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# THE FIRST THREE LARGEST NUMBERS OF SUBUNIVERSES OF SEMILATTICES

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*Abstract.* Let $(L,\vee)$ be a finite n-element semilattice where $n \geq 5$. We prove that the first largest number of subuniverses of an $n$-element semilattice is $2^n$, while the second largest number is $28 \cdot 2^{n-5}$ and the third one is $26 \cdot 2^{n-5}$. Also, we describe the $n$-element semilattices with exactly $2^n$, $28 \cdot 2^{n-5}$ or $26 \cdot 2^{n-5}$ subuniverses.

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*Keywords:* finite lattice, sublattice, subuniverse, finite semilattice, number of subuniverses

## 1. Introduction and our result

For a semilattice $(L,\vee)$, $\mathrm{Sub}(L,\vee)$ will denote its *subuniverse-lattice*. By a subuniverse, we mean a subsemilattice or the emptyset. All semilattices occurring in this paper will be assumed to be finite. On a semilattice $(L,\leq)$, we have a natural partial ordering defined by

$$x \leq y \iff x \vee y = y.$$

Conversely, if $(L,\leq)$ is partial order in which any two elements $x,y$ have a least upper bound $x \vee y$, then $(L,\vee)$ is a semilattice. For any $x,y$ in a join-semilattice, $x \wedge y$ is defined by their infimum provided it exists; if this infimum does not exist, then $x \wedge y$ is undefined. Let $P$ and $Q$ be posets with disjoint underlying sets. Then the *ordinal sum* $P +_{ord} Q$ is the poset on $P \cup Q$ with $s \leq t$ if either $s,t \in P$ and $s \leq t$; or $s,t \in Q$ and $s \leq t$; or $s \in P$ and $t \in Q$. To draw the Hasse diagram of $P +_{ord} Q$, we place the Hasse diagram of $Q$ above that of $P$ and then connect any minimal element of $Q$ with any maximal element of $P$; see Figure 1. If $K$ with 1 and $L$ with 0 are finite posets, then their glued sum $K +_{\mathrm{glu}} L$ is the ordinal sum of the posets $K \setminus \{1_K\}$, the singleton poset, and $L \setminus \{0_L\}$, in this order; see Figure 2. Note that $+_{\mathrm{glu}}$ is an associative but not a commutative operation.

The semilattices $H_3$ and $H_4$ will be used later, see Figure 3.

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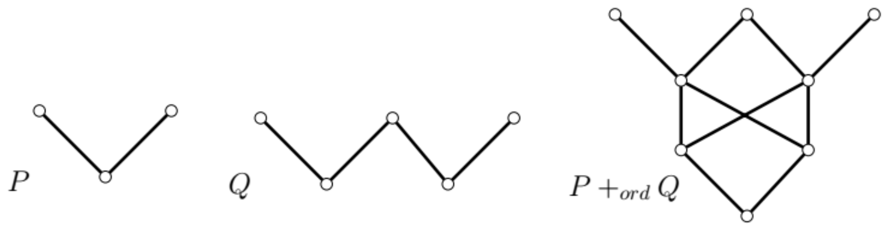


FIGURE 1. The ordinal sum $P +_{ord} Q$ of $P$ and $Q$

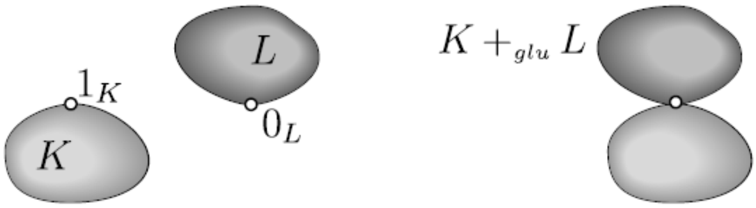


FIGURE 2. The glued sum $K +_{\mathrm{glu}} L$ of $K$ and $L$

Our result is motivated by similar results, see Ahmed and Horváth [1], Czédli [3–7] and Czédli and Horváth [8]. To obtain more information about lattice theory and semilattices we direct the reader to the bibliography indicated in [9, 10] and [2], respectively.

For a natural number $n \in \mathbb{N}^+ := \{1,2,3,\dots\}$, let

$$\mathrm{NS}(n) := \{|\operatorname{Sub}(L)| : L \text{ is a semilattice of size } |L| = n\}.$$

**Theorem 1.** *If $5 \le n \in \mathbb{N}^+$, then the following assertions hold.*

*(i) The largest number in $\mathrm{NS}(n)$ is $2^n = 32 \cdot 2^{n-5}$. Furthermore, an $n$-element semilattice $(L,\vee)$ has exactly $2^n$ subuniverses if and only if $(L,\vee)$ is a chain.*

*(ii) The second largest number in $\mathrm{NS}(n)$ is $28 \cdot 2^{n-5}$. Furthermore, an $n$-element semilattice $(L,\vee)$ has exactly $28 \cdot 2^{n-5}$ subuniverses if and only if $(L,\vee) \cong H_3 +_{\mathrm{glu}} C_1$ or $(L,\vee) \cong C_0 +_{ord} H_3 +_{\mathrm{glu}} C_1$, where $C_0$ and $C_1$ are finite chains.*

*(iii) The third largest number in $\mathrm{NS}(n)$ is $26 \cdot 2^{n-5}$. Furthermore, an $n$-element semilattice $(L,\vee)$ has exactly $26 \cdot 2^{n-5}$ subuniverses if and only if $(L,\vee) \cong H_4 +_{\mathrm{glu}} C_1$ or $(L,\vee) \cong C_0 +_{ord} H_4 +_{\mathrm{glu}} C_1$, where $C_0$ and $C_1$ are finite chains.*

## 2. TWO PREPARATORY LEMMAS

An element $u$ of a semilattice $L$ is called a narrow element, or a *narrows* for short, if $u \ne 1_L$ and $L = \uparrow u \cup \downarrow u$. That is, if $u \ne 1_L$ and $x \| u$ holds for no $x \in L$.

The notion of a binary partial algebra is well known, but the reader can refresh his/her knowledge from [4]. Let $\mathcal{A}$ be a finite $n$-element binary partial algebra. A *subuniverse* of $\mathcal{A}$ is a subset $X$ of $\mathcal{A}$ such that $X$ is closed with respect to all partial operations. The set of subuniverses of $\mathcal{A}$ will be denoted by $\mathrm{Sub}(\mathcal{A})$. The *relative number of subuniverses of* $\mathcal{A}$ denoted by $\sigma_k(\mathcal{A})$ is defined as follows:

$$\sigma_k(\mathcal{A}) = |\,\mathrm{Sub}(\mathcal{A})| \cdot 2^{k-n}.$$

In order to comply with [8], will use $k = 5$. The original definition of $\sigma_k$ is given in the paper of Czédli [4], there he used $k = 8$.

**Lemma 1.** *If $(K,\vee)$ is a subsemilattice and $H$ is a subset of a finite semilattice $(L,\vee)$, then the following three assertions hold.*

*(i) With the notation $t := |\{H \cap S : S \in Sub(L,\vee)\}|$, we have that*

$$\sigma_k(L,\vee) \le t \cdot 2^{k-|H|}.$$

*(ii) $\sigma_k(L,\vee) \le \sigma_k(K,\vee)$.*

*(iii) Assume, in addition, that $(K,\vee)$ has no narrows. Then $\sigma_k(L,\vee) = \sigma_k(K,\vee)$ if and only if $(L,\vee)$ is (isomorphic to) $C_0 +_{ord} (K,\vee) +_{\mathrm{glu}} C_1$, where $C_1$ is a chain, and $C_0$ is a chain or the emptyset.*

*Proof.* Parts (i) and (ii) can be extracted from the proof of in Lemma 2.3 of [4]. The argument there yields a bit more than stated in (i) and (ii); namely, for later reference, note the following.

$$\text{If } \sigma_k(L,\vee) = \sigma_k(K,\vee), \text{ then for every } S \in \mathrm{Sub}(K,\vee) \text{ and every subset } X \text{ of } L\backslash K \text{ we have that } S \cup X \in \mathrm{Sub}(L,\vee). \tag{2.1}$$

Next, to prove part (iii), let $n := |(L,\vee)|$ and $m := |(K,\vee)|$. Let $k := 5$, the case of another $k$ is analogous. Assume that $(K,\vee)$ has no narrows. First, let $(L,\vee) = C_0 +_{ord} (K,\vee) +_{\mathrm{glu}} C_1$. It is obvious that whenever $X \subseteq L\backslash K$ and $S \in \mathrm{Sub}(K,\vee)$, then $S \cup X \in \mathrm{Sub}(L,\vee)$. Since $L\backslash K$ has $2^{|L|-|K|}$ subsets, $|\,\mathrm{Sub}(L,\vee)| \ge |\,\mathrm{Sub}(K,\vee)| \cdot 2^{|L|-|K|}$. Dividing this inequality by $2^{n-5} = 2^{m-5} \cdot 2^{|L|-|K|}$ we obtain the required equality, as the converse inequality given in part (ii).

Conversely, assume the equality in (iii). We claim that

$$\text{for all } y \in K \text{ and for all } x \in L \setminus K,\ y \nparallel x. \tag{2.2}$$

Suppose the contrary. If $y \in K$, then $\{y\} \in \mathrm{Sub}(K)$. If $x \in L \setminus K$ and $y||x$, then $\{y,x\}$ is not a subuniverse of $L$, contradicting (2.1).

We claim that

$$\text{for all } x \in L \setminus K,\ x \not> 1_K \text{ implies that for all } y \in K,\ x < y. \tag{2.3}$$

Suppose the contrary and pick $x$ in $L \setminus K$ and $y \in K$ such that $x \not> 1_K$ and $x \not< y$. Using (2.2) and $x \neq y$, we have that $y < x < 1_K$. Let $p := \bigvee\{s \in K : s < x\}$; this exists by finiteness and $y \le p \le x$. In fact, $p \in K$ as $K$ is a subsemilattice of $L$ but $x \notin K$, so $y \le p < x$. Now let $u \in K$ such that $u \not\le p$. We know from (2.2) that $u \nparallel x$.

If we had $u \leq x$ (in fact, $u < x$ since $x \notin K$), then $u$ would be one of the joinands defining $p$ and so $u \leq p$ would be a contradiction. Hence $x < u$, and so $p < x < u$ implies $p < u$. We have seen that, for any $u \in K$, $u \not\leq p$ implies $p < u$. In other words, $K = \uparrow_K p \cup \downarrow_K p$, which means $p$ is a narrows, contradicting our assumption on $K$. Thus, (2.3) holds. Finally, we show that $L \setminus K$ is a chain. Indeed, if $L \setminus K$ is not a chain, say $a||b$, $a \in L \setminus K$ and $b \in L \setminus K$, then $\varnothing \in \mathrm{Sub}(K)$ extended by $\{a,b\} \notin \mathrm{Sub}(L)$ would contradict (2.1). Define $C_1 = \{x \in L \setminus K : x \geq 1_K\}$; it is a chain (a subchain of $L \setminus K$). Let $C_0 = (L \setminus K) \setminus C_1$; it is either a chain or empty. If $C_0$ is empty, then $L$ is $K +_{glue} C_1$, as required. If $C_0$ is nonempty, then its elements are less than any element of $K$ by (2.3), and so $L = C_0 +_{ord} K +_{glue} C_1$, as required. □

The following lemma can be proved by a computer program, but for the reader's convenience, we give its proof.

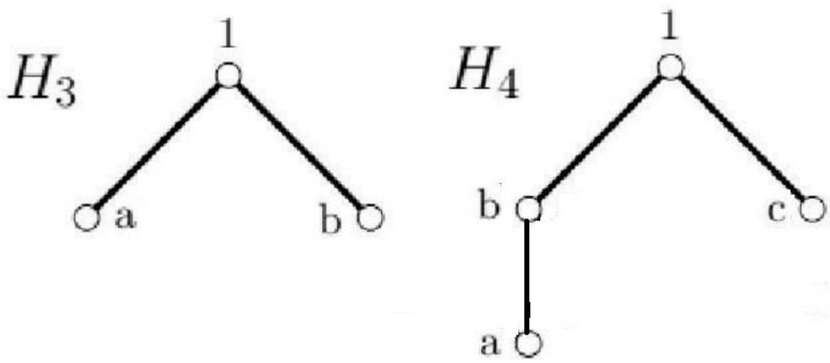


FIGURE 3. $H_3$ and $H_4$

**Lemma 2.** *For the join-semilattices given in Figure 3 the following assertions hold.*

*(i)* $\sigma_5(H_3) = 28$,
*(ii)* $\sigma_5(H_4) = 26$,

*Proof.* The notations given by Figure 3 will be used. For later reference, note that if $(L,\vee)$ is a chain then $|Sub(L,\vee)| = 2^{|(L,\vee)|}$.

For (i), observe that

$$|\{S \in \mathrm{Sub}(H_3,\vee) : a \notin S\}| = 4, \qquad \text{(S is chain)},$$
$$|\{S \in \mathrm{Sub}(H_3,\vee) : a \in S, \{b\} \cap S = \varnothing\}| = 2, \text{ and}$$
$$|\{S \in \mathrm{Sub}(H_3,\vee) : a \in S, \{b\} \cap S \neq \varnothing\}| = 1,$$

whereby $|\mathrm{Sub}(H_3,\vee)| = 4+2+1 = 7 = 28 \cdot 2^{3-5}$, note that $\sigma_5(H_3) = 28$ proves (i).

For (ii), let us compute

$$
\begin{aligned}
&|\{S \in \mathrm{Sub}(H_4,\vee) : c \notin S\}| = 7, \qquad \text{by (i)},\\
&|\{S \in \mathrm{Sub}(H_4,\vee) : c \in S, \{a,b\} \cap S = \varnothing\}| = 2, \text{ and}\\
&|\{S \in \mathrm{Sub}(H_4,\vee) : c \in S, \{a,b\} \cap S \neq \varnothing\}| = 4.
\end{aligned}
$$

Hence, $|\mathrm{Sub}(H_4,\vee)| = 7+2+4 = 13 = 26\cdot 2^{4-5}$, while $\sigma_5(H_4) = 26$ proves (ii).

*Remark* 1. For counting subsemilattices, a computer program is available on G. Czédli's webpage: http://www.math.u-szeged.hu/~czedli/

□

## 3. The rest of the proof

*Proof of Theorem 1.* Part (i) is trivial. For part (ii) let $(L,\vee)$ be an $n$-element semilattice. We know from Lemma 1 (iii) that if

$$(L,\vee) \cong H_3 +_{\mathrm{glu}} C_1 \text{ or } (L,\vee) \cong C_0 +_{ord} H_3 +_{\mathrm{glu}} C_1, \text{ where } C_0 \text{ and } C_1 \text{ are chains,} \tag{3.1}$$

then $\sigma_5(L) = \sigma_5(H_3) = 28$, indeed. Conversely, assume that $\sigma_5(L) = 28$. Then it follows from part (i) that $L$ is not a chain. So $L$ has two incomparable elements, $a$ and $b$. Clearly, $\{a,b,a\vee b\}$ is a join-subsemilattice isomorphic to $H_3$. But $\sigma_5(H_3)$ is also 28 by Lemma 2 (i). Thus, Lemma 1 (iii) immediately yields that $L$ is of the desired form. By this we completed the proof of part (ii) of Theorem 1.

We prove part (iii).

Assume that $(L,\vee)$ is of the given form, then $\sigma_5(L,\vee) = 26$ is clear from Lemma 2 (ii) and Lemma 1 (iii). In order to prove the converse, that is, the nontrivial implication, assume that $\sigma_5(L,\vee) = 26$. By Theorem 1 (i), $(L,\vee)$ has two incomparable elements, $a$ and $b$. By part Theorem 1 (ii), $\{a,b\}$ is not the only 2-element antichain in $L$ since otherwise $\sigma_5(L,\vee)$ would be 28. To complete the proof, consider the following cases.

**Case 1:** There is an antichain $\{c,d\}$ disjoint from $\{a,b\}$, where the elements $a$, $b$, $c$, $d$ are distict. Let $x := a\vee b$ and $y := c\vee d$. There are cases depending on $t := |\{a,b,c,d,x,y\}|$, which is 4, 5, or 6. The number of possible cases can be reduced by symmetry: $a$ and $b$ play a symmetric role, so do $c$ and $d$, and so do $\{a,b\}$ and $\{c,d\}$ and thus x and y. We consider three sub-cases as we will see below:

**Sub-case 1a:** Now $t = 6$. Take the partial algebra $U_1 = \{a,b,c,d,x,y\}$ with $a\vee b = x$ and $c\vee d = y$. This six-element partial algebra has $\sigma_5(U_1) = 24.5$, this can be checked by the mentioned computer program. By Lemma 2.3 from [4] we obtain that $\sigma_5(L) \le \sigma_5(U_1) \le 24.5$, contradicting $\sigma_5(L) = 26$. Thus, this case is excluded.

**Sub-case 1b:** Now $t = 5$. By symmetry, $y = c\vee d$ is not a new element, so $y = c\vee d$ is either $x$ or $a$. The case $y = b$ need not be considered because $a$ is symmetric to $b$. Therefore, this sub-case 1b is split in two cases as follows:

First, where $y = x$, this case is captured by taking the partial algebra $U_2 = \{a,b,c,d,x\}$ with $a \vee b = x$, $c \vee d = x$. It has $\sigma_5(U_2) = 25$ by the mentioned computer program. Like above, this implies $\sigma_5(L) \leq \sigma_5(U_2) \leq 25$, contradicting $\sigma_5(L) = 26$.

Second, where $y = a$, this case is captured by taking the partial algebra $U_3 = \{a,b,c,d,x\}$ with $a \vee b = x$, $c \vee d = a$. This five-element partial algebra has $\sigma_5(U_3) = 24 < 26$, and we get a contradiction as above.

By the above we can note that the sub-case 1b is excluded since so are both of its subcases.

**Sub-case 1c:** Here $t = 4$. Then $x = a \vee b$ is one of $c$ and $d$. By symmetry, we can assume that $a \vee b = c$. However, then $a < c < c \vee d$, $b < c < c \vee d$, whereby $y = c \vee d$ is none of the elements $a,b,c,d$, contradicting $t = 4$. So this case is excluded.

After having all of its sub-cases excluded, we obtain that Case 1 is excluded. That is, no two-element antichain is disjoint from $\{a,b\}$. But remember that there is another two-element antichain, whereby, by a-b symmetry, we consider **Case 2:** there is an element $c$ such that $a$ and $c$ are incomparable. Again there are two sub-cases according to the position of $b$ and $c$.

**Sub-case 2a:** Here $b$ and $c$ are also incomparable. Here, we have to investigate, how many of $a \vee b$, $a \vee c$, and $b \vee c$ are equal to $a \vee b \vee c$. Since the answer could be $0,1,2$ or $3$ (and using symmetry), it suffices to consider only the following four join-semilattices. The first join-semilattice is $K_0 = \{a,b,c,z,x,y,1\}$ with edges $ax$, $bx$, $by$, $cy$, $az$, $cz$, $x1$, $y1$, $z1$ and equalities $a \vee b = x$, $b \vee c = y$, $a \vee c = z$; this gives $\sigma_5(K_0) = 15.25$. The second join-semilattice is $K_1 = \{a,b,c,x,y,1\}$ with edges $ax$, $bx$, $by$, $cy$, $x1$, $y1$ and equalities $a \vee b = x$, $b \vee c = y$, $a \vee c = 1$; this gives $\sigma_5(K_1) = 18.5$. The third is $K_2 = \{a,b,c,x,1\}$ with edges $ax, bx, x1, c1$ and constraints $a \vee b = x$, $a \vee c = 1$, $b \vee c = 1$ ; this gives $\sigma_5(K_2) = 22$. The fourth is $K_3 = \{a,b,c,1\}$ with edges $a1, b1, c1$, and equalities $a \vee b = 1$, $a \vee c = 1$, $b \vee c = 1$; this gives $\sigma_5 = 24$. Since one of $K_0$, $K_1$, $K_2$, and $K_3$ is a subsemilattice of $L$ and all the four $\sigma_5$ values of these join-semilattices are smaller than 26, therefore sub-case 2a is excluded.

**Sub-case 2b:** Here $b$ and $c$ are comparable in addition to that $a$ and $c$ are incomparable and so do $a$ and $b$. At present, $b$ and $c$ play a symmetric role. So we can assume that $b < c$. Suppose, for contradiction, that $x := a \vee b < a \vee c =: 1$. By the incomparabilities assumed, $|\{a,b,c,x,1\}| = 5$; for example if $x = a \vee b = c$ is impossible since it would yield $a < c$. The mentioned constraints are defining $K$. Now $\sigma_5(K) = 23 < 26$ gives a contradiction. So $a \vee b < a \vee c$ fails but $a \vee b \leq a \vee c$ since $b < c$. Therefore, with $1 = a \vee b = a \vee c$, $\{a,b,c,1\}$ is a subsemilattice (isomorphic to) $H_4$.

Now that all other possibilities have been excluded, we know that $H_4$ is a join-subsemilattice of $(L, \vee)$. Observe that $H_4$ has no narrows. Therefore, by Lemma 1 (iii), $(L, \vee)$ is of the desired form. By this, the proof of Theorem 1 is completed.

$\square$

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