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Analysis of solutions of differential equations of vibratory systems with varied external forces

Iuri Hermes Muller[I], Eduardo Silva Carlos[II], Aline Brum Loreto[III], AnaLuisaSoubhia[IV], Camila Becker Picoloto[V]

ABSTRACT

Several Engineering problems can be modeled from differential equations, analytical and numerical methods can be employed to determine the solutions. Among these problems, applied in the area of Mechanics, there are those that involve the analysis of vibratory systems. The present paper aims to conduct a study on the solutions of the second order ordinary differential equations that model these vibratory systems, seeking to solve these equations analytically from the application of different external forces. In order to solve analytically each of the equations that describe these systems, the homogeneous equation solution is first determined. Then, depending on the type of external force that acts on the system, the particular solution is obtained using the methods of Indeterminate Coefficients or Parameter Variation. The general solution is then obtained from the linear combination of homogeneous and the particular solutions. The analysis of the solutions shows that the displacements of the masses according to time, depending on the external force applied in the system, present varied behaviors among themselves. Over time, the homogeneous solution, characterized as transient response, becomes negligible, remaining only the particular solution, characterized as the permanent response.

Keywords: Differencialequations;Vibratory systems; Analyticalmethods

[I]Universidade Federal de Santa Maria-Campus Cachoeira do Sul, RS, Brasil - iurihmuller@hotmail.com
[II]Universidade Federal de Santa Maria-Campus Cachoeira do Sul, RS, Brasil - 96eduardoc@gmail.com
[III]Universidade Federal de Santa Maria-Campus Cachoeira do Sul, RS, Brasil - aline.loreto@ufsm.br
[IV]Universidade Federal de Santa Maria-Campus Cachoeira do Sul, RS, Brasil - ana.soubhia@ufsm.br
[V]Universidade Federal de Santa Maria-Campus Cachoeira do Sul, RS, Brasil– camila.picoloto@ufsm.br

1 INTRODUCTION

Several engineering problems can be modeled by differential equations, which a solution can be found by employing analytical or numerical methods. Among these problems, applied both to the area of Mechanics and Electrical (electrical circuit analysis), there are those that involve the analysis of vibratory systems, which are constituted by elements of mass, spring and damping, and there may be external forces. Physically, the solutions of differential equations that model these vibratory systems represent the displacement of the masses as a function of time. It is noteworthy that the analysis of the behavior of these solutions plays an important role for the design of machines, structures and other engineering systems.

Several studies work related to the study of mechanical vibrations start from the analysis of vibratory systems of various degrees of freedom, usually applying computational numerical methods to obtain the results. Delgado and Varanis (2018) analyze a vibratory model under a two-degree base oscillation of two degrees of freedom. In order to reduce vibration amplitude, a vibration absorber was attached to the system mass and the main parameters of the system were varied. The results were obtained from computational simulation. Barros (2007) analyzes forced harmonic and parametric oscillators, seeking to solve it analytically through the parameter variation method and the Green method. Freitas. et al. (2018)studies a free vibratory system under different cases of damping, applying the constant coefficient method to obtain the general solution of the equation that models the system. The different results were obtained by adopting different parameters to the equation.

The goal of this paper is to conduct a study on the solutions of the second-order ordinary differential equations that model vibratory systems of single degree of freedom, seeking to solve the equations analytically with the application of different external forces. For this, the harmonic force, which involves sine and cosine functions, the force in the form of exponential function and the force characterized by a combination of both the aforementioned, are considered as external forces. As a focus of this work, we seek to present the methodology applied to determine the solutions of the differential equations obtained and, finally, interpreting how the behavior of such solutions represent in practice for Engineering.

This article is structured as follows: In section 2 the theoretical about differential equation used on the present work is presented. In section 3the vibratory system is modeled by means of a differential equation and are presented the homogeneous, particular and general solutions of the differential equation associated. In section 4 it is presented an analysis of the behavior of the solutions of the differential equations under different external forces. Furthermore, some examples are provides in order to highlight the relation between real solutions and physical systems.

2 BIBLIOGRAPHIC REVIEW

In this section, a bibliographic review will be carried out that will serve as a basis for the further development of the work. Fundamentals and concepts about second-order differential equations will be addressed, as well as methods of obtaining their solutions.

2.1 Differential equations

Equations containing derivatives are called differential equations. Several principles, or laws, governing the physical world can be mathematically modeled from these equations, which represents rates of changes as things happens (BOYCE and DIPRIMA, 2012).

Differential equations can be classified according to the type of their derivatives, into ordinary (derived from functions of an independent variable) or partial (derived from functions of two or more independent variables). In the group of ordinary equations, these can be classified according to the order of derivatives (first order, second order or higher order) and according to the coefficients that multiply these derivatives (constant or variable coefficients). In addition, differential equations can be also classified according to its linearity. A differential equation is said linearif it can be written according to (2.1),

$$a_n(t)\frac{d^n x}{dt^n}+a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}}+\ldots+a_1(t)\frac{dx}{dt}+a_0(t)x=g(t). \tag{2.1}$$

Linear equations present as properties the fact that the power of each term with the dependent variable *x* be unitary (all terms with *x* are always of the first degree – elevated to exponent 1) and each coefficient *a(t)* depend only on the independent variable *t* (ZILL, 2009).

2.1.1Second-order equations

According to Boyce and Diprima (2012), second-order ordinary differential equations can be written, in general, according to equation (2.2),

$$P(t)x''+Q(t)x'+R(t)x=G(t). \tag{2.2}$$

With *P*, *Q*, *R* and *G* being specified functions of variable *t*. In case the coefficients of the equation (2.2) are constant, it can be written according to (2.3),

$$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=G(t), \tag{2.3}$$

With *a*, *b* and *c* being coefficients of the equation.

According to Zill (2009), to determine the solution of an equation of type of (2.3), it should initially be considered homogeneous, that is, it is considered *G(t)=0*, and find the homogeneous solution. After obtaining the homogeneous solution, a particular solution should be determined from the *G(t)* function. The general solution of the equation (2.3) is given by the linear combination of homogeneous and particular solutions.

For the determination of the homogeneous solution, a function is assumed in the exponential form of type $x(t)=e^{rt}$. Replacing the function and its respective derivative of first and second order, that is, $x'(t)-re^{rt}$ and $x''(t)=r^2e^{rt}$, in the corresponding homogeneous equation, a second-degree characteristic equation is obtained, given according to the equation (2.4),

$$ar^2+br+c=0, \tag{2.4}$$

whose roots, given by *r*, determine the shape of the homogeneous solution (BOYCE and DIPRIMA, 2012).

The three possible cases of roots that can be obtained are: real and distinct roots, real and equal roots and conjugated complex roots.

For the case where there are real and distinct roots, r_1 and r_2, for equation (2.4), two solutions are obtained: $x_1(t)=e^{r_1t}$ and $x_2(t)=e^{r_2t}$. These solutions are linearly independent to each other in the range $(-\infty,\infty)$, according to Zill (2009), and form a general solution of type type $x(t)=C_1e^{r_1t}+C_2e^{r_2t}$.

For real and equal roots, $r_1=r_2=r$, the corresponding solutions are $x_1(t)=e^{rt}$ and $x_2(t)=te^{rt}$, being the general solution of the form $x(t)=C_1e^{rt}+C_2te^{rt}$.

For conjugated complex roots, $r_1=\alpha-\beta i$ and $r_2=\alpha+\beta i$, the homogeneous solution of the differential equation is $x(t)=C_1e^{\alpha t}\cos(\beta t)+C_2e^{\alpha t}\sin(\beta t)$ when applying Euler formula (ZILL, 2009).

Considering the equation (2.3), not homogeneous, with $G(t)\neq 0$, a particular solution can be determined through two methods: Indeterminate Coefficients and Parameter Variation.

According to the method of the Indeterminate Coefficients, to determine the particular solution of the equation (2.3), an expression is assumed with the same shape as the function *G(t)*, with unspecified coefficients. Then, it is necessary to replace the hypothetical expression in equation (2.3) in order to determine its coefficients so that the equation is satisfied. This expression, with the coefficients determined, will be a particular solution for the differential equation. This method, however, is only applicable for *G(t)*function that is composed of constant, polynomial, exponential, sine and cosine and/or sums and products of these, in addition, the differential equation must have constant coefficients (BOYCE and DIPRIMA, 2012).

The parameter variation method is intended to replace the constants C_1 and C_2, present in the homogeneous solution previously determined, with functions e respectively. So, the particular solution will be in the form $x_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, being $u_1(t) = -\int \frac{x_2(t)G(t)}{W(x_1, x_2)(t)}dt$, $u_2(t) = \int \frac{x_1(t)G(t)}{W(x_1, x_2)(t)}dt$ and $W[x_1,x_2](t)$ the wronskian related to the homogeneous solutions, given by $W[x_1, x_2](t) = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$ (BOYCE and DIPRIMA, 2012).

Finally, the general solution of the equation (2.3) is given by the linear combination of homogeneous and particular solutions previously determined.

3 METHODOLOGICAL PROCEDURES

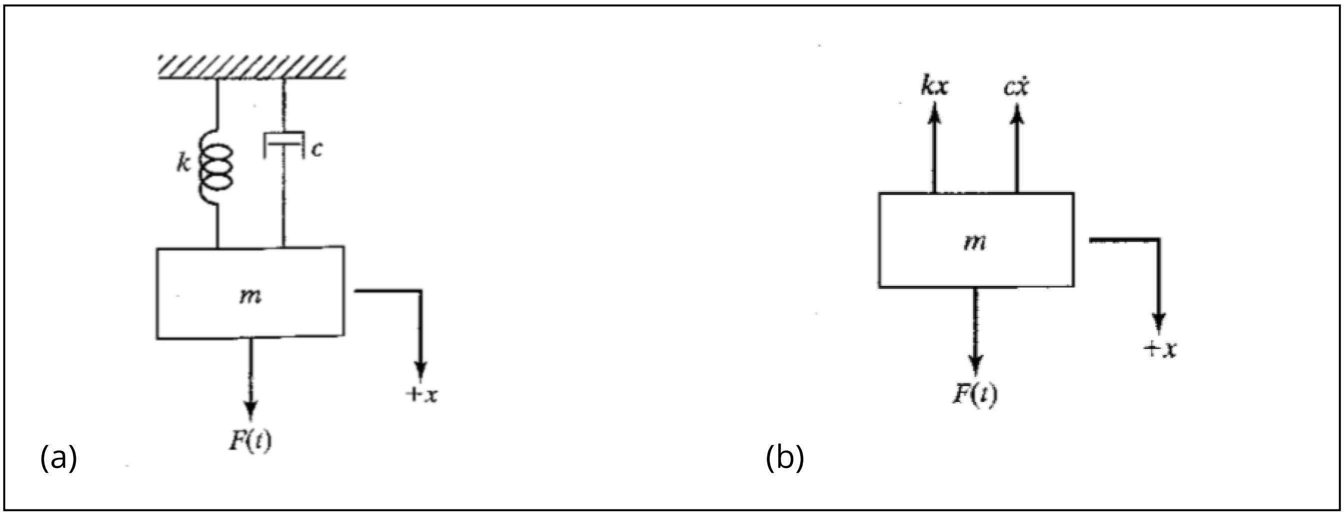
In this section of the present study, the methods used are presented to obtain the results, based on the bibliographic references. The calculation methodologies used here are applied in an analysis of vibratory models, in which, starting from the proposed vibratory system, the second-order differential equation that models it (equation of motion) is obtained, which is subsequently solved using the analytical procedures presented, with different parameters assigned.

3.1 Obtaining the ordinary differential equation of the system

Figure 1(a) shows a vibrational system of single degree of freedom composed of a mass *m* in suspension, a damper with damping coefficient *c* and a spring coefficient of rigidity *k*, with an external force *F(t)* applied according to time. To determine the forces acting on the system, the free body diagram of the mass can be constructed (Figure 1(b)). Applying newton's second law (RAO, 2011), the equation of movement is determined, as in (3.1),

$$m\frac{d^2x(t)}{dt^2}+c\frac{dx(t)}{dt}+kx(t)=F(t). \quad (3.1)$$

Figure 1 – Representation of the vibratory system (a) and the free body diagram (b)



Source: RAO, 2011.

The equation (3.1)is characterized as a second order linear non-homogeneous ordinary differential equation with constant coefficients. Thus, in order to solve it, the analytical methods described in section 2 are applied.

3.2 Differential equation solution

Considering the homogeneous equation (attributing that *F(t)=0*), when applying a solution of exponential form to the differential equation, a characteristic equation given by (3.2) is obtained,

$$mr^2+cr+k=0. \quad (3.2)$$

The three possible root types of the equation (3.2) determine the shape of the homogeneous solution. For the present study, different values are assigned to the constants *m*, *c* and *k* of the equation (3.1) in order to obtain the three different cases of roots of the characteristic equation to, consequently, obtain the different homogeneous responses of the system, similar to the work done by Freitas et al.

(2018). Table 1 presents these different cases of roots, besides showing the respective classification of the vibratory system regarding the type of damping according to these roots.

Table 1 – Homogeneoussolutions

Root types	Homogeneoussolution	Classification
Real and distinct roots: c^2-4mk>0	$x_h(t)=C_1e^{\left(\frac{-c+\sqrt{c^2-4mk}}{2m}\right)t}+C_2e^{\left(\frac{-c-\sqrt{c^2-4mk}}{2m}\right)t}$	Overdamped system.
Real and equal roots: c^2-4mk=0	$x_h(t)=C_1e^{-ct/2m}+C_2te^{-ct/2m}$	Critically dampedsystem
Conjugated complex roots: c^2-4mk<0	$x_h(t)=C_1e^{-ct/2m}\cos\left(\frac{\sqrt{4mk-c^2}}{2m}t\right)+C_2e^{-ct/2m}\sin\left(\frac{\sqrt{4mk-c^2}}{2m}t\right)$	Underdampedsystem

The calculation of the particular solution will depend on the shape of the external force, F(t), applied in the system. For the present work, problems whose external forces in the form of cosine function, exponential function and a combination of sine and exponential functions will be given, for a comparative analysis of the responses obtained from the system. The external forces considered are:$F(t)=10\cos(5t)$,$F(t)=10e^{-5t}$ and $F(t)=e^{-t}\sin(4t)$.

The use of the methods of the Indeterminate Coefficients or the Parameter Variation is optional, as the results obtained are equivalent. However, the application of the method of Indeterminate Coefficients, as mentioned above, is permitted only for certain types of non-homogeneous functions, such as constants, polynomials, sine and cosine functions, exponential functions and combination of sum and/or product of these. The Parameter Variation method is applicable for any type of non-homogeneous function, but the calculation of parameters also involves the calculation of integralsthat can hinder the process of finding the parameters.In the present work, the method of the Indeterminate Coefficients is used, because the forced functions considered are among the functions allowed for the application of this method.

Finally, the general solution is obtained from the linear combination of homogeneous and particular solutions calculated and, when considering an initial value problem (IVP) with the initial conditions$x(t)|_{t=0}=0$ and$\frac{dx(t)}{dt}|_{t=0}=0$, the value of the constants C_1 and C_2 of the homogeneous solution are obtained.

In the next section, the graphsof the functions of the calculated solutions were plotted with the aid of the *Maple 18* software, in order to obtain a visual representation, facilitating the interpretation of the results.

4 RESULTS AND DISCUSSION

In this section of the present study, the calculated solutions for the differential equation that models the vibratory system submitted todifferent external forcesare presented. Initially, the results of general homogeneous solutions are presented. Then, the results obtained for the force $F(t)=e^{-t}\sin(4t)$ are presented, showing in detail the procedures for calculating the method of the Indeterminate Coefficients. Subsequently, the results for the $F(t)=10\cos(5t)$ and $F(t)=10e^{-5t}$ forces are shown, where different general solutions are obtained. Together with the given solutions, their respective graphs are displayed.

4.1 Homogeneous solutions

The calculation of homogeneous solutions was performed by adopting the values for the coefficients $m = 1kg$ and $k = 16N/m$ and varying the damping values in order to obtain the different types of homogeneous solutions. Thus, $c_1 = 24N{\cdot}s/m$, $c_2 = 8N{\cdot}s/m$ and $c_3 = 4N{\cdot}s/m$. The solutions obtained for the three different cases are presented in Table 2.

Table 2 – Homogeneoussolutions for differentcoefficientsofdifferentialequation

Homogeneousequation	Homogeneoussolution	Classification
$x''(t)+24x'(t)+16x(t)=0$	$x(t)=C_1e^{(-12-8\sqrt{2})t}+C_2e^{(-12+8\sqrt{2})t}$	Real and distinct roots →Overdamped
$x''(t)+8x'(t)+16x(t)=0$	$x(t)=C_1e^{-4t}+C_2te^{-4t}$	Real and equal roots →Critically damped
$x''(t)+4x'(t)+16x(t)=0$	$x(t)=C_1e^{-2t}\cos(2\sqrt{3}t)+C_2e^{-2t}\sin(2\sqrt{3}t)$	Conjugated complex roots →Underdamped

The calculation of constants $C_1$and C_2 of homogeneous solutions is carried out after the determination of the particular solutions for each forced function applied in the system, when solving the initial value problem (IVP) with the initial conditions considered.

4.2 Solutions for $F(t)=e^{-t}\sin(4t)$

For the calculation of the solution of the equation (3.1), considering the forced function of the form$F(t)=e^{-t}\sin(4t)$, the values for the coefficients of the equation were considered as$m$ = *1kg*, *c* = *8N•s/m* and*k* =*16 N/m*, thus obtaining as a homogeneous solution $x_h(t) = c_1e^{-4t} + c_2te^{-4t}$.

The calculation of the particular solution was carried out using the method of Indeterminate Coefficients. The method procedure begins with an initial assumption with the same form as the forced function, in this case $x_p(t)=e^{-t}(A\cdot\cos 4t+B\cdot\sin 4t)$, whose constants *A* and *B* will be determined later. Deriving and reorganizing the terms of each expression, $x_p'(t)=e^{-t}\cos 4t\,(-A+4B)+e^{-t}\sin 4t\,(-4A-B)$ and $x_p''(t)=e^{-t}\cos 4t\,(-15A-8B)+e^{-t}\sin 4t\,(8A-15B)$ are obtained. By replacing the derivatives in equation (3.1), the equation $e^{-t}\cos 4t\,(-7A+24B)+e^{-t}\sin 4t\,(-24A-7B)=e^{-t}\sin 4t$. By matching the similar terms on both sides of the equation, a system of two equations with two variablesis obtained (coefficients *A* and *B* to be determined): $-7A+24B=0$ and

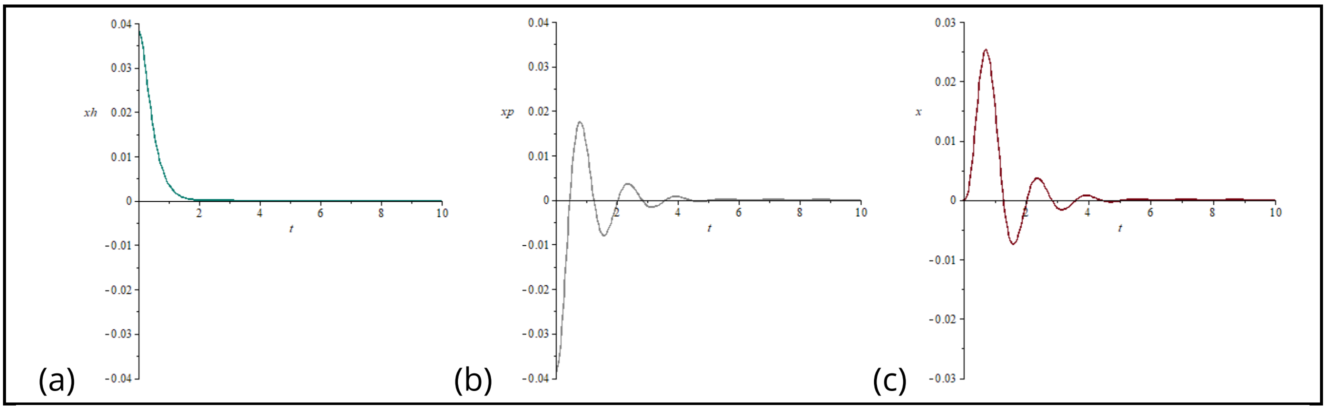
$-24A-7B=1$, from which $A=-\frac{24}{625}$ and $B=-\frac{7}{625}$ are obtained. Thus, the particular solution can be written in the form $x_p(t)=e^{-t}\left(-\frac{24}{625}\cdot\cos 4t-\frac{7}{625}\cdot\sin 4t\right)$.

The general solution was obtained from the linear combination of homogeneous and particular solutions and constants C_1 and C_2 of the homogeneous solution were determined by applying the initial conditions $x(t)\,|_{t=0}=0$ and $\frac{dx(t)}{dt}|_{t=0}=0$, of which $C_1=\frac{24}{625}$ and $C_2=\frac{4}{25}$ were obtained. In this way, the general solution is expressed according to (4.2):

$$x(t)=\frac{24}{625}e^{-4t}+\frac{4}{25}te^{-4t}-\frac{24}{625}e^{-t}\cos(4t)-\frac{7}{625}e^{-t}\sin(4 \quad (4.2)$$

The Figure 3 presents the graphs of homogeneous, particular and generalsolutions, allowing a visual representation of their behaviors as a function of time. These graphs were obtained with the aid of the Maple 18 software.

Figure 3 – Homogeneous solution (a), particular solution (b) and general solution (c)



The graph presented in Figure 3(a), related to homogeneous solution, represents the transient response of displacement of the mass of the vibratory system characterized as critically damped. This means that, as time *t* tends to infinity, the displacement *x(t)* tends to zero. Figure 3(b), relative to the particular solution graph, shows the permanent response of mass displacement, that is, as time *t* goes by, the disturbance of mass displacement *x(t)* will be caused only by the forced function applied in the system.

4.3 Solutions for $F(t)=10\cos(5t)$ and $F(t)=10e^{-5t}$

The calculation of equation solutions (3.1) considering the forced functions $F(t)=10\cos(5t)$ and $F(t)=10e^{-5t}$ was performed by adopting the different values for the coefficients *m*, *c* and *k*, according to that presented in section 4.1, in order to obtain the different types of homogeneous solutions, as shown in Table 2.

The determination of the particular solutions was carried out according to the values adopted for the coefficients of the differential equation and according to the external forces considered. The method of Indeterminate Coefficients was applied, assuming $x_p(t)=A\cos(\omega t)+B\sin(\omega t)$ for equation (3.1) under the forced function $F(t)=10\cos(5t)$ and $x_p(t)=Ae^{\alpha t}$ for the equation (3.1) under forced function $F(t)=10e^{-5t}$. After the algebraic procedures applied, the particular solutions arranged in Table 3 were obtained.

Table 3 – Particular solutions for different coefficients of differential equation and external forces

Differential equations with the different coefficients adopted	External force functions applied to the system	
	$F(t)=10\cos 5t$	$F(t)=10e^{-5t}$
$x''(t)+24x'(t)+16x(t)=F(t)$	$x_p(t)=-\frac{10}{1609}\cos(5t)+\frac{400}{4827}\sin(5t)$	$x_p(t)=-\frac{10}{79}e^{-5t}$
$x''(t)+8x'(t)+16x(t)=F(t)$	$x_p(t)=-\frac{90}{1681}\cos(5t)+\frac{400}{1681}\sin(5t)$	$x_p(t)=10\,e^{-5t}$
$x''(t)+4x'(t)+16x(t)=F(t)$	$x_p(t)=-\frac{90}{481}\cos(5t)+\frac{200}{481}\sin(5t)$	$x_p(t)=\frac{10}{21}e^{-5t}$

Finally, general solutions are given by the sum of homogeneous and particular solutions. The constants of each homogeneous solution were again determined from the same initial conditions considered for the problem of initial value ($x(t)|_{t=0}=0$ and $\frac{dx(t)}{dt}|_{t=0}=0$). Table 4 presents the general solutions of each differential equation, with the C_1 and C_2 constants of each solution already determined.

Table 4 - General solutions for different coefficients of differential equation and external forces

General solutions	
$x''(t)+24x'(t)+16x(t)=10\cos 5t$	$x''(t)+24x'(t)+16x(t)=10e^{-5t}$
$x(t)=\left(\frac{5}{1609}+\frac{205}{19308}\sqrt{2}\right)e^{(-12-8\sqrt{2})t} + \left(\frac{5}{1609}-\frac{205}{19308}\sqrt{2}\right)e^{(-12+8\sqrt{2})t} \frac{10}{1609}\cos(5t)+\frac{400}{4827}\sin(5t)$	$x(t)=\left(\frac{5}{79}-\frac{35}{1264}\sqrt{2}\right)e^{(-12-8\sqrt{2})t} + \left(\frac{5}{79}+\frac{35}{1264}\sqrt{2}\right)e^{(-12+8\sqrt{2})t} - \frac{10}{79}e^{-5t}$
$x''(t)+8x'(t)+16x(t)=10\cos 5t$	$x''(t)+8x'(t)+16x(t)=10e^{-5t}$
$x(t)=\frac{90}{1681}e^{-4t}-\frac{40}{41}te^{-4t}-\frac{90}{1681}\cos(5t)+\frac{400}{1681}\sin(5t)$	$x(t)=-10\,e^{-4t}+10\,te^{-4t}+10\,e^{-5t}$
$x''(t)+4x'(t)+16x(t)=10\cos 5t$	$x''(t)+4x'(t)+16x(t)=10e^{-5t}$
$x(t)=\frac{90}{481}e^{-2t}\cos(2\sqrt{3}t)-\frac{410}{1443}e^{-2t}\sin(2\sqrt{3}t)+\frac{90}{481}\cos(5t)+\frac{200}{481}\sin(5t)$	$x(t)=-\frac{10}{21}e^{-2t}\cos(2\sqrt{3}t)+\frac{5\sqrt{3}}{21}e^{-2t}\sin(2\sqrt{3}t)+\frac{10}{21}e^{-5t}$
$x(t)=\frac{90}{481}e^{-2t}\cos(2\sqrt{3}t)-\frac{410}{1443}e^{-2t}\sin(2\sqrt{3}t)+\frac{90}{481}\cos(5t)+\frac{200}{481}\sin(5t)$	$x(t)=-\frac{10}{21}e^{-2t}\cos(2\sqrt{3}t)+\frac{5\sqrt{3}}{21}e^{-2t}\sin(2\sqrt{3}t)+\frac{10}{21}e^{-5t}$

The graphs of homogeneous solutions and general solutions are presented, respectively, in Figures 4 and 5, for each of the two forced functions applied.

In Figure 4 presents the solutions in transient regime of displacement of the mass, for each type of damping. Figure 4(a) refers to function F(t)=10cos(5t) while Figure 4(b) refers to function F(t)=10e^{-5t} . Figure 5 presents the general solutions, as a combination of homogeneous and particular solutions for function F(t)=10cos(5t) (Figure 5(a)) and for function F(t)=10e^{-5t}(Figure 5(b)). After a certain interval of time *t*, it is observed that only permanent responses (particular solutions) prevail, showing that mass displacement depends only on the forced function acting in the system.

Figure 4 – Transient solutions for the function $F(t)=10\cos(5t)$ (a) and for de function $F(t)=10e^{-5t}$ (b)

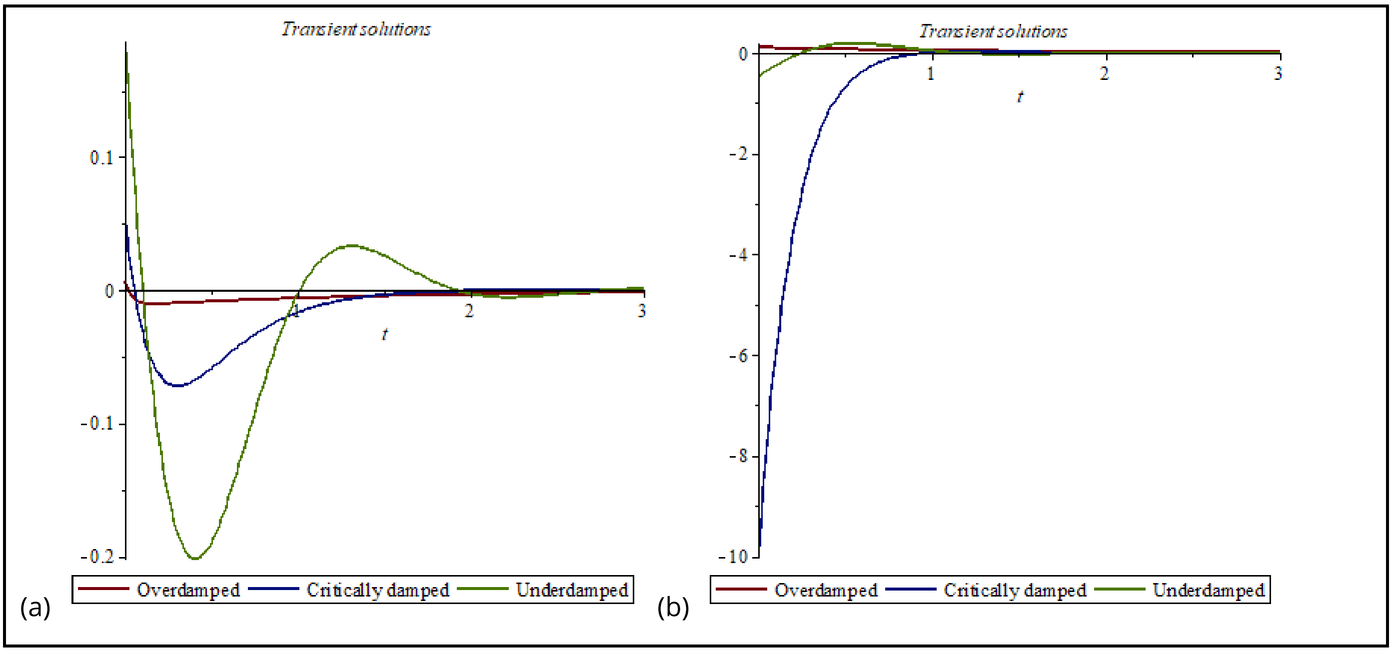
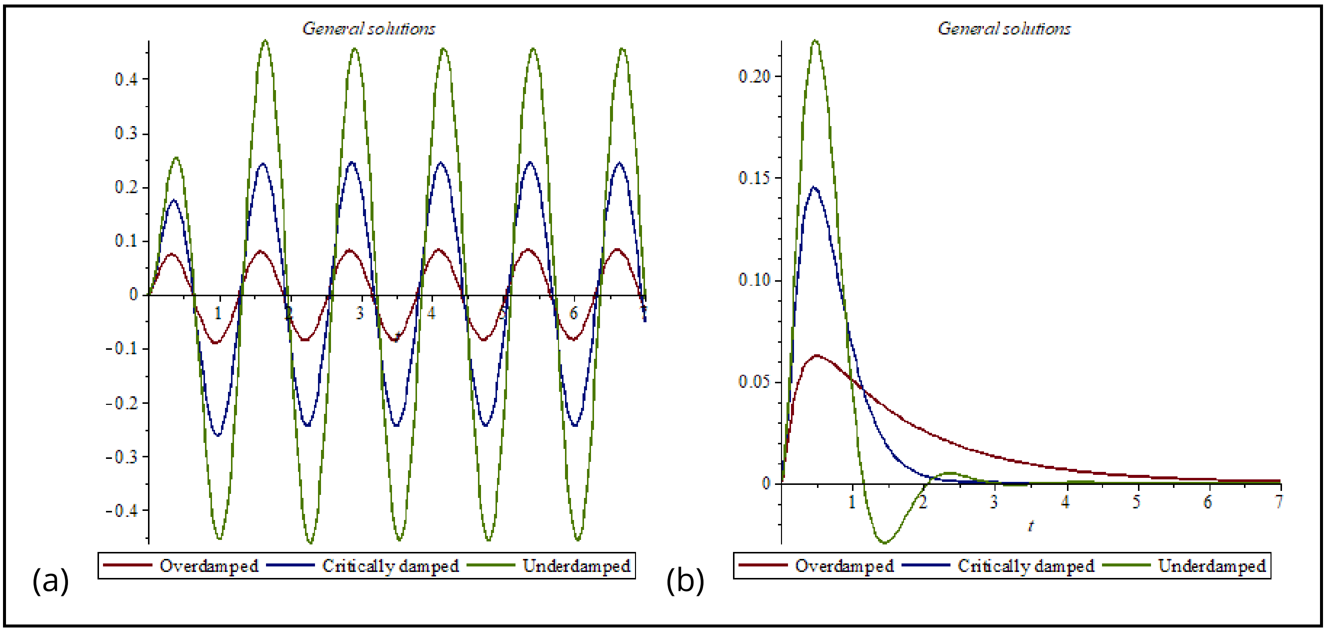


Figure 5 – General solution for the function $F(t)=10\cos(5t)$ (a) and for the function $F(t)=10e^{-5t}$ *(b)*



5 CONCLUSION

The determined solutions represent, for the analysis of vibratory models, the displacements of the masses as a function of time. Thus, depending on the external force applied in the system and the coefficients adopted for the equation of motion, these system responses present varied behaviors among themselves.

When analyzing the behaviors of the solutions, it is observed that after a certain time interval, the homogeneous solution (transient response) becomes negligible, remaining only the particular solution (permanent response) representing the displacement of the mass of the vibratory system. Thus, the displacement of the mass as a function of time has greater dependence on the particular response, which is subject to the external force applied on the vibratory system.

The interpretation of these results obtained, from the general solution of the differential equation, has great importance for studies involving engineering projects of machines, foundations, structures, motors, turbines, in addition to control systems.

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