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UNIT AND UNITARY CAYLEY GRAPHS FOR THE RING OF EISENSTEIN INTEGERS MODULO n

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Abstract: Let E_n be the ring of Eisenstein integers modulo n. We denote by $G(E_n)$ and G_{E_n}, the unit graph and the unitary Cayley graph of E_n, respectively. In this paper, we obtain the value of the diameter, the girth, the clique number and the chromatic number of these graphs. We also prove that for each $n > 1$, the graphs $G(E_n)$ and G_{E_n} are Hamiltonian.

Keywords: Unit graph, Unitary Cayley graph, Eisenstein integers, Hamiltonian graph.

1. Introduction

Associating a graph with an algebraic object is an active research subject in algebraic graph theory, an area of mathematics in which methods of abstract algebra are employed in studying various graph invariants and tools in graph theory are used in studying various properties of the associated algebraic structure. The Cayley graph of a finite group was first considered in 1878 by Arthur Cayley [12]. Research on graphs associated with rings was started in 1988 by I. Beck [10]. In the literature, there are some other graphs associated with rings, such as the Cayley graph of a commutative ring [1], the unitary Cayley graph of a ring [3], the total graph of a ring [5], the zero divisor graph of a ring [6], the unit graph of a ring [7] and the comaximal graph of a ring [16].

Let R be a commutative ring with non-zero identity. We denote by $U(R)$, $J(R)$ and $Z(R)$ the group of units of R, the Jacobson radical of R and the set of zero divisors of R, respectively. The *unitary Cayley graph* of a ring R, denoted by G_R, is the graph whose vertex set is R, and in which $\{a, b\}$ is an edge if and only if $a - b \in U(R)$. In 1995 this graph was initially introduced by Dejter and Giudici [14] for $\mathbb{Z}_n$, the ring of integers modulo n. In 2009, Akhtar et al. [3] generalized the unitary Cayley graph $G_{\mathbb{Z}_n}$ to G_R for a finite ring R. The *unit graph* of a ring R, denoted by $G(R)$, is a graph whose vertices are elements of R and two distinct vertices a and b are adjacent if and only if $a + b$ in $U(R)$. In 1990, the unit graph was first investigated by Chung [13] and Grimaldi [16] for $\mathbb{Z}_n$. In 2010, Ashrafi, et al. [7] generalized the unit graph $G(\mathbb{Z}_n)$ to $G(R)$ for an arbitrary ring R. Numerous results about unit and unitary Cayley graphs were obtained, see for examples [3, 7, 18, 19, 21, 22].

The following facts are well known, see for example [4] and [17]. Let ω be a primitive third root of unity. Then the set of all complex numbers $a+b\omega$, where a and b are integers, forms an Euclidean domain with the usual complex number operations and Euclidean norm $N(a + b\omega) = a^2 + b^2 - ab$. This domain will be denoted by E and will be called the *ring of Eisenstein integers*. The units of E are ± 1, $\pm\omega$ and $\pm\overline{\omega}$. The primes of E (up to a unit multiple) are the usual prime integers that are congruent to 2 modulo 3 and Eisenstein integers whose norm is a usual prime integer. Let n be a natural number and let $\langle n\rangle$ be the principal ideal generated by n in E. Then the factor ring

$E/\langle n\rangle$ is isomorphic to the ring $_n = \{a + b\omega | a, b \in \mathbb{Z}_n\}$, where $\mathbb{Z}_n$ is the ring of integers modulo n. Thus E_n is a principal ideal ring. This ring is called the *ring of Eisenstein integers modulo n*. In [4] this ring is studied and its properties are investigated, its units are characterized and counted. It is easy to see that $a + b\omega$ is a unit in E_n if and only if $N(a + b\omega)$ is a unit in $\mathbb{Z}_n$. Recall that a ring is *local* if it has a unique maximal ideal. It is shown that

(1) if p is a prime integer, then the ring E_{p^k} is local if and only if $p = 3$ or $p \equiv 2 \pmod 3$;

(2) let $\varphi(R)$ denote the number of units in a ring R, then $\varphi(E_{3^k}) = 2 \times 3^{2k-1}$ and

$$\varphi(E_{p^k}) = \begin{cases} p^{2k-2}(p^2-1) & \text{if} \quad p \equiv 2 \pmod 3, \\ (p^k - p^{k-1})^2 & \text{if} \quad p \equiv 1 \pmod 3. \end{cases}$$

In this article, some properties of the graphs $G(E_n)$ and G_{E_n} are studied. The diameter, the girth, chromatic number and clique number, in terms of n, are found. Also, we prove that for each $n > 1$, the graphs $G(E_n)$ and G_{E_n} are *Hamiltonian* and the independence number of G_{E_n} is calculated. An earlier study was carried out for the unit and unitary graphs for the ring of Gaussian integers modulo n, see [9].

Throughout the article, by a *graph* G we mean a fnite undirected graph without loops or multiple edges. If the degree of each vertex in G is equal to k, where k is a positive integer, then G is called *k-regular graph*. For a graph G and for any two vertices a and b of G, we recall that a *walk* between a and b is an alternating sequence $a = v_0$, e_1, v_1, e_2, ..., e_k, $v_k = b$ of vertices and edges of G, denoted by

$$a = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \dots \xrightarrow{e_k} v_k = b,$$

such that for every i with $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. Also, a *path* between a and b is a walk between a and b without repeated vertices. A *cycle* of a graph is a path such that the start and end vertices are the same. The number of edges (counting repeats) in a walk, path or a cycle, is called its *length*. A *Hamiltonian path (cycle)* in G is a path (cycle) in G that visits every vertex of G exactly once. A graph is called *Hamitonian* if it contains a Hamiltonian cycle. For vertices a and b of G, we define $d(a, b)$ to be the length of a shortest path from a to b ($d(a, a) = 0$ and $d(a, b) = \infty$ if there is no such path). The *diameter* of G is

$$\operatorname{diam}(G) = \sup\{d(a, b) \mid \ a, b \in V(G)\}.$$

The *girth* of G, denoted by $gr(G)$ is the length of a shortest cycle in G, ($gr(G) = \infty$ if G contains no cycle). For a positive integer r, a graph is called *r-partite* if the vertex set admits a partition into r classes such that vertices in the same partition class are not adjacent. A r-partite graph is called *complete* if every two vertices in different parts are adjacent. The complete 2-partite graph (also called the *complete bipartite* graph) with exactly two partitions of size n and m, is denoted by $K_{n,m}$. A complete graph on the n vertices, denoted by K_n, is a graph such that every two of distinct vertices are adjacent. A *clique* in G is a set of pairwise adjacent vertices of G. A clique of the maximum size is called a *maximum clique*. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices of a maximum clique in G. We color the vertices of G so that no two joined vertices have the same color. If we color the vertices, we call it a coloring of G. The *chromatic number* $\chi(G)$ of the graph G is the minimum number of colors of colorings of G. The *tensor product* or *Kronecker product* $G \otimes H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which (a, b) is adjacent to (c, d) if and only if a is adjacent to c in G and b is adjacent to d in H. For other notions not mentioned in this introduction, one can refer to [11, 15].

Throughout this article, the integers p and p_i are used implicitly to denote primes congruent to 2 modulo 3, while q and q_j likewise denote prime integers congruent to 1 modulo 3. For classical theorems and notations in commutative algebra, the interested reader is referred to [8].

2. The Unit and Unitary Cayley Graphs of E_n

In this section, we determine diameter and girth of the unit and unitary Cayley graphs of E_n. The case when n is a power of a prime is considered first. Then the general case is considered.

2.1. The Unit and Unitary Cayley Graphs of E_{t^n}

For the sake of compeleteness, we mention here two important results that will be kept throghout the paper.

Proposition 1 [3, Proposition 2.2].

(a) *Let R be a ring. Then G_R is a regular graph of degree $|U(R)|$.*

(b) *Let R be a local ring with the maximal ideal $\underline{m}$. Then G_R is a complete mutipartite graph whose partite sets are the cosets of $\underline{m}$ in R. In paticular, G_R is a compelete graph if and only if R is a field.*

Theorem 1 [2, Theorem 3.1]. *Let R be a ring. Then $G(R)$ is a complete r-partite graph if and only if R is a local ring with the maximal ideal $\underline{m}$ and $r = |R/\underline{m}| = 2^n$, for some $n \in \mathbb{N}$ or R is a finite field.*

Theorem 2. *Let n be a positive integer. Then the following statements hold*

(1) $\operatorname{diam}(G(E_{3^n})) = \operatorname{diam}(G_{E_{3^n}}) = 2$;

(2) $\operatorname{gr}(G(E_{3^n})) = \operatorname{gr}(G_{E_{3^n}}) = 3$.

P r o o f. For each positive integer n, E_{3^n} is a local ring with the maximal ideal $\langle 2+\omega\rangle$, see [4]. Since $\varphi(E_{3^n}) = 2 \times 3^{2n-1}$, we have

$$\left|\frac{E_{3^n}}{\langle 2+\omega\rangle}\right| = 3.$$

Therefore, by Proposition 1, $G_{E_{3^n}}$ is a complete 3-partite graph and hence $\operatorname{diam}(G_{E_{3^n}}) = 2$ and $\operatorname{gr}(G_{E_{3^n}}) = 3$. Also, by Theorem 1, $G(E_{3^n}/\langle 2+\omega\rangle)$ is a complete bipartite graph. Thus $\operatorname{diam}(G(E_{3^n})) = 2$ and $\operatorname{gr}(G(E_{3^n})) = 3$. □

Theorem 3. *Let n be a positive integer and q be a prime integer congruent to 1 modulo 3. Then the following statements hold,*

(1) $\operatorname{diam}(G(E_{q^n})) = \operatorname{diam}(G_{E_{q^n}}) = 2$;

(2) $\operatorname{gr}(G(E_{q^n})) = \operatorname{gr}(G_{E_{q^n}}) = 3$.

P r o o f. Since q is a prime integer congruent to 1 modulo 3, the ring E_{q^n} is the product of the two local rings $E/\langle(a+b\omega)^n\rangle$ and $E/\langle(a+b\overline{\omega})^n\rangle$ that have the same number of elements. The ideals $\langle a+b\omega\rangle$ and $\langle a+b\overline{\omega}\rangle$ are the only maximal ideals of E_{q^n}, see [4]. Therefore, by [18, Theorem 3.5], we have

$$\operatorname{diam}(G(E_{q^n})) = \operatorname{diam}(G_{E_{q^n}}) = 2.$$

On the other hand, in view of the proof of [7, Proposition 5.10] and [3, Theorem 3.2], we obtain

$$\operatorname{gr}(G(E_{q^n})) = \operatorname{gr}(G_{E_{3^n}}) = 3.$$

□

Lemma 1 [19, Lemma 4.1]. *Let R be a finite ring and $a \in R$. The following statements are equivalent*:

(1) $a \in J(R)$;

(2) $a + u \in U(R)$ *for any* $u \in U(R)$.

Theorem 4 [9, Theorem 2.6]. *Let $R \cong R_1 \times R_2 \times .... \times R_n$ be a finite ring, where $(R_i, \underline{m}_i)$ is a local ring, for each $i = 1, \cdots, n$. Then the following statements are equivalent*:

(1) $2 \in J(R)$;

(2) $G_R = G(R)$;

(3) *for every* $i = 1, ..., n$, $|R_i|$ *is even.*

Theorem 5. *Let n be a positive integer and p be a prime integer congruent to 2 modulo 3. Then the following statements hold*

(1) $\operatorname{diam}(G_{E_{p^n}}) = \begin{cases} 1 & \text{if} \quad n = 1, \\ 2 & \text{if} \quad n > 1; \end{cases}$

(2) $\operatorname{diam}(G(E_{p^n})) = 2$;

(3) $\operatorname{gr}(G(E_{p^n})) = \operatorname{gr}(G_{E_{p^n}}) = 3$.

P r o o f. Since p is a prime integer congruent to 2 modulo 3, p is an Eisenstein prime integer. Hence E_p is a field. If $n > 1$, then the ring E_{p^n} is a local ring with the maximal ideal $\langle p \rangle$. Since

$$\varphi(E_{p^n}) = p^{2k-2}(p^2 - 1),$$

we obtain that (see [4]):

$$\left| \frac{E_{p^n}}{\langle p \rangle} \right| = p^2.$$

If $p = 2$, then it follows from Theorem 4 that $G(E_{p^n}) = G_{E_{p^n}}$. In this case, by [3, Theorem 3.1] and [3, Theorem 3.2], we obtain $\operatorname{gr}(G(E_{p^n})) = \operatorname{gr}(G_{E_{p^n}}) = 3$ and

$$\operatorname{diam}(G(E_{p^n})) = \operatorname{diam}(G_{E_{p^n}}) = \begin{cases} 1 & \text{if} \quad n = 1, \\ 2 & \text{if} \quad n > 1. \end{cases}$$

We now assume that $p \neq 2$. Then $G_{E_{p^n}}$ is a complete p^2-partite graph. Therefore, $\operatorname{diam}(G_{E_{p^n}}) = 2$ and $\operatorname{gr}(G_{E_{p^n}}) = 3$. Since $G(E_{p^n}/\langle p \rangle)$ is a complete $(p^2+1)/2$-partite graph, we obtain that

$$\operatorname{diam}(G(E_{p^n})) = 2, \quad \operatorname{gr}(G(E_{p^n})) = 3.$$

□

2.2. Diameter and Girth for the Graphs G_{E_n} and $G(E_n)$

The general case is now investigated. It is well known [8, Theorem 8.7] that every finite commutative ring can be expressed as a direct product of finite local rings, and this decomposition is unique up to permutations of such local rings. Throughout this section we assume that $R \cong R_1 \times R_2 \times .... \times R_t$ is a finite commutative ring, where each R_i is a finite commutative local ring with the maximal ideal $\underline{m}_i$. Since $(u_1, ..., u_t)$ is a unit of R if and only if each u_i is a unit in R_i, we see immediately that $G_R \cong G_{R_1} \otimes G_{R_2} ... \otimes G_{R_t}$ and $G(R) \cong G(R_1) \otimes G(R_2) ... \otimes G(R_t)$. We denote by K_i the (finite) residue field $R_i/\underline{m}_i$ and $f_i = |K_i|$. We also assume (after appropriate permutation of factors) that $f_1 \leq f_2 \leq ... \leq f_t$.

Theorem 6. *Let $n > 1$ be an integer with at least two distinct prime factors. Then*

$$\operatorname{diam}(G_{E_n}) = \operatorname{diam}(G(E_n)) = 2.$$

P r o o f. Let

$$n = 3^k \times \prod_{i=1}^{m} p_i^{\alpha_i} \times \prod_{j=1}^{l} q_j^{\beta_j},$$

where p_i and q_j are prime integers such that $p_i \equiv 2 \pmod 3$ and $q_j \equiv 1 \pmod 3$, then

$$E_n \cong E_{3^k} \times \prod_{i=1}^{m} E_{p_i^{\alpha_i}} \times \prod_{j=1}^{l} E_{q_j^{\beta_j}},$$

see [4]. This shows that, E_n is isomorphic to a direct product of finite local rings $(R_i, \underline{m}_i)$, such that for every i, $|R_i/\underline{m}_i| = 3$ or p_i^2 or q_j. By [3, Theorem 3.5 (b)], we conclude that

$$\operatorname{diam}(G_{E_n}) = \operatorname{diam}(G(E_n) = 2.$$

□

Theorem 7. *Let $n > 1$ be an integer with at least two distinct prime factors. Then*

$$\operatorname{gr}(G_{E_n}) = \operatorname{gr}(G(E_n)) = 3.$$

P r o o f. By the argument similar to that above, we conclude that

$$E_n \cong E_{3^k} \times \prod_{i=1}^{m} E_{p_i^{\alpha_i}} \times \prod_{j=1}^{l} E_{q_j^{\beta_j}}.$$

Thus, by [3, Theorem 3.2], we obtain $\operatorname{gr}(G_{E_n}) = 3$. On the other hand, in view of the proof of [7, Theorem 5.10], we have $gr(G(E_n)) \in \{3, 4\}$.

Since n is an integer with at least two distinct prime factors, we can assume that $n = ab$ with $\gcd(a, b) = 1$. It is clear that

$$\gcd(a^2 + b^2 - ab, n) = 1.$$

Thus, $N(a + b\omega)$ is a unit in $\mathbb{Z}_n$, and so $a + b\omega$ is a unit in E_n. This showes that $x = a$ and $y = b\omega$ are adjacent. Now, by taking $z = b + a\omega$, we have $x + z = (a + b) + a\omega$ and $y + z = b + (a + b)\omega$. Clearly, $N(x + z) = N(y + z) = a^2 + b^2 + ab$ is a unit in $\mathbb{Z}_n$ which implies that $x + z$ and $y + z$ are unit elements of E_n. Therefore, we obtain the cycle

$$x \longrightarrow y \longrightarrow z \longrightarrow x.$$

This implies that $\operatorname{gr}(G(E_n)) = 3$. □

2.3. Some Graph Invariants of Graphs G_{E_n} and $G(E_n)$

In the sequel, we obtain the clique number and the chromatic number for the graphs G_{E_n} and $G(E_n)$.

Theorem 8. *Let $n > 1$ be an integer and*

$$n = 3^k \times \prod_{i=1}^{m} p_i^{\alpha_i} \times \prod_{j=1}^{l} q_j^{\beta_j}.$$

Then the following statements hold

(1) *if $3 \mid n$, then $\chi(G_{E_n}) = \omega(G_{E_n}) = 3$ and $\alpha(G_{E_n}) = n^2/3$;*

(2) *if $3 \nmid n$, then*

$$\chi(G_{E_n}) = \omega(G_{E_n}) = \min\left\{p_i^2, q_j \mid 1 \le i \le m,\ 1 \le j \le l,\ p_i \mid n,\ q_j \mid n\right\}$$

and

$$\alpha(G_{E_n}) = \frac{n^2}{\min\left\{p_i^2,\ q_j \mid 1 \le i \le m,\ 1 \le j \le l,\ p_i \mid n,\ q_j \mid n\right\}}.$$

P r o o f. 1. Let k be the biggest positive integer such that $3^k \mid n$. Then

$$E_n = E_{3^k} \times \prod_{i=1}^{m} E_{p_i^{\alpha_i}} \times \prod_{j=1}^{l} E_{q_j^{\beta_j}}.$$

Since E_{3^k} is a local ring with the maximal ideal $\langle 2+\omega \rangle$ and

$$\left| \frac{E_{3^k}}{\langle 2+\omega \rangle} \right| = 3,$$

it follows from [3, Proposition 6.1] that $\chi(G_{E_n}) = \omega(G_{E_n}) = 3$ and $\alpha(G_{E_n}) = n^2/3$.

2. If $3 \nmid n$, then it yields that E_n is isomorphic to a direct product of finite local rings $(R_i, \underline{m}_i)$, such that for every i, $|R_i/\underline{m}_i| = p_i^2$ *or* q_j. Thus by [3, Proposition 6.1], we have

$$\chi(G_{E_n}) = \omega(G_{E_n}) = k = \min\left\{\ p_i^2, q_j \ \mid\ 1 \le i \le m,\ 1 \le j \le l\right\}$$

and $\alpha(G_{E_n}) = n^2/k$. □

Theorem 9. *Let $n > 1$ be an integer and*

$$n = 3^k \times \prod_{i=1}^{m} p_i^{\alpha_i} \times \prod_{j=1}^{l} q_j^{\beta_j}.$$

Then the following statements hold

(1) *if $2 \mid n$, then $\chi(G(E_n)) = \omega(G(E_n)) = 4$;*

(2) *if $2 \nmid n$, then*

$$\chi(G(E_n)) = \omega(G(E_n)) = \frac{1}{2^{1+m+l}} \times \prod_{i=1}^{m} (p_i^{2\alpha_i} - p_i^{2\alpha_i - 2}) \times \prod_{j=1}^{l} (q_j^{\beta_j} - q_j^{\beta_j - 1})^2 + m + 2l + 1.$$

P r o o f. Since $n = 3^k \times \prod_{i=1}^{m} p_i^{\alpha_i} \times \prod_{j=1}^{l} q_j^{\beta_j}$, we have

$$E_n = E_{3^k} \times \prod_{i=1}^{m} E_{p_i^{\alpha_i}} \times \prod_{j=1}^{l} E_{q_j^{\beta_j}}.$$

1. If $2 \mid n$, then $2 \notin U(E_n)$. Hence, in view of the proof of [21, Theorem 2.2], we have

$$\chi(G(E_n)) = \omega(G(E_n)) = 4.$$

2. If $2 \nmid n$, Then $2 \in U(E_n)$. By an argument similar to that above, we conclude that

$$\chi(G(E_n)) = \omega(G(E_n)) = \frac{1}{2^{1+m+l}} \times \prod_{i=1}^{m}(p_i^{2\alpha_i} - p_i^{2\alpha_i-2}) \times \prod_{j=1}^{l}(q_j^{\beta_j} - q_j^{\beta_j-1})^2 + m + 2l + 1.$$

□

We now state our final result.

Theorem 10. *For each integer $n > 1$, the graphs $G(E_n)$ and G_{E_n} are Hamitonian.*

P r o o f. Let $n > 1$ be an integer. By Theorem 2, Theorem 3, Theorem 5 and Theorem 6, the graphs $G(E_n)$ and G_{E_n} are connected. Thus $G(E_n)$ is Hamiltonian graph, by [22, Theorem 2.1]. Also, it follows from [20, Lemma 4] that G_{E_n} is Hamiltonian graph. □

3. Concluding Remarks

In this article, the diameter, the girth, the chromatic number and the clique number of $G(E_n)$ and G_{E_n} are studied. We also prove that for each $n > 1$, the graphs $G(E_n)$ and G_{E_n} are *Hamiltonian* and the independence number of G_{E_n} is calculated. We end our paper with the following two open questions:

Question 1. Is there any closed formula for $\alpha(G(E_n))$?

Question 2. When are $G(E_n)$ and G_{E_n} Eulerian?

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