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HERMITE-HADAMARD TYPE INEQUALITIES AND RELATED INEQUALITIES FOR SUBADDITIVE FUNCTIONS

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Abstract. In this paper, we establish Hermite-Hadamard inequalities for subadditive functions, and we give some related inequalities according to Hermite-Hadamard inequalities, which generalized the previously published results.

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1. INTRODUCTION

The principal work on the general theory of subadditive functions is that of Hille and Phillips [6]. This reference also includes a part of the work of Rosenbaum [11] on subadditive functions of several variables. Additivity, subadditivity and superadditivity are important concepts both in measure theory and in several fields of mathematics and mathematical inequalities. Especially, there are a lot of examples of additive, subadditive and superadditive functions in various areas of mathematics such as norms, square roots, error function, growth rates, differential equations and integral means. Inequalities and especially subadditive functions theory is one of the most extensively developing fields not only in theoretical and applied mathematics but also physics and the other applied sciences. Here, we mention the results of [1, 6–8, 11] and the corresponding references cited therein.

Definition 1. A function f defined on a set H of real numbers and with range contained in the set $\mathbb{R}^+$ of all positive real numbers, is subadditive on H if, for all elements x and y of H such that $x+y$ is an element of H

$$f(x+y) \leq f(x)+f(y).$$

If equality holds, f is called additive; if the inequality is reversed, f is superadditive. A function f is convex on the (possibly infinite) interval D if, for all x and y in D and all t which satisfy $0 \leq t \leq 1$,

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y).$$

If this inequality is reversed, f is concave on D.

Remark 1. If f is convex and subadditive on H and if $f(0)=0$, then f is additive on H.

Definition 2. The function $f:[0,b]\to\mathbb{R}$, $b>0$ is said to be starshaped if for every $x\in[0,b]$ and $t\in[0,1]$ we have $f(tx)\leq tf(x)$.

According to above definitions, if a subadditive function $f:A\subset[0,\infty)\to\mathbb{R}$ is also starshaped, then f is a convex function.

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [4], [10]):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$.

The inequalities (1.1) have grown into a significant pillar for mathematical analysis and optimization, besides, by looking into a variety of settings, these inequalities are found to have a number of uses. What is more, for a specific choice of the function f, many inequalities with special means are obtainable. Hermite Hadamard's inequality (1.1), for example, is significant in its rich geometry and hence there are many studies on it to demonstrate its new proofs, refinements, extensions and generalizations. You can check ([2–5, 12–21]) and the references included there.

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities. Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors. Involving to the products of convex functions, Pachpatte gave two important new Hermite-Hadamard type inequalities as follows in [9]:

Theorem 1. *Let f and g be real-valued, nonnegative, and convex functions on $[a,b]$. Then,*

$$\begin{aligned}&2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\&\qquad\leq\frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{M(a,b)}{3}+\frac{N(a,b)}{6}\end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx\leq\frac{M(a,b)}{3}+\frac{N(a,b)}{6}$$

where

$$\begin{aligned}M(a,b)&=f(a)g(a)+f(b)g(b)\\N(a,b)&=f(a)g(b)+f(b)g(a).\end{aligned} \tag{1.2}$$

In this article, using subadditive functions, we obtained new inequalities of Hermite-Hadamard type and related. The results presented here would provide extensions of those given in earlier works.

2. Main Results

Hermite-Hadamard type inequalities for subadditive functions are given the following form:

Theorem 2. *If $f : I = [0,\infty) \to \mathbb{R}$ be a continuous subadditive function, $a,b \in I^\circ$ with $a < b$, then the following inequalities hold*

$$\frac{1}{2}f(a+b) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{a}\int_0^a f(x)\,dx + \frac{1}{b}\int_0^b f(x)\,dx.$$

Proof. Let $x \in [a,b]$, and for $t \in [0,1]$ $x = at + (1-t)b \in [a,b]$ (or $x = a(1-t) + tb \in [a,b]$). By using subadditivity of the function f, we can write

$$f(at + (1-t)b) \le f(at) + f((1-t)b) \tag{2.1}$$

and

$$f((1-t)a + tb) \le f(a(1-t)) + f(tb). \tag{2.2}$$

Adding (2.1) and (2.2), and by using subadditivity of the function f, we have

$$\begin{aligned} f(a+b) &\le f(at + (1-t)b) + f((1-t)a + tb) \\ &\le f(at) + f((1-t)b) + f(a(1-t)) + f(tb). \end{aligned} \tag{2.3}$$

Integrating both sides of (2.3), it follows that

$$\begin{aligned} f(a+b) &\le \int_0^1 f(at + (1-t)b)\,dt + \int_0^1 f((1-t)a + tb)\,dt \\ &\le 2\int_0^1 f(at)\,dt + 2\int_0^1 f(tb)\,dt. \end{aligned} \tag{2.4}$$

Thus, by the change of variable we obtain that

$$\frac{1}{2}f(a+b) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{a}\int_0^a f(x)\,dx + \frac{1}{b}\int_0^b f(x)\,dx$$

which completes the proof of Theorem. ☐

Corollary 1. *Under the conditions of Theorem 2, if we take $f(tx) \le tf(x)$, then we get*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}f(a+b) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{2.5}$$

Proof. Since $f(tx) \leq t f(x)$, then by using (2.4) we have

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right) &\leq f(a+b) \\ &\leq \int_0^1 f(at+(1-t)b)\,dt + \int_0^1 f((1-t)a+tb)\,dt \\ &\leq 2\int_0^1 tf(a)\,dt + 2\int_0^1 tf(b)\,dt = f(a)+f(b). \end{aligned}$$

By the change of variable, the desired inequalities (2.5) are achieved. □

Theorem 3. *If $f,\ g : I = [0,\infty) \to \mathbb{R}$ be two continuous subadditive functions, $a,b \in I^\circ$ with $a<b$, then the following inequalities hold*

$$\begin{aligned} &\frac{2}{b-a}\int_a^b f(x)g(x)dx \\ &\quad \leq \frac{2}{a}\int_0^a f(x)g(x)dx + \frac{2}{b}\int_0^b f(x)g(x)dx \\ &\qquad + 2\int_0^1 f(at)g((1-t)b)dt + 2\int_0^1 f((1-t)a)g(tb)dt \\ &\quad \leq \frac{1}{a}\int_0^a \left[(f(x))^2 + (g(x))^2\right]dx + \frac{1}{b}\int_0^b \left[(f(x))^2 + (g(x))^2\right]dx \\ &\qquad + 2\int_0^1 f(ta)f((1-t)b)dt + 2\int_0^1 g(ta)g((1-t)b)dt. \end{aligned} \tag{2.6}$$

Proof. By using subadditivity of the function f and g, we can write

$$f(ta+(1-t)b) \leq f(ta) + f((1-t)b) \tag{2.7}$$
$$g(ta+(1-t)b) \leq g(ta) + g((1-t)b). \tag{2.8}$$

Multiplying the above inequalities (2.7) and (2.8), we get

$$\begin{aligned} &f(ta+(1-t)b)g(ta+(1-t)b) \\ &\quad \leq (f(ta)+f((1-t)b))(g(ta)+g((1-t)b)) \\ &\quad = f(ta)g(ta) + f(ta)g((1-t)b) \\ &\qquad + f((1-t)b)g(ta) + f((1-t)b)g((1-t)b) \\ &\quad \leq \frac{1}{2}\left[(f(ta)+f((1-t)b))^2 + (g(ta)+g((1-t)b))^2\right]. \end{aligned} \tag{2.9}$$

and taking the integral with respect to t on $[0,1]$ and changing the variables of integration, we obtain our desired inequalities (2.6). □

Corollary 2. *Under the conditions of Theorem 3, if we take $f(tx) \le tf(x)$, then we get the following inequalities*

$$\begin{aligned}\frac{6}{b-a}\int_a^b f(x)g(x)dx &\le 2M(a,b)+N(a,b)\\ &\le (f(a)+f(b))^2+(g(a)+g(b))^2\\ &\quad -(f(a)f(b)+g(a)g(b))\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are defined by (1.2).

Theorem 4. *If f, $g : I=[0,\infty)\to\mathbb{R}$ be two continuous subadditive functions, $a,b\in I^\circ$ with $a<b$, then the following inequalities hold*

$$\begin{aligned}&\frac{1}{2}f(a+b)g(a+b)\\ &\quad\le \frac{1}{b-a}\int_a^b f(x)g(x)dx\\ &\qquad +\int_0^1 [f(at)g((1-t)a)+f((1-t)b)g(tb)]dt\\ &\qquad +\int_0^1 [f(at)g(tb)dt+f(tb)g(ta)]dt \end{aligned}\tag{2.10}$$

and

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)dx\\ &\quad\le \frac{1}{a}\int_0^a f(x)g(x)dx+\frac{1}{b}\int_0^b f(x)g(x)dx\\ &\qquad +\frac{1}{2}\int_0^1 f(at)g((1-t)b)dt+\frac{1}{2}\int_0^1 f(tb)g((1-t)a)dt. \end{aligned}\tag{2.11}$$

Proof. By using subadditivity of the function f and g, we can write

$$\begin{aligned}f(a+b) &= f(at+(1-t)b+(1-t)+tb)\\ &\le [f(at+(1-t)b)+f((1-t)a+tb)]\end{aligned}$$

and

$$g(a+b)\le [g(at+(1-t)b)+g((1-t)a+tb)].$$

Multiplying the above inequalities, we have:

$$\begin{aligned}&f(a+b)g(a+b)\\ &\quad\le [f(at+(1-t)b)g(at+(1-t)b)\\ &\qquad +f((1-t)a+tb)g((1-t)a+tb)]\\ &\qquad +[f(at+(1-t)b)g((1-t)a+tb)\\ &\qquad +f((1-t)a+tb)g(at+(1-t)b)]\end{aligned}$$

$$\begin{aligned}
&\leq [f(at+(1-t)b)g(at+(1-t)b) \\
&\qquad +f((1-t)a+tb)g((1-t)a+tb)] \\
&\qquad +[f(at)+f((1-t)b)][g((1-t)a)+g(tb)] \\
&\qquad +[f((1-t)a)+f(tb)][g(at)+g((1-t)b)] \\
&= [f(at+(1-t)b)g(at+(1-t)b) \\
&\qquad +f((1-t)a+tb)g((1-t)a+tb)] \\
&\qquad +[f(at)g((1-t)a)+f(at)g(tb) \\
&\qquad +f((1-t)b)g((1-t)a)+f((1-t)b)g(tb)] \\
&\qquad +[f((1-t)a)g(ta)+f((1-t)a)g((1-t)b) \\
&\qquad +f(tb)g(ta)+f(tb)g((1-t)b)]. \qquad (2.12)
\end{aligned}$$

Taking the integral of both sides in (2.12) with respect to t on $[0,1]$ and changing the variables of integration, we get

$$\begin{aligned}
&\frac{1}{2}f(a+b)g(a+b) \\
&\quad \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx \\
&\qquad +\int_0^1 [f(at)g((1-t)a)+f(tb)g((1-t)b)]dt \\
&\qquad +\int_0^1 [f(at)g(tb)dt+f(tb)g(ta)]dt
\end{aligned}$$

which completes the inequality (2.10). Since f and g are subadditive function, we can write

$$\begin{aligned}
f(at+(1-t)b) &\leq f(at)+f((1-t)b) \\
g(at+(1-t)b) &\leq g(at)+g((1-t)b),
\end{aligned}$$

and

$$\begin{aligned}
f((1-t)a+tb) &\leq f((1-t)a)+f(tb) \\
g((1-t)a+tb) &\leq g((1-t)a)+g(tb).
\end{aligned}$$

Multiplying the above inequalities, we get

$$\begin{aligned}
f(at &+(1-t)b)g(at+(1-t)b) \\
&\quad +f((1-t)a+tb)g((1-t)a+tb) \\
&\leq f(at)g(at)+f(at)g((1-t)b) \\
&\quad +f((1-t)b)g(at)+f((1-t)b)g((1-t)b) \\
&\quad +f((1-t)a)g((1-t)a)+f((1-t)a)g(tb)
\end{aligned}$$

$$+f(tb)\,g((1-t)\,a)+f(tb)\,g(tb)$$

and taking the integral with respect to t on $[0,1]$ and changing the variables of integration, we obtain the inequality (2.11). □

Corollary 3. *Under the conditions of Theorem 4, if we take $f(tx)\le tf(x)$, then we get*

$$\begin{aligned}
2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)&\\
&\le \frac{1}{2}f(a+b)\,g(a+b)\\
&\le \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx+\frac{M(a,b)}{3}+\frac{N(a,b)}{6}
\end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\le \frac{M(a,b)}{3}+\frac{N(a,b)}{6}$$

where $M(a,b)$ and $N(a,b)$ are defined by (1.2).

Proof. Since $f(tx)\le tf(x)$, then by using this inequality in the right and left hand sides of (2.4), we have

$$\begin{aligned}
\frac{1}{2}f(a+b)g(a+b)&\\
&\le \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\\
&\quad+\int_0^1\left[f(at)\,g((1-t)\,a)+f((1-t)\,b)\,g(tb)\right]dt\\
&\quad+\int_0^1\left[f(at)\,g(tb)\,dt+f(tb)\,g(ta)\right]dt\\
&\le \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\\
&\quad+\left[f(a)\,g(a)+f(b)\,g(b)\right]\int_0^1 t\,(1-t)\,dt\\
&\quad+\left[f(a)\,g(b)\,dt+f(b)\,g(a)\right]\int_0^1 t^2dt\\
&=\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\\
&\quad+\frac{1}{6}\left[f(a)\,g(a)+f(b)\,g(b)\right]+\frac{1}{3}\left[f(a)\,g(b)+f(b)\,g(a)\right].
\end{aligned}$$

With the same procedure as above, the other inequality is obtained. □

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