Volume 3 (2022), Article 1 https://research.library.kutztown.edu/contact/

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COMMUNICATIONS ON NUMBER THEORY AND COMBINATORIAL THEORY

Generating b-nomial numbers

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First submitted: September 3, 2021
Accepted: December 26, 2021
Published: January 7, 2022

Abstract

This paper presents three new ways to generate each type of b-nomial numbers: We develop ordinary generating functions, we find a whole new set of recurrence relations, and we identify each b-nomial number as a single binomial coefficient or as an alternating sum of products of two binomial coefficients.

Keywords: b-nomial, generalized binomial, indispensable
MSC 2020: 05A10, 11A63

1 Introduction

Throughout this paper, we let b be an integer greater than 1 and $g = b-1$. We let Σ_b be the set $\{0, 1, 2, \ldots, g\}$ and Σ_b^* be the set of all finite strings consisting of digits in Σ_b, including the *empty string*, which contains no digits, denoted by ϵ [8].

Definition 1.1. [1] For any string x and any digit $a_i \in \Sigma_b$ with $x = a_n a_{n-1} \cdots a_1$, the digit a_i is called *indispensable* in x if $a_i = a_{i-1} = a_{i-2} = \cdots = a_{i-k+1} > a_{i-k}$ for some positive integer $k \leq i+1$, considering $a_0 = 0$, and *dispensable*, otherwise.

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For example, in the string $2\dot{3}\dot{3}\dot{1}022\dot{3}$ in Σ_4^*, the dotted digits 3, 3, 1, and 3 are indispensable and the digits 2, 0, 2, and 2 are dispensable.

Definition 1.2. For any nonnegative integers n and k, we define $\binom{n}{k}_{b3}$ as the number of strings in Σ_b^* of length n with k indispensable digits.

The sequences A330381 and A330509 in [9] are for $\binom{n}{k}_{33}$ and $\binom{n}{k}_{43}$, respectively.

When $b = 2$, $\Sigma_b = \{0, 1\}$. Since $1 > 0$, the digit 0 is dispensable and the digit 1 is indispensable in any binary string. Hence, the number $\binom{n}{k}_{23}$ counts the number of binary strings of length n with k digit 1's, so $\binom{n}{k}_{23}$ is the (n, k)-th binomial coefficient:

$$\binom{n}{k}_{23} = \binom{n}{k}.$$

Hence, we can easily generate the number $\binom{n}{k}_{b3}$ when $b = 2$. However, when $b > 2$, we have to use the definition to find the number. For example, when $b = 3$, we have 9 ternary strings of length 3 with 1 indispensable digits:

$$00\dot{1}, 00\dot{2}, 0\dot{1}0, 01\dot{2}, 0\dot{2}0, \dot{1}00, 11\dot{2}, 1\dot{2}0, \text{ and } \dot{2}00,$$

and thus, the number $\binom{3}{1}_{33} = 9$. Since it is not ideal to sort and count such strings for large n, we look for other ways to generate this number $\binom{n}{k}_{b3}$.

To simplify further discussion, we denote $l_b(m)$ and $s_b(m)$ as the length and the digit sum of the base-b representation of a nonnegative integer m, respectively. For any nonnegative integer m, the digit sum of the base b-representation of the multiple $g \cdot m$ satisfies the following equation:

$$s_b(g \cdot m) = g \cdot k,$$

where k is the number of indispensable digits in the base-b representation of m [1]. Hence, we can redefine the number $\binom{n}{k}_{b3}$ as (iii) in the following definition.

Definition 1.3. [2] Let n and k be any nonnegative integers.

(i) The (n, k)-th b-nomial number of type 1, denoted by $\binom{n}{k}_{b1}$, is the number of nonnegative integers m's with $l_b(m) \le n$ and $s_b(m) = k$.

(ii) The (n, k)-th b-nomial number of type 2, denoted by $\binom{n}{k}_{b2}$, is the number of nonnegative integers m's with $l_b(m) = n + 1$ and $s_b(m) = k + 1$.

(iii) The (n, k)-th b-nomial number of type 3, denoted by $\binom{n}{k}_{b3}$, is the number of nonnegative integers m's with $l_b(m) \le n$ and $s_b(gm) = gk$.

(iv) The (n, k)-th b-nomial number of type 4, denoted by $\binom{n}{k}_{b4}$, is the number of nonnegative integers m's with $l_b(m) = n + 1$ and $s_b(gm) = g(k + 1)$.

For a convention, we define $\binom{-1}{k}_{bp} = 1$ if $k = -1$ and $\binom{-1}{k}_{bp} = 0$ otherwise for $p = 2$ and 4.

Since the b-nomial numbers of types 3 and 4 are relatively new, we do not have a good tool to generate those numbers. In this paper, we find a generating function and a recurrence relation for each type of b-nomial numbers, and we express each b-nomial number in terms of binomial coefficients.

In Section 2, we provide a combinatorial interpretation to construct a recurrence relation for b-nomial numbers of type 3. In Section 3, we summarize the previous studies on b-nomial numbers and binomial coefficients. In Section 4, we find an ordinary generating function for each type of b-nomial numbers. In Section 5, we identify each b-nomial number as a single binomial coefficient or as an alternating sum of products of two binomial coefficients. In Section 6, we find a recurrence relation for each type of b-nomial numbers algebraically.

2 Preview

Since the digit 0 is the smallest and the digit g is the largest in Σ_b, we have the following dispensability of the digit 0 and indispensability of the digit g:

Note 2.1. The digit 0 is dispensable and the digit g is indispensable in any string in Σ_b^*.

To find a recurrence relation for b-nomial numbers of type 3, we investigate a string x in Σ_b^* of length n with k indispensable digits. Let $a_i \in \Sigma_b$ such that $x = a_1a_2a_3 \dots a_n$. Then, we have two cases:

[1A] The digit a_1 is dispensable in x so the string $a_2a_3 \cdots a_n$ has k indispensable digits;

[1B] The digit a_1 is indispensable in x so the string $a_2a_3 \cdots a_n$ has $k-1$ indispensable digits.

By the definition, the number of choices for the substring $a_2a_3 \cdots a_n$ in [1A] and [1B] are $\binom{n-1}{k}_{b3}$ and $\binom{n-1}{k-1}_{b3}$, respectively. By Note 2.1, the digit $a_1 \neq g$ for [1A] and the digit $a_1 \neq 0$ for [1B]. That is,

$$a_1 \text{ is in } \{0, 1, 2, \dots, g-1\} \text{ for [1A]}; \qquad a_1 \text{ is in } \{1, 2, \dots, g\} \text{ for [1B]}.$$

Thus, the leading digit a_1 has g choices for each case. Hence, if we can ignore the digit a_1's dispensability or indispensability, the following expression counts these two cases:

$$\binom{g}{1}\left[\binom{n-1}{k}_{b3} + \binom{n-1}{k-1}_{b3}\right]. \tag{1}$$

For example, when $b = 2$, $g = 1$ so we always have $a_1 = 0$ for [1A] and $a_1 = 1$ for [1B]. Thus, the digit a_1's dispensability and indispensability are automatically satisfied. Hence, we find a recurrence relation for 2-nomial numbers of type 3 as follows:

$$\binom{n}{k}_{23} = \binom{1}{1}\left[\binom{n-1}{k}_{23} + \binom{n-1}{k-1}_{23}\right].$$

The digit a_1's dispensability or indispensability depends on the relation between a_1 and the following string $a_2a_3 \cdots a_n$. Hence, in general,

$$\binom{n}{k}_{b3} \leq \binom{g}{1}\left[\binom{n-1}{k}_{b3} + \binom{n-1}{k-1}_{b3}\right].$$

To change this to an equation, we have to subtract the following cases:

[2A] a_1 is in $\{0, 1, \ldots, g-1\}$, a_1 is indispensable, and $a_2a_3 \cdots a_n$ has k indispensable digits;

[2B] a_1 is in $\{1, 2, \ldots, g\}$, a_1 is dispensable, and $a_2a_3 \cdots a_n$ has $k-1$ indispensable digits.

By Note 2.1, the digit a_1 is in $\{1, 2, \ldots, g-1\}$ for both cases. Hence, considering the digit a_2's indispensability, we subtract the following cases instead: The digit a_1 is in $\{1, 2, \ldots, g-1\}$ and

[2A$_1$] a_1 is indispensable, a_2 is dispensable, and $a_3 \cdots a_n$ has k indispensable digits;

[2A$_2$] a_1 is indispensable, a_2 is indispensable, and $a_3 \cdots a_n$ has $k-1$ indispensable digits;

[2B$_1$] a_1 is dispensable, a_2 is dispensable, and $a_3 \cdots a_n$ has $k-1$ indispensable digits;

[2B$_2$] a_1 is dispensable, a_2 is indispensable, and $a_3 \cdots a_n$ has $k-2$ indispensable digits;

If a_1 is indispensable, $a_1 \geq a_2$, and if a_1 is dispensable, $a_1 \leq a_2$. Since a digit cannot be dispensable and indispensable at the same time, $a_1 \neq a_2$ for [2A$_1$] and [2B$_2$]. Hence, we can rewrite each case as follows: The digit a_1 is in $\{1, 2, \ldots, g-1\}$ and

[2A$_1$] $a_1 > a_2$, a_2 is dispensable, and $a_3 \cdots a_n$ has k indispensable digits;

[2A$_2$] $a_1 \geq a_2$, a_2 is indispensable, and $a_3 \cdots a_n$ has $k-1$ indispensable digits;

[2B$_1$] $a_1 \leq a_2$, a_2 is dispensable, and $a_3 \cdots a_n$ has $k-1$ indispensable digits;

[2B$_2$] $a_1 < a_2$, a_2 is indispensable, and $a_3 \cdots a_n$ has $k-2$ indispensable digits;

We count the cases [2A$_1$] and [2B$_2$] first. By the definition, the number of choices for the substring $a_3 \cdots a_n$ in [2A$_1$] and [2B$_2$] are $\binom{n-2}{k}_{b3}$ and $\binom{n-2}{k-2}_{b3}$, respectively. To count strings for a_1a_2, we notice the following conditions:

$$g-1 \geq a_1 > a_2 \geq 0 \text{ for [2A}_1\text{]}; \qquad 1 \leq a_1 < a_2 \leq g \text{ for [2B}_2\text{]}.$$

Thus, we choose two distinct digits from $\{0, 1, 2, \ldots, g-1\}$ for [2A$_1$] and $\{1, 2, \ldots, g\}$ for [2B$_2$], and then, assign them to a_1 and a_2 according to each inequality. Hence, there are $\binom{g}{2}$ choices for the string a_1a_2 in each case. If we can ignore the digit a_2's dispensability or indispensability, the following expression counts the cases [2A$_1$] and [2B$_2$]:

$$\binom{g}{2}\binom{n-2}{k}_{b3} + \binom{g}{2}\binom{n-2}{k-2}_{b3}. \tag{2}$$

The other two cases [2A$_2$] and [2B$_1$] have the same number of choices for the substring $a_3 \cdots a_n$ as $\binom{n-2}{k-1}_{b3}$, and the substring a_1a_2 satisfies the following conditions:

for [2A$_2$],	$g-1 \geq a_1 \geq a_2$ and a_2 is indispensable so	$g-1 \geq a_1 \geq a_2 \geq 1$;
for [2B$_1$],	$1 \leq a_1 \leq a_2$ and a_2 is dispensable so	$1 \leq a_1 \leq a_2 \leq g-1$.

Thus, we choose two digits from the set $\{1, 2, \ldots g-1\}$ with repetition allowed and arrange them to a_1 and a_2 according to each inequality. Since the number of multisets of size 2 from a $(g-1)$-set is $\left(\binom{g-1}{2}\right) = \binom{g-1+2-1}{2}$, we have $\binom{g}{2}$ choices for the string $a_1 a_2$ in each case. However, we do not double this number to count these two cases, because [2A$_2$] requires the digit a_2 to be indispensable and [2B$_1$] requires the digit a_2 to be dispensable. If the digit $a_1 = a_2$, these two cases are complementary to each other so that $\binom{g}{2}$ is the exact number of choices for the strings $a_1 a_2$ in [2A$_2$] and [2B$_1$] together. Hence, if $a_1 = a_2$, the following expression counts the cases [2A$_2$] and [2B$_1$] together:

$$\binom{g}{2}\binom{n-2}{k-1}_{b3}. \tag{3}$$

Therefore, if this property $a_1 = a_2$ always holds for [2A$_2$] and [2B$_1$] and if we can ignore the digit a_2's dispensability or indispensability for [2A$_1$] and [2B$_2$], the following expression counts all of the four cases to subtract from (1):

$$\binom{g}{2}\left[\binom{n-2}{k}_{b3} + \binom{n-2}{k-1}_{b3} + \binom{n-2}{k-2}_{b3}\right]. \tag{4}$$

For example, when $b = 3$, $g = 2$ so we have only one choice for the string $a_1 a_0$ in each case:

$$a_1 a_0 = 10 \text{ for [2A}_1\text{]}; \qquad a_1 a_2 = 11 \text{ for [2A}_2\text{] and [2B}_1\text{]}; \qquad a_1 a_2 = 12 \text{ for [2B}_2\text{]}.$$

Then, the digit a_2's dispensability and indispensability are automatically satisfied for [2A$_1$] and [2B$_2$] by Note 2.1, and the property $a_1 = a_2$ holds for [2A$_2$] and [2B$_1$]. Hence, by subtracting (4) from (1), we find a recurrence relation for 3-nomial numbers of type 3 as follows:

$$\binom{n}{k}_{33} = \binom{2}{1}\left[\binom{n-1}{k}_{33} + \binom{n-1}{k-1}_{33}\right] - \binom{2}{2}\left[\binom{n-2}{k}_{33} + \binom{n-2}{k-1}_{33} + \binom{n-2}{k-2}_{33}\right].$$

If $a_1 \neq a_2$, we have restrictions on the digit a_3: The digit a_3 cannot be equal to a_2 and dispensable for [2A$_2$], because if so, the digit a_2's dispensability does not hold. Similarly, the digit a_3 cannot be equal to a_2 and indispensable for [2B$_1$]. Moreover, if the difference $|a_1 - a_2| > 1$, there is a digit a such that $a_1 > a > a_2$ for [2A$_2$] and $a_1 < a < a_2$ for [2B$_1$]. Then, the digit a_3 cannot be equal to a, because if so, the digit a_2's dispensability or indispensability holds for neither [2A$_2$] nor [2B$_1$]. Since the number $\binom{n-2}{k-1}_{b3}$ counts the strings of length $n-2$ with $k-1$ indispensable digits without any restriction, this expression (3) counts more than we want. Since we also have a restriction on the digit a_2's dispensability and indispensability for [2A$_1$] and [2B$_2$], the expression (2) counts more than we want. Hence, in general,

$$\binom{n}{k}_{b3} \geq \binom{g}{1}\left[\binom{n-1}{k}_{b3} + \binom{n-1}{k-1}_{b3}\right] - \binom{g}{2}\left[\binom{n-2}{k}_{b3} + \binom{n-2}{k-1}_{b3} + \binom{n-2}{k-2}_{b3}\right].$$

We continue this process to find the recurrence relation for b-nomial numbers of type 3 in Theorem 6.4 (iii).

3 Previous studies

By the definition, the binomial coefficients are generalized by any type of b-nomial numbers, because

$$\binom{n}{k}_{2p} = \binom{n}{k} \text{ for any } p = 1, 2, 3, \text{ and } 4,$$

and the b-nomial numbers of type 1 are identified as the *extended binomial coefficients* or *polynomial coefficients* [2]. Hence, we can generate b-nomial numbers of type 1 by the following generating function and identity [4]:

$$(x^g + x^{g-1} + \cdots + x + 1)^n = \sum_k \binom{n}{k}_{b1} x^k; \qquad \binom{n}{k}_{b1} = \sum_{i \geq 0} (-1)^i \binom{n}{i} \binom{n+k-ib-1}{n-1}. \quad (5)$$

We can also generate b-nomial numbers of type 1 and 2 using the following recurrence relation [3, 2]:

$$\binom{n}{k}_{bp} = \sum_{i=0}^{g} \binom{n-1}{k-i}_{bp} \text{ for } p = 1, 2. \quad (6)$$

The following relations among b-nomial numbers are to simplify further discussion.

Lemma 3.1. *[2] For any nonnegative integers n and k,*

(i) $\binom{n}{k}_{b2} = \binom{n+1}{k+1}_{b1} - \binom{n}{k+1}_{b1}$;

(ii) $\binom{n}{k}_{b3} = \binom{n+1}{gk+1}_{b1} - \binom{n}{gk+1}_{b1}$;

(iii) $\binom{n}{k}_{b4} = \binom{n+1}{k+1}_{b3} - \binom{n}{k+1}_{b3}$;

(iv) $\binom{n}{k}_{b1} = \binom{n}{gn-k}_{b1}$;

(v) $\binom{n}{k}_{b2} = \binom{n}{g(n+1)-(k+1)}_{b2}$;

(vi) $\binom{n}{k}_{b4} = \binom{n}{n-k}_{b4}$.

By Lemma 3.1 (i) and (ii), we can obtain the following relation.

Lemma 3.2. *For any nonnegative integers n and k,*

$$\binom{n}{k}_{b3} = \binom{n}{gk}_{b2}.$$

The following identities of binomial coefficients are to simplify calculations throughout this paper.

Lemma 3.3. *For any nonnegative integers r, m, n, and k,*

(i) $\sum_{i=r}^{k} \binom{i}{r} = \binom{k+1}{r+1}$;

(ii) $\sum_{i=0}^{m} \binom{m}{i}\binom{n}{r-i} = \binom{m+n}{r}$;

(iii) $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$;

(iv) $\binom{r}{m}\binom{m}{k} = \binom{r}{r-(m-k)}\binom{r-(m-k)}{k}$;

(v) $\binom{r}{m}\binom{r-m}{k} = \binom{r}{k}\binom{r-k}{m}$;

(vi) $\binom{r}{m}\binom{r-m}{k} = \binom{r}{m+k}\binom{m+k}{m}$;

(vii) $\sum_{i=0}^{g}(-1)^i\binom{g}{i}\binom{n-i}{k} = \binom{n-g}{k-g}$.

Proof. (i) is the hockey stick identity [5]; (ii) is Vandermonde's identity [6]; (iii) is also well-known [6]. (iv) is obtained by (iii):

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k} = \binom{r}{r-k}\binom{r-k}{m-k}$$
$$= \binom{r}{m-k}\binom{r-(m-k)}{r-m} = \binom{r}{r-(m-k)}\binom{r-(m-k)}{k}.$$

(v) and (vi) are obtained, because

$$\frac{r!}{k!(r-k!)}\cdot\frac{(r-k!)}{m!(r-k-m)!} = \frac{r!}{m!(r-m)!}\cdot\frac{(r-m)!}{k!(r-m-k)!} = \frac{r!}{(m+k)!(r-m-k)!}\cdot\frac{(m+k)!}{m!k!}.$$

(vii) is obtained by applying $m = g$, $j = g - i$, and $s = n - g$ to the following alternating sum identity of the product of binomial coefficients [6]:

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{s+j}{k} = \binom{s}{k-m}.$$

□

If $n - g \geq 0$ and $k - g < 0$, the number $\binom{n-g}{k-g} = 0$. Hence, Lemma 3.3 (vii) provides the following identity.

Corollary 3.4. *For any nonnegative integers $n \geq g$ and $k < g$,*

$$\sum_{i=0}^{g}(-1)^i\binom{g}{i}\binom{n-i}{k} = 0. \tag{7}$$

4 Generating functions

It is straightforward to find a generating function for b-nomial numbers of type 2 from type 1 and type 4 from type 3, respectively, by Lemma 3.1 (i) and (iii). However, for b-nomial numbers of type 3, we need elaborate preparations.

When $b = 3$, $g = 2$. Suppose $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ is a generating function for 3-nomial numbers of type 2 so that the coefficient $a_k = \binom{n}{k}_{32}$. Then, by Lemma 3.2,

$$\sum_k \binom{n}{k}_{33} x^k = \sum_k \binom{n}{2k}_{32} x^k = a_0 + a_2x + a_4x^2 + a_6x^3 + \cdots.$$

Thus, to find a generating function for 3-nomial numbers of type 3, we need to remove all odd terms from $f(x)$ and adjust the power of x. Hence, we calculate the following average:

$$\frac{1}{2}\left(f(x^{\frac{1}{2}})+f(-1\cdot x^{\frac{1}{2}})\right)=a_0+a_2x+a_4x^2+a_6x^3+\cdots=\sum_k\binom{n}{k}_{33}x^k.$$

In general, Lemma 3.2 indicates to eliminate all terms except the terms with x^{gk} from the generating function for b-nomial numbers of types 2 for type 3. Hence, we consider a g-th primitive complex root of unity, $\xi=e^{\frac{2\pi}{g}i}$, since it has the following property [7]:

$$\sum_{i=1}^{g}\xi^{si}=\begin{cases}0, & \text{if } s\not\equiv 0 \pmod g;\\ g, & \text{if } s\equiv 0 \pmod g).\end{cases} \tag{8}$$

We present an ordinary generating function for each type of b-nomial numbers as follows.

Theorem 4.1. *Let $\Phi_g(x)=x^{g-1}+x^{g-2}+\cdots+x+1$, $\Phi_b(x)=x^g+x^{g-1}+\cdots+x+1$, and $\xi=e^{\frac{2\pi}{g}i}$. Then, for any nonnegative integer n,*

(i) $$\sum_{k\geq 0}\binom{n}{k}_{b1}x^k=[\Phi_b(x)]^n=\frac{(x^b-1)^n}{(x-1)^n};$$

(ii) $$\sum_{k\geq 0}\binom{n}{k}_{b2}x^k=\Phi_g(x)\,[\Phi_b(x)]^n=\frac{(x^g-1)(x^b-1)^n}{(x-1)^{n+1}};$$

(iii) $$\sum_{k\geq 0}\binom{n}{k}_{b3}x^k=\frac{1}{g}\sum_{t=0}^{g-1}\Phi_g(\xi^t x^{\frac{1}{g}})\left[\Phi_b(\xi^t x^{\frac{1}{g}})\right]^n=\frac{x-1}{g}\sum_{t=0}^{g-1}\frac{(\xi^t x^{\frac{b}{g}}-1)^n}{(\xi^t x^{\frac{1}{g}}-1)^{n+1}};$$

(iv) $$\sum_{k\geq 0}\binom{n}{k}_{b4}x^k=\frac{1}{gx}\sum_{t=0}^{g-1}\xi^t x^{\frac{1}{g}}\left[\Phi_g(\xi^t x^{\frac{1}{g}})\right]^2\left[\Phi_b(\xi^t x^{\frac{1}{g}})\right]^n=\frac{(x-1)^2}{gx}\sum_{t=0}^{g-1}\frac{\xi^t x^{\frac{1}{g}}(\xi^t x^{\frac{b}{g}}-1)^n}{(\xi^t x^{\frac{1}{g}}-1)^{n+2}}.$$

Proof. (i) is obtained by the generating function in (5).

(ii) is obtained by (i) and Lemma 3.1 (i): Since $\binom{n+1}{0}_{b1}=1=\binom{n}{0}_{b3}$,

$$\begin{aligned}\sum_{k\geq 0}\binom{n}{k}_{b2}x^k&=\sum_{k\geq 0}\left[\binom{n+1}{k+1}_{b1}-\binom{n}{k+1}_{b1}\right]x^k=\sum_{k\geq 1}\left[\binom{n+1}{k}_{b1}-\binom{n}{k}_{b1}\right]x^{k-1}\\&=\frac{1}{x}\sum_{k\geq 0}\left[\binom{n+1}{k}_{b1}-\binom{n}{k}_{b1}\right]x^k=\frac{1}{x}\left\{[\Phi_b(x)]^{n+1}-[\Phi_b(x)]^n\right\}\\&=\frac{1}{x}\left[\Phi(x)_b-1\right][\Phi_b(x)]^n=\frac{1}{x}\left[x\cdot\Phi(x)_g\right][\Phi_b(x)]^n=\cdot\Phi(x)_g\,[\Phi_b(x)]^n.\end{aligned}$$

(iii) Let $f(x)=\Phi_g(x)\,[\Phi_b(x)]^n$ and $a_k=\binom{n}{k}_{b2}$. Then, by (ii),

$$f(x)=\sum_{i\geq 0}a_ix^i=a_0+a_1x+a_2x^2+\cdots+a_gx^g+\cdots+a_{2g}x^{2g}+\cdots.$$

Since $\binom{n}{k}_{b3} = \binom{n}{gk}_{b2}$ by Lemma 3.2, we want to make $a_i = 0$ if $i \neq gk$ for any integer k and a_{gk} as the coefficient of x^k so that

$$\sum_{i\geq 0} \binom{n}{k}_{b3} x^i = a_0 + a_g x + a_{2g} x^2 + a_{3g} x^3 + \cdots .$$

Thus, we replace x with $\xi^t x^{\frac{1}{g}}$ in $f(x)$:

$$f(\xi^t x^{\frac{1}{g}}) = \sum_{i\geq 0} a_i \xi^{it} x^{\frac{i}{g}} = a_0 + a_1 \xi^t x^{\frac{1}{g}} + a_2 \xi^{2t} x^{\frac{2}{g}} + \cdots + a_g \xi^{gt} x^{\frac{g}{g}} + \cdots + a_{2g} \xi^{2gt} x^{\frac{2g}{g}} + \cdots .$$

Then, we calculate the average of all $f(\xi^t x^{\frac{1}{g}})$ for $t = 0, 1, 2, \ldots, g-1$:

$$\frac{1}{g} \sum_{t=0}^{g-1} f(\xi^i x^{\frac{1}{g}}) = \frac{1}{g} \sum_{t=0}^{g-1} \sum_{i\geq 0} a_i \xi^{it} x^{\frac{i}{g}} = \sum_{i\geq 0} a_i \left(\frac{1}{g} \sum_{t=0}^{g-1} \xi^{it} \right) x^{\frac{i}{g}}.$$

By (8), we have

$$\frac{1}{g} \sum_{t=0}^{g-1} \xi^{it} = \begin{cases} 1, & \text{if } i = gk \text{ for any integer } k; \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\frac{1}{g} \sum_{t=0}^{g-1} f(\xi^i x^{\frac{1}{g}}) = \sum_{k\geq 0} a_{gk} \cdot 1 \cdot x^{\frac{gk}{g}} = \sum_{k\geq 0} a_{gk} x^k = \sum_{k\geq 0} \binom{n}{gk}_{b2} x^k = \sum_{k\geq 0} \binom{n}{k}_{b3} x^k.$$

(iv) is obtained by (iii) and Lemma 3.1 (iii): Since $\binom{n+1}{0}_{b3} = 1 = \binom{n}{0}_{b3}$,

$$\begin{aligned}
\sum_{k\geq 0} \binom{n}{k}_{b4} x^k &= \frac{1}{x} \sum_{k\geq 0} \left[\binom{n+1}{k+1}_{b3} - \binom{n}{k+1}_{b3} \right] x^{k+1} = \frac{1}{x} \sum_{k\geq 0} \left[\binom{n+1}{k}_{b3} - \binom{n}{k}_{b3} \right] x^k \\
&= \frac{1}{gx} \sum_{t=0}^{g-1} \Phi_g(\xi^t x^{\frac{1}{g}}) \left[\left[\Phi_b(\xi^t x^{\frac{1}{g}}) \right]^{n+1} - \left[\Phi_b(\xi^t x^{\frac{1}{g}}) \right]^n \right) \\
&= \frac{1}{gx} \sum_{t=0}^{g-1} \Phi_g(\xi^t x^{\frac{1}{g}}) \left(\Phi_b(\xi^t x^{\frac{1}{g}}) - 1 \right) \left[\Phi_b(\xi^t x^{\frac{1}{g}}) \right]^n \\
&= \frac{1}{gx} \sum_{t=0}^{g-1} \Phi_g(\xi^t x^{\frac{1}{g}}) \left(\xi^t x^{\frac{1}{g}} \cdot \Phi_g(\xi^t x^{\frac{1}{g}}) \right) \left[\Phi_b(\xi^t x^{\frac{1}{g}}) \right]^n .
\end{aligned}$$

□

Example 4.2. When $b = 3$, we have $g = 2$ and $\xi = -1$. Hence, we have

$$\sum_{k\geq 0}\binom{n}{k}_{31}x^k = \left(x^2+x+1\right)^n;$$

$$\sum_{k\geq 0}\binom{n}{k}_{32}x^k = (x+1)\left(x^2+x+1\right)^n;$$

$$\sum_{k\geq 0}\binom{n}{k}_{33}x^k = \frac{1}{2}\left[(x^{\frac{1}{2}}+1)(x+x^{\frac{1}{2}}+1)^n + (-x^{\frac{1}{2}}+1)(x-x^{\frac{1}{2}}+1)^n\right];$$

$$\sum_{k\geq 0}\binom{n}{k}_{34}x^k = \frac{1}{2x^{\frac{1}{2}}}\left[(x^{\frac{1}{2}}+1)^2(x+x^{\frac{1}{2}}+1)^n - (-x^{\frac{1}{2}}+1)^2(x-x^{\frac{1}{2}}+1)^n\right].$$

5 Using binomial coefficients

We identify each b-nominal number as an alternating sum of products of binomial coefficients as follows.

Theorem 5.1. *For any nonnegative integers n and k,*

(i) $$\binom{n}{k}_{b1} = \sum_{i=0}^{\lfloor \frac{k}{b}\rfloor}(-1)^i\binom{n}{i}\binom{n+k-ib-1}{n-1};$$

(ii) $$\binom{n}{k}_{b2} = \sum_{i=0}^{\lfloor \frac{k+1}{b}\rfloor}(-1)^i\left[\binom{n}{i}\binom{n+k-ib}{n} + \binom{n}{i-1}\binom{n+k-ib+1}{n}\right];$$

(iii) $$\binom{n}{k}_{b3} = \sum_{i=0}^{\lfloor \frac{kg+1}{b}\rfloor}(-1)^i\left[\binom{n}{i}\binom{n+kg-ib}{n} + \binom{n}{i-1}\binom{n+kg-ib+1}{n}\right];$$

(v) $$\binom{n}{k}_{b4} = \sum_{i=0}^{\lfloor \frac{kg+b}{b}\rfloor}(-1)^i\left[\binom{n}{i}\binom{n+(k+1)g-ib}{n+1} + 2\binom{n}{i-1}\binom{n+(k+1)g-ib+1}{n+1}\right.$$
$$\left. + \binom{n}{i-2}\binom{n+(k+1)g-ib+2}{n+1}\right].$$

Proof. (i) is obtained by the identity in (5).

(ii) is obtained by (i) and Lemma 3.1 (i):

$$\begin{aligned}\binom{n}{k}_{b2} &= \binom{n+1}{k+1}_{b1} - \binom{n}{k+1}_{b1}\\ &= \sum_{i=0}^{\lfloor \frac{k+1}{b}\rfloor}(-1)^i\left[\binom{n+1}{i}\binom{n+k-ib+1}{n} - \binom{n}{i}\binom{n+k-ib}{n-1}\right],\end{aligned}$$

and by the recurrence relation for the binomial coefficients,

$$\begin{aligned}
&\binom{n+1}{i}\binom{n+k-ib+1}{n} - \binom{n}{i}\binom{n+k-ib}{n-1}\\
&= \left[\binom{n}{i} + \binom{n}{i-1}\right]\binom{n+k-ib+1}{n} - \binom{n}{i}\binom{n+k-ib}{n-1}\\
&= \binom{n}{i}\left[\binom{n+k-ib+1}{n} - \binom{n+k-ib}{n-1}\right] + \binom{n}{i-1}\binom{n+k-ib+1}{n}\\
&= \binom{n}{i}\binom{n+k-ib}{n} + \binom{n}{i-1}\binom{n+k-ib+1}{n}.
\end{aligned}$$

(iii) is obtained by (ii) and Lemma 3.2.

(iv) is obtained by (iii) and Lemma 3.1 (iii): Since $(k+1)g+1 = kg+b$,

$$\begin{aligned}
&\binom{n}{k}_{b4} = \binom{n+1}{k+1}_{b3} - \binom{n}{k+1}_{b3}\\
&= \sum_{t=0}^{\lfloor \frac{kg+b}{b} \rfloor} (-1)^i \left[\binom{n+1}{i}\binom{n+(k+1)g-ib+1}{n+1} + \binom{n+1}{i-1}\binom{n+(k+1)g-ib+2}{n+1}\right]\\
&\quad - \sum_{t=0}^{\lfloor \frac{kg+b}{b} \rfloor} (-1)^i \left[\binom{n}{i}\binom{n+(k+1)g-ib}{n} + \binom{n}{i-1}\binom{n+(k+1)g-ib+1}{n}\right].
\end{aligned}$$

Since $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$ and $\binom{n+1}{i-1} = \binom{n}{i-1} + \binom{n}{i-2}$,

$$\begin{aligned}
\binom{n}{k}_{b4} = \sum_{t=0}^{\lfloor \frac{kg+b}{b} \rfloor} (-1)^i \Bigg\{ &\binom{n}{i}\left[\binom{n+(k+1)g-ib+1}{n+1} - \binom{n+(k+1)g-ib}{n}\right]\\
&+ \binom{n}{i-1}\left[\binom{n+(k+1)g-ib+2}{n+1} - \binom{n+(k+1)g-ib+1}{n}\right]\\
&+ \binom{n}{i-1}\binom{n+(k+1)g-ib+1}{n+1} + \binom{n}{i-2}\binom{n+(k+1)g-ib+2}{n+1} \Bigg\},
\end{aligned}$$

which is simplified as (iv). □

By this theorem and the symmetric properties of b-nomial numbers, we identify each of the following nonzero b-nomial numbers as a single binomial coefficient.

Corollary 5.2. *For any nonnegative integers n and k,*

(**i**) $\binom{n}{k}_{b1} = \binom{n}{gn-k}_{b1} = \binom{n+k-1}{k}$ *for $k \le g$;*

(**ii**) $\binom{n}{k}_{b2} = \binom{n}{g(n+1)-(k+1)}_{b2} = \binom{n+k}{k}$ *for $k < g$;*

(**iii**) $\binom{n}{n}_{b3} = \binom{n+g-1}{g-1}$ *and* $\binom{n}{0}_{b3} = \binom{n}{0}$;

(**iv**) $\binom{n}{0}_{b4} = \binom{n}{n}_{b4} = \binom{n+g}{g-1}$.

Proof. (i) and (ii) are obtained by Theorem 5.1 and 3.1 (iv) and (v), respectively. (iii) is obtained by (ii) and Lemma 3.2:

$$\binom{n}{n}_{b3} = \binom{n}{gn}_{b2} = \binom{n}{g(n+1)-(gn+1)}_{b2} = \binom{n}{g-1}_{b2} = \binom{n+g-1}{g-1}.$$

(iv) is obtained by Theorem 5.1 and 3.1 (vi): Since $\binom{n-1}{n+1} = 0 = \binom{n}{n+1}$,

$$\binom{n}{0}_{b4} = \binom{n}{0}\binom{n+g}{n+1} - \left[\binom{n}{1}\binom{n+g-b}{n+1} + 2\binom{n}{0}\binom{n+g-b+1}{n+1}\right] = \binom{n+g}{n+1}.$$

□

6 Recurrence relations

In Section 2, we previewed a recurrence relation for b-nomial numbers of type 3 combinatorially. In this section, we find and justify a recurrence relation for each type of b-nomial numbers algebraically.

To simplify further discussion, we construct the following identity.

Lemma 6.1. *For any $0 \le n \le g$ and $k > 0$,*

$$\sum_{i=0}^{g}\binom{g}{i}\binom{n-i}{t-i}\binom{(k-t-1)g+n-t}{n-i-1} = \sum_{i=0}^{g}(-1)^i\binom{g}{i}\binom{n-i}{t}\binom{(k-t)g+n-t-i}{n-i-1}. \tag{9}$$

Proof. Since the left-hand side of the equation (9) is the same as

$$\sum_{i=0}^{g}\binom{g}{i}\left[\binom{n-i-1}{n-t} + \binom{n-i-1}{n-t-1}\right]\binom{(k-t-1)g+n-t}{n-i-1},$$

we use Lemma 3.3 (iii) to have

$$\binom{(k-t-1)g+n-t}{n-i-1}\binom{n-i-1}{n-t} = \binom{(k-t-1)g+n-t}{n-t}\binom{(k-t-1)g}{t-1-i};$$
$$\binom{(k-t-1)g+n-t}{n-i-1}\binom{n-i-1}{n-t-1} = \binom{(k-t-1)g+n-t}{n-t-1}\binom{(k-t-1)g+1}{t-i}.$$

By Vandermonde's identity in Lemma 3.3 (ii), we have

$$\sum_{i=0}^{g}\binom{g}{i}\binom{(k-t-1)g}{t-1-i} = \binom{(k-t)g}{t-1}; \qquad \sum_{i=0}^{g}\binom{g}{i}\binom{(k-t-1)g+1}{t-i} = \binom{(k-t)g+1}{t}.$$

Since $\binom{(k-t-1)g+n-t}{n-t} = \binom{(k-t-1)g+n-t}{(k-t-1)g}$ and $\binom{(k-t-1)g+n-t}{n-t-1} = \binom{(k-t-1)g+n-t}{(k-t-1)g+1}$, the left-hand side of (9) is equal to

$$\binom{(k-t-1)g+n-t}{(k-t-1)g}\binom{(k-t)g}{t-1} + \binom{(k-t-1)g+n-t}{(k-t-1)g+1}\binom{(k-t)g+1}{t}. \tag{10}$$

Since the right-hand side of the equation (9) is the same as

$$\sum_{i=0}^{g}(-1)^i\binom{g}{i}\left[\binom{n-i-1}{t-1}+\binom{n-i-1}{t}\right]\binom{(k-t)g+n-t-i}{n-i-1},$$

we use Lemma 3.3 (iv) to have

$$\binom{(k-t)g+n-t-i}{n-i-1}\binom{n-i-1}{t-1}=\binom{(k-t)g+n-t-i}{(k-t)g}\binom{(k-t)g}{t-1};$$
$$\binom{(k-t)g+n-t-i}{n-i-1}\binom{n-i-1}{t}=\binom{(k-t)g+n-t-i}{(k-t)g+1}\binom{(k-t)g+1}{t}.$$

By Lemma 3.3 (vii), we have

$$\sum_{i=0}^{g}(-1)^i\binom{g}{i}\binom{(k-t)g+n-t-i}{(k-t)g}=\binom{(k-t-1)g+n-t}{(k-t-1)g};$$
$$\sum_{i=0}^{g}(-1)^i\binom{g}{i}\binom{(k-t)g+n-t-i}{(k-t)g+1}=\binom{(k-t-1)g+n-t}{(k-t-1)g+1}.$$

Hence, the right-hand side of (9) is also equal to (10). □

Then, we find the following identity for some particular b-nomial numbers of type 1.

Lemma 6.2. *For any integers n and k, if $n=g$ or $k=k'g+1$ for some positive integer k',*

$$\sum_{i=0}^{g}(-1)^i\binom{g}{i}\binom{n-i}{k}_{b1}=\sum_{i=0}^{g}(-1)^i\binom{g}{i}\binom{n-i}{k-(i+1)g}_{b1}. \tag{11}$$

Proof. By Theorem 5.1 (i) and Lemma 6.1, if $k>0$,

$$\begin{aligned}
\sum_{i=0}^{g}(-1)^i\binom{g}{i}\binom{n-i}{kg+1}_{b1} &= \sum_{i}(-1)^i\binom{g}{i}\sum_{t}(-1)^t\binom{n-i}{t}\binom{n-i+kg+1-tb-1}{n-i-1}\\
&= \sum_{t}(-1)^t\left[\sum_{i}(-1)^i\binom{g}{i}\binom{n-i}{t}\binom{(k-t)g+n-t-i}{n-i-1}\right]\\
&= \sum_{t}(-1)^t\left[\sum_{i}\binom{g}{i}\binom{n-i}{t-i}\binom{(k-t-1)g+n-t}{n-i-1}\right]\\
&= \sum_{t}(-1)^{t+i}\sum_{i}\binom{g}{i}\binom{n-i}{t}\binom{(k-t-i-1)g+n-t-i}{n-i-1}\\
&= \sum_{i}(-1)^i\binom{g}{i}\sum_{t}(-1)^t\binom{n-i}{t}\binom{n-i+(k-i-1)g-tb}{n-i-1}\\
&= \sum_{t}(-1)^i\binom{g}{i}\binom{n-i}{kg+1-(i+1)g}_{b1}.
\end{aligned}$$

Hence, when $k = k'g + 1$ for some positive integer k', the identity (11) holds.

Now we consider $n = g$. Then, by Theorem 5.1 (i) and Lemma 3.3 (v) and (vii), we have

$$\begin{aligned}\sum_{i=0}^{g}(-1)^i\binom{g}{i}\binom{n-i}{k}_{b1} &= \sum_i\sum_t(-1)^i(-1)^t\binom{g}{i}\binom{g-i}{t}\binom{g-i+k-tb-1}{k-tb}\\ &= \sum_i\sum_t(-1)^i(-1)^t\binom{g}{t}\binom{g-t}{i}\binom{g-i+k-tb-1}{k-tb}\\ &= \sum_t(-1)^t\binom{g}{t}\sum_i(-1)^i\binom{g-t}{i}\binom{g+k-tg-t-1-i}{k-tg-t}\\ &= \sum_t(-1)^t\binom{g}{t}\binom{k-tg-1}{k-tg-g}.\end{aligned}$$

By Theorem 5.1 (i) and Lemma 3.3 (vi) and (ii), we have

$$\begin{aligned}\sum_{i=0}^{g}(-1)^i\binom{g}{i}\binom{n-i}{k-ig-g}_{b1} &= \sum_i\sum_t(-1)^{t+i}\binom{g}{i}\binom{g-i}{t}\binom{g-i+k-ig-g-tb-1}{g-i-1}.\\ &= \sum_i\sum_t(-1)^{t+i}\binom{g}{t+i}\binom{t+i}{i}\binom{k-(t+i)g-(t+i)-1}{g-i-1}\\ &= \sum_i\sum_t(-1)^{t}\binom{g}{t}\binom{t}{i}\binom{k-tg-t-1}{g-i-1}\\ &= \sum_t(-1)^t\binom{g}{t}\sum_i\binom{t}{i}\binom{k-tg-t-1}{g-1-i}\\ &= \sum_t(-1)^t\binom{g}{t}\binom{k-tg-1}{g-1} = \sum_t(-1)^t\binom{g}{t}\binom{k-tg-1}{k-tg-g}.\end{aligned}$$

Therefore, the identity (11) holds for $n = g$ as well. □

Adding more terms to both sides in the identity (11), we find the following identity.

Corollary 6.3. *For any nonnegative integers n and k, if $n = g$ or $k = k'g + 1$ for some positive integer k',*

$$\sum_{i=0}^{g}(-1)^i\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-jg}_{b1} = 0. \tag{12}$$

Proof. The proof is done by mathematical induction on $\left\lfloor\frac{k}{g}\right\rfloor$. If $\left\lfloor\frac{k}{g}\right\rfloor = 0$, $0 \le k < g$. Thus, $k \neq k'g + 1$ for any $k' > 0$ so $n = g$. Hence, the base case holds by Corollary 5.2 and Corollary 3.4.

Induction Hypothesis: Assume $\displaystyle\sum_{i=0}^{g}(-1)^i\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-jg}_{b1} = 0$ for $\displaystyle\left\lfloor\frac{k}{g}\right\rfloor < m$.

Suppose $\left\lfloor \frac{k}{g} \right\rfloor = m$. Then, $\left\lfloor \frac{k-g}{g} \right\rfloor = m-1$ so $\sum_{i=0}^{g}(-1)^i\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-g-jg}_{b1} = 0$ by the induction hypothesis. Hence, by Lemma 6.2, we have

$$\begin{aligned}
&\sum_{i=0}^{g}(-1)^i\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-jg}_{b1}\\
&=\sum_{i=0}^{g}(-1)^i\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-jg}_{b1}-\sum_{i=0}^{g}(-1)^i\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-g-jg}_{b1}\\
&=\sum_{i=0}^{g}(-1)^i\binom{g}{i}\left[\binom{n-i}{k}_{b1}+\sum_{j=1}^{i}\binom{n-i}{k-jg}_{b1}-\sum_{j=0}^{i-1}\binom{n-i}{k-g-jg}_{b1}-\binom{n-i}{k-(i+1)g}_{b1}\right]\\
&=\sum_{i=0}^{g}(-1)^i\binom{g}{i}\binom{n-i}{k}_{b1}-\sum_{i=0}^{g}(-1)^i\binom{g}{i}\binom{n-i}{k-(i+1)g}_{b1}=0.
\end{aligned}$$

□

By solving for $\binom{n}{k}_{b1}$ from (12), we find a recurrence relation for some particular b-nomial numbers of type 1:

$$\binom{n}{k}_{b1}=\sum_{i=0}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-jg}_{b1} \text{ for } n=g \text{ or } k=k'g+1>g. \tag{13}$$

We extend the range of n in (13), and find a recurrence relation for each type of b-nomial numbers as follows.

Theorem 6.4. *For any nonnegative integers n and k,*

(**i**) $\displaystyle\binom{n}{k}_{b1}=\sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-jg}_{b1}$ *for $n\geq g$ or $k=k'g+1>g$;*

(**ii**) $\displaystyle\binom{n}{k}_{b2}=\sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-jg}_{b2}$ *for $n\geq g$ or $k=k'g\geq g$;*

(**iii**) $\displaystyle\binom{n}{k}_{b3}=\sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-j}_{b3}$ *for $n\geq g$ or $k>0$;*

(**iv**) $\displaystyle\binom{n}{k}_{b4}=\sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-j}_{b4}.$

Proof. (i) Because of (13), we just need to show when $n\geq g$. The proof is done by mathematical induction on n. The base case is shown in (13).

Induction Hypothesis: Assume that (i) holds when $n<N$.

Supposes $n = N$. Then, (i) is obtained by the recurrence relation in (6) and the induction hypothesis:

$$\binom{n}{k}_{b1} = \sum_{t=0}^{g} \binom{n-1}{k-t}_{b1} = \sum_{t=0}^{g}\sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n-1-i}{k-t-jg}_{b1}$$
$$= \sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\left(\sum_{t=0}^{g}\binom{n-i-1}{k-jg-t}_{b1}\right)$$
$$= \sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-jg}_{b1}.$$

(ii) is obtained by (i) and Lemma 3.1 (i): If $k = k'g \geq g$, $k+1 = k'g+1 > g$. Hence, for any integers n and k with $n \geq g$ or $k = k'g \geq g$,

$$\binom{n}{k}_{b2} = \binom{n+1}{k+1}_{b1} - \binom{n}{k+1}_{b1}$$
$$= \sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n+1-i}{k+1-jg}_{b1} - \sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k+1-jg}_{b1}$$
$$= \sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\left[\binom{n-i+1}{k-jg+1}_{b1} - \binom{n-i}{k-jg+1}_{b1}\right]$$
$$= \sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-jg}_{b2}.$$

(iii) is obtained by (ii) and Lemma 3.2: If $k > 0$, $kg \geq g$. Hence, for $n \geq g$ and $k > 0$,

$$\binom{n}{k}_{b3} = \binom{n}{kg}_{b2} = \sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{kg-jg}_{b2} = \sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-j}_{b3}.$$

(iv) is obtained by (iii) and Lemma 3.1 (iii): If $k \geq 0$, the inequality $k+1 > 0$ always holds. Hence, for any nonnegative integers n and k,

$$\binom{n}{k}_{b4} = \binom{n+1}{k+1}_{b3} - \binom{n}{k+1}_{b3}$$
$$= \sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n+1-i}{k+1-j}_{b3} - \sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k+1-j}_{b3}$$
$$= \sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\left[\binom{n-i+1}{k-j+1}_{b3} - \binom{n-i}{k-j+1}_{b3}\right]$$
$$= \sum_{i=1}^{g}(-1)^{i+1}\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-j}_{b4}.$$

□

Notice that when $b = 2$, every recurrence relation in Theorem 6.4 is identified as the famous recurrence relation for the binomial coefficients:

$$\binom{n}{k}_{2p} = \binom{n-1}{k}_{2p} + \binom{n-1}{k-1}_{2p} \text{ for all } p = 1, 2, 3, \text{ and } 4.$$

We can also simplify the recurrence relations as the following identities.

Corollary 6.5. *For any nonnegative integers n and k,*

(i) $$\sum_{i=0}^{g}(-1)^i\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-jg}_{b1} = 0 \quad \textit{for } n \geq g \textit{ or } k = k'g + 1 > g;$$

(ii) $$\sum_{i=0}^{g}(-1)^i\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-jg}_{b2} = 0 \quad \textit{for } n \geq g \textit{ or } k = k'g \geq g;$$

(iii) $$\sum_{i=0}^{g}(-1)^i\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-j}_{b3} = 0 \quad \textit{for } n \geq g \textit{ or } k > 0;$$

(iv) $$\sum_{i=0}^{g}(-1)^i\binom{g}{i}\sum_{j=0}^{i}\binom{n-i}{k-j}_{b4} = 0.$$

References

[1] J. Choi, Indispensable digits for digit sums, *Notes Number Theory Discrete Math.* **25** no. 2 (2019), 40–48.

[2] J. Choi, Digit sums generalizing the binomial coefficients, *J. Integer sequences* **22** (2019), Article 19.8.3.

[3] J. E. Freund, Restricted occupancy theory—a generalization of Pascal's triangle, *Amer. Math. Monthly* **63** (1956), 20–27.

[4] S. Eger, Restricted weighted integer compositions and extended binomial coefficients, *J. Integer Sequences* **16** (2013), Article 13.1.3.

[5] C. H. Jones, Generalized Hockey Stick Identities and N-Dimensional Block Walking, *Fibonacci Quart.* **34** (1996), 280–288.

[6] D. E. Knuth, *The Art of Computer Programming: Volume 1: Fundamental Algorithms* (1997), Addison-Wesley.

[7] W. Ladermann, *Complex Numbers* (1960), Routledge & Kegan Paul.

[8] J. Shallit, *A Second Course in Formal Languages and Automata Theory* (2009), Cambridge University Press.

[9] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`.