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AN ITERATIVE PROCESS OF GENERALIZED LIPSCHITIZIAN MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. In this article, three generalized Lipschitizian mappings in the uniformly convex Banach spaces are considered. By defining an iterative process, the existence of a common fixed point of this iteration is proved. An example is presented to guarantee the convergence of the iteration.

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1. INTRODUCTION

In computational mathematics, an iterative method is a mathematical procedure that uses an initial value to generate a sequence of improving approximate solutions for a class of problems, in which the $n-th$ approximation is derived from the previous ones. If an equation can be put into the form $f(x) = x$, and a solution x is an attractive fixed point of the function f, then one may begin with a point x_1 in the basin of attraction of x, and let $x_{n+1} = f(x_n)$ for $n = 1$, and the sequence $\{x_n\}_{n\geq 1}$ will converge to the solution x. Here x_n is the n-th approximation or iteration of x and x_{n+1} is the next or $n+1$ iteration of x (see [4, 7, 9–16], and reference therein).

Let C be a nonempty subset of a Banach space E, $T : C \longrightarrow C$ be a mapping and

$$F(T) = \{x \in C : Tx = x\}$$

denotes the set of fixed points of T. A mapping T is said to be asymptotically nonexpansive, if there exists a sequence $\{k_n\}$ of positive numbers with $\lim_{n\longrightarrow\infty} k_n = 1$ such that for $x, y \in C$ and $n \geq 1$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|.$$

The study of iterative construction for fixed points of asymptotically nonexpansive mappings began in 1975. Baillon [3] proved, if C is a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and $T : C \to C$ is a nonexpansive mapping such that

$F(T) \neq \emptyset$, then for every $x \in C$, the Cesáro means

$$T_n x = \frac{1}{n+1} \sum_{j=0}^{n} T^j x$$

is weakly convergent to a fixed point of T. Then, Shimizu et al. [18] studied the convergence of the following approximated sequence for an asymptotically nonexpansive mapping in Hilbert space,

$$x_0 = x \in C,$$
$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{n+1} \sum_{j=0}^{n} T^j x_n,$$

where $\{\alpha_n\}$ is a real sequence satisfying $0 < \alpha_n \leq 1$ and $\alpha_n \longrightarrow 0$ as $n \longrightarrow \infty$. They proved that $\{x_n\}$ is strongly convergent to an element of $F(T)$. We recall the definition of uniformly convex space (see [2, 6] for more details).

Definition 1. A Banach space E is said to be strictly convex if

$$\|x+y\| < 2$$

for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. We recall that a Banach space E is called uniformly convex, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|x\| = \|y\| = 1$ then

$$\|\frac{(x+y)}{2}\| \leq 1-\delta.$$

It is obvious that uniform convexity implies strict convexity.

In 1991, [23] proved the characterization of uniform convexity as follows.

Theorem 1. *A Banach space E is uniformly convex if and only if for each fixed number $r > 0$, there exists a continuous function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(s) = 0 \Longleftrightarrow s = 0$, such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\varphi(\|x-y\|)$$

for all $\lambda \in [0,1]$ and all $x, y \in E$ such that $\|x\| \leq r$ and $\|y\| \leq r$.

Shioji et al. [19] studied the strongly convergence of the sequence

$$x_0 = x \in C,$$
$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{n+1} \sum_{j=0}^{n} T^j x_n,$$

in uniformly convex Banach spaces with uniformly Gâteaux differentiable norms.

Also, for an asymptotically nonexpansive mapping T, Tan et. al [22] defined the modified Ishikawa iterations process

$$x_{n+1} = t_n T^n(s_n T^n x_n + (1-s_n)x_n) + (1-t_n)x_n,$$

where $\{t_n\}$ and $\{s_n\}$ are real sequences such that $\{t_n\}$ is bounded away from 0 and 1 and $\{s_n\}$ bounded away from 1. They proved the sequence $\{x_n\}$ is weakly convergent to a fixed point of T.

Definition 2. Let E be a Banach space. E is said to satisfy Opial's condition if for each sequence $\{x_n\}$ in E the condition $x_n \rightharpoonup x$ implies

$$\limsup_{n\longrightarrow\infty} \|x_n - x\| < \limsup_{n\longrightarrow\infty} \|x_n - y\|$$

for all $y \in E$ and $y \neq x$.

Definition 3. Let E be an arbitrary real Banach space with norm $\|.\|$ and E^* be the dual space of E. The duality mapping $J : E \longrightarrow E^*$ is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\},$$

where $\langle x, f\rangle$ denotes the value of the continuous linear function $f \in E^*$ at $x \in E$.

Lemma 1 ([21])**.** *Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers satisfying the recursive inequality*

$$\delta_{n+1} \leq \beta_n \delta_n + \gamma_n \ \textit{for all } n \geq 1,$$

if $\beta_n \geq 1$, $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$, then $\lim_{n\to\infty} \delta_n$ exists.

Lemma 2 ([17])**.** *Let E be a uniformly convex Banach space. Assume $\{t_n\}$ is a sequence of real numbers in $(0,1)$ bounded away from 0 and 1. If $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that for some $a \geq 0$*

$$\limsup_{n\longrightarrow\infty} \|x_n\| \leq a,\ \limsup_{n\longrightarrow\infty} \|y_n\| \leq a \textit{ and } \limsup_{n\longrightarrow\infty} \|t_n x_n + (1-t_n)y_n\| = a,$$

then $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$.

Lemma 3 ([1])**.** *Let $\{x_n\}$ be a bounded sequence in a uniformly convex Banach space E. If $w_w(\{x_n\}) = \{x\}$, then $x_n \rightharpoonup x$, where $w_w(\{x_n\}) = \{x \in E : \exists x_{nj} \rightharpoonup x\}$ denotes the weak w-limit set of $\{x_n\}$.*

In 2001, Jung et al. [8] introduced the following class of mappings.

Definition 4. A mapping $T : C \to C$, where C is a nonempty subset of Banach space E, is said to be a generalized Lipschitzian mapping if

$$\|T^n x - T^n y\| \leq a_n\|x-y\| + b_n(\|x - T^n x\| + \|T^n y - y\|) + c_n(\|x - T^n y\| + \|T^n x - y\|) \tag{1.1}$$

for each $x, y \in C$ and $n \geq 1$, where a_n, b_n and c_n are nonnegative constants such that there exists an integer n_0 such that $b_n + c_n < 1$, for all $n > n_0$.

Definition 5. A mapping $T : C \to C$, where C is a nonempty subset of Banach space E, is said to be a uniformly generalized Lipschitzian mapping if

$$\|T^n x - T^n y\| \leq a\|x-y\| + b(\|x-T^n x\| + \|T^n y - y\|) + c(\|x - T^n y\| + \|T^n x - y\|) \tag{1.2}$$

for each $x, y \in C$ where a, b and c are nonnegative constants and $b + c < 1$.

Clearly, every Lipschitzian mapping is a generalized Lipschitzian mapping. But the vice versa is not necessarily true. See the following example.

Example 1. Let $E = \mathbb{R}$ be the set of real numbers and $C = [0, \infty)$. For each $x \in C$, we define

$$Tx = \begin{cases} \dfrac{rx}{1+x} & \text{if} \quad x \in [0, \dfrac{1}{4}], \\ 0 & \text{if} \quad x \in (\dfrac{1}{4}, \infty), \end{cases} \tag{1.3}$$

where $0 < r < \frac{1}{4}$. Then $T : C \longrightarrow C$ is not continuous at $x = \frac{1}{4}$ and hence T is not a Lipschitzian mapping. Set $C_1 = [0, \frac{1}{4}]$ and $C_2 = (\frac{1}{4}, \infty)$. In order to prove $T : C \longrightarrow C$ is a generalized Lipschitzian mapping, we need the following steps: for all $x, y \in C_1$ and $n \geq 1$,

$$|Tx - Ty| = |\frac{rx}{1+x} - \frac{ry}{1+y}| = |\frac{rx(1+y) - ry(1+x)}{(1+x)(1+y)}| \leq r|x-y|$$

and

$$|T^2 x - T^2 y| = |\frac{rTx}{1+Tx} - \frac{rTy}{1+Ty}| \leq r|Tx - Ty| \leq r^2|x-y|.$$

By induction, for all $n \geq 1$

$$|T^n x - T^n y| \leq r^n|x-y| + r^n(|x - T^n x| + |y - T^n y|) + r^n(|x - T^n y| + |y - T^n x|). \tag{1.4}$$

For all $x, y \in C_2$ and $n \geq 1$,

$$|T^n x - T^n y| = 0 \leq |x-y|. \tag{1.5}$$

For $x \in C_1$ and $y \in C_2$,

$$|Tx - Ty| = |\frac{rx}{1+x} - 0|.$$

By induction, for $n \in \mathbb{N}$,

$$T^{n+1}x = \frac{rT^n x}{1+T^n x} \leq rT^n x \leq r^{n+1}x,$$

and by a computation

$$\begin{aligned} |T^n x - T^n y| &= |T^n x - 0| \\ &\leq r^n|(x-y) + (y - T^n x) + (T^n x - x) + x| \\ &\leq r^n(|x-y| + |T^n x - x| + |y - T^n x| + |x + y - 0|) \\ &\leq r^n|x-y| + r^n(|x - T^n x| + |y - T^n y|) + r^n(|x - T^n y| + |y - T^n x|). \end{aligned} \tag{1.6}$$

Thus inequalities (1.4), (1.5) and (1.6) imply that $T : C \longrightarrow C$ is a generalized Lipschitzian mapping.

Lemma 4. *Let C be a bounded, closed and convex subset of a uniformly convex Banach space E and T be a uniformly generalized Lipschitzian mapping of C into itself. Then, for any $\varepsilon > 0$, there exists a positive number $\xi(\varepsilon)$ such that $\|Tx - x\| < \varepsilon$ for all $x \in [x_0, x_1]$ whenever for $x_0, x_1 \in C$, $\|Tx_0 - x_0\| \le \xi(\varepsilon)$ and $\|Tx_1 - x_1\| \le \xi(\varepsilon)$, where*

$$[x_0, x_1] = \{\lambda x_1 + (1-\lambda)x_0 : 0 \le \lambda \le 1\}.$$

Proof. Fix $\varepsilon > 0$ and $x \in [x_0, x_1]$. Then $x = \lambda x_1 + (1-\lambda)x_0$ for some λ with $0 \le \lambda \le 1$. We consider two cases, $\|x_1 - x_0\| < \varepsilon_1$ and $\|x_1 - x_0\| \ge \varepsilon_1$.

If $\|x_1 - x_0\| < \varepsilon_1$ then it is obvious that $\|x - x_0\| < \varepsilon_1$, where $\varepsilon_1 = \frac{1-(b+c)}{1+a+2c}\frac{\varepsilon}{2}$. Since

$$\begin{aligned}
\|Tx - x\| &\le \|Tx - Tx_0\| + \|Tx_0 - x_0\| + \|x_0 - x\| \\
&\le a\|x_0 - x\| + b(\|Tx - x\| + \|Tx_0 - x_0\|) \\
&\quad + c(\|x_0 - Tx\| + \|Tx_0 - x\|) + \|Tx_0 - x_0\| + \|x_0 - x\| \\
&\le a\|x_0 - x\| + b(\|Tx - x\| + \|Tx_0 - x_0\|) \\
&\quad + c(\|x - Tx\| + \|Tx_0 - x_0\| + 2\|x_0 - x\|) + \|Tx_0 - x_0\| + \|x_0 - x\|
\end{aligned}$$

and so

$$\|Tx - x\| \le \frac{1+a+2c}{1-(b+c)}\|x_0 - x\| + \frac{b+c+1}{1-(b+c)}\|x_0 - Tx_0\|. \tag{1.7}$$

Thus if $\|Tx_0 - x_0\| \le \xi(\varepsilon) < \frac{1-(b+c)}{1+b+c}\frac{\varepsilon}{2}$, then $\|Tx - x\| < \varepsilon$.

If $\|x_0 - x_1\| \ge \varepsilon_1$, let d_0 denote the diameter of C. For any nonnegative number $0 \le \lambda \le 1$, we consider three cases:

Case 1) If $0 \le \lambda < \frac{\varepsilon_1}{d_0}$, then

$$\|x_0 - x\| = \lambda\|x_1 - x_0\| < \frac{\varepsilon_1}{d_0}\|x_1 - x_0\| = \varepsilon_1\frac{\|x_1 - x_0\|}{d_0} < \varepsilon_1.$$

Thus similar to above, if $\xi(\varepsilon) < \frac{1-(b+c)}{1+b+c}\frac{\varepsilon}{2}$, then inequation (1.7) implies $\|Tx - x\| < \varepsilon$.

Case 2) If $1 - \frac{\varepsilon_1}{d_0} < \lambda \le 1$ or $0 \le (1-\lambda) < \frac{\varepsilon_1}{d_0}$, so

$$\|x_1 - x\| = (1-\lambda)\|x_1 - x_0\| < \varepsilon_1,$$

and similar to above we have $\|Tx - x\| < \varepsilon$.

Case 3) If $\frac{\varepsilon_1}{d_0} \le \lambda \le 1 - \frac{\varepsilon_1}{d_0}$, $y = Tx$ implies

$$\|y - x_0\| = \|Tx - x_0\| \le \|Tx - Tx_0\| + \|Tx_0 - x_0\|.$$

By inequalities (1.2) and (1.7),

$$\|y - x_0\| \le \frac{a+b+c}{1-(b+c)}\|x - x_0\| + \frac{1+b+c}{1-(b+c)}\|Tx_0 - x_0\| \le r\lambda\|x_1 - x_0\| + h\xi(\varepsilon), \tag{1.8}$$

where $r = \frac{a+b+c}{1-(b+c)}$ and $h = \frac{1+b+c}{1-(b+c)}$, and also

$$\|y - x_1\| \le \frac{a+b+c}{1-(b+c)}\|x - x_1\| + \frac{1+b+c}{1-(b+c)}\|Tx_1 - x_1\| \\ \le r(1-\lambda)\|x_1 - x_0\| + h\xi(\varepsilon). \tag{1.9}$$

Set

$$z_0 = \frac{y - x_0}{\lambda\|x_1 - x_0\|} \text{ and } z_1 = \frac{x_1 - y}{(1-\lambda)\|x_1 - x_0\|}. \tag{1.10}$$

Then

$$\|z_0\| \le \frac{r\lambda\|x_1 - x_0\| + h\xi(\varepsilon)}{\lambda\|x_1 - x_0\|} \le r + \frac{h\xi(\varepsilon)d_0}{\varepsilon_1^2},$$

and similarly

$$\|z_1\| \le r + \frac{h\xi(\varepsilon)d_0}{\varepsilon_1^2}.$$

On the other hand, for λ with $\frac{\varepsilon_1}{d_0} \le \lambda \le 1 - \frac{\varepsilon_1}{d_0}$,

$$\|\lambda z_0 + (1-\lambda)z_1\| = \frac{\|x_1 - x_0\|}{\|x_1 - x_0\|} = 1.$$

Therefore using uniform convexity of E, we can choose $\xi(\varepsilon)$ so small that $\|z_1 - z_0\| < \frac{\varepsilon}{d_0}$. Thus by $x = \lambda x_1 + (1-\lambda)x_0$ and (1.10) we have

$$\|y - x\| = \|(1-\lambda)(y - x_0) - \lambda(x_1 - y)\| \\ = \lambda(1-\lambda)\|x_1 - x_0\|\|z_1 - z_0\| \\ < \varepsilon. \tag{1.11}$$

Notice that $y = Tx$, and this means $\|Tx - x\| < \varepsilon$. □

Lemma 5 ([20]). *Let C be a bounded, closed and convex subset of a uniformly convex Banach space E and T be a uniformly generalized Lipschitzian mapping of C into itself. If $\{x_j\}$ is a sequence of C such that $x_j \rightharpoonup x_0$ and $x_j - Tx_j \longrightarrow 0$, then x_0 is a fixed point of T.*

Proof. Since $\|(I - T)x_j\| = \varepsilon_j \longrightarrow 0$, for each j we assume $\varepsilon_j \le \xi(\varepsilon_{j-1}) < \varepsilon_{j-1}$, where $\xi(\varepsilon)$ for any $\varepsilon > 0$ is the constant described in the conclusion of Lemma 4. Hence if $x \in \overline{Co}\{x_j : j \ge k\}$ then $\|x - Tx\| \le \varepsilon_{k-1}$. Since $\overline{Co}\{x_j : j \ge k\}$ is weakly compact, $x_0 \in \overline{Co}\{x_j : j \ge k\}$, $k = 1, 2, 3, \cdots$ and hence $\|x_0 - Tx_0\| \le \varepsilon_j$, $j = 1, 2, 3, \cdots$. This implies $\|Tx_0 - x_0\| = 0$. □

Let C be a bounded, closed and convex subset of a Banach space E and $\{T_n\}$ be a sequence of uniformly generalized Lipschitizian self-mappings of C, such that the set F of common fixed points of $\{T_n\}$ is nonempty. Let $k_n = \frac{a_n+b_n+c_n}{1-(b_n+c_n)}$ for T_n and $k_n \ge 1$ for all $n \ge 1$. For a given $x_1 \in C$, we define the sequence $\{x_n\}$ by $x_{n+1} = T_n x_n$ for $n \ge 1$.

Lemma 6. *Let E be a normed linear space and C be a nonempty closed and convex subset of E. Let $T : C \longrightarrow C$ be a uniformly generalized Lipschitzian mapping. Let $\{x_n\}$ be defined by $x_{n+1} = T_n x_n$. If $\lim_{n\longrightarrow\infty} \|x_n - T^n x_n\| = 0$ and $\lim_{n\longrightarrow\infty} \|x_n - x_{n+1}\| = 0$, then $\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0$.*

Proof. Let $r_n = \|T^n x_n - x_n\|$. Since T is a uniformly generalized Lipschitzian mapping,

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\leq \|x_{n+1} - T^{n+1}x_{n+1}\| + \|Tx_{n+1} - T^{n+1}x_{n+1}\| \\
&\leq r_{n+1} + a\|x_{n+1} - T^n x_{n+1}\| + b(\|x_{n+1} - Tx_{n+1}\| \\
&\quad + \|T^n x_{n+1} - T^{n+1}x_{n+1}\|) + c(\|x_{n+1} - T^{n+1}x_{n+1}\| \\
&\quad + \|T^n x_{n+1} - Tx_{n+1}\|) \\
&\leq r_{n+1} + a\|x_{n+1} - T^n x_{n+1}\| + b(\|T^n x_{n+1} - x_{n+1}\| \\
&\quad + \|x_{n+1} - Tx_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\|) \\
&\quad + c(\|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^n x_{n+1} - x_{n+1}\| + \|x_{n+1} - Tx_{n+1}\|),
\end{aligned}$$

and

$$\|x_{n+1} - Tx_{n+1}\| \leq \frac{1+b+c}{1-(b+c)} r_{n+1} + \frac{a+b+c}{1-(b+c)} \|x_{n+1} - T^n x_{n+1}\|. \tag{1.12}$$

Now we obtain $\|x_{n+1} - T^n x_{n+1}\|$ as

$$\begin{aligned}
\|x_{n+1} - T^n x_{n+1}\| &\leq \|x_n - T^n x_n\| + \|x_{n+1} - x_n\| + \|T^n x_{n+1} - T^n x_n\| \\
&\leq r_n + \|x_{n+1} - x_n\| + a\|x_{n+1} - x_n\| + b(\|T^n x_n - x_n\| \\
&\quad + \|T^n x_{n+1} - x_{n+1}\|) + c(\|x_{n+1} - T^n x_n\| + \|T^n x_{n+1} - x_n\|).
\end{aligned}$$

So

$$\|T^n x_{n+1} - x_{n+1}\| \leq \frac{1+b+c}{1-(b+c)} r_n + \frac{1+a+2c}{1-(b+c)} \|x_{n+1} - x_n\|. \tag{1.13}$$

By relations (1.12) and (1.13),

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\leq \frac{1+b+c}{1-(b+c)} r_{n+1} + \frac{a+b+c}{1-(b+c)} \|T^n x_{n+1} - x_{n+1}\| \\
&\leq \frac{1+b+c}{1-(b+c)} r_{n+1} + \frac{a+b+c}{1-(b+c)} \Big(\frac{1+b+c}{1-(b+c)} r_n \\
&\quad + \frac{1+a+2c}{1-(b+c)} \|x_{n+1} - x_n\|\Big).
\end{aligned}$$

This completes the proof. □

In the next section, based on [19] and [22], a modified version of the iterative process for three generalized Lipschitizian mappings is presented. Then the existence of a common fixed point for these three maps is proved.

2. An iterative approximation of three generalized Lipschitizian mappings

In this section, we prove, if $T_i : C \longrightarrow C$, $i = 1,2,3$, are three generalized Lipschitzian mappings, then the sequence $\{x_n\}$ which is defined by equation (2.1) converges to $q \in \bigcap_{i=1}^{3} F(T_i)$ in the uniformly convex Banach space.

The result presented in this section generalizes and improves the corresponding results, in [5, 19] and [22].

Let C be a nonempty subset of real Banach space E and $T_i : C \longrightarrow C$, $i = 1,2,3$, be three generalized Lipschitizian mappings. Consider the following iterative sequence $\{x_n\}$ which is defined by

$$\begin{cases} x_1 \in C, \\ u_n = \dfrac{1}{n+1} \displaystyle\sum_{j=0}^{n} T_1^j x_n, \\ z_n = (1-\lambda_n)x_n + \lambda_n u_n & n \geq 1, \\ y_n = (1-\beta_n)x_n + \beta_n T_2^n z_n & n \geq 1, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_3^n y_n & n \geq 1, \end{cases} \tag{2.1}$$

where $0 < \alpha_n, \beta_n, \lambda_n < 1$. We prove, if $T_i : C \longrightarrow C$, $i = 1,2,3$, are three generalized Lipschitzian mappings, then the sequence $\{x_n\}$ which is defined by equation (2.1) converges to $q \in \bigcap_{i=1}^{3} F(T_i)$, in a uniformly convex Banach space.

Theorem 2. *Let E be a uniformly convex Banach space, satisfying Opial's condition and C be a nonempty, closed and convex subset of E. $T_i : C \longrightarrow C$ are three generalized Lipschitizian mappings for $i = 1,2,3$ satisfying*

$$\|T_i^n x - T_i^n y\| \leq a_n^i \|x-y\| + b_n^i(\|x - T_i^n x\| + \|T_i^n y - y\|) + c_n^i(\|x - T_i^n y\| + \|T_i^n x - y\|)$$

for all $x, y \in C$ and $n \geq 1$, where $k_n^i = \frac{a_n^i + b_n^i + c_n^i}{1-(b_n^i + c_n^i)}$ and $\{k_n^i\} \subset [1,\infty)$ for $i = 1,2,3$ satisfy $k_n^i \longrightarrow 1$ as $n \longrightarrow \infty$. Also $\sum_{n=1}^{\infty}(\eta_n k_n^2 k_n^3 - 1) < \infty$ where $\eta_n = \max\{k_j^1, 1 \leq j \leq n\}$. Let $\{x_n\}$ be the sequence defined by (2.1). If the following statements hold

(I) $F = \bigcap_{i=1}^{3} F(T_i) \neq \phi$
(II) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty} \alpha_n < 1$
(III) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$
(IV) $0 < \liminf_{n\longrightarrow\infty} \lambda_n \leq \limsup_{n\longrightarrow\infty} \lambda_n < 1$

then

(1) $\lim_{n\longrightarrow\infty} \|x_n - q\|$ *exists, for all* $q \in F$.
(2) $\lim_{n\longrightarrow\infty} \|x_n - u_n\| = 0$ *and* $\lim_{n\longrightarrow\infty} \|x_n - T_i x_n\| = 0$ $(i = 2,3)$.
(3) *The sequence $\{x_n\}$ is weakly convergent to a common fixed point of T_i $(i = 1,2,3)$.*

Proof. For any $q \in F$ and $i = 1,2,3$, we get

$$\begin{aligned}\|T_i^n x_n - q\| &\le a_n^i\|x_n - q\| + b_n^i\|T_i^n x_n - x_n\| + c_n^i(\|T_i^n x_n - q\| + \|x_n - q\|)\\ &\le a_n^i\|x_n - q\| + b_n^i(\|T_i^n x_n - q\| + \|x_n - q\|) + c_n^i(\|T_i^n x_n - q\| + \|x_n - q\|),\end{aligned}$$

and so

$$\|T_i^n x_n - q\| \le \frac{a_n^i + b_n^i + c_n^i}{1-(b_n^i + c_n^i)}\|x_n - q\| = k_n^i\|x_n - q\|. \tag{2.2}$$

In order to find a bound for $\|x_{n+1} - q\|$, we need to compute $\|u_n - q\|$, $\|z_n - q\|$ and $\|y_n - q\|$ as follows:

(i) Suppose $1 \le j \le n$, by inequality (2.2)

$$\|u_n - q\| \le \frac{1}{n+1}(\|x_n - q\| + \sum_{j=1}^{n} k_j^1\|x_n - q\|) \le \eta_n\|x_n - q\|. \tag{2.3}$$

(ii) The definition of z_n in (2.1) implies

$$\begin{aligned}\|z_n - q\| &= \|(1-\lambda_n)x_n + \lambda_n u_n - q\|\\ &\le (1-\lambda_n)\|x_n - q\| + \lambda_n\eta_n\|x_n - q\|\\ &\le (1+\lambda_n(\eta_n - 1))\|x_n - q\|\\ &\le \eta_n\|x_n - q\|.\end{aligned} \tag{2.4}$$

(iii) The definition of y_n in (2.1) implies

$$\begin{aligned}\|y_n - q\| &= \|(1-\beta_n)x_n + \beta_n T_2^n z_n - q\|\\ &\le (1-\beta_n)\|x_n - q\| + \beta_n k_n^2\|z_n - q\|\\ &\le (1+\beta_n(k_n^2\eta_n - 1))\|x_n - q\|\\ &\le k_n^2\eta_n\|x_n - q\|.\end{aligned} \tag{2.5}$$

By (i), (ii) and (iii) one can have

$$\begin{aligned}\|x_{n+1} - q\| &= \|(1-\alpha_n)(x_n - q) + \alpha_n(T_3^n y_n - q)\|\\ &\le (1-\alpha_n)\|x_n - q\| + \alpha_n k_n^3\|y_n - q\|\\ &\le (1-\alpha_n)\|x_n - q\| + \alpha_n k_n^3 k_n^2\eta_n\|x_n - q\|\\ &\le (1+\alpha_n(k_n^3 k_n^2\eta_n - 1))\|x_n - q\|.\end{aligned} \tag{2.6}$$

Since $\sum_{n=1}^{\infty}(\eta_n k_n^2 k_n^3 - 1) < \infty$, Lemma 1 and inequality (2.6) imply $\lim\limits_{n\longrightarrow\infty}\|x_n - q\| = r$ exists. Furthermore,

$$\lim_{n\longrightarrow\infty}\|x_{n+1} - q\| = \lim_{n\longrightarrow\infty}\|(1-\alpha_n)(x_n - q) + \alpha_n(T_3^n y_n - q)\| = r.$$

Also, by (2.5)

$$\limsup_{n\longrightarrow\infty}\|T_3^n y_n - q\| \le \limsup_{n\longrightarrow\infty}(k_n^3\|y_n - q\|)$$

$$\leq \limsup_{n\longrightarrow\infty}(k_n^3k_n^2\eta_n\|x_n-q\|)$$
$$\leq \lim_{n\longrightarrow\infty}\|x_n-q\| = r.$$

Lemma 2 shows that $\lim_{n\longrightarrow\infty}\|T_3^ny_n - x_n\| = 0$. Furthermore, by (2.6)

$$\frac{\|x_{n+1}-q\| - \|x_n-q\| + \alpha_n\|x_n-q\|}{\alpha_nk_n^3} \leq \|y_n-q\|, \tag{2.7}$$

and by taking the limit $n\longrightarrow\infty$ in (2.7), we get

$$\lim_{n\longrightarrow\infty}\|x_n-q\| \leq \liminf_{n\longrightarrow\infty}\|y_n-q\|.$$

By (2.5),

$$\limsup_{n\longrightarrow\infty}\|y_n-q\| \leq \limsup_{n\longrightarrow\infty}(k_n^2\eta_n\|x_n-q\|) = \lim_{n\longrightarrow\infty}\|x_n-q\|. \tag{2.8}$$

Notice that (2.7) and (2.8) show

$$\lim_{n\longrightarrow\infty}\|x_n-q\| = \lim_{n\longrightarrow\infty}\|y_n-q\| = r.$$

Thus

$$\limsup_{n\longrightarrow\infty}\|(1-\beta_n)(x_n-q)+\beta_n(T_2^nz_n-q)\| = \limsup_{n\longrightarrow\infty}\|y_n-q\| = r.$$

On the other hand, (2.4) implies

$$\limsup_{n\longrightarrow\infty}\|T_2^nz_n-q\| \leq \limsup_{n\longrightarrow\infty}(k_n^2\|z_n-q\|)$$
$$\leq \limsup_{n\longrightarrow\infty}(k_n^2\eta_n\|x_n-q\|)$$
$$\leq \lim_{n\longrightarrow\infty}\|x_n-q\| = r,$$

and Lemma 2 shows that $\lim_{n\longrightarrow\infty}\|T_2^nz_n-x_n\| = 0$. Using the same technique, we have $\lim_{n\longrightarrow\infty}\|z_n-q\| = r$ and $\lim_{n\longrightarrow\infty}\|u_n-x_n\| = 0$.

Now, we prove $\|x_n - T_2^nx_n\| \longrightarrow 0$. From (2.1),

$$\|z_n-x_n\| = \|(1-\lambda_n)x_n+\lambda_nu_n-x_n\| = \lambda_n\|u_n-x_n\| \longrightarrow 0, \text{ as } n\longrightarrow\infty.$$

Notice that $\|x_n-T_2^nx_n\| \leq \|x_n-T_2^nz_n\| + \|T_2^nz_n-T_2^nx_n\|$, where

$$\begin{aligned}\|T_2^nz_n-T_2^nx_n\| &\leq a_n^2\|z_n-x_n\| + b_n^2(\|z_n-T_2^nz_n\| + \|T_2^nx_n-x_n\|)\\ &\quad + c_n^2(\|T_2^nz_n-x_n\| + \|T_2^nx_n-z_n\|)\\ &\leq a_n^2\|x_n-z_n\| + b_n^2(\|z_n-x_n\| + \|T_2^nz_n-x_n\| + \|T_2^nx_n-x_n\|)\\ &\quad + c_n^2(\|T_2^nz_n-x_n\| + \|x_n-z_n\| + \|x_n-T_2^nx_n\|)\\ &\leq (a_n^2+b_n^2+c_n^2)\|x_n-z_n\| + (b_n^2+c_n^2)(\|T_2^nz_n-x_n\| + \|T_2^nx_n-x_n\|).\end{aligned} \tag{2.9}$$

Since

$$\|x_n - T_2^n x_n\| \leq \|x_n - T_2^n z_n\| + (a_n^2 + b_n^2 + c_n^2)\|x_n - z_n\| + (b_n^2 + c_n^2)(\|T_2^n z_n - x_n\| + \|T_2^n x_n - x_n\|)$$

and

$$\|x_n - T_2^n x_n\| \leq k_n^2 \|z_n - x_n\| + (1 + k_n^2)\|x_n - T_2^n z_n\|, \tag{2.10}$$

by inequality (2.10), $\|x_n - T_2^n x_n\| \longrightarrow 0$.

Similarly, $\|x_n - T_3^n x_n\| \longrightarrow 0$. Therefore, by Lemma 6, $\|x_n - T_i x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, for $i = 2, 3$.

Since $\lim_{n \longrightarrow \infty} \|x_n - q\|$ exists and by the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $x_{nk} \rightharpoonup x$. Lemma 5 shows $x \in F$. Notice $w_w(\{x_n\})$ is a singleton. To do this, suppose there exists another subsequence $\{x_{nj}\}$ of $\{x_n\}$ which is weakly convergent to some $z \neq x$ such that $z \in F$. The existence of $\lim_{n\to\infty} \|x_n - q\|$ implies the existence of $\lim_{n\to\infty} \|x_n - x\|$ and $\lim_{n\to\infty} \|x_n - z\|$. Since E satisfies the Opial's condition, we have

$$\lim_{n\to\infty} \|x_n - x\| = \lim_{k\to\infty} \|x_{nk} - x\| < \lim_{k\to\infty} \|x_{nk} - z\| = \lim_{n\to\infty} \|x_n - z\|,$$

and

$$\lim_{n\to\infty} \|x_n - z\| = \lim_{j\to\infty} \|x_{nj} - z\| < \lim_{j\to\infty} \|x_{nj} - x\| = \lim_{n\to\infty} \|x_n - x\|,$$

which leads to a contradiction, and $w_w(\{x_n\})$ is a singleton. Therefore by Lemma 3 $\{x_n\}$ is weakly convergent to x. □

With respect to Theorem 2, the following example is presented.

Example 2. Let $E = \mathbb{R}$, $C = [0, \infty)$. Assume $\{x_n\}$ is the sequence defined by (2.1), where $T_1 x = \frac{x}{100}$, $T_2 x = \frac{x}{1000}$ and $T_3 x = \frac{x}{10000}$. Also $\alpha_n = \frac{n}{3n+1}$, $\beta_n = \frac{n}{4n+1}$ and $\lambda_n = \frac{n}{5n+1}$. We have

$$\begin{cases} u_n = \dfrac{1}{n+1} \displaystyle\sum_{j=0}^{n} \dfrac{x_n}{100^j}, \\ z_n = \dfrac{4n+1}{5n+1} x_n + (\dfrac{n}{5n+1}) u_n, \\ y_n = \dfrac{3n+1}{4n+1} x_n + (\dfrac{n}{4n+1})(\dfrac{1}{1000^n}) z_n, \\ x_{n+1} = \dfrac{2n+1}{3n+1} x_n + (\dfrac{n}{3n+1})(\dfrac{1}{10000^n}) y_n. \end{cases}$$

Let $x_1 = 1$, then by iteration we can have

$$x_{10} = 0.03842, \cdots, x_{20} = 0.00074, \cdots, x_{30} = 0.00001, \cdots$$

This shows $x_n \to 0$. Thus 0 is a common fixed point of T_1, T_2 and T_3 or $\{0\} = \bigcap_{i=1}^{3} F(T_i)$.

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