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Modified almost unbiased two-parameter estimator for the Poisson regression model with an application to accident data

Mustafa I. Alheety[1], Muhammad Qasim[2], Kristofer Månsson[2] and B.M. Golam Kibria[3]

Abstract

Due to the large amount of accidents negatively affecting the wellbeing of the survivors and their families, a substantial amount of research is conducted to determine the causes of road accidents. This type of data come in the form of non-negative integers and may be modelled using the Poisson regression model. Unfortunately, the commonly used maximum likelihood estimator is unstable when the explanatory variables of the Poisson regression model are highly correlated. Therefore, this paper proposes a new almost unbiased estimator which reduces the instability of the maximum likelihood estimator and at the same time produce smaller mean squared error. We study the statistical properties of the proposed estimator and a simulation study has been conducted to compare the performance of the estimators in the smaller mean squared error sense. Finally, Swedish traffic fatality data are analyzed to show the benefit of the proposed method.

MSC: 62J05; 62J07.

Keywords: Applied traffic modeling, Maximum likelihood estimator, Mean squared error matrix, Poisson regression, Simulation study, Traffic fatality.

1. Introduction

According to the World Health Organization (2015), fatalities caused by motor vehicle collisions leads to more than 1.2 million deaths worldwide. This large amount of acci-

[1] Department of Mathematics,College of Education for Pure Sciences, University of Anbar, Iraq.

[2] Department of Economics, Finance and Statistics, Jönköping University, Sweden.

[3] Department of Mathematics and Statistics, Florida International University, Miami, FL 33199, USA, Email: kibriag@fiu.edu

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dents negatively affects the wellbeing of the survivors and their families (Donaldson, Brooke and Faux, 2009). Therefore a great interest exists in developing new models and methods to estimate the causes of accidents. Examples of previous research where new methods are suggested have appeared in Ivan, Wang and Bernardo (2000), Lyon et al. (2003), Lord, Manar and Vizioli (2005b), Chiou and Fu (2013) and Shi, Abdel-Aty and Lee (2016) among others. This paper is motivated by the work of Shi et al. (2016) and focuses on the issue of multicollinearity which is defined as the situation when two or more explanatory variables are highly correlated.

The problem of multicollinearity has significant impact on the performance of ordinary least squares (OLS) estimation of unknown regression coefficients. Furthermore, it leads to instability and a high variance of the parameters estimated by OLS and eventually provides the wrong sign of the regression coefficients. Another consequence of multicollinearity is the wider confidence interval, decreased statistical power which result in increased probability of type II error in hypothesis testing in terms of the parameters. As a solution to this problem for linear regression models, Hoerl and Kennard (1970a, 1970b) proposed the ridge regression (RR) method, which is a biased or shrinkage estimator, as an alternative to ordinary least squares. They showed that one may reduce the variance of the estimated coefficients substantially by introducing a small amount of bias. This method was generalized in order to be used for models estimated by maximum likelihood estimator (MLE) such as the logit and Poisson models by Schaefer, Roi and Wolfe (1984) and Månsson and Shukur (2011), among others. Kibria, Månsson and Shukur (2015) proposed several estimators for estimating the ridge parameter k based on Poisson ridge regression (PRR) model. Liu (1993) by taking the advantage of ridge regression and Stein estimator (1956), proposed a new biased estimator and showed its merit for the linear regression model. The ridge (Hoerl and Kennard, 1970a), Liu (1993) and Liu-type estimators have been developed for other generalized linear models such as negative binomial regression, Poisson regression, zero inflated Poisson regression, gamma regression and beta regression models, for instances, see Månsson (2011), Månsson (2013), Asar and Genç (2018), Cetinkaya and Kaciranlar (2019), Toker, Ustundağ and Qasim (2019), Qasim et al. (2020a, 2020b), Kibria, Månsson and Shukur (2013), Huang and Yang (2014), Kurtoglu and Ozkale (2016), Qasim, Amin and Amanullah (2018), Lukman et al. (2020), Amin, Qasim and Amanullah (2019), Amin et al. (2020a, 2020b), Karlsson, Månsson and Kibria (2020), Qasim, Månsson and Kibria (2021) among others.

In this paper, we propose a new general biased estimator for Poisson regression model, which will be called the modified almost unbiased two-parameter Poisson estimator (MAUTPPE). The previous methods suggested by Månsson and Shukur (2011) and Shi et al. (2016) have disadvantages of inducing much bias. This is an unattractive property to applied researchers of these estimators and therefore, in this paper, we suggest a bias correction that substantially reduces the bias and still solves the problem of multicollinearity. As an illustration of this new method, we model traffic fatality data of Sweden. We show a substantial increase of predictive power of this new method as compared to MLE and the standard ridge regression method.

The organization of the paper is as follows. The proposed estimator and its superiority are given in Section 2. The estimation of the shrinkage parameters are outlined in Section 3. To compare the performance of the estimators, a simulation study has been conducted in Section 4. An application about the traffic fatalities in Sweden is given in Section 5. Finally some concluding remarks are given in Section 6.

2. Statistical methodology

2.1. Maximum likelihood estimator for the Poisson regression model

The Poisson regression model is used when the dependent variable (y_i) comes in the form of count data and distributed as $\mathrm{P}(\mu_i)$, where μ_i is a parameter of the Poisson distribution and it can be written as $\mu_i = \exp(x_i\beta)$ as mean response function for the Poisson regression model, where x_i is the i-th row of X which is a $n\times(p+1)$ data matrix with p explanatory variables and β is a $(p+1)\times 1$ vector of coefficients. The traditional MLE is used to estimate β. The log likelihood of this model corresponds to:

$$L(\beta;y) = \sum_{i=1}^{n}\exp(x_i\beta) + \sum_{i=1}^{n} y_i \log(\exp(x_i\beta)) + \log(\prod_{i=1}^{n} y_i!) \tag{1}$$

Solving $L(\beta;y)$ with respect to β results in:

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{n}(y_i - \exp(x_i\beta))x_i = 0$$

Now, we use the iteratively re-weighted least squares (IRLS) algorithm to get the MLE which can be written as follows:

$$\hat{\beta} = (X^{\mathsf{T}}\hat{W}X)^{-1}X^{\mathsf{T}}\hat{W}Z = (S)^{-1}X^{\mathsf{T}}\hat{W}Z, \tag{2}$$

where $S = X^{\mathsf{T}}\hat{W}X$, $\hat{W} = \mathrm{diag}(\hat{\mu}_i)$ and Z is the column vector with

$$Z_i = \log(\hat{\mu}_i)\frac{y_i - \hat{\mu}_i}{\hat{\mu}_i}$$

The MLE of $\hat{\beta}$ is asymptotically unbiased estimator of β. When the explanatory variables are suffering for high correlation, the matrix S is ill-conditioned and the MLE becomes unstable with high variance. To solve this problem, Månsson and Shukur (2011) introduced the Poisson ridge estimator (PRE) as follows:

$$\hat{\beta}_{\mathrm{PRR}} = (S + kI_p)^{-1}S\hat{\beta}, k > 0 \tag{3}$$

Also, Månsson et al. (2012) and Qasim et al. (2019) proposed the Poisson Liu regression estimator (PLRE) as:

$$\hat{\beta}_{\mathrm{PLE}} = (S + I_p)^{-1}(S + dI_p)\hat{\beta}$$

$$= [I_p - (1-d)(S+I_p)^{-1}]\hat{\beta}, \qquad 0 < d < 1 \tag{4}$$

In order to get an estimator that performs better than the PRE and PLRE, Asar and Genç (2018) proposed the following two-parameter Poisson estimator (TPPE) as:

$$\hat{\beta}_{\text{TPE}} = T_{k,d}\hat{\beta}, \qquad k > 0, \quad 0 < d < 1 \tag{5}$$

where $T_{k,d} = (S+kI_p)^{-1}(S+kdI_p)$.

2.2. The proposed estimator

The TPPE (Asar and Genç, 2018) is the biased estimator and it has disadvantage of inducing considerable bias. This is an unattractive property to applied researchers. Therefore, in this section, we propose a bias correction that substantially reduces the bias and is more efficient than TPPE as well as improved estimators. The new estimator, which we called the modified almost unbiased two-parameters Poisson estimator, denoted by $\hat{\beta}_{\text{MAUTPPE}}$ and defined as follows:

$$\hat{\beta}_{\text{MAUTPPE}} \quad = \quad F_{k,d}\hat{\beta}, \qquad k > 0, \quad 0 < d < 1 \tag{6}$$

where $F_{k,d} = [I_p - (1-d)^2(S+I_p)^{-2}](I_p + kS^{-1})^{-1}, \quad 0 < d < 1, k > 0.$

The estimator in (6) is motivated from the following fact: The bias of $\hat{\beta}_{\text{PLE}}$ in Eq. (4) is given as

$$\text{Bias}(\hat{\beta}_{\text{PLE}}) = -(1-d)(S+I_p)^{-1}\beta.$$

Hence, by following Kadiyala (1984), the biased corrected of $\hat{\beta}_{\text{PLE}}$ can be defined as

$$\tilde{\beta}_{\text{PLE}} = \hat{\beta}_{\text{PLE}} + (1-d)(S+I_p)^{-1}\hat{\beta}.$$

Therefore, by following Ohtani (1986), we replace the $\hat{\beta}$ by $\hat{\beta}_{\text{PLE}}$ to get the almost unbiased PLRE, $\tilde{\beta}_{\text{PLE}}$:

$$\begin{aligned}\tilde{\beta}_{\text{PLE}} &= [I_p - (1-d)(S+I_p)^{-1}]\hat{\beta}_{\text{PLE}} \\ &= [I_p - (1-d)^2(S+I_p)^{-2}](I_p + kS^{-1})^{-1}\hat{\beta}\end{aligned} \tag{7}$$

Now, if we replace $\hat{\beta}$ in Eq. (7) by $\hat{\beta}_{\text{PRE}}$ from Eq. (3), we get the proposed estimator in Eq. (6).

The properties of the MAUTPPE are obtained as follows:

$$E(\hat{\beta}_{\text{MAUTPPE}}) = F_{k,d}\beta$$

The bias of the MAUTPPE:

$$\begin{aligned}\text{Bias}(\hat{\beta}_{\text{MAUTPPE}}) &= (F_{k,d} - I_p)\beta \\ &= [(I_p - (1-d)^2(S+I_p)^{-2})(I_p + kS^{-1})^{-1} - I_p]\beta \\ &= = S^{-1}\{-k(S+I_p)^2 - S(1-d)^2\}(S+I_p)^{-2}(S+kI_p)^{-1}S\beta \\ &= B_1^*.\end{aligned} \tag{8}$$

The variance covariance matrix of the MAUTPPE is given as:

$$\text{Cov}\Big(\hat{\beta}_{\text{MAUTPPE}}\Big) = F_{k,d}S^{-1}F_{k,d}. \tag{9}$$

2.3. Properties of the estimators

We use the spectral decomposition in order to find the matrix mean square error (MMSE) and scalar mean squared error (SMSE). So, we can rewrite the matrix S as $S = P\Lambda P^{\intercal}$, where P and Λ are the eigenvectors and eigenvalues of the matrix S, respectively, such that $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p)$. Since MAUTPPE is the biased estimator, we have to use the MMSE as a criterion for goodness of fit where it is containing all relevant information regarding the estimators (such as, variance and biased). The MMSE of an estimator $\tilde{\beta}$ of β can be written as:

$$\begin{aligned} \text{MMSE}(\tilde{\beta}) &= E(\tilde{\beta}-\beta)(\tilde{\beta}-\beta)^{\intercal} \\ &= \text{Var}(\tilde{\beta}) + (\text{Bias}(\tilde{\beta}))(\text{Bias}(\tilde{\beta}))^{\intercal} \end{aligned}$$

$$\begin{aligned} \text{MMSE}(\hat{\beta}_{\text{MAUTPPE}}) &= P(I_p + k\Lambda^{-1})^{-1}(I_p - (1-d)^2(\Lambda + I_p)^{-2}) \\ & \quad \Lambda^{-1}(I_p - (1-d)^2(\Lambda+I_p)^{-2})(I_p + k\Lambda^{-1})^{-1}P^{\intercal} + B_1 B_1^{\intercal}, \end{aligned} \tag{10}$$

where k and d are the biasing parameters and $B_1 = \text{Bias}(\hat{\beta}_{\text{MAUTPPE}}) = (F_{k,d} - I_p)\alpha$, where $\alpha = P^{\intercal}\beta$.

If we take the trace of MMSE, then we get SMSE as follows:

$$\text{SMSE}(\tilde{\beta}) = \text{tr}(\text{MMSE}(\tilde{\beta})) \tag{11}$$

So,

$$\text{MMSE}(\hat{\beta}_{\text{MAUTPPE}}) = \sum_{j=1}^{p} \frac{\lambda_j \{(\lambda_j+1)^2 - (1-d)^2 \quad {}^2 + \alpha^2 \{k(\lambda_j+1)^2 + \lambda_j(1-d)^2 \quad {}^2}{(\lambda_j+k)^2(\lambda_j+1)^2} \tag{12}$$

Asar and Genç (2018) computed the MMSE and SME of the TPPE as:

$$\text{MMSE}(\hat{\beta}_{\text{TPE}}) = P(\Lambda+k)^{-1}(\Lambda+kdI_p)\Lambda^{-1}(\Lambda+kdI_p)(\Lambda+k)^{-1}P^{\intercal} + B_2B_2^{\intercal},$$

$$\text{SMSE}(\hat{\beta}_{\text{TPE}}) = \sum_{j=1}^{p}\Big(\frac{(\lambda_j+kd)^2}{\lambda_j(\lambda_j+k)^2}\Big) + \sum_{j=1}^{p}\Big(\frac{\alpha_j^2(d-1)^2k^2}{(\lambda_j+k)^2}\Big)$$

where $B_2 = \text{Bias}(\hat{\beta}_{\text{TPE}}) = P(\Lambda + kI_p)^{-1}\alpha(d-1)k$.

The MMSE and SMSE of the MLE are defined respectively as follows:

$$\text{MMSE}(\hat{\beta}) = S^{-1} = P\Lambda^{-1}P^{\intercal}.$$

$$\text{SMSE}(\hat{\beta}) = \sum_{j=1}^{p}\frac{1}{\lambda_j}$$

2.4. The performance of the proposed estimator

2.4.1. The comparison between the MLE and MAUTPPE

The comparison between MLE and MAUTPPE are illustrated using matrix mean squared error (MMSE):

$$\mathrm{MMSE}(\hat{\beta}) = S^{-1}.$$

$$\mathrm{MMSE}(\hat{\beta}_{\mathrm{MAUTPPE}}) = F_{k,d}S^{-1}F_{k,d} + B_1B_1^{\mathsf{T}},$$

We state the following theorem to demonstrate the comparison between MLE and MAUTPPE.

Theorem 2.1. *Under MMSE criterion, the MAUTPPE ($\hat{\beta}_{MAUTPPE}$) is superior to the MLE ($\hat{\beta}$), namely, $MMSE(\hat{\beta}) - MMSE(\hat{\beta}_{MAUTPPE}) \geq 0$ if and only if:*

$$B_1^{\mathsf{T}}[S^{-1} - F_{k,d}S^{-1}F_{k,d}]^{-1}B_1 \leq 1$$

Proof. The difference of MMSE values between MLE and MAUTPPE can be found as

$$\begin{aligned}\Delta_1 = \mathrm{MMSE}(\hat{\beta}) - \mathrm{MMSE}(\hat{\beta}_{\mathrm{MAUTPPE}}) &= S^{-1} - (F_{k,d}S^{-1}F_{k,d} + B_1B_1^{\mathsf{T}}) \\ &= D_1 - B_1B_1^{\mathsf{T}},\end{aligned}$$

where $D_1 = S^{-1} - F_{k,d}S^{-1}F_{k,d}$.

Let $D_1 = P\Upsilon P^{\mathsf{T}} = P\mathrm{diag}\{\gamma_1, \ldots, \gamma_p\}P^{\mathsf{T}}$ by using the spectral decomposition, where

$$\Upsilon = \Lambda^{-1} - (I+k\Lambda^{-1})^{-1}(I-(1-d)^2(\Lambda+I)^{-2})\Lambda^{(-1)}(I-(1-d)^2(\Lambda+I)^{-2})(I+k\Lambda^{-1})^{-1}.$$

Therefore,

$$\gamma_j = \frac{1 - \left[\left(1+\frac{k}{\lambda_j}\right)^{-2}\left(1-\frac{(1-d)^2}{(\lambda_j+1)^2}\right)\left(1-\frac{(1-d)^2}{(\lambda_j+1)^2}\right)\right]}{\lambda_j}, \quad j = 1, \ldots, p$$

Since $\frac{(1-d)^2}{(\lambda_j+1)^2} < 1$ and $\left(1+\frac{k}{\lambda_j}\right)^{-2} < 1$ for $k > 0, 0 < d < 1$ and $\lambda_j > 0$. Then

$$\left(1+\frac{k}{\lambda_j}\right)^{-2}\left(1-\frac{(1-d)^2}{(\lambda_j+1)^2}\right)\left(1-\frac{(1-d)^2}{(\lambda_j+1)^2}\right) > 1;$$

and that means $\gamma_j > 0, \forall j$.

This implies that D_1 is positive definite Now, in order to find the conditions that make Δ_1 is positive definite, we have to introduce the Lemma 2.1:

Lemma 2.1 (See Farebrother, 1976)**.** *Let M be a positive definite matrix and α be a vector, then $M - \alpha\alpha^{\mathsf{T}} \geq 0$ if and only if $\alpha^{\mathsf{T}}M^{-1}\alpha \leq 1$.*

Therefore, by applying Lemma 2.1, the proof is completed.

2.4.2. The comparison between the TPPE and MAUTPPE estimators

The properties of TPPE are obtained as follows:

$$\begin{aligned}\text{Bias}(\hat{\beta}_{\text{TPPE}}) &= k(d-1)(S+kI_p)^{-1}\beta \\ &= B_2\end{aligned}$$

and

$$\text{Cov}(\hat{\beta}_{\text{TPPE}}) = T_{k,d}S^{-1}T_{k,d}.$$

The MMSE of TPPE is given as follows:

$$\text{MMSE}(\hat{\beta}_{\text{TPPE}}) = T_{k,d}S^{-1}T_{k,d} + B_2B_2^{\mathsf{T}}. \tag{13}$$

The following theorem is demonstrated the comparison between TPPE and MAUTPPE.

Theorem 2.2. *For $0 < d < 1$ for fixed k, under Poisson regression model, the MAUTPPE $\hat{\beta}_{MAUTPPE}$ is superior to TPPE $\hat{\beta}_{TPPE}$ in the sense of MMSE if and only if*

$$B_1^{\mathsf{T}} D_2^{-1} B_1 \leq 1.$$

Proof. The difference of MMSE values between them can be given by:

$$\begin{aligned}\Delta_2 &= \text{MMSE}(\hat{\beta}_{\text{TPPE}}) - \text{MMSE}(\hat{\beta}_{\text{MAUTPPE}}) \\ &= PD_2P^{\mathsf{T}} + B_2B_2^{\mathsf{T}} - B_1B_1^{\mathsf{T}} \\ &= P\,\text{diag}\left\{\frac{(\lambda_j+kd)^2}{\lambda_j(\lambda_j+k)^2} - \frac{\left(1+\frac{k}{\lambda_j}\right)^{-2}\left(1-\frac{(1-d)^2}{(\lambda_j+1)^2}\right)\left(1-\frac{(1-d)^2}{(\lambda_j+1)^2}\right)}{\lambda_j}\right\}_{j=1}^{p} P^{\mathsf{T}} + B_2B_2^{\mathsf{T}} - B_1B_1^{\mathsf{T}},\end{aligned}$$

where

$$\begin{aligned}D_2 &= (\Lambda+kI_p)^{-1}(\Lambda+kd)\Lambda_{-1}(\Lambda+kd)(\Lambda+kI_p)^{-1} \\ &- (I_p-(1-d)^2(\Lambda+I_p)^{-2})(I+k\Lambda^{-1})^{-1}\Lambda^{-1}(I_p+k\Lambda^{-1})^{-1} \\ &\quad (I_p-(1-d)^2(\Lambda+I_p)^{-2})\end{aligned}$$

Since $B_2B_2^{\mathsf{T}}$ is nonnegative definite, we focus upon the quantity

$$\frac{(\lambda_j+kd)^2}{\lambda_j(\lambda_j+k)^2} - \frac{\left(1+\frac{k}{\lambda_j}\right)^{-2}\left(1-\frac{(1-d)^2}{(\lambda_j+1)^2}\right)\left(1-\frac{(1-d)^2}{(\lambda_j+1)^2}\right)}{\lambda_j}$$

for searching on the condition or conditions that make Δ_2 is positive definite.

Therefore, Δ_2 is positive definite if

$$\frac{(\lambda_j+kd)^2}{(\lambda_j+k)^2} \geq \left(1+\frac{k}{\lambda_j}\right)^{-2}\left(1-\frac{(1-d)^2}{(\lambda_j+1)^2}\right)\left(1-\frac{(1-d)^2}{(\lambda_j+1)^2}\right)$$

Let k be fixed, then after some simplifications for the above expression, we get:

$$(1-d)^2+(\lambda+1)^{-2}\frac{k}{\lambda_j}d \geq 0$$

Since $0 < d < 1$, $k > 0$ and $\lambda_j > 0$, the above inequality is hold and after applying Lemma 2.1, the proof is completed.

Also, we can state the following theorem:

Theorem 2.3. *For $k > 0$ and let d be fixed, under Poisson regression model, the MAUTPPE is superior to TPPE in the MMSE if and only if*

$$B_1^{\mathrm{T}} D_2^{-1} B_1 \leq 1.$$

Proof. Same proof of Theorem 2.2.

Since the proposed estimator depends on the unknown parameters, d and k, we discuss their estimation techniques in the section follow.

3. New estimating methods for selection of *k* and *d*

It is a complicated challenge for practitioners to choose an appropriate value of k and d. Based on the work of Hoerl and Kennard (1970a), Alkhamisi et al. (2006), Kibria (2003), we propose some estimation methods for the selection of k and d.

Asar and Genç (2018) provided optimal values of k and d. Now, we derive the optimal value of k by taking derivative of SMSE($\hat{\beta}_{\mathrm{MAUTPPE}}$) with respect to k and equating the resulting function to zero and solve for k. The procedure of estimating the optimal value is stated as:

$$\frac{\partial\left\{\mathrm{SMSE}(\hat{\beta}_{\mathrm{MAUTPPE}})\right\}}{\partial k} = \sum_{j=1}^{p}\left(\frac{2\alpha_j^2\left\{k(\lambda_j+1)^2+(1-d)^2\lambda_j\right.}{(\lambda_j+1)^2(\lambda_j+k)^2}\right)$$
$$-\sum_{j=1}^{p}\left(\frac{2\left\{\alpha_j^2((\lambda_j+1)^2k+(1-d)^2\lambda_j)^2+\lambda_j((\lambda_j+1)^2-(1-d)^2)^2\right\}}{(\lambda_j+1)^4(\lambda_j+k)^3}\right)$$

Equating the above equation to zero and solve for k:

$$k_j = \frac{\lambda_j^2+\left\{2-\alpha_j^2(1-d)^2\right\}\lambda_j-d^2+2d}{\alpha_j^2(\lambda_j+1)^2}, \forall \;\; j=1,2,\ldots,p.$$

Since the parameter k is positive, therefore, we suggest to apply absolute $|\ .\ |$ as

$$\hat{k}_j = | k_j |$$

. We propose the following new estimating methods for choosing the value of k based on the work of Hoerl and Kennard (1970a), Alkhamisi et al., (2006) and Kibria (2003).

$$\hat{k}_1 = \min(| k_j |).$$

$$\hat{k}_2 = \max(| k_j |).$$

$$\hat{k}_3 = \text{mean}(| k_j |).$$

$$\hat{k}_1 = \text{median}(| k_j |).$$

In addition, we derive the optimal value of d by taking derivative of $\hat{\beta}_{\text{MAUTPPE}}$ with respect to d and equating the resulting function to zero and solve for d:

$$d_j = \frac{(k\lambda_j)^{1/2}\left(\alpha_j^2\lambda_j + \alpha_j^2\right)\left(\alpha_j^2\lambda_j^2 + 1\right)^{1/2} + \alpha_j^2\lambda_j^2 + 1}{\alpha_j^2\lambda_j^2 + 1}$$

Since the value of d_j is limited between 0 and 1, therefore, we should use following estimating methods with min operator to get the value of d_j as follows:

$$\hat{d}_j = \frac{\hat{\alpha}_j^2\lambda_j^2 + 1}{(\hat{k}\lambda_j)^{1/2}\left(\hat{\alpha}_j^2\lambda_j + \hat{\alpha}_j^2\right)\left(\hat{\alpha}_j^2\lambda_j^2 + 1\right)^{1/2} + \hat{\alpha}_j^2\lambda_j^2 + 1}, \tag{14}$$

where $\lambda_j > 0$, $\alpha_j^2 >\ 0$ and $\hat{k} >\ 0$ which implies that the value of estimator $\hat{d}$ is between 0 and 1.

Now, we use the following algorithm to estimate parameters k and d.

1. Since $\hat{k}_1 - \hat{k}_4$ needs an initial value of d, we start by setting d equals some number between 0 and 1 and obtain $\hat{k}$.

2. By using Eq. (14), we estimate parameter d by plugging-in the value of k found in the first step.

3. In order to get a suitable value of $\hat{k}$, we use one of the $\hat{k}_1 - \hat{k}_4$ estimators by plugging-in the value of $\hat{d}$ found in the second step.

4. Finally, to choose the best estimate of the parameter d using one of the $\hat{k}_1 - \hat{k}_4$ from step 3 in Eq. (14) and then compute the $\hat{d}$ estimator.

4. A Simulation Study

In this section, we study the performance of the estimators using Monte Carlo simulation under different factors such as degrees of multicollinearity, different values of the shrinkage parameter d and number of explanatory variables. Different parameters are used with some specified value, illustrated in Table 1.

4.1. The design of an experiment

Following is the design of an experiment for the Poisson regression model:

1. The correlated explanatory variables are generated by considering the work of McDonald and Galarneau (1975).

$$x_{ij} = (1-\rho^2)^{0.50} w_{ij} + \rho w_{ip+1}\ ;\ \ j = 1,...,p;\ \ i = 1,\dots,n, \tag{15}$$

   where w_{ij} are the independent standard normal pseudo-random numbers, ρ is quantified correlation between any two explanatory variables is stated as ρ^2 and x_{ij} is the number of explanatory variables. After generated correlated explanatory variables, we standardized these variables using length scaling.

2. The response variable, $Y_i (i = 1,\dots,n)$ are generated from the Poisson distribution $P_o(\mu_i)$:

$$Y_i \sim P_o(\mu_i),$$

   where

$$\mu_i = E(Y_i) = \exp(\beta_0 + \beta_1 x_{i1} + ... + \beta_p x_{ip});\ j = 1,2,...,p+1.$$

3. The parameter vectors corresponding to $p = 3,\ p = 6$ and $p = 9$ are selected by imposing the restriction on the coefficients $\beta_1, \beta_2, ..., \beta_p$ as normalized eigenvectors corresponding to the largest eigenvalues of the matrix $X^{\mathrm{T}}X$ so that $\sum_{j=1}^{p} \beta_j^2 = 1$ (see for more details; Kibria, 2003).

4. We use different estimators that given in Eq. (3) , (5) and (6) in this experiment. The $\widehat{\beta_{\mathrm{TPE}}}$ is estimated with the best shrinkage parameter

$$k_{\max} = \max\left[\frac{\lambda_j}{\lambda_j(1-d)\alpha_j^2 - d}\right],$$

and it was suggested by Asar and Genç (2018). For $\hat{\beta}_{\text{MAUTPPE}}$, we propose an algorithm for choosing values of the shrinkage parameters k and d. In addition, we consider initial value of d which are 0.10, 0.50 and 0.99. These values are chosen due to $0 < d < 1$ (e.g. see, Asar, Erişoğlu and Arashi, 2017).

5. In order to investigate the performance of the proposed estimators, we use MSE and bias.

$$\text{MSE}(\hat{\beta}) = \frac{\sum_{r=1}^{5000}\left[(\hat{\beta}_r - \beta)^{\mathsf{T}}(\hat{\beta}_r - \beta)\right]}{5000}. \tag{16}$$

$$\text{Bias}(\hat{\beta}) = \frac{\sum_{r=1}^{5000} \mid E(\hat{\beta}_r) - \beta \mid}{5000}, \tag{17}$$

where $\hat{\beta}_r$ is the estimated value of any estimator.

Table 1. *Factors, notations and values are used in the simulation.*

Factors	Notations	Values
Multicollinearity	ρ^2	0.85, 0.90, 0.95, 0.99
Number of explanatory variables	p	3, 6, 9
Initial value of shrinkage parameter	d	0.10, 0.50, 0.90
Sample size	n	25, 50, 100, 150, 200, 500
Number of Replications	R	5000

4.2. Results and Discussion

The estimated MSE and bias of the estimators are computed under different effective parameters such as sample size (n), degrees of correlation (ρ^2), initial value of the shrinkage parameter (d) and number of explanatory variables (p) and summarized them in Tables 2 to 5. All together, we created six simulation tables where we analyze the performance of MLE, TPPE and MAUTPPE by assuming different initial value of d which are 0.10, 0.50 and 0.99 (e.g. see for more details, Asar et al., 2017). To summarize the results and reduce the length of the paper, four representative tables (2-5) are included in the study. From the simulation results, it is perceived that proposed estimator MAUTPPE has the best performance as compared to the MLE and TPPE in sense of smaller MSE and bias. The MSE and bias of the MAUTPPE with $(\hat{d}, \hat{k}_2)$is minimized as compared to other shrinkage parameters ($\hat{k}_1, \hat{k}_3$ and $\hat{k}_4$).

Table 2. *Estimated MSE and bias of the estimators when $p = 3$.*

		Estimated MSE						Estimated Bias				
		MLE	TPPE	MAUTPPE				TPPE	MAUTPPE			
ρ^2	n			$(\hat{d}, \hat{k}_1)$	$(\hat{d}, \hat{k}_2)$	$(\hat{d}, \hat{k}_3)$	$(\hat{d}, \hat{k}_4)$		$(\hat{d}, \hat{k}_1)$	$(\hat{d}, \hat{k}_2)$	$(\hat{d}, \hat{k}_3)$	$(\hat{d}, \hat{k}_4)$
0.85	25	4.3700	3.6225	1.6484	1.5851	1.8466	3.0288	0.9101	0.6309	0.6387	0.6803	0.8034
	50	4.0496	3.5229	1.3957	1.4171	1.6462	2.0619	0.8912	0.6065	0.5756	0.6081	0.7142
	100	4.0184	3.3228	1.0442	1.2850	1.5154	2.0482	0.8686	0.4958	0.5541	0.5874	0.6465
	150	3.8890	3.1757	1.0151	1.2524	1.4106	2.0450	0.8444	0.4898	0.5253	0.5469	0.6337
	200	3.0153	2.0234	0.9514	0.8246	0.6644	0.1358	0.4607	0.4180	0.0182	0.3577	0.0750
	500	0.0114	0.0094	0.0071	0.0117	0.0116	0.0109	0.0071	0.0045	0.0047	0.0042	0.0038
0.90	25	4.4092	3.9826	1.7537	1.6586	2.2018	2.8577	0.9440	0.6548	0.6347	0.6954	0.7851
	50	4.0614	3.9435	1.5955	1.6100	1.9665	2.4220	0.9195	0.6114	0.6201	0.6882	0.7501
	100	4.0314	3.6039	1.4783	1.5610	1.9043	2.3502	0.9039	0.6066	0.6027	0.6506	0.7369
	150	4.0044	3.4187	1.2395	1.4477	1.8472	2.2683	0.8891	0.5597	0.5921	0.6483	0.6963
	200	3.8303	3.4145	1.2090	1.2460	1.4398	1.5617	0.8760	0.5489	0.5762	0.6040	0.6217
	500	0.0179	0.0112	0.0081	0.0170	0.0176	0.0171	0.0061	0.0044	0.0042	0.0038	0.0036
0.95	25	4.5039	3.9561	3.0520	1.8006	2.3236	3.0005	0.9703	0.7548	0.6258	0.6971	0.7840
	50	4.1655	3.8989	2.4713	1.6244	2.0000	2.7034	0.9695	0.7485	0.6182	0.6758	0.7560
	100	4.0666	3.8809	1.4129	1.5621	1.9589	2.6450	0.9651	0.5776	0.6161	0.6701	0.7466
	150	4.0547	3.7768	1.3792	1.5208	1.9571	2.3203	0.9406	0.5661	0.5757	0.6301	0.7066
	200	4.0317	3.5604	1.3585	1.5114	1.9229	2.2465	0.9067	0.5525	0.5592	0.6199	0.6848
	500	0.0734	0.0335	0.0169	0.0375	0.0541	0.0643	0.0061	0.0036	0.0060	0.0041	0.0034
0.99	25	5.5687	3.8433	3.7017	1.6010	1.9174	3.0569	1.0205	0.9152	0.6104	0.6565	0.8338
	50	4.9333	3.7302	3.5376	1.4350	1.7870	2.6267	1.0127	0.9133	0.5784	0.6214	0.7447
	100	4.2879	3.5995	3.3014	1.4221	1.7789	2.4104	0.9755	0.8361	0.5725	0.6148	0.7334
	150	4.1000	3.5868	3.2060	1.4197	1.6664	2.4072	0.9561	0.8241	0.5710	0.6019	0.6612
	200	3.9870	3.3639	3.0154	1.2841	1.5114	1.9330	0.9219	0.7374	0.5346	0.5821	0.6558
	500	0.7039	0.2528	0.0122	0.0538	0.1486	0.4296	0.0043	0.0045	0.0075	0.0045	0.0033

Table 3. *Estimated MSE of the estimators when $p = 6$ under consider different values of d.*

		$d = 0.10$						$d = 0.50$					$d = 0.99$				
		MLE	TPPE	MAUTPPE				TPPE	MAUTPPE				TPPE	MAUTPPE			
ρ^2	n			$(\hat{d}, \hat{k}_1)$	$(\hat{d}, \hat{k}_2)$	$(\hat{d}, \hat{k}_3)$	$(\hat{d}, \hat{k}_4)$		$(\hat{d}, \hat{k}_1)$	$(\hat{d}, \hat{k}_2)$	$(\hat{d}, \hat{k}_3)$	$(\hat{d}, \hat{k}_4)$		$(\hat{d}, \hat{k}_1)$	$(\hat{d}, \hat{k}_2)$	$(\hat{d}, \hat{k}_3)$	$(\hat{d}, \hat{k}_4)$
0.85	25	4.600	3.771	2.193	1.313	1.731	2.529	3.880	2.193	1.313	1.731	2.529	4.559	2.193	1.313	1.731	2.529
	50	4.205	3.610	1.507	1.262	1.654	2.437	3.790	1.507	1.262	1.654	2.437	4.069	1.507	1.262	1.654	2.437
	100	4.084	3.602	1.419	1.201	1.522	2.393	3.782	1.419	1.201	1.522	2.393	4.029	1.419	1.201	1.522	2.393
	150	4.040	2.993	1.107	1.143	1.318	2.175	3.518	1.107	1.143	1.318	2.175	4.001	1.107	1.143	1.318	2.175
	200	4.006	2.518	1.033	1.121	1.299	1.893	3.510	1.033	1.121	1.299	1.893	3.978	1.033	1.121	1.299	1.893
	500	0.0071	0.0910	0.0071	0.0095	0.0077	0.0071	0.0357	0.0061	0.0069	0.0066	0.0065	0.0357	0.0061	0.0069	0.0066	0.0065
0.90	25	5.135	3.997	2.628	1.339	1.739	3.127	3.882	2.628	1.339	1.739	3.127	4.894	2.628	1.339	1.739	3.127
	50	4.213	3.677	1.936	1.254	1.596	2.671	3.868	1.936	1.254	1.596	2.671	4.805	1.936	1.254	1.596	2.671
	100	4.096	3.451	1.756	1.206	1.466	2.566	3.733	1.756	1.206	1.466	2.566	4.797	1.756	1.206	1.466	2.566
	150	4.044	3.326	1.254	1.193	1.462	2.484	3.713	1.254	1.193	1.462	2.484	4.208	1.254	1.193	1.462	2.484
	200	4.003	2.744	1.065	1.126	1.408	2.165	3.656	1.065	1.126	1.408	2.165	3.997	1.065	1.126	1.408	2.165
	500	0.0141	0.0181	0.0135	0.0133	0.0134	0.0136	0.0139	0.0069	0.0078	0.0077	0.0077	0.0139	0.0069	0.0078	0.0077	0.0077
0.95	25	5.536	3.899	3.401	1.424	2.119	3.292	3.959	3.365	1.420	2.110	3.290	5.246	3.385	1.422	2.115	2.530
	50	4.322	3.834	2.800	1.371	1.836	3.280	3.942	2.784	1.381	1.844	3.279	4.289	2.797	1.376	1.832	2.915
	100	4.137	3.694	2.366	1.316	1.832	2.896	3.932	2.353	1.315	1.829	2.899	4.141	2.378	1.322	1.828	2.904
	150	4.054	3.508	1.577	1.283	1.649	2.893	3.901	1.589	1.278	1.644	2.898	4.063	1.595	1.279	1.650	3.284
	200	4.008	2.985	1.156	1.220	1.600	2.554	3.865	1.157	1.216	1.594	2.521	4.010	1.156	1.216	1.593	3.286
	500	0.0606	0.0491	0.0598	0.0474	0.0541	0.0602	0.0234	0.0071	0.0105	0.0103	0.0103	0.0234	0.0071	0.0105	0.0103	0.0103
0.99	25	9.054	3.917	5.368	1.606	2.109	3.675	5.244	5.331	1.603	2.101	3.669	8.290	5.416	1.607	2.107	3.667
	50	4.993	3.900	3.873	1.525	2.105	3.628	4.195	3.879	1.510	2.083	3.641	4.931	3.882	1.520	2.099	3.633
	100	4.541	3.845	3.836	1.374	2.039	3.606	4.084	3.836	1.370	2.025	3.601	4.498	3.839	1.369	2.029	3.611
	150	4.228	3.767	2.765	1.364	1.904	3.594	4.037	2.743	1.367	1.916	3.558	4.217	2.766	1.364	1.906	3.596
	200	4.082	3.619	2.573	1.241	1.762	3.560	3.976	2.565	1.237	1.754	3.539	4.074	2.572	1.241	1.763	3.556
	500	0.0492	0.0353	0.0186	0.0315	0.0355	0.0391	0.0353	0.0186	0.0315	0.0355	0.0391	0.0353	0.0186	0.0315	0.0355	0.0391

Table 4. *Estimated bias of the estimators when $p = 6$ under consider different values of d.*

		$d = 0.10$					$d = 0.50$					$d = 0.99$				
		TPPE	MAUTPPE				TPPE	MAUTPPE				TPPE	MAUTPPE			
ρ^2	n		$(\hat{d},\hat{k}_1)$	$(\hat{d},\hat{k}_2)$	$(\hat{d},\hat{k}_3)$	$(\hat{d},\hat{k}_4)$		$(\hat{d},\hat{k}_1)$	$(\hat{d},\hat{k}_2)$	$(\hat{d},\hat{k}_3)$	$(\hat{d},\hat{k}_4)$		$(\hat{d},\hat{k}_1)$	$(\hat{d},\hat{k}_2)$	$(\hat{d},\hat{k}_3)$	$(\hat{d},\hat{k}_4)$
0.85	25	0.759	0.592	0.459	0.515	0.607	0.782	0.592	0.459	0.515	0.607	0.880	0.592	0.459	0.515	0.607
	50	0.751	0.483	0.454	0.497	0.600	0.778	0.483	0.454	0.497	0.600	0.817	0.483	0.454	0.497	0.600
	100	0.742	0.480	0.446	0.485	0.585	0.774	0.480	0.446	0.485	0.585	0.813	0.480	0.446	0.485	0.585
	150	0.687	0.428	0.442	0.479	0.578	0.772	0.428	0.442	0.479	0.578	0.809	0.428	0.442	0.479	0.578
	200	0.646	0.415	0.440	0.466	0.557	0.765	0.415	0.440	0.466	0.557	0.799	0.415	0.440	0.466	0.557
	500	0.0694	0.0070	0.0095	0.0077	0.0070	0.0310	0.0081	0.0077	0.0073	0.0071	0.0310	0.0081	0.0077	0.0073	0.0071
0.90	25	0.798	0.642	0.463	0.516	0.674	0.789	0.642	0.463	0.516	0.674	0.890	0.642	0.463	0.516	0.674
	50	0.757	0.530	0.462	0.503	0.628	0.782	0.530	0.462	0.503	0.628	0.838	0.530	0.462	0.503	0.628
	100	0.723	0.525	0.457	0.497	0.621	0.780	0.525	0.457	0.497	0.621	0.820	0.525	0.457	0.497	0.621
	150	0.713	0.449	0.431	0.470	0.592	0.778	0.449	0.431	0.470	0.592	0.806	0.449	0.431	0.470	0.592
	200	0.670	0.422	0.429	0.467	0.590	0.758	0.422	0.429	0.467	0.590	0.798	0.422	0.391	0.467	0.590
	500	0.0163	0.0065	0.0068	0.0066	0.0065	0.0055	0.0023	0.0024	0.0023	0.0023	0.0055	0.0023	0.0024	0.0023	0.0023
0.95	25	0.779	0.718	0.479	0.568	0.696	0.794	0.715	0.478	0.567	0.695	0.890	0.717	0.479	0.567	0.696
	50	0.777	0.628	0.475	0.530	0.691	0.794	0.627	0.475	0.531	0.692	0.835	0.628	0.476	0.530	0.692
	100	0.741	0.593	0.471	0.526	0.656	0.789	0.592	0.472	0.526	0.656	0.813	0.595	0.471	0.525	0.658
	150	0.738	0.492	0.452	0.519	0.639	0.787	0.493	0.451	0.518	0.639	0.803	0.494	0.451	0.519	0.640
	200	0.692	0.439	0.436	0.488	0.635	0.773	0.439	0.436	0.487	0.631	0.799	0.439	0.436	0.487	0.632
	500	0.0360	0.0068	0.0105	0.0083	0.0068	0.0183	0.0073	0.0066	0.0061	0.0059	0.0183	0.0073	0.0066	0.0061	0.0059
0.99	25	0.783	0.844	0.510	0.573	0.753	0.811	0.841	0.509	0.572	0.749	0.880	0.846	0.510	0.573	0.755
	50	0.778	0.773	0.487	0.559	0.741	0.797	0.774	0.485	0.556	0.740	0.827	0.775	0.486	0.558	0.742
	100	0.763	0.761	0.463	0.550	0.741	0.794	0.761	0.462	0.548	0.740	0.809	0.762	0.463	0.548	0.740
	150	0.763	0.634	0.461	0.530	0.734	0.788	0.631	0.462	0.531	0.735	0.802	0.634	0.461	0.530	0.735
	200	0.745	0.619	0.453	0.525	0.729	0.782	0.618	0.453	0.524	0.729	0.798	0.619	0.453	0.525	0.729
	500	0.0279	0.0055	0.0075	0.0063	0.0057	0.0279	0.0055	0.0075	0.0063	0.0057	0.0279	0.0055	0.0075	0.0063	0.0057

Table 5. *Estimated MSE and bias of the estimators when $p = 9$.*

		Estimated MSE						Estimated Bias				
		MLE	TPPE	MAUTPPE				TPPE	MAUTPPE			
ρ^2	n			$(\hat{d},\hat{k}_1)$	$(\hat{d},\hat{k}_2)$	$(\hat{d},\hat{k}_3)$	$(\hat{d},\hat{k}_4)$		$(\hat{d},\hat{k}_1)$	$(\hat{d},\hat{k}_2)$	$(\hat{d},\hat{k}_3)$	$(\hat{d},\hat{k}_4)$
0.85	25	4.9867	3.9254	3.3671	1.2180	1.6399	3.7194	0.6837	0.6095	0.3820	0.4310	0.6537
	50	4.4573	3.8356	1.6960	1.1597	1.5815	3.5295	0.6735	0.4555	0.3808	0.4290	0.6320
	100	4.1008	3.5133	1.5471	1.1421	1.5178	3.1586	0.6383	0.4285	0.3775	0.4221	0.5877
	150	4.0432	3.3801	1.3994	1.1229	1.5028	2.7216	0.6138	0.4080	0.3724	0.4204	0.5519
	200	4.0187	2.7509	1.3574	1.0992	1.3052	2.1693	0.5777	0.4034	0.3688	0.4087	0.5087
	500	0.0157	0.0130	0.0121	0.0159	0.0160	0.0145	0.0189	0.0151	0.0154	0.0153	0.0147
0.90	25	5.1892	3.8377	3.6209	1.2633	1.7033	3.4648	0.6238	0.6320	0.3792	0.4333	0.6107
	50	4.2100	3.7621	2.6786	1.2155	1.6710	3.4088	0.6521	0.5576	0.3870	0.4411	0.6248
	100	4.1743	3.6540	1.7841	1.2143	1.6565	3.3917	0.6500	0.4694	0.4043	0.4586	0.5801
	150	4.0396	3.5072	1.6078	1.1832	1.6184	3.0263	0.6710	0.4380	0.3754	0.4338	0.6278
	200	4.0243	3.4173	1.1155	1.1415	1.5880	2.8179	0.6613	0.3694	0.3824	0.4362	0.5802
	500	0.0240	0.0163	0.0087	0.0234	0.0238	0.0208	0.0051	0.0052	0.0054	0.0054	0.0053
0.95	25	5.9292	3.9792	4.1619	1.3290	2.2126	3.8283	0.6920	0.6726	0.4023	0.5082	0.6664
	50	4.1767	3.8952	2.3592	1.3283	1.9382	3.6988	0.6786	0.5121	0.3945	0.4637	0.6503
	100	4.1462	3.7097	2.1484	1.2887	1.8386	3.6626	0.6576	0.5014	0.3942	0.4592	0.6375
	150	4.0412	3.7050	2.1469	1.1858	1.6857	3.4699	0.6423	0.4952	0.3781	0.4410	0.6182
	200	4.0278	3.6487	2.1459	1.1145	1.4582	3.2969	0.6413	0.4951	0.3671	0.4106	0.6094
	500	0.1204	0.0599	0.0185	0.0828	0.0960	0.0808	0.0203	0.0164	0.0252	0.0200	0.0143
0.99	25	78.7139	78.7125	9.3212	1.9733	2.7194	78.4760	0.9057	0.6977	0.4703	0.5470	0.9037
	50	11.8039	4.0722	5.4770	1.3071	1.9804	3.9385	0.6820	0.6891	0.4045	0.4825	0.6748
	100	5.6412	3.9143	4.3348	1.2536	1.9150	3.8780	0.6812	0.6647	0.3895	0.4693	0.6699
	150	4.3063	3.9019	3.8198	1.1253	1.5446	3.8354	0.6689	0.6526	0.3716	0.4268	0.6673
	200	4.0859	3.8468	3.6952	1.0334	1.5223	3.7788	0.6641	0.0491	0.3449	0.2861	0.6609
	500	0.9683	0.4018	0.0077	0.1194	0.2088	0.1181	0.0067	0.0058	0.0080	0.0063	0.0052

Table 6. *Estimated Coefficients and SMSE of the MLE, TPPE and MAUTPPE.*

Estimators	MLE	TPPE	MAUTPPE				λ_j [1]
			$(\hat{d},\hat{k}_1)$	$(\hat{d},\hat{k}_2)$	$(\hat{d},\hat{k}_3)$	$(\hat{d},\hat{k}_4)$	
Intercept	2.240	2.254	2.291	2.295	2.290	2.288	2010
Unemployment	0.075	0.067	0.051	0.046	0.045	0.045	223.2
Cars	-9.559	-6.799	-0.062	-0.282	-0.647	-0.875	186.54
Trucks	4.018	3.123	0.669	1.072	1.446	1.566	14.157
15-24 years	1.971	1.460	0.042	0.209	0.464	0.560	0.581
25-64 years	2.004	1.424	-0.030	0.101	0.202	0.242	0.224
> 64 years	1.979	1.185	-0.280	-0.755	-1.113	-1.138	0.047
MSE	27.749	14.969	8.699	7.520	6.390	6.059	

[1] $\lambda_j(\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 > \lambda_6 > \lambda_7)$ are the eigenvalues and Condition Index = $\sqrt{\frac{\lambda_{max}}{\lambda_{min}}} = 207.77$

Increasing the degree of correlation has an adverse effect on the estimators in terms of MSE. However, the estimated bias of the estimators are decreasing when the degrees of correlation is increased particular especially for MAUTPPE with $\hat{k}_2$ and $\hat{k}_3$. When the sample size increases the estimated MSE and bias are decreased. The sample size makes a good effect on the estimators in sense of large sample size. An increase in the number of explanatory variables has a negative effect in sense of estimated MSE and positive effect in sense of estimated bias for some cases. The estimated bias of the TPPE is reduced when the number of explanatory variables are increased. However, the proposed MAUTPPE has lowest bias in all cases than the TPPE. It is also noted that the estimated bias of all the estimators are reduced when the $p = 6$ and then slight increase in the estimated bias when $p = 9$ and $\rho^2 = 0.99$ for only TPPE and MAUTPPE $(\hat{d},\hat{k}_4)$. The performance of MAUTPPE $(\hat{d},\hat{k}_2)$ is significant in terms of estimated MSE and MAUTPPE $(\hat{d},\hat{k}_1)$is almost unbiased when the $\rho^2 = 0.99, n = 200$ and $p = 9$.

In addition, we analyzed the performance of TPPE and MAUTPPE by assuming different initial value of d which are 0.10, 0.50 and 0.99 (e.g. see for more details, Asar et al., 2017). These results are illustrated in Tables 3-4. The performance of TPPE and MAUTPPE do not change substantially when we consider the different initial values of d and one can see this findings in Table 3 and Table 4. The estimated MSE and bias values of the MAUTPPE are approximately same when the $\rho^2 = 0.85$ and $\rho^2 = 0.90$. One can see the insignificant change in the estimated MSE and bias of the MAUTPPE when the $\rho^2 = 0.95$ and $\rho^2 = 0.99$. Meanwhile, the estimated MSE and bias values of the TPPE are increased when the d rises. The performance of TPPE is near to MLE when the $d = 0.99$, $\rho^2 = 0.99$ and $n = 200$. For large sample size ($n = 500$), the bias of

MAUTPPE is close to zero which indicate the benefit of the proposed estimator in the sense of bias correction. Simulation results demonstrate that a bias correction estimator (MAUTPPE) substantially reduces the bias and more efficient than TPPE as well as improved estimators under certain conditions.

We can conclude that the performance of MLE is worsted in almost all condition. The MLE is not good choice in the presence of multicollinearity. The proposed MAUTPPE has quite good performance as compared to the TPPE and MLE under different conditions. However, the MAUTPPE with $(\hat{d}, \hat{k}_2)$ has better performance than the other estimators in almost all conditions.

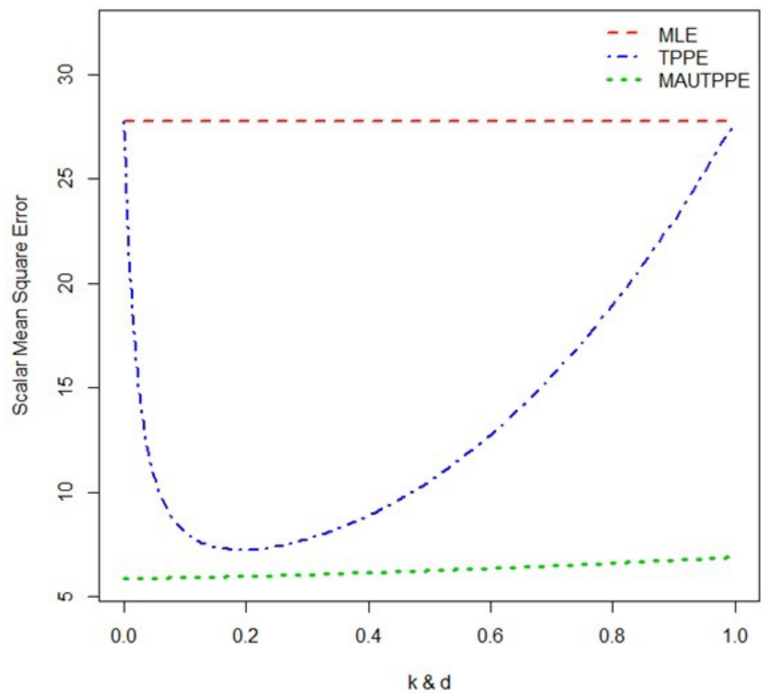

Figure 1. *Emperical estimated SMSE of MLE, TPPE and MAUTPPE.*

5. Application

To illustrate the findings of the paper, Swedish traffic fatality data for the year 2019 are analyzed in this section. The data are taken from the Statistics Sweden and Swedish transport administration. The aim of this case study is to see the impact of external factors on the traffic fatalities in Sweden, where the number of traffic fatalities is considered as dependent variable. As discussed by Wiklund, Simonsson and Forsman (2012), the main factors are economic conditions defined as unemployment rate, traffic exposure that we measure as number of vehicles (cars and trucks), and demographic variables, all of which are considered as explanatory variables. By following the study of Stipdonk et al., (2013), we divide all individuals into three different age groups (age 15-24 years, age 25-64 years and more than 64 years). The estimated results of the model are presented in Table 6. The eigenvalues of $X^{\intercal}X$ matrix are 2010, 223.2, 186.54, 14.157, 0.581, 0.224 and 0.047. The condition index, $CI = \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} = 207.77$, which confirmed that there are serious problems of multicollinearity. Therefore, we used TPPE and MAUTPPE to combat the multicollinearity problem. The unemployment rate coefficient is positive which shows that the number of fatalities increases and this impact is considerable low for MAUTPPE $(\hat{d},\hat{k}_4)$. The traffic exposure variables (cars and trucks) have negative and positive coefficients. This shows that more accidents occur when trucks are used and less accidents occur when cars are used. Age group 15-24 and 25-64 year's parameters are positive except MAUTPPE $(\hat{d},\hat{k}_1)$. Age group more than 64 years is positive when we use MLE and TPPE but it is negative for MAUTPPE which shows the robust results. The number of fatalities decreases when the drivers have more experience and this result can be seen only by using proposed estimator (MAUTPPE). The SMSE of MLE is inflated due to multicollinearity problem and biased estimation methods (TPPE and MAUTPPE) have lower SMSE than the MLE. One can see that a substantial decrease of the SMSE when applying MAUTPPE than the MLE and TPPE. Figure 1 illustrates the empirical SMSE of MLE, TPPE and MAUTPPE. The SMSE of MAUTPPE is smaller than the MLE and TPPE. In summary, the application shows the benefits of the proposed estimator. Program code in R for analyzing this application data set is given in Supplementary Material.

6. Some concluding Remarks

This paper proposes a new almost unbiased estimator for the parameters of the Poisson regression model. The MSE properties of the proposed estimator is investigated and a comparison is made with some existing estimators. Furthermore, a simulation study has been conducted to compare the performance of the estimators under several parametric conditions. Finally, an example illustrates the benefit of the new MAUTPPE. The overall results of the paper show the benefit of the new estimator as compared to previously suggested estimators such as TPPE and MLE. Based on both the simulation study and empirical application, we may recommend MAUTPPE with parameter combinations $(\hat{d},\hat{k}_2)$ and $(\hat{d},\hat{k}_4)$ to researchers.

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