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Convergence Properties of Solutions of a Length-Structured Density-Dependent Model for Fish

By *Geigh Zollicoffer*[*][†]

Abstract. We numerically study solutions of a length-structured matrix model for fish populations in which the probability that a fish grows into the next length class is a decreasing nonlinear function of the total biomass of the population. We make conjectures about the convergence properties of solutions, and give numerical simulations which support these conjectures. We also study the distribution of biomass in the different length classes as a function of the total biomass.

1 Introduction and Background

In this paper we numerically analyze a discrete-time, nonlinear length-based model for a fish population. This model has been mathematically studied (with proofs) in Callahan, et. al. [1], with a restrictive condition on the survival probabilities. This condition is that the survival of fish is a nondecreasing function of the length of the fish. However, in practice, survival probabilities may not always satisfy this condition. For instance, if angling is allowed in the habitat, larger fish can have lower survival than smaller fish. In this paper we numerically study the solutions of this model when this condition on the survival probabilities is removed. We make conjectures that are stronger than the results proved in [1], and give numerical examples that make these conjectures plausible.

In [1] references are given for papers that give fish population models which incorporate length structure. These papers give justifications for incorporating length structure into a population model. For instance, it is known that many fish species are less likely to grow when their habitat is crowded. Thus it makes sense to include in this model the condition that the probability that a fish moves to a larger length class in one time step is a decreasing function of the total amount of fish biomass in the habitat. This function should be positive, should take the value 1 when there is no biomass (since

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with no crowding fish are guaranteed to grow), and should approach 0 as the biomass gets infinite (since with “infinite" crowding fish cannot grow). To keep the modeling manageable, in these models it is assumed that there is only one fish species in the habitat. Incorporating such a nonlinearity into the model means that when the population gets too big, there’s pressure to keep it down, and when the population is small, it grows unimpeded. Thus it is plausible that there are circumstances under which we expect the population to converge. In [1] it is shown that under certain conditions the biomass converges uniformly (i.e. independent of the initial conditions), and we conjecture that in fact under those circumstances the population vectors converges uniformly, even when the survival hypotheses used in [1] are not satisfied. In this paper we give conjectures about uniform convergence of the population vectors, and give numerical simulations which make these conjectures plausible.

The paper is organized as follows. In Sect. 2 we describe the discrete-time nonlinear length-structured model, taken directly from [1], which we are studying. In this model the probability that a fish will grow into the next length class in one time step is a nonlinear function of the total biomass of the population. In Sect. 3 we make conjectures about the convergence properties of solutions to the model. In Sect. 4 we give numerical results that are consistent with these conjectures. We use as a case study an invasive white perch species studied in [2, 1]. For the purposes of angling, a long-term population which is dominated by fish in the larger length classes is desirable, and in Sect. 5 we make connections between the distribution of biomass and the survival rates and total biomass.

Notation: Denote transpose by a T superscript, a row vector by $^{\mathrm{T}}$, $[v_1, v_1, \ldots, v_n]$ or $[v_j]$. Denote a column vector by , $[v_1, v_1, \ldots, v_n]^{\mathrm{T}}$ or $[v_j]^{\mathrm{T}}$. The spectral radius of a square matrix A is denoted by $\rho(\mathrm{A})$, and is the modulus of the largest eigenvalue of A. The 1-norm of a vector is denoted by $\| \|$, and is the sum of the absolute values of the entries.

2 Mathematical Model

This length-based model was introduced in [1], and is similar to models in [3, 5], and is related to the age-structured models in [2, 6]. We begin by using the following model design pulled directly from [1]. There are n length classes of reproductively viable fish, and we denote the population in each class after t time steps by $\mathrm{P}_1(t)$, $\mathrm{P}_2(t)$, …, $\mathrm{P}_n(t)$. We define the population vector to be $\mathbb{P}(t) = [\mathrm{P}_0(t), \mathrm{P}_1(t), \ldots, \mathrm{P}_n(t)]^{\mathrm{T}} \in \mathbb{R}^{n+1}$. We assume that newborn fish cannot reproduce in their first time step of life and place them in a zeroth class with population $\mathrm{P}_0(t)$. After one time step, surviving newborn fish enter class 1 with population $\mathrm{P}_1(t)$. Let L_i be the average length of fish in class $i = 0, 1, \ldots n$, so L_0 is the average length of newborn fish. The time step size is constant and might be determined by the behavior of the species or by the timing of the data collection. We assume that in each time step a surviving fish either stays in its length class or grows into the next length class but cannot skip beyond the next length class, so $\mathrm{L}_{i+1} - \mathrm{L}_i$ is the maximum a fish in class $i = 1, 2, \dots n-1$ can grow in one time step.

Let s_i be the survival rate of fish in class $i = 0, 1, \dots n$ each year and f_i the fecundity of fish in class $i = 1, 2, \dots n$ each year. We assume that the time step in the model is less than or equal to the time needed to reach maturation size. In [1] the following assumptions are made:

A: The survival rates satisfy $0 < s_i \le 1$ and the fecundities $f_i \ge 0$ and not all $f_i = 0$. The sequences $(f_j)_{j=0}^n$ and $(s_j)_{j=0}^n$ are nondecreasing.

The condition on $(f_j)_{j=0}^n$ is satisfied for most species since the fecundity of larger fish is greater because larger fish can hold more eggs. The assumption that the survivals $(s_j)_{j=0}^n$ are nondecreasing, is not as plausible, since angling can decrease the population of large fish more than the population of small fish. One of the purposes of the current paper is to study what happens when the survival condition in Assumption **A** is removed. Therefore, in this paper we will work with the modified assumptions:

A′: The survival rates satisfy $0 < s_i \le 1$ and the fecundities $f_i \ge 0$ and not all $f_i = 0$. The sequence $(f_j)_{j=0}^n$ is nondecreasing.

Let p_t be the probability at time step t that a fish grows into the next length class in one time step. We use a model of the form

$$\mathbb{P}(t+1) = \mathrm{A}_{p_t}\mathbb{P}(t)$$

where

$$\mathrm{A}_{p_t} = \begin{bmatrix} 0 & f_1 & f_2 & f_3 & \cdots & f_{n-2} & f_{n-1} & f_n \\ s_0 & s_1(1-p_t) & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & s_1 p_t & s_2(1-p_t) & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & s_2 p_t & s_3(1-p_t) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & s_{n-2}p_t & s_{n-1}(1-p_t) & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & s_{n-1}p_t & s_n \end{bmatrix}. \tag{1}$$

Thus the population at time $t+1$ is a A_{p_t} times the population at time t. This equation is solvable once an initial population $\mathbb{P}(0)$ is specified. A matrix of this type is known as a “population projection matrix". We now give an interpretation of this matrix. When the top row is multiplied by the population vector, we get

$$\sum_{j=1}^{n} f_j \mathrm{P}_j(t),$$

which can be interpreted as the number of newborns generated by the current population. Hence the 0th entry of $\mathrm{A}_{p_t}\mathbb{P}(t)$ is the number of newborns generated from the current population. The jth entry of $A_{p_t}\mathbb{P}(t)$ for $j = 2, \dots n$ is

$$s_{j-1}p_t P_{j-1} + s_j(1-p_t)P_j.$$

Since p_t is the probability that a fish will move to the next length class, this can be interpreted as the number of fish that survived and got bigger from the $(j-1)$th length class, plus the number of fish that survived and didn't get bigger from the jth length class. When $j = 1$, the jth entry of $A_{p_t}\mathbb{P}(t)$ is just how many survive from the 0th class.

As in [1, 2, 6], we assume that p_t is a strictly decreasing function of the population biomass. That is, the more biomass there is in the population, the less likely a fish is to move to the next length class. Furthermore, we expect the probability to be near zero when the biomass is very large, and we expect the probability to be 1 when the biomass is zero. Let $B(t)$ denote the population biomass at time step t. We need to identify a function g so that $p_t = g(B(t))$.

In [1, 2, 6] that function is

$$g(y) = \frac{1}{1 + b_{growth}\, y}$$

where b_{growth} is a scaling parameter. Note that that g is strictly decreasing on $[0,\infty)$ (since more biomass means lower probability), $g(0) = 1$ (since no biomass means probability 1) and $\lim_{y\to\infty} g(y) = 0$ (since large biomass means small probability).

Following [1, 2, 6], we assume that the mass of a fish of length L_i is $W_i = \alpha L_i^3$ where α is the mass-length coefficient. The population biomass at time step t is then approximated by

$$B(t) = \sum_{i=0}^{n} W_i P_i(t) \tag{2}$$

We study the nonlinear dynamical system

$$\mathbb{P}(t+1) = A_{p_t}\mathbb{P}(t), \;\; p_t = g(B(t)), \;\; \mathbb{P}(0) = \mathbb{P}_0. \tag{3}$$

This system gives the population at time step $t+1$ as the matrix multiplication of matrix A_{p_t} with the population $\mathbb{P}(t)$, where the probability of growth p_t is updated at each time step.

3 Mathematical Conjectures

In this section we give some conjectures about the solutions to (3).

Definition 3.1. A vector is *positive* if all of its entries are positive. A vector is *nonnegative* if all its entries are nonnegative. A nonnegative vector is *nonzero* if not all its entries are zero.

In the study of dynamical systems, the *equilibria* play a central role. An equilibrium is a vector that is unchanged by the system from a time step to the next.

Definition 3.2. 1. We say that a vector $\mathbb{P}^* = [\mathrm{P}_1^*, \mathrm{P}_2^*, \dots \mathrm{P}_n]^{\mathrm{T}}$ is an equilibrium for (3) if

$$\mathbb{P}^* = \mathrm{A}_{p^*}\mathbb{P}^*,\ p^* = g(\mathrm{B}^*), \mathrm{B}^* = \sum_{i=0}^{n} \mathrm{W}_i \mathrm{P}_i^*$$

2. We say that an equilibrium $\mathbb{P}^*$ for (3) is globally attracting if for every nonnegative nonzero $\mathbb{P}_0$,

$$\lim_{t\to\infty} \mathbb{P}_{p_t} = \mathbb{P}^*.$$

The goal of this paper is to use simulations to make the following conjecture plausible.

Conjecture 3.1. 1. If $\rho(\mathrm{A}_1) < 1$, then the zero population is globally attracting.

2. If $\rho(\mathrm{A}_0) > 1$, then $\lim_{t\to\infty} \|\mathbb{P}(t)\| = \infty$ for all nonzero nonnegative initial states $\mathbb{P}(0)$.

3. If $\rho(\mathrm{A}_0) < 1 < \rho(\mathrm{A}_1)$, then the system has a unique nonzero positive equilibrium $\mathbb{P}^*$ which is globally attracting.

Before discussing what these conjectures mean, we will discuss the role of the spectral radius in predicting population dynamics. Consider first the simpler linear model

$$\mathbb{P}(t+1) = \mathrm{A}\mathbb{P}(t). \tag{4}$$

for a constant matrix A. It is well known from linear algebra that the long term behavior of $\mathbb{P}(t)$ is determined by $\rho(\mathrm{A})$. Roughly speaking, if $\rho(\mathrm{A}) < 1$, then all solutions of (4) go to zero; if $\rho(\mathrm{A}) > 1$, then all solutions of (4) go to infinity in norm; if $\rho(\mathrm{A}) = 1$, all solutions converge, but not uniformly. The moral is that the growth, decay or convergence of the population is determined by $\rho(\mathrm{A})$ in the linear case. In our nonlinear model, the analysis uses $\rho(\mathrm{A}_0)$ and $\rho(\mathrm{A}_1)$:

For Case 1 of Conjecture 3.1: A_1 describes what happens when the probability of a fish moving to the next length class in one time step is always 1 - this is in some sense the “best case scenario" for the fish. If $\rho(\mathrm{A}_1) < 1$, that means that in the best case scenario the population goes to zero, so it is plausible that the solution of (3) goes to zero as well.

For Case 2 of Conjecture 3.1: A_0 describes what happens when the probability of a fish moving to the next length class in one time step is always 0 - this is in some sense the “worst case scenario" for the fish. If $\rho(\mathrm{A}_0) > 1$, that means that in the worst case scenario the population goes to infinity, so it is plausible that the solution of (3) goes to infinity as well.

In Case 1 the population is endangered, while Case 2 is unlikely to happen in a physical habitat. If the population is not endangered and the population is known to stay finite, the nonlinear probability of growth kicks in, and Case 3 is likely to occur. The conjectures for Case 1 and Case 2 were proved in [1] when the survivals are a nondecreasing function of size class (i.e. under Assumption A). In this paper we give numerical examples which help support Cases 1 and 2 of the conjecture under the weaker Assumption A′. In this paper we give numerical examples that help support Case 3 of the conjecture under Assumption A′.

4 Numerical Simulations

We have done extensive simulations which support Conjecture 3.1 under Assumption A′. We will give examples to illustrate each of the cases in this Conjecture. As a case study we will use the parameters for the white perch population studied in [1, 2]. Please see these papers for a discussion of this species. We will use the simulation parameters L_i, f_i, and α given in Section 4 of [1] (with $n = 8$, so there are nine stages in the population vector). We will be varying the survival probabilities s_j to get the three cases in Conjecture 3.1. Since Cases 1 and 2 of Conjecture 3.1 have been established in [1] when (s_j) is nondecreasing, our examples will use survivals that are not nondecreasing, so Assumption A′ holds but Assumption A does not.

1. For case 1, we'll use survival rates

   $$[.2, .8, .2, .8, .2, .8, .2, .8, .2].$$

   In this case $\rho(A_1) = 0.777272 < 1$, so this satisfies the hypotheses of case 1. The population dynamics for this model (with an arbitrarily chosen initial population) are given in Figure 1. The top graph represents the first size class, and the bottom graph represents the last size class.

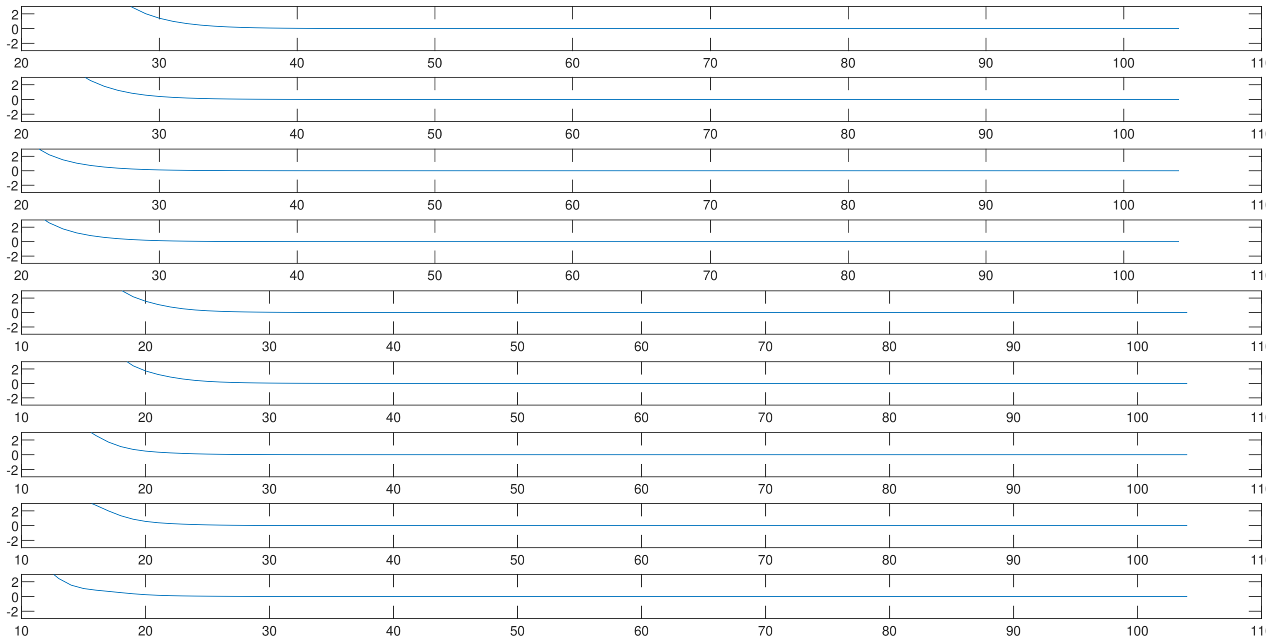

Figure 1: Example for case 1 of Conjecture 1, $\rho(A_1) < 1$; the population appears to converge to the predicted zero population. The number of time steps is given on the X-axis, and the current population count is given on the Y-axis. The top graph represents the first size class, and the bottom graph represents the last size class.

For us to determine that the population converges to zero, the population for each size class is compared after each time step with the previous time step and the absolute value of the difference is calculated. If the difference was less than 2^{-1021} (which is the smallest float in Matlab), then our criteria for convergence is met and we conclude convergence. We repeated this simulation for many randomly chosen initial populations, and for every initial condition the population appeared to converge to zero.

2. For case 2, we use the following survival rates:

$$[.99,\ 1,\ .97,\ .96,\ .95,\ .94,\ .93,\ .92,\ .91]$$

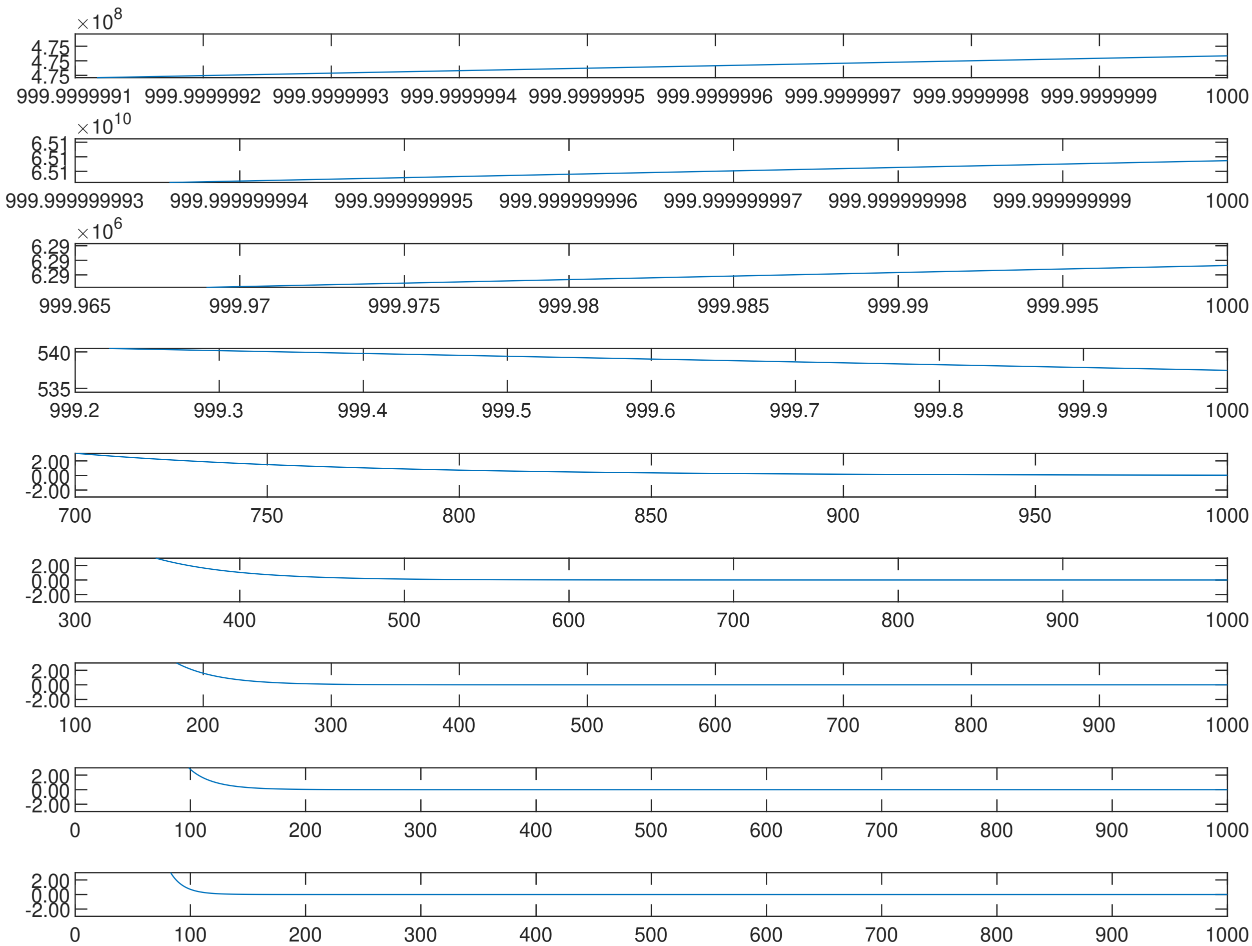

Figure 2: Example for case 2, $\rho(A_0) > 1$; $\lim_{t\to\infty} \|\mathbb{P}(t)\| = \infty$ and p_t approaches 0. The number of time steps is given on the X-axis, and the current population count is given on the Y-axis. The top graph represents the first size class, and the bottom graph represents the last size class.

In this case we see that $\rho(A_0) = 1.007207 > 1$. We determine for this case convergence to infinity by setting a large upper bound, and showing that the total population eventually exceeds that upper bound. We did simulations with 100,000 time steps, and found clear evidence that the population in the first two stages goes to infinity, the population in the third stage goes to 6.2985×10^6, and the population in the larger stages goes to zero. We only show 1000 time steps in Figure 2, in order to get a clearer picture. As expected, the probability p_t appears to converge to zero, but does not get to zero because the biomass does not get to infinity. Also, we see in this example an extreme case of *stunting*, where the population of the smaller fish dominates as the population gets more and more crowded.

3. For case 3, we use survival rates:

$$[.7\ .7\ .7\ .7\ .4\ .4\ .4\ .4\ .4]$$

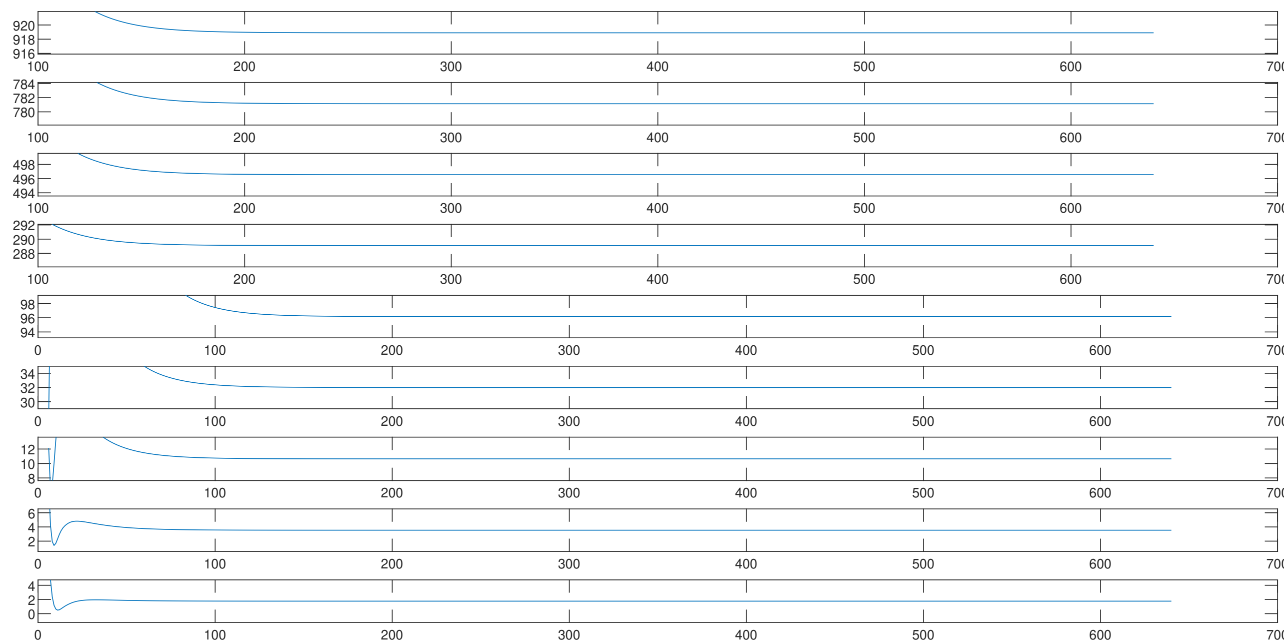

Figure 3: Example for case 3, $\rho(A_0) < 1 < \rho(A_1)$; the population converges to a unique limiting population. The number of time steps is given on the X-axis, and the current population count is given on the Y-axis. The top graph represents the first size class, and the bottom graph represents the last size class.

In this case $\rho(A_0) = 0.707258$ and $\rho(A_1) = 1.149390$, so $\rho(A_0) < 1 < \rho(A_1)$, and the hypotheses for case 3 are satisfied. We determined and tested convergence to the limiting population in the same way that we tested convergence to zero in case 1, using the tolerance 2^{-1021}. It is easy to check that the limiting population is in fact an equilibrium.

Part of cases 1 and 3 of Conjecture 3.1 is that the convergence is independent of the initial population. To illustrate this, we give in the Appendix two charts - one for Case 1 and one for Case 3 - where the top rows gives 15 initial populations vectors, and the bottom row gives the corresponding limiting population vectors. We see from these that the limiting population is indeed independent of the initial population (up to at least seven significant figures) in these examples. Thus in Case 1 it appears that the zero vector is in fact globally attracting, and in Case 3 it appears that the positive equilbrium is in fact globally attracting. We found the same type of results for other examples.

We have automatically generated approximately 3000+ simulations, each with survivals randomly generated from a uniformly random distribution, and random initial conditions also from a uniformly random distribution. All of them are consistent with Conjecture 3.1. Of course, that is not a proof of Conjecture 3.1, but it does make the Conjecture very plausible.

5 Distribution of biomass in the limiting population

[1] studied (in Case 3 and examples where the survivals were nondecreasing) the relationship between the limiting total population biomass and the distribution of biomass in the nine stages. It was found, and proved mathematically, that as the total biomass got larger, the biomass became more concentrated in the lower stages. In this paper we would like to illustrate that we expect this to happen even when the survivals are not nondecreasing. In our case study, we use all of the parameter values from the previous section, except we use the survivals

$$(.99-\alpha,\ .98-\alpha,\ .97-\alpha,\ .96-\alpha,\ .95-\alpha,\ .94-\alpha,\ .93-\alpha,\ .92-\alpha,\ .91-\alpha)$$

where α is a parameter which varies from 0 to .91. The total limiting population biomass is very large when α is very close to 0, and is very close to 0 when α is close to .91; since we cannot take infinitely many time steps, the total biomass never gets to zero. We show the total biomass on the curve in Figure 4, with value of α on the x-axis and the biomass scale shown on the right y-axis. To illustrate the biomass distribution with a heat map, we take 9 equally-spaced values of α, and above each value of α there are colored boxes which indicate what fraction of the biomass is in each of the nine length stages. The stages are shown on the left y-axis, the the smallest stage at the top and the largest stage at the bottom (note that we label the stages 1 through 9 instead of 0 through 8). The scale for the heat map is on the top of Figure 4.

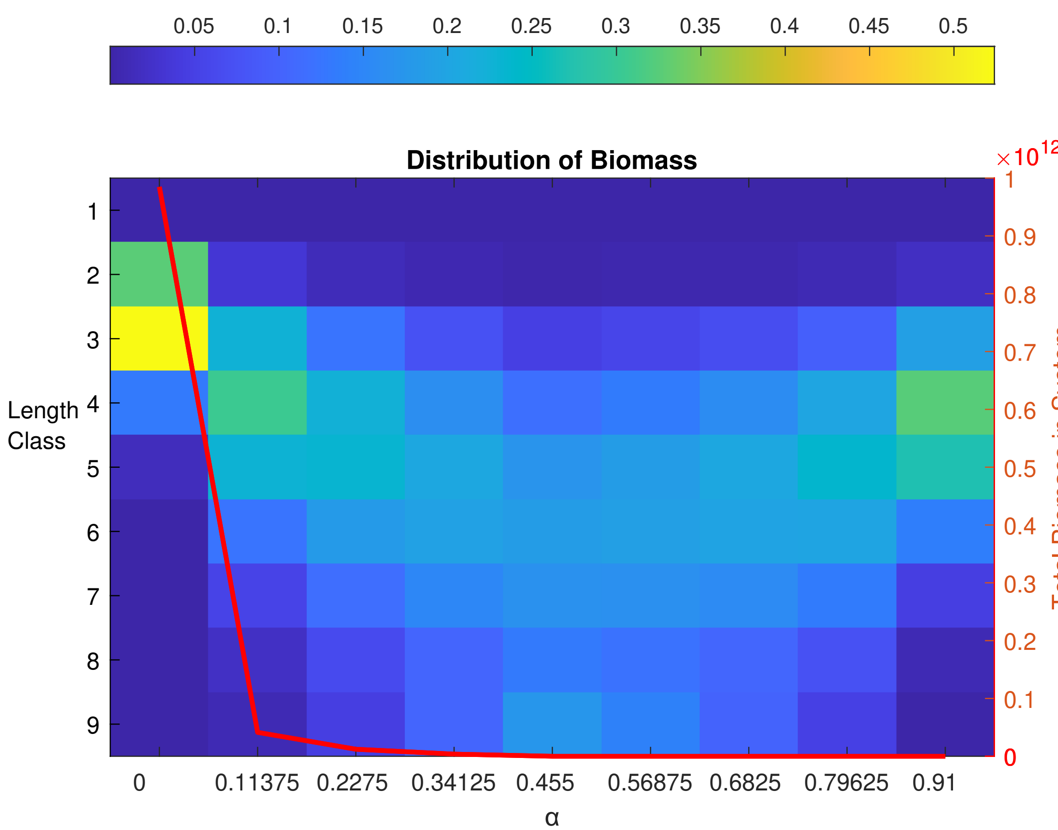

Figure 4: Heat map for biomass distribution as a function of α. The values of α are shown on the X-axis, and the survivals decrease with increasing α. The left side Y-axis shows the stages, and the right side Y-axis is total biomass scale. The curve gives the biomass for each choice of α, and the color gives what fraction of the biomass is in each of the length stages.

In all cases a small fraction of the biomass is in the newborn fish class. When α is near zero, the total biomass is very high, and is skewed towards the P_1 class. As α increase, we see that the biomass percentage shifts towards the larger classes of fish until $\alpha = .455$. Between $\alpha = .455$ and $\alpha = .56875$, the total biomass in the system falls to near zero, and stays there for larger values of α. Calculating the spectral radius of A_1 and A_0 where $\alpha = .455$, yields $\rho(A_1) = 1.000507$ and $\rho(A_0) = 0.532369$, which is Case 3. When $\alpha = .568750$, $\rho(A_1) = 0.807852$ and $\rho(A_0) = 0.418629$, which is Case 1. Thus the system changes from Case 3 to Case 1 for some α between .455 and .56875. Just looking at α from 0 to .455, we're in the most interesting case, and we see that smaller biomass corresponds to a larger percentage of large fish. This is numerical evidence that the relationship between size distribution and population biomass found in [1] still holds (at least until biomass near zero) even when Assumption A is weakened to Assumption A′.

For larger values of α, we're in Case 1, where where the population crashes; however, in practice, we'll never get to zero biomass, so it is still of theoretical interest how the population is distributed, and the heat maps shows a limiting biomass distribution. We find that as α gets closer to 1 (so the survivals get smaller), the population starts moving towards smaller fish, which was not expected.

6 Conclusions

In this paper we study a discrete-time length-structured model for invasive fish which incorporates a nonlinear growth probability. This model was proposed in [1], and studied there with the restrictive condition that the survival parameters be nondecreasing (i.e. Assumption A). We numerically studied the case where the survival parameters do not satisfy this restrictive condition (i.e. Assumption A′. We give Conjecture 3.1 about the long-term behavior of the solutions - this conjecture is stronger than the results in [1]. We then do numerical simulations to test this conjecture, and all simulations are consistent with the conjecture. Finally, we study the effect of the total biomass on the length distribution of the limiting population. This paper gives numerical evidence that the nondecreasing condition on the survivals needed in [1] (i.e. Assumption A) is not really important in the long-term dynamics the model.

These results are significant because population managers care about the long-term behavior of populations, both for predicting and managing fish populations. In nature, survival rates are not necessarily nondecreasing with length, so it is important to consider cases where Assumption A is not satisfied. This study makes it plausible that if the populations are bounded (which is expected in real habitats), then the population either goes to zero, or the population approaches some equilibrium which is independent of the initial population. The independence of initial population is important because it implies that the limiting population can be computed even if the initial population is unknown. In the future, we'd like to prove Conjecture 3.1 under Assumption A ′ holding.

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7 Appendix

In this Appendix we illustrate the part of Conjecture 3.1 that says that in Cases 1 and 3 the limiting population is independent of the initial population. Figure 5 illustrates the convergence to zero in Case 1, and Figure 6 illustrates the convergence to an equilibrium in Case 3. The top rows represent the randomly chosen initial populations vectors, and the bottom rows represent the corresponding limiting population vector.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9	Test 10	Test 11	Test 12	Test 13	Test 14	Test 15
	Initial Population vectors														
Population Class 0	353.4217	2353.124	1812.296	4904.518	2883.791	1862.671	3334.658	1944.419	4498.567	614.075	3474.026	93.06387	1878.461	2000.399	2699.525
Population Class 1	4613.723	2803.567	3940.567	1433.102	129.2874	2965.923	4668.628	2273.709	2251.968	2036.592	4171.845	3373.882	2732.769	4159.357	476.8635
Population Class 2	4001.86	1345.458	3901.479	4004.101	2232.655	4362.763	4054.75	1233.436	1028.362	1376.435	3048.148	2192.544	2809.601	671.6917	732.5743
Population Class 3	1429.734	3745.092	3342.561	4480.557	3231.51	4667.508	2422.741	3922.115	4498.255	3583.349	2873.686	2189.101	1979.111	302.3339	3155.706
Population Class 4	2718.316	2519.439	667.5193	2987.633	2606.015	3342.321	3783.746	4414.188	3812.928	1416.922	1630.211	585.1841	1990.654	421.2353	4296.602
Population Class 5	4923.881	3234.048	107.7794	4420.084	1861.563	1033.882	2085.237	4568.558	4412.432	4480.994	2282.123	4073.408	2576.836	819.4916	4871.108
Population Class 6	3578.39	1538.728	2799.204	4718.658	4685.673	3269.253	4858.93	2791.425	1424.751	4132.894	3568.978	1624.277	3287.653	1621.1	2854.192
Population Class 7	4194.848	693.6232	1504.095	2745.79	4147.664	360.2578	4939.874	2994.341	3366.13	1950.133	4422.025	1231.141	4754.576	1508.634	4984.251
Population Class 8	2166.303	2377.865	4697.049	3641.934	4245.427	2033.635	4320.738	744.3836	3321.4	2489.515	3604.278	1713.566	3611.743	58.40496	2767.708
	Limiting Population Vectors														
Population Class 0	1.81E-12	1.83E-12	1.71E-12	1.64E-12	2.22E-12	1.67E-12	1.81E-12	1.59E-12	1.58E-12	1.68E-12	1.61E-12	2.08E-12	2.25E-12	2.22E-12	1.67E-12
Population Class 1	5.22E-13	5.27E-13	4.92E-13	4.74E-13	6.41E-13	4.82E-13	5.24E-13	4.58E-13	4.56E-13	4.85E-13	4.66E-13	6.01E-13	6.50E-13	6.42E-13	4.83E-13
Population Class 2	1.51E-13	1.52E-13	1.42E-13	1.37E-13	1.85E-13	1.39E-13	1.51E-13	1.32E-13	1.32E-13	1.40E-13	1.35E-13	1.74E-13	1.88E-13	1.85E-13	1.39E-13
Population Class 3	1.74E-13	1.76E-13	1.64E-13	1.58E-13	2.14E-13	1.61E-13	1.75E-13	1.53E-13	1.52E-13	1.62E-13	1.56E-13	2.01E-13	2.17E-13	2.14E-13	1.61E-13
Population Class 4	5.03E-14	5.08E-14	4.74E-14	4.56E-14	6.17E-14	4.64E-14	5.04E-14	4.41E-14	4.39E-14	4.67E-14	4.49E-14	5.79E-14	6.26E-14	6.18E-14	4.65E-14
Population Class 5	5.81E-14	5.87E-14	5.48E-14	5.27E-14	7.13E-14	5.36E-14	5.82E-14	5.10E-14	5.08E-14	5.40E-14	5.19E-14	6.69E-14	7.23E-14	7.14E-14	5.37E-14
Population Class 6	1.68E-14	1.69E-14	1.58E-14	1.52E-14	2.06E-14	1.55E-14	1.68E-14	1.47E-14	1.47E-14	1.56E-14	1.50E-14	1.93E-14	2.09E-14	2.06E-14	1.55E-14
Population Class 7	1.94E-14	1.96E-14	1.83E-14	1.76E-14	2.38E-14	1.79E-14	1.94E-14	1.70E-14	1.69E-14	1.80E-14	1.73E-14	2.23E-14	2.41E-14	2.38E-14	1.79E-14
Population Class 8	7.87E-15	7.94E-15	7.42E-15	7.14E-15	9.66E-15	7.26E-15	7.89E-15	6.90E-15	6.87E-15	7.31E-15	7.02E-15	9.06E-15	9.79E-15	9.67E-15	7.28E-15

Figure 5: 15 initial populations and their respective limiting populations of case 1 of conjecture 1, all initial populations lead to limiting populations that are indistinguishable from zero.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9	Test 10	Test 11	Test 12	Test 13	Test 14	Test 15
	Initial Population vectors														
Population Class 0	353.4217	2353.124	1812.296	4904.518	2883.791	1862.671	3334.658	1944.419	4498.567	614.075	3474.026	93.06387	1878.461	2000.399	2699.525
Population Class 1	4613.723	2803.567	3940.567	1433.102	129.2874	2965.923	4668.628	2273.709	2251.968	2036.592	4171.845	3373.882	2732.769	4159.357	476.8635
Population Class 2	4001.86	1345.458	3901.479	4004.101	2232.655	4362.763	4054.75	1233.436	1028.362	1376.435	3048.148	2192.544	2809.601	671.6917	732.5743
Population Class 3	1429.734	3745.092	3342.561	4480.557	3231.51	4667.508	2422.741	3922.115	4498.255	3583.349	2873.686	2189.101	1979.111	302.3339	3155.706
Population Class 4	2718.316	2519.439	667.5193	2987.633	2606.015	3342.321	3783.746	4414.188	3812.928	1416.922	1630.211	585.1841	1990.654	421.2353	4296.602
Population Class 5	4923.881	3234.048	107.7794	4420.084	1861.563	1033.882	2085.237	4568.558	4412.432	4480.994	2282.123	4073.408	2576.836	819.4916	4871.108
Population Class 6	3578.39	1538.728	2799.204	4718.658	4685.673	3269.253	4858.93	2791.425	1424.751	4132.894	3568.978	1624.277	3287.653	1621.1	2854.192
Population Class 7	4194.848	693.6232	1504.095	2745.79	4147.664	360.2578	4939.874	2994.341	3366.13	1950.133	4422.025	1231.141	4754.576	1508.634	4984.251
Population Class 8	2166.303	2377.865	4697.049	3641.934	4245.427	2033.635	4320.738	744.3836	3321.4	2489.515	3604.278	1713.566	3611.743	58.40496	2767.708
	Limiting Population Vectors														
Population Class 0	1.81E-12	1.83E-12	1.71E-12	1.64E-12	2.22E-12	1.67E-12	1.81E-12	1.59E-12	1.58E-12	1.68E-12	1.61E-12	2.08E-12	2.25E-12	2.22E-12	1.67E-12
Population Class 1	5.22E-13	5.27E-13	4.92E-13	4.74E-13	6.41E-13	4.82E-13	5.24E-13	4.58E-13	4.56E-13	4.85E-13	4.66E-13	6.01E-13	6.50E-13	6.42E-13	4.83E-13
Population Class 2	1.51E-13	1.52E-13	1.42E-13	1.37E-13	1.85E-13	1.39E-13	1.51E-13	1.32E-13	1.32E-13	1.40E-13	1.35E-13	1.74E-13	1.88E-13	1.85E-13	1.39E-13
Population Class 3	1.74E-13	1.76E-13	1.64E-13	1.58E-13	2.14E-13	1.61E-13	1.75E-13	1.53E-13	1.52E-13	1.62E-13	1.56E-13	2.01E-13	2.17E-13	2.14E-13	1.61E-13
Population Class 4	5.03E-14	5.08E-14	4.74E-14	4.56E-14	6.17E-14	4.64E-14	5.04E-14	4.41E-14	4.39E-14	4.67E-14	4.49E-14	5.79E-14	6.26E-14	6.18E-14	4.65E-14
Population Class 5	5.81E-14	5.87E-14	5.48E-14	5.27E-14	7.13E-14	5.36E-14	5.82E-14	5.10E-14	5.08E-14	5.40E-14	5.19E-14	6.69E-14	7.23E-14	7.14E-14	5.37E-14
Population Class 6	1.68E-14	1.69E-14	1.58E-14	1.52E-14	2.06E-14	1.55E-14	1.68E-14	1.47E-14	1.47E-14	1.56E-14	1.50E-14	1.93E-14	2.09E-14	2.06E-14	1.55E-14
Population Class 7	1.94E-14	1.96E-14	1.83E-14	1.76E-14	2.38E-14	1.79E-14	1.94E-14	1.70E-14	1.69E-14	1.80E-14	1.73E-14	2.23E-14	2.41E-14	2.38E-14	1.79E-14
Population Class 8	7.87E-15	7.94E-15	7.42E-15	7.14E-15	9.66E-15	7.26E-15	7.89E-15	6.90E-15	6.87E-15	7.31E-15	7.02E-15	9.06E-15	9.79E-15	9.67E-15	7.28E-15

Figure 6: 15 initial populations and their respective limiting populations of case 1 of conjecture 1, all initial populations lead to limiting populations that are indistinguishable from zero.

1		Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9	Test 10	Test 11	Test 12	Test 13	Test 14	Test 15
2		Initial Population vectors														
3	Population Class 0	2233.919	2664.128	2354.617	4616.898	2974.48	2539.291	1186.418	1977.576	1676.784	4518.603	4523.611	143.3708	212.1557	2592.975	866.9431
4	Population Class 1	1531.747	1753.636	1152.441	2151.037	1311.059	427.579	2294.244	1837.183	3398.64	4454.613	3049.333	2449.507	357.2273	4864.873	1954.689
5	Population Class 2	2542.543	4695.008	4221.544	924.0816	3014.215	1312.411	4815.443	4939.91	682.7657	1670.815	3088.332	839.6357	2608.249	3244.957	4156.899
6	Population Class 3	2553.858	4379.714	973.8214	4524.405	3556.079	4005.073	2734.029	188.6943	3606.137	3493.729	4297.212	4893.403	483.6501	4001.653	4016.822
7	Population Class 4	4088.139	2750.782	1129.609	4898.742	1108.734	146.1014	2605.679	4425.84	533.8093	989.0491	4027.447	3563.472	4090.743	2268.989	302.3559
8	Population Class 5	3974.157	3112.375	853.5402	2194.35	587.0883	4644.271	1157.972	4566.434	3268.787	152.7047	2883.608	2502.358	4087.735	2161.958	1996.289
9	Population Class 6	3221.591	2935.224	1138.321	555.5961	1483.379	3651.654	2444.489	3980.919	2470.87	3720.371	914.6123	2355.442	3612.198	4126.569	2634.379
10	Population Class 7	1893.047	1038.711	2178.493	1290.323	1593.892	2443.045	3120.3	493.5614	3895.259	2500.112	1199.66	298.0943	749.3272	417.3491	2083.997
11	Population Class 8	4057.902	1506.232	1555.511	2043.599	2120.834	2892.625	3395.678	1309.356	3575.185	2399.611	4432.56	3409.86	3298.026	665.855	3284.299
12		Limiting Population vectors														
13	Population Class 0	388192.4	388192.4	388192.4	388192.4	388192.4	388192.4	388192.4	388192.4	388192.4	388192.4	388192.4	388192.4	388192.4	388192.4	388192.4
14	Population Class 1	11037584	11037584	11037584	11037584	11037584	11037584	11037584	11037584	11037584	11037584	11037584	11037584	11037584	11037584	11037584
15	Population Class 2	1187950	1187950	1187950	1187950	1187950	1187950	1187950	1187950	1187950	1187950	1187950	1187950	1187950	1187950	1187950
16	Population Class 3	103528	103528	103528	103528	103528	103528	103528	103528	103528	103528	103528	103528	103528	103528	103528
17	Population Class 4	7554.574	7554.574	7554.574	7554.574	7554.574	7554.574	7554.574	7554.574	7554.574	7554.574	7554.574	7554.574	7554.574	7554.574	7554.574
18	Population Class 5	472.728	472.728	472.728	472.728	472.728	472.728	472.728	472.728	472.728	472.728	472.728	472.728	472.728	472.728	472.728
19	Population Class 6	25.82288	25.82288	25.82288	25.82288	25.82288	25.82288	25.82288	25.82288	25.82288	25.82288	25.82288	25.82288	25.82288	25.82288	25.82288
20	Population Class 7	1.248516	1.248516	1.248516	1.248516	1.248516	1.248516	1.248516	1.248516	1.248516	1.248516	1.248516	1.248516	1.248516	1.248516	1.248516
21	Population Class 8	0.062708	0.062708	0.062708	0.062708	0.062708	0.062708	0.062708	0.062708	0.062708	0.062708	0.062708	0.062708	0.062708	0.062708	0.062708

Figure 7: 15 initial populations and their respective limiting populations of case 3 of conjecture 1, all initial conditions lead to the same limiting vector to seven significant figures.

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