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TIME SERIES MODEL BUILDING WITH FOURIER AUTOREGRESSIVE MODEL

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This paper presents time series model building using Fourier autoregressive models. This model is capable of modelling and forecasting time series data that exhibit periodic and seasonal movements. From the implementation of the model, FAR(1), FAR(2) and FAR(3) models were chosen based on the periodic autocorrelation function (PeACF) and periodic partial autocorrelation function. The coefficients of the tentative model were estimated using a discrete Fourier transform estimation method. The FAR(1) model was chosen as the optimal model based on the smallest value of periodic Akaike and Bayesian information criteria, and the residuals of the fitted models were diagnosed to be white noise using the periodic residual autocorrelation function. The out-sample forecasts were obtained for the Nigerian monthly rainfall series from January 2018 to December 2019 using the FAR(1) and SARIMA$(1, 1, 1)x(1, 1, 1)_{12}$ models. The results exhibited a continuous periodic and seasonal movement but the periodic movement in the forecasted rainfall series was better with FAR(1) because its values showed a close reflection of the original series. The values of the forecast evaluation for both models showed that the forecast was consistent and accurate but the FAR(1) model forecast was more accurate since its forecast evaluation values were relatively lower. Hence, the Fourier autoregressive model is adequate and suitable for modelling and forecasting periodicity and seasonality in Nigerian rainfall time series data and any part of the world with rainfall series that are mostly characterised with periodic variation.

Key words: Forecasting, Fourier autoregressive process, Periodicity, Rainfall series, Seasonality.

1. Introduction

Cyclic and seasonal movements are found and seen in numerous fields. The periodic and occasional qualities of several phenoma in our immediate local environment can be defined in the form of time and space. These movements happen contingent upon every day, month to month, yearly or other periodic changes (Bloomfield, 2004). Recently, there has been broad research work in the improvement of time series analysis models for cyclic and occasional time series data. The most noticeable is the growth of the class of autoregressive moving average models by Box and Jenkins

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(1970) and the routinely used models are seasonal autoregressive integrated moving average models by Box et al. (2015), and periodic autoregressive moving average models by Tiao and Grupe (1980). Seasonal autoregressive integrated moving average models have been utilised to examine time series data that display periodic variation over the years by many researchers such as Olatayo and Taiwo (2015), Mohamed and Etuk (2015), Godwin and Fakiyesi (2016), Lidiema (2017) and Zhou et al. (2018). However, this appeared to be not absolutely accurate since most periodic time series data exhibit intermittent conduct in the mean, standard deviation and skewness (Tesfaye et al., 2006). In addition to these periodicities, they exhibited a period relationship structure which might be either consistent or intermittent (Iqelan, 2011).

Basically, periodic autoregressive moving average models are capable of dealing with the attributes of cyclic time series data and have been used by many researchers over the years. These are reflected in the works of Vecchia (1985), Anderson and Vecchia (1993), Ula and Smadi (2003), McCall and Jeraj (2007), H. I. El Shekh and Al-Awar (2014), Iwok and Udoh (2016) and Taiwo (2017), where positive advancements have been recorded. However, the main obstacle with the utilisation of these models has to do with the means associated with model structure. In accordance with Dudek et al. (2016), techniques for obtaining the coefficients of periodic autoregressive moving average (PARMA) framework when the periodic coefficients are expressed in term of Fourier series, are still used in practice and remain a vital sphere for further research.

As discussed by early researchers on PARMA models, such as Hannan (1955) and Jones and Brelsford (1967), if the periodic movements are smooth inside the crucial period, as may be normal in numerous physical time series, a significant decrease in the quantity of assessed parameters might be acknowledged by setting a significant number of the Fourier coefficients to be zero and this confines the evaluated answers for a subspace. The identified challenges turn into how to determine the vital lags and frequencies with crucial amplitudes. In spite of the noteworthy advancement that has been set aside in modelling periodic time series with suitable time series models, this research work proposes a simple and robust model for periodic and seasonal time series data. The proposed model is named the Fourier autoregressive (FAR) model and it is based on Fourier-PAR techniques. This will include model identification, estimation, error diagnostics and forecasting. In order to verify the performance of the FAR model, the out-sample forecast and forecast evaluation measures of the FAR model will be compared with the seasonal autoregressive integrated moving average model result that is regularly used for modelling and forecasting seasonal and periodic time series data.

2. Materials and methods

2.1 Fourier autoregressive model

The Fourier autoregressive (FAR) model is formed by defining a different periodic autoregressive for each period of the year. If $y_{k\omega+\nu} = \{y_{k\omega+\nu} \in Z\}$ is a periodic stationary stochastic process then FAR is given by

$$y_{k\omega+\nu} = \varphi_0 + \sum_{i=1}^{p(\nu)} \left[\varphi_i(\nu) \cos 2\pi \frac{k}{\omega} + \varphi_i^*(\nu) \sin 2\pi \frac{k}{\omega} \right] y_{k\omega+\nu-i} + \epsilon_{k\omega+\nu}, \tag{1}$$

where ν is the period index $(\nu = 1, 2, \ldots, \omega), k$ is the year index $(k = 0, \pm1, \pm2, \ldots)$, $\varphi_i(\nu)$ is the periodic autoregressive coefficient, ω is the number of seasons, and $\epsilon_{k\omega+\nu}$ is white noise with mean zero and periodic variance $\sigma^2_\epsilon(\nu)$. Given that the mean and autocovariance functions are periodic functions of time with periodic ω, then the first and second order moments of the process are

$$\begin{aligned} E[y_t] &= \mu(t) = \mu[t + k\omega], \\ cov(y_t y_s) &= \gamma(t, s) = \gamma(t + k\omega, s + k\omega). \end{aligned}$$

2.2 Autoregressive model building

2.2.1 Model identification for the Fourier autoregressive (FAR) model

The identification of the FAR model to be estimated will be based on the periodic autocorrelation function (PeACF) and periodic partial autocorrelation function (PePACF). For a univariate periodic stationary process $\{y_{k\omega+\nu}\}$ defined by equation (1) in which the white noise terms $\{\epsilon_{k\omega+\nu}\}$ are assumed to be independent, the periodic autocovariance function is defined as

$$\gamma_{k\omega+\nu}(l) = cov(y_{k\omega+\nu} y_{k\omega+\nu-l}) = E\left[(y_{k\omega+\nu} - \mu_\nu)(y_{k\omega+\nu-i} - \mu_{\nu-l})\right]$$

for season ν at backward lag $l \geq 0$. Then, the PeACF for period ν at backward lag $l \geq 0$ is defined as

$$\rho_i(\nu) = \frac{\gamma_l}{\sqrt{\gamma_0(\nu)\gamma_0(\nu - l)}}, \quad l \geq 0,$$

where γ_0 is the variance for the νth season.

Let $\{y_1, y_2, \ldots, y_{n\omega}\}$ be a series of size n_ω of periodic stationary process $\{y_{k\omega+\nu}\}$. Then the sample estimate of $\rho_l(\nu)$ is called the sample periodic autocorrelation function and it is given as

$$\Gamma_l(\nu) = \frac{\hat{\gamma}_l}{\sqrt{\hat{\gamma}_0(\nu)\hat{\gamma}_0(\nu - l)}}, \quad l \geq 0,$$

where $\hat{\gamma}_l(\nu)$ is the sample periodic autocorrelation function calculated from

$$\hat{\gamma}_l(\nu) = \frac{1}{N}\sum_{k=0}^{N-1}(y_{k\omega+\nu} - \bar{y}_\nu)(y_{k\omega+\nu-l} - \bar{y}_{\nu-l}),$$

in which

$$\bar{y}_\nu = \frac{1}{N}\sum_{k=0}^{N-1}(y_{k\omega+\nu})$$

is the sample mean for season ν.

The periodic partial autocorrelation function $\phi_{ll}(\nu)$ is viewed as a proportion of the careful connection between $y_{k\omega+\nu}$ and $y_{k\omega+\nu-l}$ after removing the effect of the intervening observation. It is defined for integers $l \geq 1$ as

$$\phi_{ll} = corr\left[y_{k\omega+\nu}, y_{k\omega+\nu-l} \,\middle|\, y_{k\omega+\nu-1}, \ldots, y_{k\omega+\nu-l+1}\right].$$

Denote the periodic partial autocorrelation function by

$$\phi_{ll} = \frac{cov[(y_{k\omega+\nu} - \hat{y}_{k\omega+\nu}), (y_{k\omega+\nu-l} - \hat{y}_{k\omega+\nu-l})]}{\sqrt{var(y_{k\omega+\nu} - \hat{y}_{k\omega+\nu})}\sqrt{var(y_{k\omega+\nu-l} - \hat{y}_{k\omega+\nu-l})}}.$$

The sample partial autocorrelation function will be obtained using a recursive method by substituting the value of $\hat{\rho}_i(\nu) = \hat{\phi}_{ll}$ to find $\hat{\phi}_{ll}$ so that

$$\hat{\phi}_{l+1,l+1} = \frac{\hat{\Gamma}(\nu) - \sum_{j=1}^{k} \hat{\phi}_{kl}\hat{\Gamma}_{k\omega+\nu-l}(\nu)}{1 - \sum_{j=1}^{k} \hat{\phi}_{kl}\hat{\Gamma}_{ll}}$$

and

$$\hat{\phi}_{k\omega+1,l} = \hat{\phi}_{kl} - \hat{\phi}_{k\omega+1,k\omega+l}\, \hat{\phi}_{kl,k+1-l}.$$

2.2.2 *Parameter estimation for the FAR model*

The Fourier autoregressive coefficients will be estimated using the discrete Fourier transform estimation method, assuming $y_{k\omega+\nu}$ is the source of the periodic time series data and the discrete Fourier transform of the periodic stationary process is

$$y_{k\omega+\nu} = \sum_{i=0}^{p(\nu)-1} \left[\phi_i(\nu)\frac{\cos 2\pi k}{\omega} y_{k\omega+\nu-i} + \phi_i^*(\nu)\frac{\sin 2\pi k}{\omega} y_{k\omega+\nu-i} \right].$$

If $t = k\omega + \nu$, for $\nu = 0, 1, 2, \ldots, \omega$ and $\omega = 12$, then the periodic autoregressive coefficients of order $p(\nu)$ for the ω seasons were obtained as follows:

$$\begin{aligned}
y_{t1} &= \phi_0 + \phi_1\frac{\cos 2\pi k}{12} y_{t1-1} + \phi_1^*\frac{\sin 2\pi k}{12} y_{t1-1} + \cdots + \phi_{p(\nu)}\frac{\cos 2\pi k}{12} y_{t1-p(\nu)} + \phi_{p(\nu)}^*\frac{\sin 2\pi k}{12} y_{t1-p(\nu)},\\
y_{t2} &= \phi_0 + \phi_1\frac{\cos 2\pi k}{12} y_{t2-1} + \phi_1^*\frac{\sin 2\pi k}{12} y_{t2-1} + \cdots + \phi_{p(\nu)}\frac{\cos 2\pi k}{12} y_{t2-p(\nu)} + \phi_{p(\nu)}^*\frac{\sin 2\pi k}{12} y_{t2-p(\nu)},\\
&\cdots\\
y_{t\omega} &= \phi_0 + \phi_1\frac{\cos 2\pi k}{12} y_{t\omega-1} + \phi_1^*\frac{\sin 2\pi k}{12} y_{t\omega-1} + \cdots + \phi_{p(\nu)}\frac{\cos 2\pi k}{12} y_{t\omega-p(\nu)} + \phi_{p(\nu)}^*\frac{\sin 2\pi k}{12} y_{t\omega-p(\nu)}.
\end{aligned} \tag{2}$$

In matrix form, the set of equations (2) is

$$\begin{pmatrix} y_{t1} \\ y_{t2} \\ \vdots \\ y_{t\omega} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\cos 2\pi k}{12} y_{t1-1} & \frac{\sin 2\pi k}{12} y_{t1-1} & \cdots & \frac{\cos 2\pi k}{12} y_{t1-p(\nu)} & \frac{\sin 2\pi k}{12} y_{t1-p(\nu)} \\ 1 & \frac{\cos 2\pi k}{12} y_{t2-1} & \frac{\sin 2\pi k}{12} y_{t2-1} & \cdots & \frac{\cos 2\pi k}{12} y_{t2-p(\nu)} & \frac{\sin 2\pi k}{12} y_{t2-p(\nu)} \\ \vdots & & & & & \\ 1 & \frac{\cos 2\pi k}{12} y_{t\omega-1} & \frac{\sin 2\pi k}{12} y_{t\omega-1} & \cdots & \frac{\cos 2\pi k}{12} y_{t\omega-p(\nu)} & \frac{\sin 2\pi k}{12} y_{t\omega-p(\nu)} \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_1^* \\ \vdots \\ \phi_{p(\nu)} \\ \phi_{p(\nu)}^* \end{pmatrix}. \tag{3}$$

By taking the inverse of equation (3), the Fourier autoregressive process coefficients will be obtained.

In order to assess the correct order of the model fitted, periodic Akaike and Bayesian information criteria will be used. The periodic Akaike information criterion is given as

$$AIC(P) = n \ln \hat{\sigma}_\epsilon^2(\nu) + 2p(\nu).$$

The optimal order of the model is chosen by the value of P for which $AIC(P)$ is a minimum. The periodic Bayesian information criterion is given by

$$BIC = \ln \hat{\sigma}_\epsilon^2 + \frac{\ln N}{N} p(\nu),$$

where $\hat{\sigma}_\epsilon^2(\nu)$ is the periodic estimator of $\sigma_\epsilon^2(\nu)$ and $p(\nu)$ is the number of periodic autoregressive coefficients in the season.

2.2.3 *Diagnostic checking in the FAR model*

The appraisal for model adequacy will be carried out by checking whether the model assumptions are fulfilled. The fundamental assumption is that the residuals $\{\epsilon_t\}$ are white noise. Hence a cautious investigation of the estimated residuals will be carried out by checking whether the residuals are white noise. This will be done by obtaining the sample PACF and PePACF of the residuals to check whether they do not form any pattern and they are found to be statistically significant within two standard deviations with $\alpha = 0.05$.

2.2.4 *Forecasting with the FAR model*

Suppose we have the FAR(1) model

$$\begin{aligned} y_{t1} &= \mu + \phi_1 \cos z(y_{t1-1}) - \phi_1^* \sin z(y_{t1-1}) + \epsilon_t, \\ (1 - \phi_1 \cos z - \phi_1^* \sin z)(y_{t1} - \mu) &= \epsilon_t, \end{aligned} \tag{4}$$

where $z = 2\pi k/\omega$ and μ is a constant. The model in (4) can be written as

$$y_{t1} - \mu = \left[(\phi_1 \cos z y_{t1-1} - \phi_1^* \sin z y_{t1-1}) - \mu\right].$$

The general form of the forecast equation is given as

$$\begin{aligned} \hat{y}_l &= \mu + \left[(\phi_1 \cos z y_{t1}(l-1) - \mu) + (\phi_1^* \sin z y_{t1}(l-1) - \mu)\right] \\ &= \mu + \left[(\phi_1^l \cos z y_{t1}(l-1) - \mu) + (\phi_1^{*l} \sin z y_{t1}(l-1) - \mu)\right], \quad l \geq 1. \end{aligned}$$

2.2.5 *Forecast evaluation for the FAR process*

Once the forecast is obtained, an evaluation is computed to determine if the actual values of the series forecast are observed. The forecast evaluation was based on the periodic root mean square error (PRMSE), the periodic mean absolute error (PMAE) and the periodic mean absolute percentage error (PMAPE). These are defined by

$$PRMSE = \sqrt{\frac{1}{t\nu + 1} \sum_{t\nu=1}^{p-1} (\hat{y}_{t\nu} - y_{t\nu})^2},$$

$$PMAPE = \left| \sum_{t\nu=1}^{p-1} \frac{\hat{y}_{t\nu} - y_{t\nu}}{\hat{y}_{t\nu}} \right|,$$

and

$$PMAE = \frac{1}{t\nu + 1} \sum_{t\nu=1}^{p-1} |\hat{y}_{t\nu} - y_{t\nu}|,$$

where $t\nu = 1, 2, \ldots, p-1$. The actual and predicted values for corresponding $t\nu$ values are denoted by $\hat{y}_{t\nu}$ and $y_{t\nu}$ respectively. The smaller the values of PRMSE, PMAPE and PMAE, the better the forecasting performance of the model.

2.2.6 Seasonal autoregressive integrated moving average (SARIMA) model

The seasonal autoregressive integrated moving average was proposed by Box et al. (2008) and it is an extension of the autoregressive integrated moving average developed by Box and Jenkins (1970). A seasonal ARIMA model can be defined as

$$\phi_p(B^S)\Phi(B)\nabla_S^D\nabla^d X_t = \Theta_q(B^S)\theta(B)\epsilon_t,$$

where $\{\epsilon_t\}$ is the usual Gaussian white noise process and S is the period of the time series. The ordinary autoregressive and moving average components are represented by polynomials $\phi(B)$ and $\theta(B)$ of orders p and q, respectively. The seasonal autoregressive and moving average components are $\Phi(B)(B^S)$ and $\Theta(B)(B^S)$ where P and Q are their orders. ∇^d and ∇_S^D are ordinary and seasonal difference components and B is the backshift operator.

3. Result and discussion

The Nigerian monthly rainfall series from January 1993 to December 2017, collected from the Nigerian Meteorological Agency (2017), Lagos, given in Figure 1, was analysed. The rainfall series exhibited seasonal and periodic variations and this informed the use of FAR and SARIMA models.

A critical look at the PeACF and PePACF for January to December showed that the PeACF was stable and the PePACF cuts off at lag 2, so tentatively FAR(1), FAR(2) and FAR(3) were chosen for the January to December rainfall series. The discrete Fourier transform estimation method was used to obtain the coefficients of the FAR(1), FAR(2) and FAR(3) models. Based on the smallest values of the periodic Akaike (PAIC) and Bayesian information criteria (PBIC) given in Table 1 after estimation, the FAR(1) model was chosen as the optimal model and estimation results are as follows:

$$\begin{aligned}
y_{jan} &= 3.163271 - 1.133662\cos(t)y_{t-1} - 0.742311\sin(t)y_{t-1},\\
y_{feb} &= 9.345211 + 2.342522\cos(t)y_{t-1} + 0.644311\sin(t)y_{t-1},\\
y_{mar} &= 25.343171 - 2.192561\cos(t)y_{t-1} + 0.184311\sin(t)y_{t-1},\\
y_{apr} &= 63.153379 + 6.193441\cos(t)y_{t-1} - 0.7420011\sin(t)y_{t-1},\\
y_{may} &= 121.365321 + 5.340560\cos(t)y_{t-1} + 1.582314\sin(t)y_{t-1},\\
y_{jun} &= 167.563271 - 8.127563\cos(t)y_{t-1} - 0.342311\sin(t)y_{t-1},\\
y_{jul} &= 321.153101 - 10.053568\cos(t)y_{t-1} - 0.043311\sin(t)y_{t-1},\\
y_{aug} &= 325.165211 - 12.135567\cos(t)y_{t-1} - 0.152311\sin(t)y_{t-1},\\
y_{sep} &= 202.261171 + 3.147566\cos(t)y_{t-1} + 0.900311\sin(t)y_{t-1},\\
y_{oct} &= 102.263471 - 6.123561\cos(t)y_{t-1} - 0.812001\sin(t)y_{t-1},\\
y_{nov} &= 98.165221 - 7.143561\cos(t)y_{t-1} - 0.122901\sin(t)y_{t-1},\\
y_{dec} &= 2.963171 + 0.823900\cos(t)y_{t-1} + 0.542311\sin(t)y_{t-1},
\end{aligned}$$

where $t = 2\pi k/\omega$. The periodic residual autocorrelation for the FAR(1) models showed the residuals are approximately white noise, hence the models can be used to forecast the Nigerian monthly rainfall series.

For the SARIMA model, the augmented Dickey–Fuller test showed that the Nigerian rainfall series was stationary at the first difference at the 1%, 5% and 10% levels of significance with a p-value of

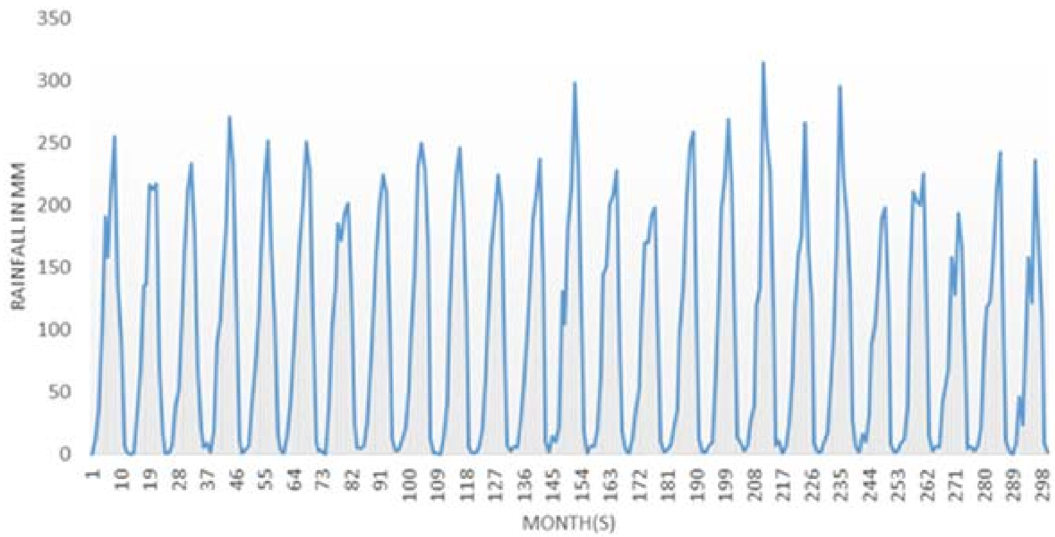

Figure 1. Rainfall series plot from January 1993 to December 2017.

< 0.001, and hence d(order of integration) = 1. The partial autocorrelation function tailed off and the autocorrelation function cut off after lag 1, hence the four suggested models for the rainfall series are SARIMA$(1,1,1)$ x $(1,1,1)_{12}$, SARIMA$(2,1,2)$ x $(1,1,1)_{12}$, SARIMA$(1,1,2)$ x $(1,1,1)_{12}$ and SARIMA$(2,1,1)$ x $(1,1,1)_{12}$. The four suggested models were estimated using ordinary least squares estimation and the optimal model for the rainfall series was SARIMA$(1,1,1)$ x $(1,1,1)_{12}$ based on the smallest values of the estimated Akaike information criterion (AIC) and Schwarz information criterion given in Table 2. The periodic residual autocorrelation test revealed that the residuals for each period are normally distributed and are white noise. Therefore, SARIMA$(1,1,1)$ x $(1,1,1)_{12}$ is adequate for forecasting Nigerian rainfall. The out-sample forecast was obtained for the Nigerian monthly rainfall series from January 2018 to December 2019 using the FAR(1) and SARIMA$(1,1,1)$ x $(1,1,1)_{12}$ models. The out-sample forecast for both models is given in Table 3 and Figure 2. The results exhibit continuous periodic and seasonal movements from January 2018 to December 2019 for both models, but the periodic movement in the rainfall series forecast is better shown with FAR(1) because the forecast is a close reflection of the original series from January 1993 to December 2017. The values of the forecast evaluation for both models in Tables 4 and 5 show that the forecasts were consistent and accurate but that the FAR(1) model forecasts were more accurate since their forecast evaluation values were relatively lower. Hence, the Fourier autoregressive model is adequate and suitable for modelling and forecasting periodicity and seasonality in the Nigerian rainfall time series data. The residual autocorrelation test revealed that the residuals are normally distribution and are white noise. Therefore, the SARIMA$(1,1,1)$x$(1,1,1)_{12}$ model is adequate to forecast Nigerian rainfall.

Out-sample forecasts were obtained for Nigerian monthly rainfall from January 2018 to December 2019 using the FAR(1) and SARIMA$(1,1,1)$x$(1,1,1)_{12}$ models. The out-sample forecast for both models are given in Table 3 and Figure 2. The results exhibit continuous periodic and seasonal

Table 1. Information criteria for FAR models.

Month(s)	**Inf. Criteria**	**FAR(1)**	**FAR(2)**	**FAR(3)**
JANUARY	PAIC	3.430102*	3.455248	3.541566
	PBIC	3.578210*	3.702095	3.887152
FEBRUARY	PAIC	5.621061*	5.684433	5.839269
	PBIC	5.769169*	5.931279	6.184854
MARCH	PAIC	7.344116*	7.479084	7.503036
	PBIC	7.492224*	7.725931	7.848621
APRIL	PAIC	8.621772*	8.7137	8.721801
	PBIC	8.769880*	8.960547	9.067386
MAY	PAIC	8.962050*	9.014371	9.144934
	PBIC	9.110158*	9.261218	9.490519
JUNE	AIC	8.862177*	8.929468	9.097053
	BIC	9.010285*	9.176315	9.442639
JULY	AIC	10.21054 *	10.34706	10.52046
	BIC	10.35865 *	10.59391	10.86605
AUGUST	AIC	9.501169*	9.638035	9.67652
	BIC	9.649277*	9.884881	10.02211
SEPTEMBER	AIC	9.421283*	9.584354	9.701643
	BIC	9.569391*	9.8312	10.04723
OCTOBER	AIC	9.011679*	9.250466	9.054205
	BIC	9.27657 *	9.497312	9.39979
NOVEMBER	AIC	9.030679*	9.250466	9.054205
	BIC	9.278787*	9.497312	9.39979
DECEMBER	AIC	4.626287*	4.75582	4.907723
	BIC	4.774395*	5.002667	5.253309

Table 2. Information criteria for SARIMA models.

	SARIMA $(1,1,1)$ **x** $(1,1,1)_{12}$	**SARIMA** $(2,1,2)$ **x** $(1,1,1)_{12}$	**SARIMA** $(1,1,2)$ **x** $(1,1,1)_{12}$	**SARIMA** $(2,1,1)$ **x** $(1,1,1)_{12}$
Akaike info criterion	9.047474	9.065214	9.053132	9.057689
Schwarz criterion	9.100221	9.144543	9.119065	9.123796

Table 3. Out-sample Forecast of Nigerian Rainfall series from January 2018 to December 2019.

S/N	Month(s)	Forecast with FAR(1)	Forecast with SARIMA$(1,1,1)$x$(1,1,1)_{12}$
1	Jan-2018	3.8648	2.94521
2	Feb-2018	13.5153	7.97545
3	Mar-2018	132.2314	23.1458
4	Apr-2018	34.0749	93.6134
5	May-2018	649.8104	121.723
6	Jun-2018	689.5175	125.624
7	Jul-2018	474.2523	165.923
8	Aug-2018	388.8888	218.061
9	Sep-2018	821.4758	245.435
10	Oct-2018	105.5695	98.9342
11	Nov-2018	95.0692	9.9567
12	Dec-2018	2.9225	2.43258
13	Jan-2019	5.7203	1.62353
14	Feb-2019	20.9143	10.6348
15	Mar-2019	133.2610	50.7535
16	Apr-2019	33.0023	29.0345
17	May-2019	689.7604	81.3456
18	Jun-2019	669.5021	165.234
19	Jul-2019	474.2523	130.192
20	Aug-2019	388.8888	255.46
21	Sep-2019	921.5758	180.43
22	Oct-2019	114.2595	115.423
23	Nov-2019	96.0132	11.5454
24	Dec-2019	3.0225	2.56701

movements from January 2018 to December 2019 for both models. But the periodic movement in the rainfall series forecast are better shown with FAR(1) because the forecast is a close reflection of the original series from January 1993 to December 2017. The values of the forecast evaluation for both models in Tables 4 and 5 show that the forecasts are consistent and accurate but the FAR(1) model forecasts are more accurate since their forecast evaluations values are relatively lower. Hence, the Fourier autoregressive model is adequate and suitable for modelling and forecasting periodicity and seasonality in the Nigeria rainfall time series data.

4. Conclusion

This research presented time series model building using the Fourier autoregressive model (FAR). A FAR model was used to analyse Nigerian monthly rainfall data collected by NIMET (2017) between January 1993 and December 2017. FAR(1), FAR(2) and FAR(3) models were chosen based on the PeACF and PeACF. The coefficients of the tentative model were estimated using a

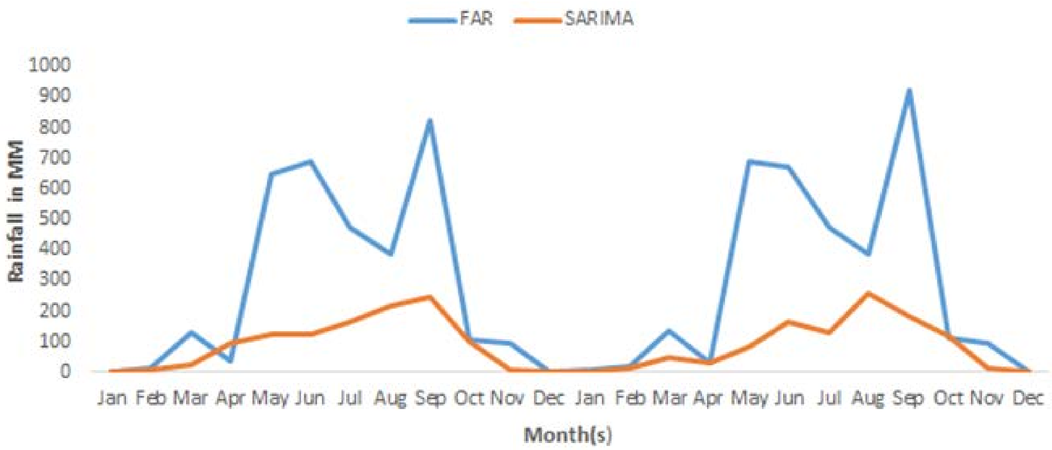

Figure 2. Monthly rainfall forecast from January 2018 to December 2019.

Table 4. Forecast evaluation for the FAR(1) model.

Month(s)	Mean Absolute Percent Error of FAR(1)	Root Mean Square Error of FAR(1)	Mean absolute error of FAR(1)
January	4.34728	2.08501	0.53143
February	34.00119	5.83105	2.34187
March	37.02215	6.08458	8.32480
April	23.16132	4.81262	11.08643
May	12.20014	3.49287	15.54210
June	9.172527	3.02862	12.80761
July	11.00311	3.31709	24.23190
August	2.09120	1.44610	17.34786
September	10.19090	3.19232	19.08512
October	17.56370	4.19091	13.52198
November	19.17569	4.37901	16.87124
December	73.32516	8.56301	0.09312

Table 5. Forecast evaluation for the SARIMA$(1,1,1)x(1,1,1)_{12}$ model.

Forecast Evaluation	Value
Mean absolute percent error	90.048
Root Mean square error	9.61
Mean absolute error	25.03

discrete Fourier transform estimation. The FAR(1) model was chosen as the optimal model based on the smallest value of the periodic Akaike (PAIC) and Bayesian information criteria (PBIC). The residuals of the fitted models were diagnosed to be white noise. For comparison purposes, the augmented Dickey–Fuller test was used to show that the Nigerian rainfall series was stationary at the first difference. SARIMA$(1,1,1)$ x $(1,1,1)_{12}$, SARIMA$(2,1,2)$ x $(1,1,1)_{12}$, SARIMA$(1,1,2)$ x $(1,1,1)_{12}$ and SARIMA$(2,1,1)$ x $(1,1,1)_{12}$ models were identified based on the ACF and PACF. The four suggested models were estimated using ordinary least squares estimation and the optimal model was SARIMA$(1,1,1)$x$(1,1,1)_{12}$ based on the smallest value of the AIC and SBIC. The residual autocorrelation test revealed the residuals are white noise. Out-sample forecasts were obtained for the Nigerian monthly rainfall series from January 2018 to December 2019 using the FAR(1) and SARIMA$(1,1,1)$ x $(1,1,1)_{12}$ models. The results exhibited continuous periodic and seasonal movements from January 2018 to December 2019 for both models, but the periodic movement in the rainfall series forecast was better shown with the FAR(1) model because the forecasted values are a close reflection of the original series from January 1993 to December 2017. The values of the forecast evaluation for both models showed the forecasted values were consistent and accurate but the FAR(1) model forecasts were more accurate since their forecast evaluation values were relatively lower. Hence, the Fourier autoregressive model is adequate and suitable for modelling and forecast periodicity and seasonality in the Nigerian rainfall time series data. Hence, this model can be applied to forecast rainfall anywhere if the series is characterised by periodic variation.

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