption. In this article, we present a novel strategy to connect a large number of wireless sensors for distributed consensus with low power consumption by combining small networks with basic topologies using the Kronecker product.

Keywords

Low power consumption, network design, large wireless sensor networks, distributed consensus, Kronecker product

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Introduction

Monitoring complex structures such as aerospace, civil and mechanical infrastructures to provide real-time information and detect damage is referred in the literature as structural health monitoring (SHM). Visual inspection and time-based maintenance procedures are replaced by damage assessment processes using new technological developments such as sensor-based SHM systems. Since large-scale sensor networks are used for SHM, traditional wire-based SHM systems require significant time and cost for cable installation and maintenance. Wireless SHM systems are therefore an alternative solution for low-cost structural monitoring (see previous works).[1,2] Moreover, when monitoring aerospace infrastructures such as plane wings, data cables significantly increase aircraft weight and therefore its fuel consumption. Hence, the use of wireless sensor networks (WSN) is a key factor for the fuel-efficiency of aircrafts. For more details on the current research progress see Noel et al.[3] and references therein.

In wireless SHM systems, it is common that each sensor requires a target value that depends on the values measured by other sensors of the WSN. When the

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WSN has a central entity that computes the target values, sensors have to transmit their measured values to the central entity and this leads to a large energy consumption. Since sensors on a WSN are usually powered with batteries, they have very limited energy resources and hence a reduction in the energy consumption due to transmission (power consumption) increases their life cycle.

The distributed computation strategy is a high-energy efficient strategy, where each sensor computes its target value by interchanging information with its neighbouring sensors.[4] Many practical applications, where the distributed computation strategy is used, rely on the *distributed averaging problem* (also known as the *distributed average consensus problem*), which is the problem of obtaining the average of the values measured in all the sensors of the network in a distributed way.[5,6]

A common approach for solving such problem is to use a linear iterative algorithm that is characterized by a matrix called *weighting matrix*. Its entries depend on the topology of the network considered. In Xiao and Boyd,[7] the authors showed that a weighting matrix that makes the algorithm the fastest possible (i.e. the algorithm with the lowest convergence time) can be obtained by numerically solving a semidefinite programme, which can also be solved in a distributed way (see Insausti et al.).[8] Unfortunately, except for certain basic network topologies, a closed-form expression for this optimal weighting matrix is not found in the literature.

In the literature, there are many works (see references)[9–11] whose goal is to minimize the power consumption in wireless consensus networks. Power consumption of the distributed averaging problem depends on the convergence time of the algorithm used for solving the problem and on the number of connections of the network. Thus, under a convergence time restriction, a reduction in the number of connections leads to a reduction of the power consumption.

If sensors are randomly connected, there is no control neither on the convergence time nor on the energy consumption. In this article, we present a strategy to connect a large number of wireless sensors for distributed consensus with low power consumption. In particular, we consider the case in which the large network is built from other smaller networks with basic topologies, which we call *basic building blocks*, using the Kronecker product. To connect a large number of wireless sensors for distributed consensus with low power consumption under a convergence time restriction, we previously derived a new mathematical result which is the key that gives a closed-form expression of the optimal weighting matrix for a large network built in such way. Moreover, we show that this optimal weighting matrix only depends on the optimal weighting matrices for the employed basic building blocks. Specifically, applying our result, we determine the basic building blocks to be used to minimize the number of connections of the resulting network. Observe that the number of connections of the network directly determines the power consumption of the sensors due to transmission when the convergence time is fixed.

The remainder of this article is organized as follows: The next section states preliminary considerations and the new mathematical result that gives a closed-form expression of the optimal weighting matrix for distributed consensus on any network modelled using the Kronecker product. In section ‘Problem formulation’, we introduce the problem of designing large wireless consensus networks with low power consumption under a convergence time restriction using the Kronecker product. In section ‘Numerical examples’, we numerically solve the considered problem for certain scenarios. Finally, we present our conclusions in section ‘Conclusion’.

Preliminaries

The distributed averaging problem

Consider a network of n nodes. The network can be viewed as an (undirected) graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the set of nodes and $\mathcal{E}$ is the set of edges. We say that the nodes v_i and v_j are connected if $\{v_i, v_j\} \in \mathcal{E}$.

Each node of the network needs to obtain the average (arithmetic mean) of the initial values stored in all the nodes of the network in a distributed way. A common approach for solving this problem is to use a linear iterative algorithm of the form

$$\mathbf{x}(t+1) = W\mathbf{x}(t) \qquad t \in \{0, 1, \ldots\} \tag{1}$$

where $[\mathbf{x}(0)]_{i,1}$ is the initial value stored in the node v_i and W is a real $n \times n$ symmetric matrix, called the weighting matrix, with $[W]_{i,j} = 0$ whenever the nodes v_i and v_j are not connected and $i \neq j$. The weighting matrix W needs to be set so that

$$\lim_{t\to\infty} [\mathbf{x}(t)]_{i,1} = \lim_{t\to\infty} [W^t\mathbf{x}(0)]_{i,1} = x_{\text{ave}} \quad \forall i \in \{1, \ldots, n\} \tag{2}$$

where x_{ave} is the sought average, that is

$$x_{\text{ave}} := \frac{1}{n}\sum_{i=1}^{n} [\mathbf{x}(0)]_{i,1}$$

Let $\mathcal{W}(G)$ be

$$\mathcal{W}(G) := \{W \in \mathbb{R}^{n\times n}, \quad W^T = W, \quad W\mathbf{1}_n = \mathbf{1}_n$$
$$[W]_{i,j} = 0 \text{ if } i \neq j \text{ and } \{v_i, v_j\} \notin \mathcal{E}\}$$

where $\mathbb{R}^{n\times n}$ is the set of all $n\times n$ real matrices, T denotes transpose, and $\mathbf{1}_n$ is the $n\times 1$ matrix of ones. From Xiao and Boyd[7] Theorem 1, if $W\in\mathcal{W}(G)$ and $\|W-P_n\|_2<1$ then, equation (2) holds, where $\|\cdot\|_2$ is the spectral norm (see Bernstein[12]) and $P_n:=(1/n)\mathbf{1}_n\mathbf{1}_n^T$. We assume that G is a connected graph (see Bollobás[13]) to guarantee the existence of such a $W\in\mathcal{W}(G)$ with $\|W-P_n\|_2<1$.

The convergence time of an algorithm of the form equation (1) only depends on the weighting matrix W and was defined in Xiao and Boyd[7] as

$$\tau(W):=\frac{-1}{\log\|W-P_n\|_2} \tag{3}$$

whenever $W\in\mathcal{W}(G)$ and $\|W-P_n\|_2<1$.

An algorithm of the form equation (1) that minimizes the convergence time is known as the fastest symmetric distributed linear averaging (FSDLA) algorithm, that is, its (optimal) weighting matrix W_{opt} satisfies

$$\|W_{\text{opt}}-P_n\|_2\leqslant\|W-P_n\|_2 \qquad \forall W\in\mathcal{W}(G)$$

We denote as $\tau_{\text{opt}}(G)$ the convergence time of the FSDLA algorithm on that graph, that is

$$\tau_{\text{opt}}(G):=\tau(W_{\text{opt}})$$

Building networks using the Kronecker product

We recall that the adjacency matrix of a graph $G=(\mathcal{V},\mathcal{E})$ with n nodes (see Bollobás[13] (Section 8.2)) is the $n\times n$ real symmetric matrix A given by

$$[A]_{i,j}:=\begin{cases}1 & \text{if } \{i,j\}\in\mathcal{E},\\ 0 & \text{otherwise,}\end{cases} \qquad i,j\in\{1,\ldots,n\}$$

In this article, we use the Kronecker product as a tool for modelling a large network. Let G_1 and G_2 be two graphs with n_1 and n_2 nodes, respectively. Using the Kronecker product, we build a larger graph $H=(\mathcal{V}_H,\mathcal{E}_H)$ with n_1n_2 nodes, whose adjacency matrix A_H is given by

$$A_H=(A_1+I_{n_1})\otimes(A_2+I_{n_2})-I_{n_1n_2} \tag{4}$$

where A_1 and A_2 are the adjacency matrices of G_1 and G_2, respectively, I_n is the $n\times n$ identity matrix, and $\otimes$ denotes the Kronecker product. For convenience, we denote the nodes of H as

$$\mathcal{V}_H=\{(k,l):\quad k\in\{1,\ldots,n_1\},\quad l\in\{1,\ldots,n_2\}\}$$

If q_1 is the number of edges of G_1 and q_2 is the number of edges of G_2 then, the number of edges of H is given by

$$q_H=\frac{(2q_1+n_1)(2q_2+n_2)-n_1n_2}{2} \tag{5}$$

A very interesting example of graphs built this way are the grids, which are built using two paths.

A new mathematical result

In this subsection, we give a new mathematical result regarding the FSDLA algorithm on networks built using the Kronecker product. This new mathematical result is key to design large wireless consensus networks with low power consumption. More precisely, we consider a graph H built using the Kronecker product of another two graphs G_1 and G_2 as explained in subsection 'Building networks using the Kronecker product'. Without loss of generality, we assume that $\tau_{\text{opt}}(G_1)\geqslant\tau_{\text{opt}}(G_2)$. Let W_1 be the weighting matrix of the FSDLA algorithm on G_1 and let W_2 be any weighting matrix in $\mathcal{W}(G_2)$ satisfying $\tau(W_2)\leqslant\tau(W_1)=\tau_{\text{opt}}(G_1)$. Theorem 1 shows that

- The convergence time of the FSDLA algorithm on H is equal to the convergence time of the FSDLA algorithm on G_1.
- $W_1\otimes W_2$ is the weighting matrix of the FSDLA algorithm on H.

Theorem 1. Let G_1 and G_2 be two connected graphs with n_1 and n_2 nodes, respectively, satisfying $\tau_{opt}(G_1)\geqslant\tau_{opt}(G_2)$. Consider $W_1\in\mathcal{W}(G_1)$ such that $\tau(W_1)=\tau_{opt}(G_1)$ and let $W_2\in\mathcal{W}(G_2)$ with $\tau(W_2)\leqslant\tau(W_1)$. Then, $W_1\otimes W_2\in\mathcal{W}(H)$, and

$$\tau_{opt}(G_1)=\tau(W_1\otimes W_2)=\tau_{opt}(H)$$

where H is the graph whose adjacency matrix A_H is defined as in equation (4).

Proof. We divide the proof into three steps.

Step 1: We show that $W_1\otimes W_2\in\mathcal{W}(H)$. Observe that $W_1\otimes W_2\in\mathbb{R}^{n_1n_2\times n_1n_2}$

$$(W_1\otimes W_2)^T=W_1^T\otimes W_2^T=W_1\otimes W_2$$

and

$$\begin{aligned}(W_1\otimes W_2)\mathbf{1}_{n_1n_2}&=(W_1\otimes W_2)(\mathbf{1}_{n_1}\otimes\mathbf{1}_{n_2})\\&=(W_1\mathbf{1}_{n_1})\otimes(W_2\mathbf{1}_{n_2})=\mathbf{1}_{n_1}\otimes\mathbf{1}_{n_2}\\&=\mathbf{1}_{n_1n_2}\end{aligned}$$

Moreover

$$
\begin{aligned}
&[W_1 \otimes W_2]_{n_2(i-1)+k,\, n_2(j-1)+l} \\
&= [W_1]_{i,j}[W_2]_{k,l} \\
&= [A_1 + I_{n_1}]_{i,j}[W_1]_{i,j}[A_2 + I_{n_2}]_{k,l}[W_2]_{k,l} \\
&= [(A_1 + I_{n_1}) \otimes (A_2 + I_{n_2})]_{n_2(i-1)+k,\, n_2(j-1)+l} \\
&\times [W_1 \otimes W_2]_{n_2(i-1)+k,\, n_2(j-1)+l} \\
&= [A_H + I_{n_1 n_2}]_{n_2(i-1)+k,\, n_2(j-1)+l} \\
&\times [W_1 \otimes W_2]_{n_2(i-1)+k,\, n_2(j-1)+l}
\end{aligned}
$$

for all $i,j \in \{1,\ldots,n_1\}$ and $k,l \in \{1,\ldots,n_2\}$.

Step 2: We prove that $\tau(W_1 \otimes W_2) = \tau(W_1)$. If $W \in \mathbb{R}^{n\times n}$ is symmetric, we denote its eigenvalues by $\lambda_k(W)$, $k \in \{1,\ldots,n\}$, with $|\lambda_1(W)| \geqslant \ldots \geqslant |\lambda_n(W)|$. Since $\tau(W_2) \leqslant \tau(W_1)$, we have $\|W_2 - P_{n_2}\|_2 \leqslant \|W_1 - P_{n_1}\|_2 < 1$. Hence, from Insausti et al.[8] (Lemma 1) we have $\|W_2 - P_{n_2}\|_2 = |\lambda_2(W_2)|$ and $\|W_1 - P_{n_1}\|_2 = |\lambda_2(W_1)|$. Consequently, as

$$
\begin{aligned}
&|\lambda_2(W_1 \otimes W_2)| \\
&= \max(|\lambda_1(W_1)\lambda_2(W_2)|, |\lambda_1(W_2)\lambda_2(W_1)|) \\
&= \max(|\lambda_2(W_2)|, |\lambda_2(W_1)|) \\
&= |\lambda_2(W_1)| = \|W_1 - P_{n_1}\|_2 < 1
\end{aligned}
$$

we obtain that

$$\|W_1 \otimes W_2 - P_{n_1 n_2}\|_2 = \|W_1 - P_{n_1}\|_2$$

Step 3: We prove that $\tau(W_1 \otimes W_2) = \tau_{\text{opt}}(H)$. Let G be a connected graph with n nodes and q edges, and let $B : \mathbb{R}^q \mapsto \mathcal{W}(G)$ be the bijection defined in Insausti et al.[8](equation (8)). Consider the function $f_G : \mathbb{R}^q \mapsto [0,\infty)$ defined as $f_G(w_1,\ldots,w_q) := \|B(w_1,\ldots,w_q) - P_n\|_2$.

Observe that proving $\tau(W_1 \otimes W_2) = \tau_{\text{opt}}(H)$ is equivalent to prove

$$\|W_1 \otimes W_2 - P_{n_1 n_2}\|_2 \leqslant \|B(w_1,\ldots,w_{q_H}) - P_{n_1 n_2}\|_2 \quad (6)$$
$$\text{for all } w_1,\ldots,w_{q_H} \in \mathbb{R}.$$

Let $\mathbf{u} \in \mathbb{R}^{n_1\times 1}$ be such that $\|\mathbf{u}\|_2 = 1$ and $W_1\mathbf{u} = (-1)^s|\lambda_2(W_1)|\mathbf{u}$ for some $s \in \{1,2\}$. Since, W_2 satisfies $W_2\mathbf{1}_{n_2} = \mathbf{1}_{n_2}$, then $(1/\sqrt{n_2})\mathbf{1}_{n_2}$ is the unit eigenvector of W_2 associated to $\lambda_1(W_2) = 1$, and therefore $\mathbf{u} \otimes (1/\sqrt{n_2})\mathbf{1}_{n_2}$ is a unit eigenvector of $W_1 \otimes W_2$ associated to the eigenvalue $(-1)^s|\lambda_2(W_1 \otimes W_2)| = (-1)^s|\lambda_2(W_1)|$.

Applying Insausti et al.[8](Theorem 1), we get a subgradient of f_H associated to the eigenvector $\mathbf{u} \otimes (1/\sqrt{n_2})\mathbf{1}_{n_2}$

$$
\mathbf{g}_H(\mathbf{u}) = \frac{(-1)^{s+1}}{n_2}\begin{pmatrix} \left([\mathbf{u}]_{k_1,1} - [\mathbf{u}]_{p_1,1}\right)^2 \\ \vdots \\ \left([\mathbf{u}]_{k_{q_H},1} - [\mathbf{u}]_{p_{q_H},1}\right)^2 \end{pmatrix}
$$

where $\{(k_m,l_m),(p_m,r_m)\} \in \mathcal{E}_H$ for $m \in \{1,\ldots,q_H\}$. Observe that from equation (4), $\{(k_m,l_m),(p_m,r_m)\} \in \mathcal{E}_H$ if and only if

$$
\begin{cases}
\{k_m,p_m\} \in \mathcal{E}_1 \text{ and } \{l_m,r_m\} \in \mathcal{E}_2, \text{ or} \\
k_m = p_m \text{ and } \{l_m,r_m\} \in \mathcal{E}_2, \text{ or} \\
\{k_m,p_m\} \in \mathcal{E}_1 \text{ and } l_m = r_m
\end{cases}
$$

where $G_1 = (\mathcal{V}_1,\mathcal{E}_1)$ and $G_2 = (\mathcal{V}_2,\mathcal{E}_2)$.

Fix $m \in \{1,\ldots,q_H\}$. If $k_m = p_m$, then

$$[\mathbf{g}_H(\mathbf{u})]_{m,1} = \frac{(-1)^{s+1}}{n_2}\left([\mathbf{u}]_{k_m,1} - [\mathbf{u}]_{p_m,1}\right)^2 = 0 \quad (7)$$

If $\{k_m,p_m\} \in \mathcal{E}_1$, then there exists $d \in \{1,\ldots,q_1\}$ such that

$$\left[\mathbf{g}_{G_1}(\mathbf{u})\right]_{d,1} = (-1)^{s+1}\left([\mathbf{u}]_{k_m,1} - [\mathbf{u}]_{p_m,1}\right)^2$$

where $\mathbf{g}_{G_1}(\mathbf{u})$ is a subgradient of f_{G_1} associated to the eigenvector $\mathbf{u}$. Therefore

$$
\begin{aligned}
[\mathbf{g}_H(\mathbf{u})]_{m,1} &= \frac{(-1)^{s+1}}{n_2}\left([\mathbf{u}]_{k_m,1} - [\mathbf{u}]_{p_m,1}\right)^2 \\
&= \frac{1}{n_2}\left[\mathbf{g}_{G_1}(\mathbf{u})\right]_{d,1}
\end{aligned} \quad (8)
$$

As W_1 is the weighting matrix of the FSDLA algorithm on G_1, $\|W_1 - P_{n_1}\|_2 \leqslant f_{G_1}(w_1,\ldots,w_{q_1})$ for all $w_1,\ldots,w_{q_1} \in \mathbb{R}$, and therefore from Insausti et al.[14] $\mathbf{0}_{q_1}$ is a subgradient of f_{G_1}, where $\mathbf{0}_{q_1}$ denotes the $q_1 \times 1$ matrix of zeros. Since the set of all the subgradients of f_{G_1} is the convex hull of the set

$$\{\mathbf{g}_{G_1}(\mathbf{u}) : W_1\mathbf{u} \in \{\pm\lambda_2(W_1)\mathbf{u}\}, \quad \|\mathbf{u}\|_2 = 1\}$$

there exist unit eigenvectors $\mathbf{u}_1,\ldots,\mathbf{u}_L$ and nonnegative scalars $\alpha_1,\ldots,\alpha_L$ with $\sum_{j=1}^{L}\alpha_j = 1$ satisfying

$$\mathbf{0}_{q_1} = \sum_{j=1}^{L}\alpha_j\mathbf{g}_{G_1}(\mathbf{u}_j)$$

Consequently, from equations (7) and (8) we get that

$$\sum_{j=1}^{L}\alpha_j\mathbf{g}_H(\mathbf{u}_j) = \mathbf{0}_{q_H}$$

and since a convex combination of subgradients is also a subgradient, from Shor[14] equation (6) holds.

Observe that, according to Theorem 1, for obtaining the closed-form expression of the weighting matrix of the FSDLA algorithm on H, there is no need to know the weighting matrix of the FSDLA algorithm on G_2.

Problem formulation

In this section, we introduce the problem of designing large wireless consensus networks with low power consumption.

The power consumption (energy consumption of the sensors of the network due to transmission) of the distributed averaging problem depends on the convergence time of the algorithm used for solving the problem and on the number of edges of the network. Hence, under a convergence time restriction, a reduction in the number of edges leads to a reduction of the power consumption.

We aim to connect a large number of sensors (nodes) so that the distributed averaging problem is solved under a convergence time restriction. We build this large network using the Kronecker product of basic building blocks. We study the problem of combining these basic building blocks (allowing repetitions) to minimize the number of edges of the resulting network.

In particular, let $\{G_1, \ldots, G_k\}$ be the basic building blocks to be combined. We want to use the Kronecker product for building a graph H that connects at least n nodes under a convergence time restriction $\tau_{\max}$. We denote with n_i the number of nodes of G_i, and with r_i the number of times that the basic building block G_i is used to build H (r_i can be zero). The goal is to obtain the values of n_i and r_i for $i \in \{1, \ldots, k\}$ such that the number of edges q of the resulting graph H is minimized. Let $n_{i_{\max}}$ be the maximum number of nodes of G_i such that $\tau_{\text{opt}}(G_i) \leqslant \tau_{\max}$ for all $i \in \{1, \ldots, k\}$. Observe that from Theorem 1, regardless of the combination of basic building blocks employed, any graph H built in this way satisfies

$$\tau_{\text{opt}}(H) = \max_{i \in \{1, \ldots, k\}} \tau_{\text{opt}}(G_i) \leqslant \tau_{\max}$$

Therefore, to connect the n nodes under a convergence time restriction $\tau_{\max}$ and with a minimum number of edges, we need to solve the following minimization problem

$$\begin{aligned} &\text{minimize} \quad q \\ &\text{subject to} \quad \prod_{i=1}^{k} n_i^{r_i} \geqslant n \\ &\qquad\qquad\quad n_i \leqslant n_{i_{\max}}, \quad i \in \{1, \ldots, k\} \end{aligned} \tag{9}$$

Observe that q can be obtained by applying equation (5) recursively. By solving equation (9), we obtain the values of n_i and r_i. Hence, from Theorem 1, the problem of designing the topology of a large consensus WSN under a convergence time restriction becomes into the minimization problem given in equation (9). Moreover, Theorem 1 provides the weighting matrix of the FSDLA algorithm on H, that is, from Theorem 1, there does not exist another weighting matrix that solves the distributed averaging problem faster on H.

For the reader's convenience, we end this section with an algorithm that summarizes the implementation of the solution of the considered problem (see Algorithm I).

Numerical examples

Here, we solve the problem introduced in section 'Problem formulation' for two different scenarios.

Scenario 1: cycles and paths as basic building blocks

We consider two building blocks with basic topologies $\{G_1, G_2\}$. In particular, G_1 is a cycle with n_1 nodes ($n_1 \geqslant 3$) and G_2 is a path with n_2 nodes ($n_2 \geqslant 2$). We want to connect at least n nodes under a convergence time restriction $\tau_{\max}$ and with a minimum number of edges by using r_1 cycles of n_1 nodes and r_2 paths of n_2 nodes. From Insausti et al.[15](Section 2.1.1) we obtain $n_{1_{\max}}$, which is the maximum number of nodes of G_1 such that $\tau_{\text{opt}}(G_1) \leqslant \tau_{\max}$. Similarly, from Insausti et al.[15](Section 2.1.3) we obtain $n_{2_{\max}}$, which is the maximum number of nodes of G_2 such that $\tau_{\text{opt}}(G_2) \leqslant \tau_{\max}$.

In this scenario, it is possible to obtain an explicit expression of q, and therefore the minimization problem in equation (9) can be rewritten as follows

$$\begin{aligned} &\text{minimize} \quad q = \frac{(3n_1)^{r_1}(3n_2-2)^{r_2} - n_1^{r_1} n_2^{r_2}}{2} \\ &\text{subject to} \quad n_1^{r_1} n_2^{r_2} \geqslant n \\ &\qquad\qquad\quad 3 \leqslant n_1 \leqslant n_{1_{\max}} \\ &\qquad\qquad\quad 2 \leqslant n_2 \leqslant n_{2_{\max}} \end{aligned} \tag{10}$$

where the function to be minimized has been obtained by applying equation (5) recursively.

Table 1 shows the numerical resolution of equation (10) when at least 10,000 nodes need to be connected under different convergence time restrictions.

Figure 1(a) and (b) shows the minimum number of edges obtained by solving equation (10) when different number of nodes need to be connected under different convergence time restrictions.

Figure 2 shows the graphs resulting from solving equation (10) when $n = 36$ and for different convergence time restrictions.

Scenario 2: stars and paths as basic building blocks

In this subsection, we study the same scenario considered in subsection 'Scenario 2: cycles and paths as basic building blocks', but G_1 is now a star with n_1 nodes

Table 1. Numerical resolution of equation (10) for at least $n = 10,000$ nodes.

τ_{max}	1	5	10	50	100	500
n_1	4	9	12	29	41	71
r_1	7	2	3	2	2	2
n_2	–	5	6	12	6	2
r_2	0	3	1	1	1	1
$\tau_{opt}(H)$	0.91	4.72	7.45	42.65	85.20	255.42

Table 2. Numerical resolution of equation (11) for at least $n = 10,000$ nodes.

τ_{max}	1	5	10	50	100	500
n_1	–	10	15	100	100	1000
r_1	0	4	3	2	2	1
n_2	2	–	3	–	–	10
r_2	14	0	1	0	0	1
$\tau_{opt}(H)$	0	4.98	7.49	50.00	50.00	500.00

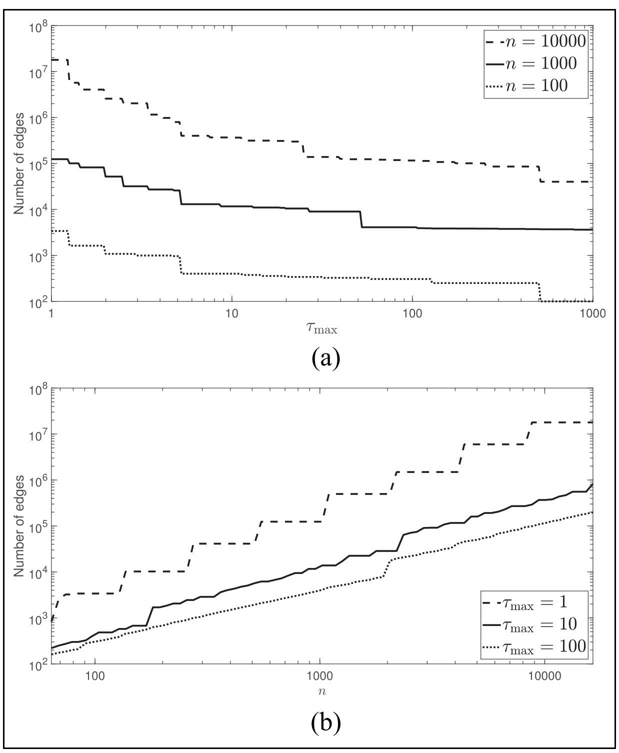

Figure 1. Minimum number of edges for connecting at least n nodes under a convergence time restriction τ_{max}. (a) Variation of the minimum number of edges required for building graphs that connect 100, 1000 and 10,000 nodes when $\tau_{max} \in (1; 1000)$. (b) Variation of the minimum number of edges required for building graphs under a convergence time restriction of 1, 10 and 100 when the number of nodes to connect $n \in (2^8; 2^{14})$.

($n_1 \geqslant 4$). Since a closed-form expression of $\tau_{opt}(G_1)$ is not found in the literature, we numerically compute it from Xiao and Boyd[7] equation (4) to obtain $n_{1_{max}}$, which is the maximum number of nodes of G_1 such that $\tau_{opt}(G_1) \leqslant \tau_{max}$.

In this scenario, the minimization problem in equation (9) can be rewritten as follows

$$\begin{aligned} \text{minimize} \quad & q = \frac{(3n_1-2)^{r_1}(3n_2-2)^{r_2} - n_1^{r_1} n_2^{r_2}}{2} \\ \text{subject to} \quad & n_1^{r_1} n_2^{r_2} \geqslant n \\ & 4 \leqslant n_1 \leqslant n_{1_{max}} \\ & 2 \leqslant n_2 \leqslant n_{2_{max}} \end{aligned} \tag{11}$$

Table 2 shows the numerical resolution of equation (11) when at least 10, 000 nodes need to be connected under different convergence time restrictions.

Algorithm 1 for the design of large wireless consensus networks from basic building blocks under a convergence time restriction.

1. Select k basic building blocks $G_1, \ldots, G_k$.
2. Set τ_{max}.
3. for $i = 1 : k$ do
4. Compute $n_{i_{max}}$ such that $\tau_{opt}(G_i) \leqslant \tau_{max}$
5. end for
6. Solve equation (9)
7. Compute the adjacency matrix of H by applying equation (4) recursively

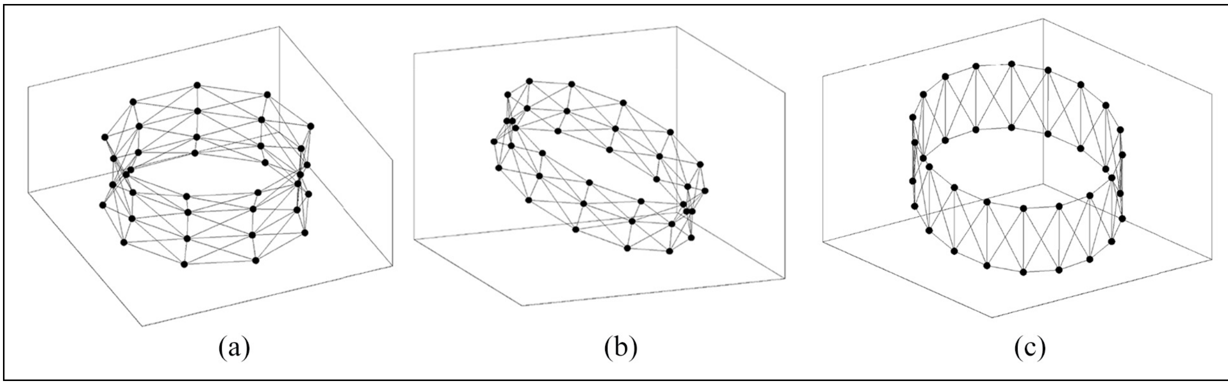

Figure 2. Graph *H* with 36 nodes built using the Kronecker product of cycles and paths. (a) $\tau_{max} = 5$: this graph is built using one cycle of nine nodes and one path of four nodes. It has 117 edges. (b) $\tau_{max} = 10$: this graph is built using one cycle of 12 nodes and one path of 3 nodes. It has 108 edges. (c) $\tau_{max} = 50$: this graph is built using one cycle of 18 nodes and one path of 2 nodes. It has 90 edges.

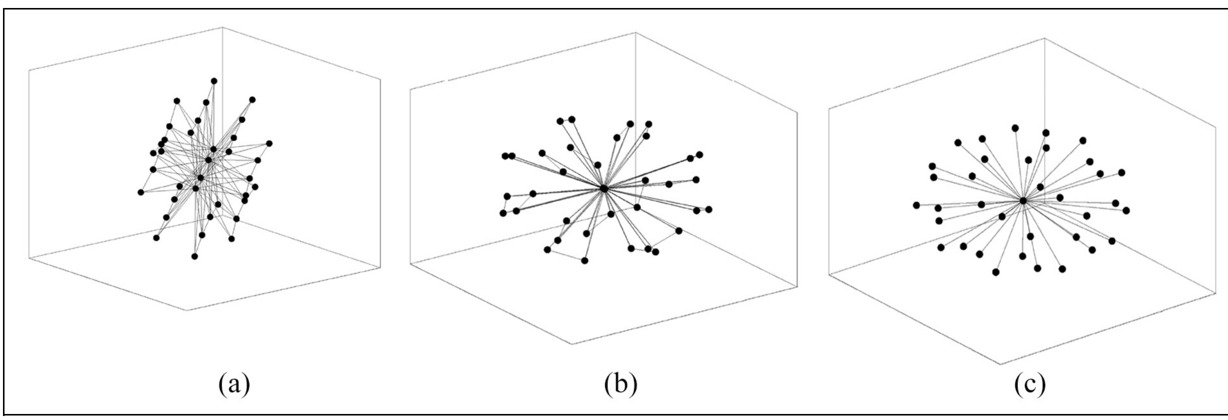

Figure 3. Graph *H* with 36 nodes built using the Kronecker product of stars and paths. (a) $\tau_{max} = 5$: this graph is built using one star of nine nodes and one path of four nodes. It has 107 edges. (b) $\tau_{max} = 10$: this graph is built using one star of 18 nodes and one path of 2 nodes. It has 86 edges. (c) $\tau_{max} = 50$: this graph is built using a single star of 18 nodes. It has 35 edges.

Figure 3 shows the graphs resulting from solving equation (11) when $n = 36$ and for different convergence time restrictions.

Conclusion

In this article, we have proposed a method for building large wireless consensus networks with low power consumption under a convergence time restriction. These large networks are built from other smaller networks with basic topologies using the Kronecker product. We observe that the advantage of building a large network this way is that it inherits the symmetries of its corresponding basic building blocks. Hence, the shape of the structure to be monitored determines the suitability of the basic building blocks to be used.

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References

1. Federici F, Graziosi F, Faccio M, et al. An integrated approach to the design of wireless sensor networks for structural health monitoring. *Int J Distrib Sens Netw* 2012; 8(3): 594842.
2. Wang P, Yan Y, Tian GY, et al. Investigation of wireless sensor networks for structural health monitoring. *J Sens* 2012; 2012: 156329.
3. Noel AB, Abdaoui A, Elfouly T, et al. Structural health monitoring using wireless sensor networks: a comprehensive survey. *IEEE Commun Surv Tut* 2017; 19(3): 1403–1423.
4. Dimakis AG, Kar S, Moura JMF, et al. Gossip algorithms for distributed signal processing. *Proc IEEE* 2010; 98(11): 1847–1864.
5. Hendrickx JM, Olshevsky A and Tsitsiklis JN. Distributed anonymous discrete function computation. *IEEE T Automat Contr* 2011; 56(10): 2276–2289.
6. Dai HK and Toulouse M. Lower-bound study for function computation in distributed networks via vertex-eccentricity. *SN Comput Sci* 2019; 1(1): 10.
7. Xiao L and Boyd S. Fast linear iterations for distributed averaging. *Syst Control Lett* 2004; 53: 65–78.
8. Insausti X, Gutiérrez-Gutiérrez J, Zárraga-Rodríguez M, et al. In-network computation of the optimal weighting matrix for distributed consensus on wireless sensor networks. *Sensors* 2017; 17(8): 1702.
9. Insausti X, Camaró F, Crespo PM, et al. Distributed pseudo-gossip algorithm and finite-length computational codes for efficient in-network subspace projection. *IEEE J Sel Top Signa* 2013; 7(2): 163–174.
10. Asensio-Marco C and Beferull-Lozano B. Energy efficient consensus over complex networks. *IEEE J Sel Top Signa* 2015; 9(2): 292–303.
11. Sardellitti S, Barbarossa S and Swami A. Optimal topology control and power allocation for minimum energy consumption in consensus networks. *IEEE T Signal Pr* 2012; 60(1): 383–399.
12. Bernstein DS. *Matrix mathematics*. Princeton, NJ: Princeton University Press, 2009, pp.350, 603.
13. Bollobás B. *Modern graph theory* (Graduate texts in mathematics). New York: Springer, 1998, p.6.
14. Shor NZ. *Minimization methods for non-differentiable functions*. New York: Springer, 1985, p.12.
15. Gutiérrez-Gutiérrez J, Zárraga-Rodríguez M and Insausti X. Analysis of known linear distributed average consensus algorithms on cycles and paths. *Sensors* 2018; 18(4): 968.