DIVERGENCE FUNCTION OF THE BRAIDED THOMPSON GROUP

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Abstract. We prove that the braided Thompson group BV has a linear divergence function. By the work of Druţu, Mozes, and Sapir, this leads none of asymptotic cones of BV has a cut-point.

1. Introduction

R. Thompson groups F, T, and V are defined by Richard Thompson in 1965. These groups have many interesting properties. For instance, F is the first example of a torsion-free group of type F_∞ but not of type F, by Brown and Geoghegan [6]. T and V are also group of type F_∞, by Brown [5], and known as the first examples of infinite simple group with finite presentation, by Thompson. These groups have been studied using not only algebra but also analysis and geometry.

On the other hand, various "Thompson-like" groups have been considered to study the relationship with Thompson groups and their own interesting properties. In this paper, we focus on the generalization of V, braided Thompson group BV (sometimes this group is written as V_{br}). This group is defined independently by Brin [4] and Dehornoy [11]. It is known that BV has similar properties to those of V. For instance, Brin [3] showed BV is finitely presented, where the generators and relations are similar to those of V, and Bux, Fluch, Marschler, Witzel, and Zaremsky [9] proved that this group is also of type F_∞. Zaremsky [17] suggests the relationship between BV and metric spaces being CAT(0) or hyperbolic.

Golan and Sapir [15] showed that Thompson groups F, T, and V have linear divergence functions. Roughly speaking, the divergence function of a finitely generated group G is a function given by the length of the path connecting two points at the same distance from the origin while avoiding a small ball with the center at the origin in the Cayley graph. Gersten [14] introduced divergence of connected geodesic metric spaces as collections of such functions. We focus on each function rather than a collection, since it corresponds to the topological characterization of the asymptotic cones of the group [12]. Golan and Sapir posed a question whether their proof can be extended to Thompson-like groups. In this paper, we give a partial answer to this question.

Theorem 1.1. *Braided Thompson group BV has a linear divergence function.*

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This paper is organized as follows. In Section 2, we summarize definitions of Thompson groups, braid groups, braided Thompson group, and in Section 3, we define the divergence functions of finitely generated groups. In Section 4, first we prepare some lemmas on the number of carets of elements in BV. Then we construct a path which satisfies the requirement for the definition of the divergence function. This path is connecting two points g in BV and the point $v(|g|)$ in $F \subset BV$ which only depend the word length of g. This is achieved in the following way: For g, we construct the element h (denoted by $w_1w_2w_3$ in Section 4) in BV such that gh and $v(|g|)$ are commute. Then, we move $g \to gh \to ghv(|g|) = v(|g|)gh \to v(|g|)$. We remark that the above paths do not work for elements having less than three carets. For those elements g, we consider gx_1, a multiplication by a generator x_1, instead of g itself.

It is also interesting to study divergence functions of other Thompson-like groups, similar to BV. For example, Brady, Burillo, Cleary, and Stein [2] defined BF (sometimes denoted by F_{br}), which is braided version of Thompson group F. Acora and Cumplido [1] defined a family of infinitely braided Thompson's groups, which contains BV as a special case. Another example is the Higman-Thompson groups, for instance.

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2. Background

2.1. **Finitely generated groups and binary words.**

A group G is said to be *finitely generated* if there exists a subset X such that every element of G can be written as a product of finitely many elements in $X \cup X^{-1}$, where $X^{-1} := \{x^{-1} \mid x \in X\}$. We call such a product a *word* in X. We use "$\equiv$" and "$=$" to express equalities as words in X and as elements of G, respectively. Let $x \equiv x_1x_2\cdots x_n$ be a word in X. A word x' is said to be a *prefix* of x, denoted by $x' \leq x$, if $x' \equiv \emptyset$ or $x' \equiv x_1\cdots x_k$ for some $1 \leq k \leq n$, where $\emptyset$ denote the empty word. A word x' is said to be a *strict prefix* of x if x' is a prefix of x and $x' \not\equiv x$.

Let w be a finite character string that consists of 0 and 1. We call such a character string *binary word* and we also use "$\equiv$" to express equality. By the similar way above, we define a *prefix* and *strict prefix* of w. For every two binary words $w_1 \equiv a_1a_2\cdots a_j$ and $w_2 \equiv b_1b_2\cdots b_k$ where a_i and $b_i \in \{0, 1\}$ for every i, w_1w_2 denotes the concatenation $a_1a_2\cdots a_jb_1b_2\cdots b_k$.

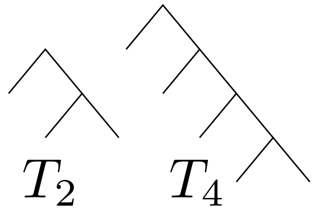

FIGURE 1. Examples of T_n

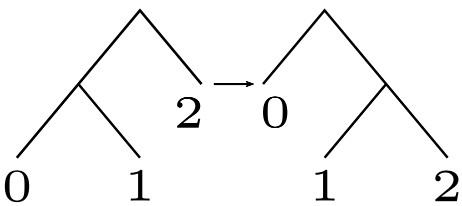

FIGURE 2. $(T_+(x_0), \sigma, T_-(x_0))$

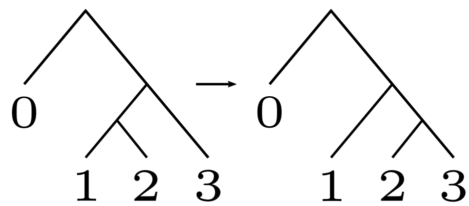

FIGURE 3. $(T_+(x_1), \sigma, T_-(x_1))$

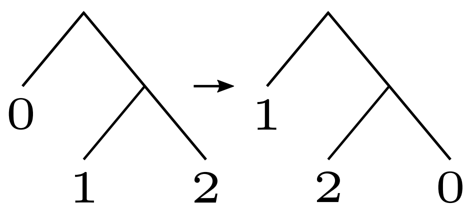

FIGURE 4. $(T_+(c_1), \sigma, T_-(c_1))$

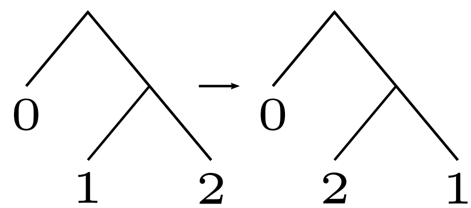

FIGURE 5. $(T_+(\pi_1), \sigma, T_-(\pi_1))$

2.2. **Thompson groups.** A *rooted binary tree* is a tree with a distinguished vertex (*root*) which has 2 edges, and any other vertex has either degree 1 (*leaves*) or degree 3. We think of a rooted binary tree as a descending tree with the root on top, and different level of vertices, with the root being the unique vertex of level 0. We define a *caret* of a rooted binary tree to be a subtree of the tree that consists a vertex together with two downward-directed edge. We write *all-right tree* T_n for the rooted binary tree that is constructed by attaching a caret to the right edge of a caret n times. Thus T_1 is a caret. See Figure 1. The number of carets play important role to estimate the word lengths of elements of Thompson groups.

Let (T_+, σ, T_-) be a triplet where T_+ and T_- be finite rooted binary trees with n caret, L be the set of $(n+1)$ leaves and σ be a permutation of L. We order the leaves of T_+ and T_- from left to right from 0 to n, respectively and use the numbers to represent the permutation σ. We call this *tree diagram.* For example, see Figure 2, 3, 4 and 5.

Let (T_+, σ, T_-) be the above tree diagram. We define a reduction of carets of a tree diagram as follows. We assume that two leaves i, $i+1$ have the same parent in T_+, two leaves $\sigma(i)$, $\sigma(i+1)$ have the same parent in T_-, and $\sigma(i+1) = \sigma(i) + 1$ holds. In that case, each pair of the leaves forms carets. Then, we get the trees T'_+ and T'_- by removing those carets. We regard the roots of the above carets as new leaves of the new trees, and

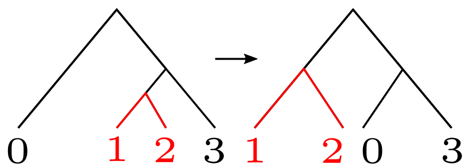

FIGURE 6. Before the reduction

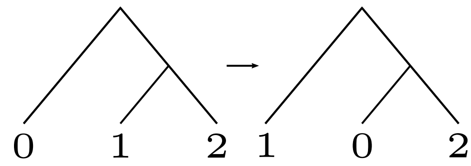

FIGURE 7. After the reduction and reordering the leaves

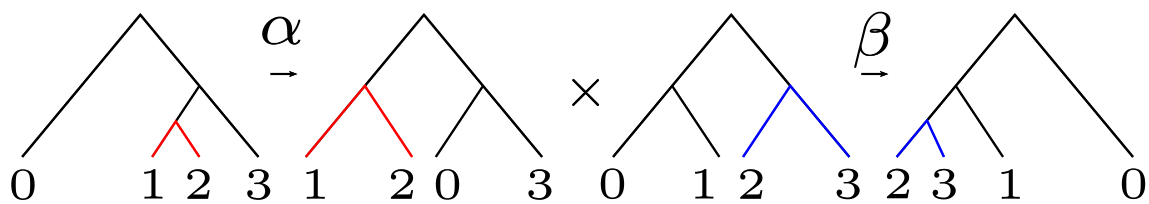

FIGURE 8. Diagrams of a and b with (colored) carets attached

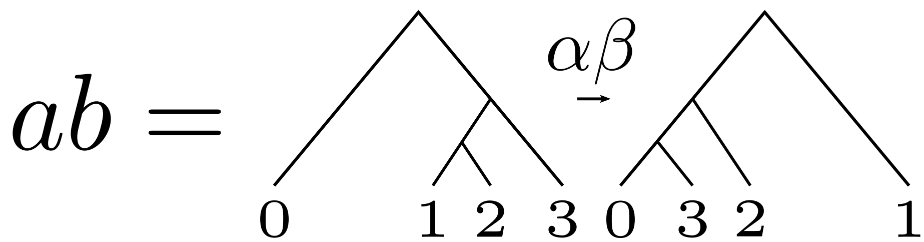

FIGURE 9. The product of the tree diagrams

we write i_+ and i_- for the new leaves of T'_+ and T'_-, respectively. By sending i_+ to i_- and sending other leaves by σ, we also get the permutation σ' on the set of n leaves. This operation and its inverse are called *reduction* and *attachment* of carets, respectively. For example, see Figure 6 and 7.

Using these operations, we define the equivalence relation on the set of tree diagrams as follows. Two tree diagrams (T_+, σ, T_-) and (T'_+, σ', T'_-) are equivalent if (T_+, σ, T_-) is obtained from (T'_+, σ', T'_-) by a finite number of reductions and attachments. The Thompson group V consists of all equivalence classes of tree diagrams. The product on V is defined in the following way.

For every two elements $a, b \in V$ represented by tree diagrams (A_+, α, A_-) and (B_+, β, B_-), by successive attachments of carets, we get diagrams (A'_+, α', A'_-) and (B'_+, β', B'_-) representing the same elements and such that $A'_- = B'_+$. Then the product $ab \in V$ is the equivalence class of $(A'_+, \alpha'\beta', B'_-)$, where the permutation $\alpha'\beta'$ is composed from left to right. For example, see Figure 8 and 9.

The group T is a subgroup of V consists of equivalence classes of tree diagrams (T_+, σ, T_-) where σ is a cyclic permutation, and the group F is a subgroup of T consists of equivalence classes of tree diagrams (T_+, σ, T_-) where σ is the identity.

For every caret of a rooted binary tree, we label its left edge by 0 and the right edge by 1. Since every leaf o of such a tree T corresponds to a unique path $s_T(o)$ from the root to the leaf, every leaf o corresponds to a binary word $\ell_T(o)$ labeling the path from the root to o. We identify the path $s_T(o)$ with the binary word $\ell_T(o)$.

By identifying the Cantor set C with the set of infinite binary words, we can associate each tree diagram (T_+, σ, T_-) to a homeomorphism from C to itself. Indeed, for every leaf o of T_+ and infinite binary word w, by mapping $\ell_{T_+}(o)w$ to $\ell_{T_-}(\sigma(o))w$, we get a homeomorphism. Hence, V is a subgroup of the homeomorphism group of C.

See [10] for details of the properties of Thompson groups.

2.3. **Braid groups.** Let $n \in \mathbb{N}$. We briefly review the definition of geometric braid groups B_n. See [16, Section 1.2] for details. Let I be the closed interval $[0,1] \subset \mathbb{R}$. We call a topological space which is homeomorphic to I *topological interval*.

DEFINITION 2.1 ([16, Definition 1.4]). A *geometric braid* on n strings is a set $b \subset \mathbb{R}^2 \times I$ formed by n disjoint topological intervals called the *strings* of b such that the projection $\mathbb{R}^2 \times I \to I$ maps each string homeomorphically onto I and

$$b \cap (\mathbb{R}^2 \times \{0\}) = \{(0,0,0), (1,0,0), \dots, (n-1,0,0)\},$$
$$b \cap (\mathbb{R}^2 \times \{1\}) = \{(0,0,1), (1,0,1), \dots, (n-1,0,1)\}.$$

We assume that every string goes from the bottom to up.

By the definition, every string of b meets each plane $\mathbb{R}^2 \times \{t\}$ with $t \in I$ in exactly one point and connects a point $(i,0,0)$ to a point $(\sigma(i),0,1)$, where σ is a permutation of $\{0,1,\dots,n-1\}$. We call the both points *endpoints* of the string, and call σ *underlying permutation* of the braid.

DEFINITION 2.2 ([16]). Two geometric braids b and b' on n strings are *isotopic* if there exists a continuous map $F\colon b \times I \to \mathbb{R}^2 \times I$ such that for each $s \in I$, the map $F_s\colon b \to \mathbb{R}^2 \times I; x \mapsto F(x,s)$ is an embedding whose image is a geometric braid on n strings, $F_0 = \mathrm{Id}\colon b \to b$, and $F_1(b) = b'$. Both the map F and the family of geometric braids $\{F_s(b)\}_{s \in I}$ are called an *isotopy* of b to b'.

The relation of isotopy is an equivalence relation on the class of geometric braids on n strings. We call the equivalence classes and each string of an equivalence class *braid* (on n strands) and *strand*, respectively. We write B_n for the set of braids on n strands.

For every two geometric braids b_1 and b_2, we define their product $b_1 b_2$ to be the set of points $(x,y,t) \in \mathbb{R}^2 \times I$ such that

$$(x,y,2t) \in b_1 \text{ if } 0 \le t \le \tfrac{1}{2},$$

and

$$(x,y,2t-1) \in b_2 \text{ if } \tfrac{1}{2} \le t \le 1.$$

It is clear that if b_1 and b_2 are isotopic to geometric braids b_1' and b_2', respectively, then the product b_1b_2 is isotopic to the product $b_1'b_2'$. Hence the product of B_n is defined by the equivalence class of products of geometric braids.

A braid can be projected onto $\mathbb{R} \times \{0\} \times I$ along the second coordinate with "crossing information" at each crossing point. Indeed, if necessary, by appropriate isotopies, we can assume that the number of intersections of strands at each point of the image of projection is at most two, every two strands meet transversely at each intersection point of the two strands, and there are only a finite number of such intersections. We call the intersections *crossing points*. For each crossing point, the one with the lesser y-coordinate is denoted by *over crossing*, and the other is denoted by the corresponding *under crossing*. Then, we draw each over crossing by a continuous line, and each under crossing by a broken line. For example, Figure 19 is the projection of the element in B_6. In this paper, we identify braids with projected braids equipped with crossing information.

We introduce an operation for braids which we use to define the product of elements of braided Thompson group.

Definition 2.3 (splitting). Let $0 \leq i \leq n-1$. Let B be a braid on n strands, $\{b_k \mid k = 0, 1, \ldots, n-1\}$ be the set of strands of B, σ be the underlying permutation of B, and $(k, 0, 0)$ and $(\sigma(k), 0, 1)$ be endpoints of each strand b_k. We define a braid on $(n+1)$ strands B' to be the following: B' is obtained by adding strand b_i' to B such that it satisfies the following:

(1) Endpoints of b_i' are $(i + 1/2, 0, 0)$ and $(\sigma(i) + 1/2, 0, 0)$, then shift all endpoints appropriately.
(2) The strand b_i' does not cross with b_i.
(3) The strand b_i' intersects with strands other than b_i in the same way that b_i intersects with strands in braid B.

In other words, B' is a braid such that b_i' is to the right of b_i and B is obtained from B' by removing b_i.

We say that b_i and b_i' are *parallel*, and we call B' the *splitting of the strand* b_i.

For example, see Figure 10.

2.4. **Braided Thompson group.**

Elements of Thompson groups V can be seen as pairs of finite rooted binary trees, with permutations from leaves to itself. Roughly speaking, by replacing permutations with braids, we get elements of BV.

Let T_+ and T_- be finite rooted binary trees with n carets and br be a braid on $n+1$ strands from bottom to up. (T_+, br, T_-) denotes a diagram where the leaves of both trees are joined by the braid with T_+ positioned upside down. We call this *tree-braid-tree diagram*. For example, see Figure 11.

Let (T_+, br, T_-) be the above tree-braid-tree diagram. Similar to Thompson groups, we define a reduction of carets of a tree-braid-tree diagram as follows. We assume that two

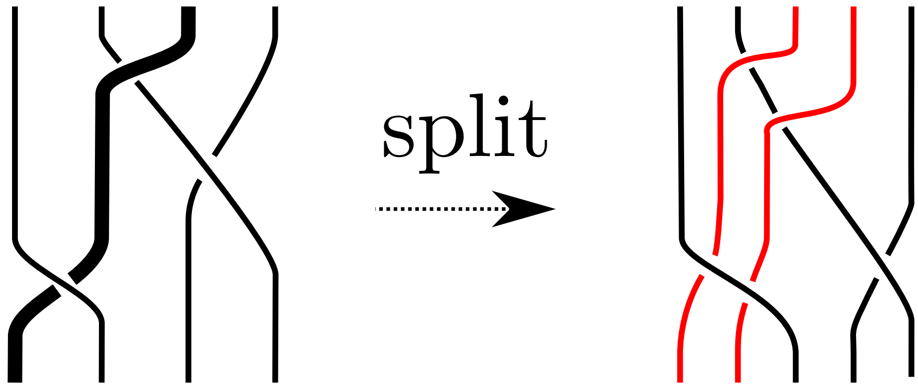

FIGURE 10. An example of splitting of the strand

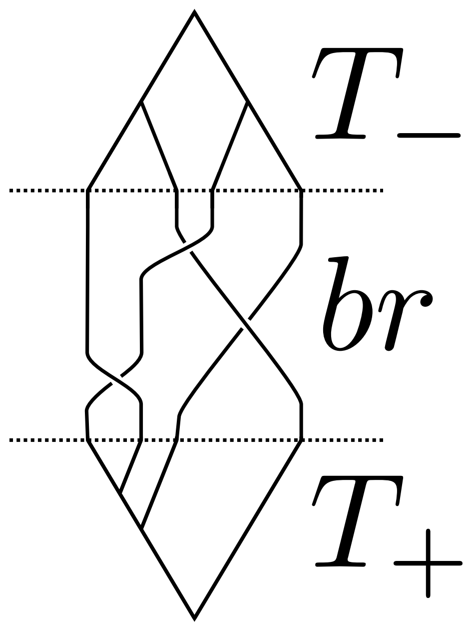

FIGURE 11. An example of tree-braid-tree diagram

braids b_i and b_i' are parallel (cf. Definition 2.3) and each endpoints of b_i and b_i' have the same parent in T_+ and T_-. In that case, each pair of the endpoints (leaves) forms carets. Then, we get the trees T_+' and T_-' by removing those carets. We regard the roots of the above carets as new leaves of the new trees, and we write i_+ and i_- for the new leaves of T_+' and T_-', respectively. By removing the strand b_i', letting the endpoints of b_i be the new leaves, and keeping the other strands, we also get the braid br' on n strands from br. This operation and its inverse operation are called *reduction* of carets and *splitting* of a strand, respectively. For example, see Figure 12.

Using these operations, we define the equivalence relation on the set of tree-braid-tree diagrams as follows. Two tree-braid-tree diagrams (T_+, br, T_-) and (T_+', br', T_-') are equivalent if (T_+, br, T_-) is obtained from (T_+', br', T_-') by finite number of reductions and

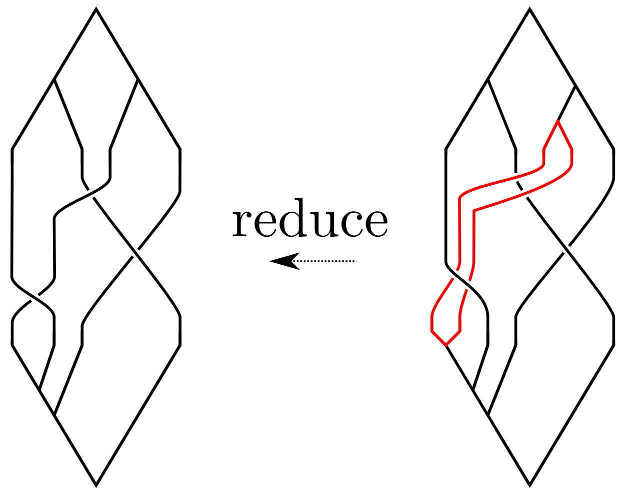

FIGURE 12. An example of reduction

splittings. Each equivalence class has a unique representative with minimal number of carets. We call this diagram a *reduced* tree-braid-tree diagram.

The braided Thompson group BV consists of all equivalence classes of tree-braid-tree diagrams. The product on BV is defined in the following way.

For every two elements $a, b \in BV$ represented by tree-braid-tree diagrams (A_+, br_A, A_-) and (B_+, br_B, B_-), by successive splittings of strands, we get diagrams (A'_+, br'_A, A'_-) and (B'_+, br'_B, B'_-) representing the same elements and such that $A'_- = B'_+$. Hence br'_A and br'_B are braids from the same braid group. Then the product $ab \in BV$ is the equivalence class of $(A'_+, br'_A br'_B, B'_-)$, where $br'_A br'_B$ is the braid that br'_A and br'_B connected from the bottom to the top, in this order. Figure 13 shows an example of a multiplication of elements of BV.

It is known that BV has the following infinite presentation.

THEOREM 2.4 ([2, Theorem 2.4]). *The group BV admits a presentation with generators:*

- x_i, *for* $i \geq 0$,
- σ_i, *for* $i \geq 1$,
- τ_i, *for* $i \geq 1$.

and relators

A $x_j x_i = x_i x_{j+1}$, *for* $j > i$
B1 $\sigma_i \sigma_j = \sigma_j \sigma_i$, *for* $j - i \geq 2$
B2 $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
B3 $\sigma_i \tau_j = \tau_j \sigma_i$ *for* $j - i \geq 2$
B4 $\sigma_i \tau_{i+1} \sigma_i = \tau_{i+1} \sigma_i \tau_{i+1}$
C1 $\sigma_i x_j = x_j \sigma_i$, *for* $i < j$
C2 $\sigma_i x_i = x_{i-1} \sigma_{i+1} \sigma_i$
C3 $\sigma_i x_j = x_j \sigma_{i+1}$, *for* $i \geq j + 2$

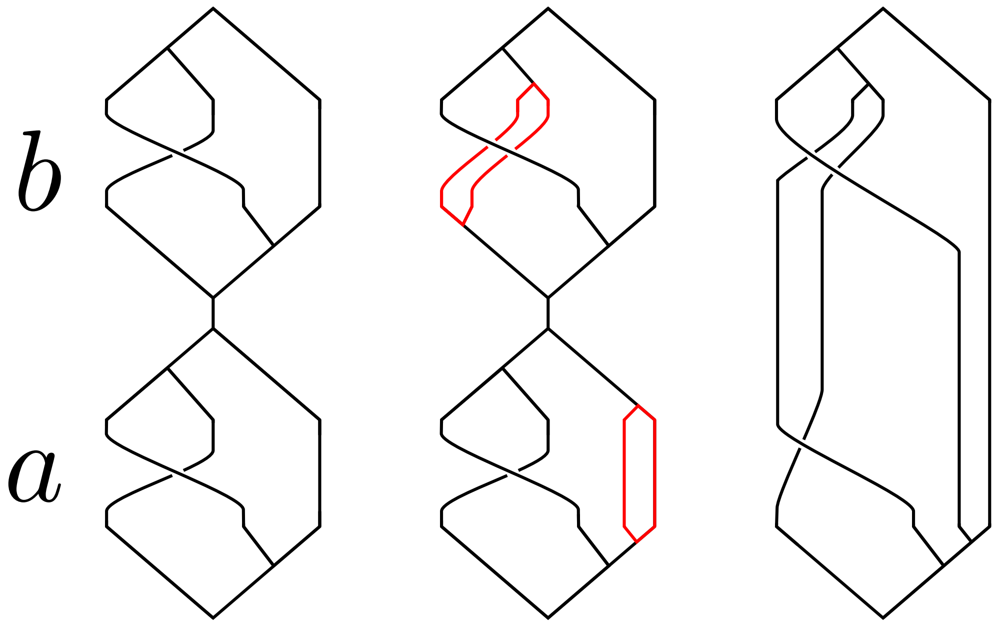

FIGURE 13. An example of the product ab in BV

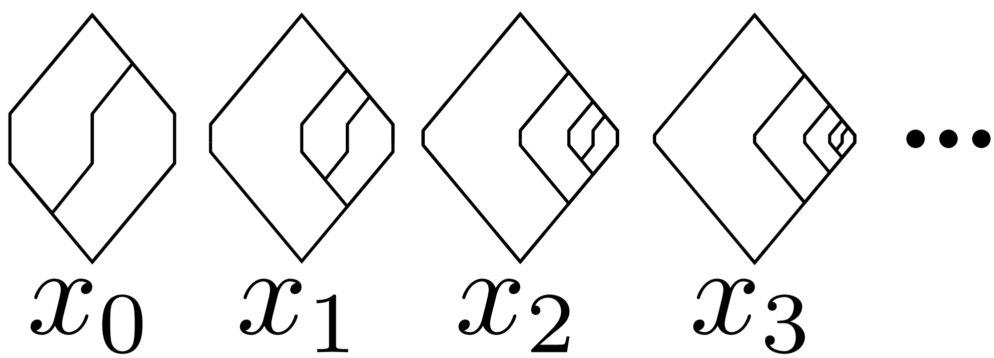

FIGURE 14. The infinite generators x_i

C4 $\sigma_{i+1}x_i = x_{i+1}\sigma_{i+1}\sigma_{i+2}$
D1 $\tau_i x_j = x_j \tau_{i+1}$, *for* $i - j \geq 2$
D2 $\tau_i x_{i-1} = \sigma_i \tau_{i+1}$
D3 $\tau_i = x_{i-1}\tau_{i+1}\sigma_i$.

The reduced diagrams of generators x_i are in Figure 14. The reduced diagrams of generators σ_i and τ_i are in Figure 15. We note that a set of the generators x_i corresponds to the standard infinite generating set of Thompson group F. Indeed, each x_i is regarded as two rooted binary trees and identical permutation (see Figure 2 and 3). Hence, BV contains F as a subgroup. Incidentally, in some papers, Thompson groups are defined by "tree-permutation-tree diagrams" similar to tree-braid-tree diagrams.

Moreover, it is also known that BV has the following finite presentation.

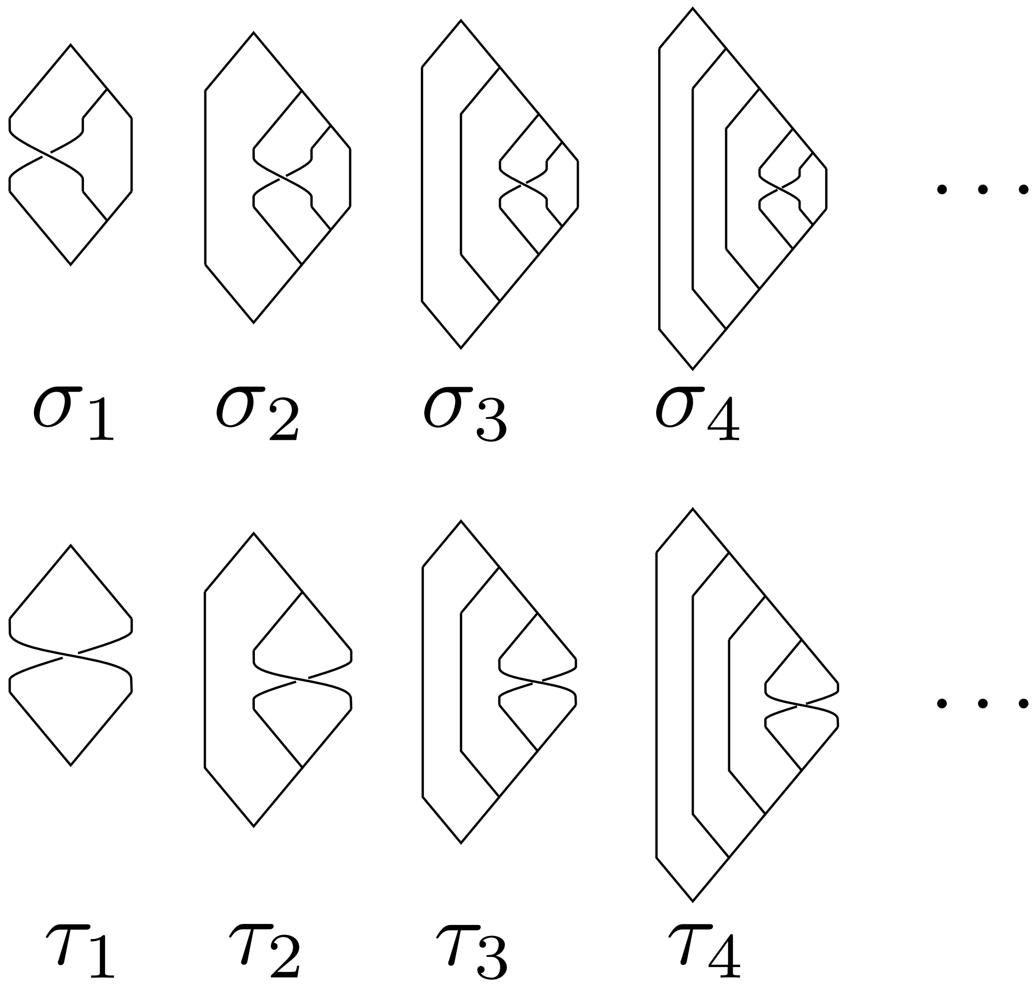

FIGURE 15. The infinite generators σ_i and τ_i

THEOREM 2.5 ([2, Theorem 3.1]). *The group BV admits a finite presentation with generators $x_0, x_1, \sigma_1, \tau_1$ and relators*

a $x_2x_0 = x_0x_3,\ x_3x_1 = x_1x_4$

c1 $\sigma_1x_2 = x_2\sigma_1,\ \sigma_1x_3 = x_3\sigma_1,\ \sigma_2x_3 = x_3\sigma_2,\ \sigma_2x_4 = x_4\sigma_2$

c3 $\sigma_2x_0 = x_0\sigma_3,\ \sigma_3x_1 = x_1\sigma_4$

c4 $\sigma_1x_0 = x_1\sigma_1\sigma_2,\ \sigma_2x_1 = x_2\sigma_2\sigma_3$

d1 $\tau_2x_0 = x_0\tau_3,\ \tau_3x_1 = x_1\tau_4$

d2 $\tau_1x_0 = \sigma_1\tau_2,\ \tau_2x_1 = \sigma_2\tau_3$

b1 $\sigma_1\sigma_3 = \sigma_3\sigma_1$

b2 $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

b3 $\sigma_1\tau_3 = \tau_3\sigma_1$

b4 $\sigma_1\tau_2\sigma_1 = \tau_2\sigma_1\tau_2$

where the letters in the relators not in the set of 4 *generators are defined inductively by* $x_{i+2} = x_i^{-1}x_{i+1}x_i$ *for* $i \geq 0$, $\sigma_{i+1} = x_{i-1}^{-1}\sigma_i x_i \sigma_i^{-1}$ *for* $i \geq 1$, *and* $\tau_{i+1} = x_{i-1}^{-1}\tau_i\sigma_i^{-1}$ *for* $i \geq 1$.

We call $\{x_0, x_1, \sigma_1, \tau_1\}$ *standard generating set* of BV.

As well as Thompson groups, for every caret of a rooted binary tree, we label its left edge by 0 and the right edge by 1. Since every leaf o of such a tree T corresponds to a unique path $s_T(o)$ from the root to the leaf, every leaf o corresponds to a binary word $\ell_T(o)$ labeling the path from the root to o. We identify the path $s_T(o)$ with the word $\ell_T(o)$. The path $\ell_T(o)$ will be called a *branch* of T. Let (T_+, br, T_-) be a tree-braid-tree diagram of $g \in BV$, o be a leaf of T_+, and o' be the corresponding leaf of T_-. We say that $\ell_{T_+}(o) \to \ell_{T_-}(o')$ is a *branch* of the tree-braid-tree diagram (T_+, br, T_-).

Let T be a rooted binary tree with n carets. Recall that T_n denotes an all-right tree. Then $(T, \mathrm{Id}, T_n) \in F$ is termed *positive element*. Because there exist $0 \leq i_1 < i_2 < \cdots < i_k$ and $r_1, r_2, \ldots, r_k > 0$ such that

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_k}^{r_k} = (T, \mathrm{Id}, T_n)$$

holds (see [10]), where each x_{i_t} is given by a diagram in Figure 14. Similarly, $(T_n, \mathrm{Id}, T) \in F$ is termed *negative element*. Since every element in F is rewritten as a product of positive element and negative element, we call the product *normal form*. Then we call the positive element and the negative element in normal form *positive part* and *negative part*, respectively.

3. Divergence functions of finitely generated groups

Let G be a finitely generated group, X be a finite generating set of G, and Γ be the Cayley graph $Cay(G, X)$. We will define the divergence functions of G. Since we consider asymptotic behavior of functions, we introduce a relation on the set of functions $\mathbb{R}_+ \to \mathbb{R}_+$ as follows. For such f and g, we define $f \preceq g$ if

$$f(x) \leq Ag(Bx + C) + Dx + E$$

for some $A, B, C, D, E \geq 0$ and all x. This defines an equivalence relation on the set of functions $\mathbb{R}_+ \to \mathbb{R}_+$, by saying $f \approx g$ if $f \preceq g$ and $g \preceq f$. We note that all linear functions and constant functions are equivalent.

Let $\delta \in (0, 1)$. Then the δ-*divergence function* of Γ is the smallest function $f_\delta(x)$ such that every two vertices of Γ at distance x from the identity can be connected by a path in Γ of length less than $f_\delta(x)$ and avoiding the ball of radius δx with a center at the identity. If no such path exists, take $f_\delta(x) = \infty$. For each $\delta \in (0, 1)$, the equivalence class of $f_\delta(x)$ is invariant under quasi-isometries, especially, it does not depend on the choice of finite generating set X. Hence the δ-*divergence function* of G is defined to be the equivalence class of the δ-divergence function of Γ.

Definition 3.1. We say that the group G has a *linear divergence function* if there exists $\delta \in (0, 1)$ such that the δ-divergence function of G is equivalent to a linear function.

By definition, if f_δ is equivalent to a linear function, then for every $0 < \delta' < \delta$, $f_{\delta'}$ is equivalent to a linear function. Indeed, since a path that avoids the ball of radius δx also avoids the ball of radius $\delta' x$, $f_\delta(x) \geq f_{\delta'}(x)$ holds for every x.

Druţu, Mozes and Sapir showed that having a linear divergence function is equivalent to the following topological property of asymptotic cones.

THEOREM 3.2 ([12,13]). *The following are equivalent.*

(1) *G has a linear divergence function.*
(2) *For every $\delta \in (0, \frac{1}{54})$, f_δ is equivalent to a linear function.*
(3) *None of asymptotic cones of G has a cut-point.*

4. Proof of Theorem 1.1

4.1. **Number of carets for elements of BV.** Let $X = \{x_0, x_1, \sigma_1, \tau_1\}$ be the standard generating set of BV. For an element $g \in BV$, $|g|$ denotes the word length of g with respect to the generating set X, and $N(g)$ denotes the number of carets in one of the trees in the reduced tree-braid-tree diagram of g. We will use the following estimate.

THEOREM 4.1 ([7, Theorem 3.6]). *For an element g of BV in tree-braid-tree diagram with k total crossings, there exists a constant C_1 for which the word length satisfies the following inequalities:*

$$C_1 \max\{N(g), \sqrt[3]{k}\} \leq |g|.$$

Here we can assume that $0 < C_1 < 1$ without loss of generality.

Let $g \in BV$ with a reduced tree-braid-tree diagram $(T_+(g), br, T_-(g))$. We call $T_+(g)$ the *domain-tree* of g, $T_-(g)$ the *range-tree* of g and br the *braid* of g. Let T be a rooted binary tree. Then, $\ell_0(T)$ denotes the length of the left most branch of T, that is, $\ell_0(T) = \ell$ if and only if 0^ℓ is a branch of T, where we define

$$i^\ell \equiv \underbrace{i \cdots i}_{\ell},$$

for $i = 0, 1$. Similarly, $\ell_1(T)$ denotes the length of the right most branch of T. For an element $g \in BV$, we define $\ell_i(g) := \ell_i(T_-(g))$, $i = 0, 1$.

We will need the following lemmas. Although the proofs of them are almost the same as in [15], we write down the proofs for reader's convenience. Note that the definition of $N(g)$ in this paper is different from that of $\mathcal{N}(g)$ in [15]. The former denotes the number of carets and the latter the number of leaves, respectively.

The following lemma corresponds to [15, Lemma 2.2].

LEMMA 4.2. *Let g be an element in BV with reduced tree-braid-tree diagram $(T_+(g), br, T_-(g))$ and assume that $N(g) \geq 3$. Then*

$$N(g) - 1 \leq N(gx_0) \leq N(g) + 1.$$

In addition,

(1) *If* $\ell_0(g) = 1$ *then* $N(gx_0) = N(g) + 1$ *and* $\ell_0(gx_0) = 1$.
(2) *If* $\ell_0(g) \neq 1$ *then* $N(gx_0) = N(g)$ *or* $N(gx_0) = N(g) - 1$. *Moreover,* $\ell_0(gx_0) = \ell_0(g) - 1$.
(3) *If* $\ell_0(g) \neq 1$ *and either* 1 *or* 01 *is a strict prefix of some branch of* $T_-(g)$*, then* $N(gx_0) = N(g)$ *and* 1 *is a strict prefix of some branch of* $T_-(gx_0)$*, where* $T_-(gx_0)$ *is a range-tree of reduced tree-braid-tree diagram of* gx_0.

Proof. We start by proving part (2). Assume that $\ell_0(g) \neq 1$. To multiply g by x_0, we replace the reduced tree-braid-tree diagram of x_0 by an equivalent tree-braid-tree diagram (R_+, Id, R_-) where $R_+ = T_-(g)$. Then $(T_+(g), br, R_-)$ is a tree-braid-tree diagram of the product gx_0. Let $u \to wv$ is a branch of $(T_+(g), br, T_-(g))$ where $w \equiv 00$, $w \equiv 01$, or $w \equiv 1$. By the construction of (R_+, Id, R_-), $u \to w'v$ is a branch of $(T_+(g), br, R_-)$ where $w' \equiv 0$, $w' \equiv 10$, or $w' \equiv 11$, respectively. All branchs of $(T_+(g), br, R_-)$ can be written in this way.

If $(T_+(g), br, R_-)$ is reduced diagram, then all assertions of part (2) hold, by the relation of $w \equiv 00$ and $w' \equiv 0$. Indeed, it follows from the domain-tree that $N(gx_0) = N(g)$. Moreover, since the reduced diagram of x_0 has a branch $00 \to 0$, $\ell_0(R_-) = \ell_0(R_+) - 1 = \ell_0(T_-(g)) - 1 = \ell_0(g) - 1$. Hence, we can assume that $(T_+(g), br, R_-)$ is not reduced, that is, this diagram has a pair of branches $x0 \to y0$ and $x1 \to y1$ such that corresponding strands are parallel. Then $y \equiv 1$. Indeed, if y is an empty word, R_- has a branch 1. This contradicts the fact that $\ell_1(x_0) \neq 1$. If 0 is a prefix of y, then the diagram $(T_+(g), br, T_-(g))$ of g has the pair of branches $x0 \to 0y0$ and $x1 \to 0y1$ such that corresponding strands are parallel, in contradiction to $(T_+(g), br, T_-(g))$ being reduced. If 10 or 11 is a prefix of y, the assumption that $(T_+(g), br, T_-(g))$ is reduced yields a contradiction in a similar way. Hence, $y \equiv 1$. So R_- has branches of form 10 and 11. Now, we reduce the carets corresponding to $x0 \to 10$ and $x1 \to 11$ of the diagram $(T_+(g), br, R_-)$. Then the obtained diagram $(T'_+(g), br', R'_-)$ of gx_0 has a branch $x \to 1$ and this diagram is reduced. Indeed, if not reduced, there exists a pair of branches $x'0 \to y'0$ and $x'1 \to y'1$ such that corresponding strands are parallel. If y' is not an empty word, 0 is a prefix of y'. This contradicts with the same way as in branches $y0$ and $y1$. If y' is an empty word, then the obtained diagram has a branch $x'0 \to 0$. Since $N(g) \geq 3$ and R_- has branches 10 and 11, 0 is a strict prefix of some branch of R_-. Hence, 0 is a strict prefix of some branch of R'_-. This is a contradiction. Since the reduction is replacing branches 10 and 11 with 1, we have $\ell_0(R'_-) = \ell_0(R_-)$. Hence, part (2) holds.

In the conditions of part (3) of the lemma, the tree-braid-tree diagram $(T_+(g), br, R_-)$ of gx_0 is reduced. Indeed, since 1 or 01 is a strict prefix of some branch of $T_-(g)$, by the relation of w and w', either 11 or 10 is a strict prefix of some branch of R_-. It follows that $(T_+(g), br, R_-)$ is reduced, because, if not, as noted above, R_- has branches both 11 and 10. In particular, 1 is a strict prefix of some branch of R_-. Hence, part (3) holds.

Now we assume the condition of part (1). To multiply g by x_0, we replace the tree-braid-tree diagram $(T_+(g), br, T_-(g))$ by an equivalent diagram $(T'_+(g), br', T'_-(g))$ by attaching carets to the leaf of the branch 0 of $T_-(g)$ and to the corresponding leaf of $T_+(g)$, and splitting of the corresponding strand. Let (R_+, Id, R_-) be the tree-braid-tree diagram of x_0 such that $R_+ = T'_-(g)$. Then we get the diagram $(T'_+(g), br', R_-)$ of gx_0 and we can proceed as part (2). To complete the proof it suffices to prove that $(T'_+(g), br', R_-)$ is reduced. Indeed, in that case, by the construction, $N(gx_0) = N(g) + 1$ and since $\ell_0(R_+) = 2$, we have $\ell_0(gx_0) = \ell_0(R_-) = 1$. If $(T'_+(g), br', R_-)$ is not reduced, there exists a pair of branches $x0 \to y0$ and $x1 \to y1$ such that corresponding strands are parallel. By the construction, the left most and second from the left branches of $T'_-(g)$ are 00 and 01. Hence the left most and second from the left branches of R_- are 0 and 10. This means y is not empty and 0 and 10 are not prefixes of y. Then we first assume that 11 is a prefix of y and let $y \equiv 11y'$ (y' is probably an empty word). From the construction of R_-, $(T'_+(g), br', T'_-(g))$ has a pair of branch $x0 \to 1y'0$ and $x1 \to 1y'1$ such that corresponding strands are parallel. On the other hand, 00 and 01 are only branches of $T'_-(g)$ that can be reduced. This is a contradiction. Finally, we assume that $y \equiv 1$. Then, (R_+, Id, R_-) is the reduced tree-braid-tree diagram of x_0. Since $R_+ = T'_-(g)$ and $N(x_0) = 2$ (cf. Figure 14) hold, by the construction of $T'_-(g)$, $N(g) = 2 - 1 = 1$. This contradicts the assumption of the lemma. Hence, part (1) holds. □

The following corollary corresponds to [15, Corollary 2.3]. The proof given here is slightly modified.

Corollary 4.3. *Let g be an element in BV with a reduced tree-braid-tree diagram $(T_+(g), br, T_-(g))$ such that $N(g) \geq 3$. Let $\ell := \ell_0(g)$. Then the following assertions hold.*

(1) *If $N(g) \geq 3 + (\ell - 1)$ then for every $i \geq 0$ we have*
$$N(gx_0^i) \geq N(g) + i - 2(\ell - 1).$$

(2) *If either 1 or 01 is a strict prefix of some branch of $T_-(g)$ then for every $i \geq 0$ we have*
$$N(gx_0^i) = \max\{N(g), N(g) + i - (\ell - 1)\}.$$

Proof. To prove part (1), we first assume that $\ell = 1$. By applying Lemma 4.2 (1) to g iteratively, we have that for every $i \geq 0$,
$$N(gx_0^i) = N(g) + i.$$

Thus, we can assume that $\ell > 1$. Since $N(g) \geq 3 + (\ell - 1)$, we can apply Lemma 4.2 (2) to g at least $(\ell - 1)$ times. Then we have $\ell_0(gx_0^{\ell-1}) = 1$ and for every $i \leq \ell - 1$
$$N(gx_0^i) \geq N(g) - i \geq N(g) - (\ell - 1) \geq N(g) + i - 2(\ell - 1). \tag{4.1}$$

Since $N(gx_0^{\ell-1}) \geq 3$ and $\ell_0(gx_0^{\ell-1}) = 1$, by applying Lemma 4.2 (1) to $gx_0^{\ell-1}$ iteratively, we have

$$N(gx_0^{\ell-1}x_0^j) = N(gx_0^{\ell-1}) + j \geq N(g) - (\ell-1) + j, \tag{4.2}$$

for every $j \geq 0$. By substituting $j = i - (\ell - 1)$ in inequality (4.2), we have that for every $i \geq \ell - 1$,

$$N(gx_0^i) \geq N(g) + i - 2(\ell-1). \tag{4.3}$$

It follows from inequalities (4.1) and (4.3) that for every $i \geq 0$,

$$N(gx_0^i) \geq N(g) + i - 2(\ell-1),$$

as required.

In the condition of part (2), we first assume that $\ell = 1$, again. By Lemma 4.2 (1) (applying iteratively), we have

$$N(gx_0^i) = N(g) + i = \max\{N(g), N(g) + i\},$$

for every $i \geq 0$. Thus, we can assume that $\ell > 1$. Since $N(g) \geq 3$ and either 1 or 01 is a strict prefix of some branch of $T_-(g)$, by Lemma 4.2 (2) and (3), $N(gx_0^i) = N(g) \geq N(g) + i - (\ell-1)$ for every $i \in \{0, \dots, \ell-1\}$ and $\ell_0(gx_0^{\ell-1}) = 1$. Thus, it suffices to prove that for every $i \geq \ell - 1$ we have $N(gx_0^i) = N(g) + i - (\ell-1) \geq N(g)$. Since $\ell_0(gx_0^{\ell-1}) = 1$ and $N(gx_0^{\ell-1}) = N(g) \geq 3$, by Lemma 4.2 (1), we have

$$N(gx_0^{\ell-1}x_0^j) = N(gx_0^{\ell-1}) + j = N(g) + j, \tag{4.4}$$

for every $j \geq 0$. Substituting $j = i - (\ell-1)$ in inequality (4.4) gives that for every $i \geq \ell - 1$,

$$N(gx_0^i) = N(g) + i - (\ell-1),$$

as required. □

The proofs of the following lemma and corollary are symmetric to those of Lemma 4.2 and Corollary 4.3. We only need to switch 0 and 1.

Lemma 4.4. *Let g be an element in BV with reduced tree-braid-tree diagram $(T_+(g), br, T_-(g))$ and assume that $N(g) \geq 3$. Then*

$$N(g) - 1 \leq N(gx_0^{-1}) \leq N(g) + 1.$$

In addition,

1. *If $\ell_1(g) = 1$ then $N(gx_0^{-1}) = N(g) + 1$ and $\ell_1(gx_0^{-1}) = 1$.*
2. *If $\ell_1(g) \neq 1$ then $N(gx_0^{-1}) = N(g)$ or $N(gx_0^{-1}) = N(g) - 1$. Moreover, $\ell_1(gx_0) = \ell_1(g) - 1$.*

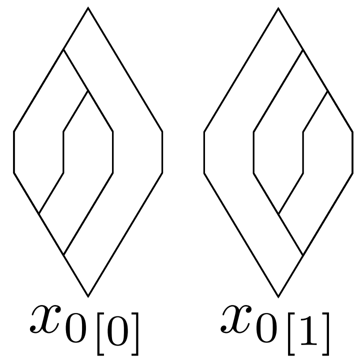

FIGURE 16. The reduced tree-braid-tree diagrams of $x_{0[0]}$ and $x_{0[1]}$

(3) *If $\ell_1(g) \neq 1$ and either 0 or 10 is a strict prefix of some branch of $T_-(g)$, then $N(gx_0^{-1}) = N(g)$ and 0 is a strict prefix of some branch of $T_-(gx_0^{-1})$, where $T_-(gx_0^{-1})$ is a range-tree of reduced tree-braid-tree diagram of gx_0^{-1}.*

COROLLARY 4.5. *Let g be an element in BV with reduced tree-braid-tree diagram $(T_+(g), br, T_-(g))$ such that $N(g) \geq 3$. Let $\ell := \ell_1(g)$. Then the following assertions hold.*

(1) *If $N(g) \geq 3 + (\ell - 1)$ then for every $i \geq 0$ we have*

$$N(gx_0^{-i}) \geq N(g) + i - 2(\ell - 1).$$

(2) *If either 0 or 10 is a strict prefix of some branch of $T_-(g)$ then for every $i \geq 0$ we have*

$$N(gx_0^{-i}) = \max\{N(g), N(g) + i - (\ell - 1)\}.$$

The next lemma describes the result of multiplying an element of BV on the right by an element of F with a following specific form. Let u be a finite binary non-empty word and $h \in F$ be an non-identity element with reduced tree-braid-tree diagram $(T_+(h), \mathrm{Id}, T_-(h))$. Let T be a minimal finite rooted binary tree which contains the branch u. We take two copies of the tree T. To the first copy, we attach the tree $T_+(h)$ at the end of the branch u, and we write R_+ for this tree. To the second copy, we attach the tree $T_-(h)$ at the end of the branch u, and we write R_- for this tree. Then the element $h_{[u]}$ is the one represented by the tree-braid-tree diagram, where domain-tree is R_+, range-tree is R_- and braid is the "identity", that is, all strands are straight. It is clear from the definition that $h_{[u]} \in F < BV$. For example, $x_{0[0]}$ and $x_{0[1]}$ are elements corresponding to the diagrams in Figure 16 or 17. Note that $x_0^2 x_1^{-1} x_0^{-1} = x_{0[0]}$ holds, see Figure 18.

The following lemma corresponds to [15, Lemma 2.6].

LEMMA 4.6. *Let $g \in BV$ be a non-identity element, $u \to v$ be a branch of g, h be a non-identity element of F. Let $h' = h_{[v]}$. Then*

$$N(gh') = N(g) + N(h).$$

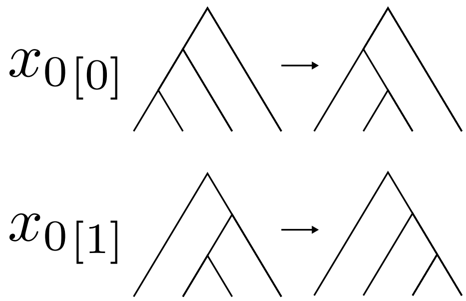

FIGURE 17. The reduced tree diagrams of $x_{0[0]}$ and $x_{0[1]}$

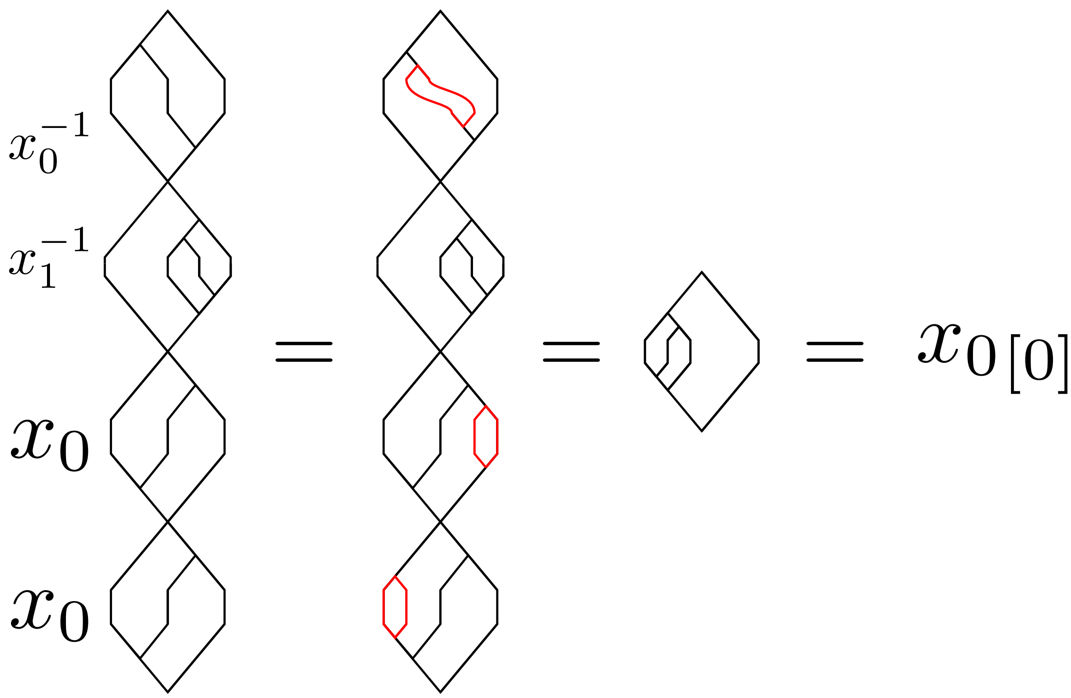

FIGURE 18. Calculation of $x_0^2 x_1^{-1} x_0^{-1} = x_{0[0]}$

Moreover, if h consists of branches $w_i \to z_i, i = 1, \ldots, k$ and B is the set of branches of g which are not equal to $u \to v$, then gh' consists of branches $uw_i \to vz_i, i = 1, \ldots, k$ along with all branches in B.

PROOF. Let $(T_+(g), br, T_-(g))$, $(T_+(h), \mathrm{Id}, T_-(h))$ and $(T_+(h'), \mathrm{Id}, T_-(h'))$ be the reduced tree-braid-tree diagrams of g, h and h', respectively. To multiply g by h', we note that the minimal refinement of $T_-(g)$ and $T_+(h')$ is the tree obtained from $T_-(g)$ by attaching the tree $T_+(h)$ at the bottom of the branch v, since $T_-(g)$ has a branch v and $T_+(h')$ is constructed from the minimal tree which has a branch v. Let S denote the described tree and (R_1, br', S) be an equivalent tree-braid-tree diagram of g. We note that R_1 is obtained from $T_+(g)$ by attaching a copy of $T_+(h)$ to the bottom of the branch u. If (S, Id, R_2) is a tree-braid-tree diagram of h', we also note that R_2 can be obtained from

$T_-(g)$ by attaching $T_-(h)$ at the bottom of the branch v. The product of the tree-braid-tree diagrams (R_1, br', S) and (S, Id, R_2) is (R_1, br', R_2). Since (R_1, br', S) has branches $uw_i \to vw_i$ and branches in B ($x \to y$ denotes these one), and (S, Id, R_2) has branches $vw_i \to vz_i$ and $y \to y$, it follows that (R_1, br', R_2) has branches $uw_i \to vz_i$ and $x \to y$. To finish the proof, it remains to prove that (R_1, br', R_2) is reduced. Since $(T_+(h), \mathrm{Id}, T_-(h))$ is reduced, $(T_+(h), \mathrm{Id}, T_-(h))$ has no pair of branches of the form $p0 \to q0$ and $p1 \to q1$ where $p0, p1 \equiv w_i$ for some i, respectively and $q0, q1 \equiv z_i$ for some i, respectively. Hence, (R_1, br', R_2) has no pair of branches of the form $up0 \to vq0$ and $up1 \to vq1$, that is, (R_1, br', R_2) has no pair of the branches of the form $uz_i \to vq_i$ such that reducible. Similarly, since $(T_+(g), br, T_-(g))$ is reduced, (R_1, br', R_2) has no pair of branches of the form $x \to y$ such that reducible.

Recall that R_1 is obtained from $T_+(g)$ by attaching a copy of $T_+(h)$. Then it is clear that $N(gh') = N(g) + N(h)$ holds, and the proof is complete. □

4.2. **Construction of the path.** If w is a word over the alphabet X, $\|w\|$ denotes the length of w. Note that any word w over the alphabet X can be regarded as an element of BV, then we have $|w| \leq \|w\|$.

Remark 4.7. Golan-Sapir constructed a path between elements whose number of carets is greater than or equal to three in [15, Proposition 2.7]. Linear divergence of Thompson groups F, T, V follow immediately from this path. However, in the case of the braided Thompson group BV, we need a little more discussion. Because the number of $g \in BV$ such that $N(g) \leq 2$ is infinite. For example, $\tau_1, \tau_1^2, \tau_1^3, \dots$ all have one caret.

First, we consider elements in BV whose number of carets are greater than or equal to three (Proposition 4.8). Next, we construct paths between elements whose number of carets are less than three and others.

The following proposition corresponds to [15, Proposition 2.7]. In [15], they constructed the path that consists of five subpaths, subpath 1, …, subpath 5. In this paper, we will take a similar process, but our subpath 3 (and therefore also the path w) is different from the original one. Our subpath 3 does not work for Thompson group T, but an almost similar approach works for Thompson groups F and V.

Proposition 4.8. *There exist constants $\delta, D > 0$ and a positive integer Q such that the following holds. Let $g \in BV$ be an element with $N(g) \geq 3$. Then there exists a path of length at most $D|g|$ in the Cayley graph $\Gamma = \mathrm{Cay}(BV, X)$ which avoids a $\delta|g|$-neighborhood of the identity and which has initial vertex g and terminal vertex $x_0^{Q|g|} x_1^{-1} x_0^{-Q|g|+1}$.*

In other words, there exists a word w in the alphabet X such that $\|w\| < D|g|$; for any prefix w' of w, we have $|gw'| > \delta|g|$ and such that

$$gw = x_0^{Q|g|} x_1^{-1} x_0^{-Q|g|+1}.$$

Proof. Let C_1 be the constant from Theorem 4.1. We give 5 subwords $w_1, \ldots, w_5$ and then let $w \equiv w_1 \cdots w_5$. Let $(T_+(g), br, T_-(g))$ be the reduced tree-braid-tree diagram of g.

Subpath 1. *If 0 is not a branch of $T_-(g)$ we let $w_1 \equiv \emptyset$ and let $g_1 = g$. Otherwise, we let $w_1 \equiv x_0^2 x_1^{-1} x_0^{-1}$ and let $g_1 = gw_1$.*

Let $(T_+(g_1), br_1, T_-(g_1))$ be the reduced tree-braid-tree diagram of g_1. The following lemma corresponds to [15, Lemma 2.8].

Lemma 4.9. *We have that 0 is not a branch of $T_-(g_1)$. Moreover, $N(g) \leq N(g_1) \leq N(g) + 2$ and for every prefix w' of w_1 we have $N(gw') \geq N(g)$.*

Proof. If 0 is not a branch of $T_-(g)$ then $g_1 = g$ and $w_1 \equiv \emptyset$, so the lemma holds. Thus, we can assume that 0 is a branch of $T_-(g)$. Let u be the binary word such that $(T_+(g), br, T_-(g))$ has the branch $u \to 0$. We recall that $w_1 = x_0^2 x_1^{-1} x_0^{-1} = x_{0[0]}$ (cf. Figure 18). Hence, by Lemma 4.6, $uv_1 \to 0v_2$ is a branch of the reduced tree-braid-tree diagram of $g_1 = gw_1 = gx_{0[0]}$ for each branch $v_1 \to v_2$ of x_0. Therefore, 0 is not a branch of $T_-(g_1)$ since it is a strict prefix of some branch.

For the second claim, by Lemma 4.6, we have

$$N(g_1) = N(gx_{0[0]}) = N(g) + N(x_0) = N(g) + 2.$$

For the last claim, we will consider the number of carets of gx_0, gx_0^2 and $gx_0^2 x_1^{-1}$. Since 0 is a branch of $T_-(g)$, we have $\ell_0(g) = \ell_0(T_-(g)) = 1$. Hence, by Lemma 4.2 (1), $N(gx_0) = N(g) + 1$ and $\ell_0(gx_0) = 1$. Again, by applying Lemma 4.2 (1) to gx_0, $N(gx_0^2) = N(gx_0) + 1 = N(g) + 2$. Finally, we note that $gx_0^2 x_1^{-1} = g_1 x_0$ and $N(g_1) = N(g) + 2$. By applying the inequality in Lemma 4.2 to g_1, we have

$$N(gx_0^2 x_1^{-1}) = N(g_1 x_0) \geq N(g_1) - 1 = N(g) + 1,$$

and the proof is complete. □

Subpath 2. *We fix an integer $M \geq 100/C_1$. Then we define a word w_2 by*

$$w_2 \equiv x_0^{-M(N(g_1)+1)} x_1 x_0^{M(N(g_1)+1)}$$

and we let $g_2 = g_1 w_2$.

Let $(T_+(g_2), br_2, T_-(g_2))$ be a reduced tree-braid-tree diagram of g_2. The following lemma corresponds to [15, Lemma 2.9].

Lemma 4.10. *The following assertions hold.*

(1) *For every prefix w' of w_2, we have $N(g_1 w') \geq N(g_1)$.*
(2) *$N(g_2) \geq MN(g_1)$.*

PROOF. We first prove part (2). It follows from the relation A in Theorem 2.4 that, as an element of BV, we have $w_2 = x_m$ where $m = M(N(g_1)+1)+1$. Let $\ell_1 = \ell_1(g_1)$ and u be a finite binary word such that $u \to 1^{\ell_1}$ is a branch of $(T_+(g_1), br_1, T_-(g_1))$. Considering a minimum tree-braid-tree diagram where some branch is 1^{ℓ_1}, $\ell_1 \leq N(g_1)$ is clear. We note that $x_m = x_{m-\ell_1[1^{\ell_1}]}$. Hence, by Lemma 4.6, we have

$$N(g_2) = N(g_1 w_2) = N(g_1 x_m) = N(g_1 x_{m-\ell_1[1^{\ell_1}]}) = N(g_1) + N(x_{m-\ell_1}).$$

By the definition of standard infinite generating set of F, we have $N(x_{m-\ell_1}) = m - \ell_1 + 2$. Hence,

$$N(g_2) = N(g_1) + m - \ell_1 + 2 = N(g_1) - \ell_1 + M(N(g_1)+1) + 3 \geq MN(g_1),$$

as $\ell_1 \leq N(g_1)$. Hence, part (2) holds.

Before proceeding the proof of part (1), we note that if $z_j \to q_j$, $j = 1, \ldots, n$ are the branches of $x_{m-\ell_1}$, then, by Lemma 4.6, the branches of the diagram $(T_+(g_2), br_2, T_-(g_2))$ of $g_2 = g_1 x_m$ are $uz_j \to 1^{\ell_1} q_j$, $j = 1, \ldots, n$ as well as all branches $a_k \to b_k$ which are branches of the diagram $(T_+(g_1), br_1, T_-(g_1))$, other than $u \to 1^{\ell_1}$. By Lemma 4.9, 0 is a strict prefix of some branch of $T_-(g_1)$. Hence, 0 is a strict prefix of some branch of $T_-(g_2)$.

Now, let w' be a prefix of w_2. Then either (a) $w' \equiv x_0^{-i}$ for some $0 \leq i \leq M(N(g_1)+1)$, or (b) $w' \equiv x_0^{-M(N(g_1)+1)} x_1 x_0^i$ for some $0 \leq i \leq M(N(g_1)+1)$.

By Lemma 4.9, $N(g_1) \geq N(g)$ and 0 is a strict prefix of some branch of $T_-(g_1)$. Hence, by Corollary 4.5 (2), for g_1 and any $i \geq 0$, $N(g_1 x_0^{-i}) \geq N(g_1)$. Hence, part (1) of the lemma holds for prefixes w' of type (a). Next, we consider the element $g_1 w'$ for w' of type (b). As an element of BV, we have

$$\begin{aligned} g_1 w' &= g_1 x_0^{-M(N(g_1)+1)} x_1 x_0^i \\ &= g_1 x_0^{-M(N(g_1)+1)} x_1 x_0^{M(N(g_1)+1)} \cdot x_0^{i-M(N(g_1)+1)} \\ &= g_1 x_m x_0^{i-M(N(g_1)+1)} \\ &= g_2 x_0^{i-M(N(g_1)+1)}. \end{aligned}$$

From the note above, 0 is a strict prefix of some branch of $T_-(g_2)$, and we have already shown that $N(g_2) \geq MN(g_1) \geq N(g)$. Hence, by Corollary 4.5 (2), for g_2 and any $s \geq 0$, we have

$$N(g_2 x_0^{-s}) \geq N(g_2) \geq MN(g_1).$$

We also note that $i - M(N(g_1)+1) \leq 0$. By substituting $-s = i - M(N(g_1)+1)$, we have

$$N(g_1 w') = N(g_2 x_0^{i-M(N(g_1)+1)}) \geq MN(g_1),$$

as required. $\square$

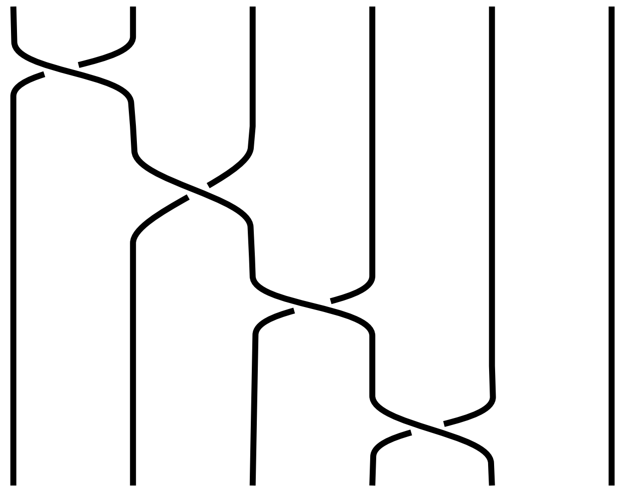

FIGURE 19. The braid of h for $N(g_1) = 5$ and $k = 4$

SUBPATH 3. *For reduced tree-braid-tree diagram* $(T_+(g_1), br_1, T_-(g_1))$ *of* g_1*, we assume that some strand of* br_1 *connects* 0*th leaf to* k*th leaf, where* $0 \leq k \leq N(g_1)$*. Let* h *be the element of* BV *given by the (maybe not reduced) tree-braid-tree diagram* $(T_-(g_1), br_h, T_+(g_1))$*, where* br_h *is following. If* $k > 0$*, the* k*th strand goes over the* $(k-1)$*-th strand,* $(k-2)$*-th strand,* $\dots$*,* 0*th strand, in order, and other strands are straight. If* $k = 0$*, all strands are straight. For example, Figure 19 illustrates the construction of braid for* $N(g_1) = 5$ *and* $k = 4$*. Now, we let* w_3 *be the minimal word over* X *such that* $h = w_3$ *in* BV *and let* $g_3 = g_2 w_3$*.*

Part (2) and (3) of the following lemma correspond to [15, Lemma 2.10].

LEMMA 4.11. *The following assertions hold.*

(1) $\|w_3\| \leq 14N(g_1)$.
(2) $N(g_3) \geq (M-1)N(g_1) + M + 3$.
(3) $\ell_0(g_3) \leq N(g_1) + 1$ *and* $0^{\ell_0(g_3)} \to 0^{\ell_0(g_3)}$ *is a branch of* g_3*.*

PROOF. Let $n = N(g_1)$. To prove part (1), we split $(T_-(g_1), br_h, T_+(g_1))$ into three diagrams by using all-right trees T_n, giving a split into a positive element $(T_-(g_1), \mathrm{Id}, T_n) \in F$, a braid element (T_n, br_h, T_n), and a negative element $(T_n, \mathrm{Id}, T_+(g_1)) \in F$, where T_n has n carets (recall Figure 1). Let p, Br_h and q be the minimal words over X such that $p = (T_-(g_1), \mathrm{Id}, T_n)$, $Br_h = (T_n, br_h, T_n)$ and $q = (T_n, \mathrm{Id}, T_+(g_1))$ in BV, respectively. We identify these words with elements of BV.

First, we prove that $|p| \leq 6n$ holds. Let $\mathcal{A}$ be the standard generating set of F such that $\mathcal{A} \subset X$ and we note that $|p|_{\mathcal{A}} \leq 6N(p)$ (see [8, Theorem 1 and Proposition 2]). We also note that tree-braid-tree diagram $(T_+(g_1), \mathrm{Id}, T_n)$ might not be reduced. Hence, we have

$$|p| \leq |p|_{\mathcal{A}} \leq 6N(p) \leq 6n.$$

Similarly, we have $|q| \leq 6n$.

Next, we prove that $|Br_h| \leq 2n$ holds. To get this upper bound, we rewrite the word Br_h by elements of infinite generator of BV. The following rewritings are obvious.

$$\begin{aligned}
k = n \quad &\Rightarrow Br_h = \tau_n \sigma_{n-1} \ldots \sigma_1, \\
k = n-1 &\Rightarrow Br_h = \sigma_{n-1} \ldots \sigma_1, \\
k = n-2 &\Rightarrow Br_h = \sigma_{n-2} \ldots \sigma_1, \\
&\vdots \\
k = 1 \quad &\Rightarrow Br_h = \sigma_1, \\
k = 0 \quad &\Rightarrow Br_h = \emptyset.
\end{aligned}$$

It suffices to consider only the case $k = n$, as we can get the following estimation. Indeed, for $n \geq 4$, then we have

$$\begin{aligned}
Br_h &= \tau_n \sigma_{n-1} \cdots \sigma_2 \sigma_1 \\
&= (x_0^{-(n-2)} \tau_2 x_0^{n-2})(x_0^{-(n-3)} \sigma_2 x_0^{n-3}) \cdots (x_0^{-1} \sigma_2 x_0)(\sigma_2 \sigma_1) \\
&= (x_0^{-(n-2)} \tau_2 x_0)(\sigma_2 x_0) \cdots (\sigma_2 x_0)(\sigma_2 \sigma_1) \\
&= (x_0^{-(n-2)} \sigma_1^{-1} \tau_1 x_0 x_0)(x_0^{-1} \sigma_1 x_1 \sigma_1^{-1} x_0) \cdots (x_0^{-1} \sigma_1 x_1 \sigma_1^{-1} x_0)(x_0^{-1} \sigma_1 x_1 \sigma_1^{-1} \sigma_1) \\
&= (x_0^{-(n-2)} \sigma_1^{-1} \tau_1 x_0)(\sigma_1 x_1 \sigma_1^{-1}) \cdots (\sigma_1 x_1 \sigma_1^{-1})(\sigma_1 x_1) \\
&= (x_0^{-(n-2)} \sigma_1^{-1} \tau_1 x_0) \sigma_1 x_1^{n-2},
\end{aligned}$$

where we rewrite $\tau_n = x_0^{-(n-2)} \tau_2 x_0^{n-2}$, $\sigma_i = x_0^{-(i-2)} \sigma_2 x_0^{i-2}$ for each $i \geq 3$, $\tau_2 = \sigma_1^{-1} \tau_1 x_0$, and $\sigma_2 = x_0^{-1} \sigma_1 x_1 \sigma_1^{-1}$ by the relations D1($j = 0$), C3($j = 0$), D2($i = 1$), and C2($i = 1$) in Theorem 2.4, respectively. Hence, we have

$$|Br_h| \leq n - 2 + 3 + 1 + n - 2 = 2n.$$

When $n = 3$, we have

$$\begin{aligned}
Br &= \tau_3 \sigma_2 \sigma_1 \\
&= (x_0^{-1} \tau_2 x_0)(\sigma_2 \sigma_1) \\
&= (x_0^{-1} \sigma_1^{-1} \tau_1 x_0 x_0)(x_0^{-1} \sigma_1 x_1) \\
&= x_0^{-1} \sigma_1^{-1} \tau_1 x_0 \sigma_1 x_1.
\end{aligned}$$

Hence, we have

$$Br_h = 6 = 2 \times 3 = 2n.$$

Therefore, we have

$$\|w_3\| \leq |p| + |Br_h| + |q| \leq 14n,$$

as required.

For part (2) and (3) we recall the form of branches of the reduced tree-braid-tree diagram $(T_+(g_2), br_2, T_-(g_2))$ of g_2 as described in the proof of Lemma 4.10. Let $\ell_1 = \ell_1(T_-(g_1))$ and let $m = M(N(g_1)+1)+1$. Then

$$g_2 = g_1 x_m = g_1 x_{m-\ell_1[1^{\ell_1}]}.$$

Now, let u be such that $u \to 1^{\ell_1}$ is a branch of $(T_+(g_1), br_1, T_-(g_1))$. If $z_j \to q_j$, $j = 1, \ldots, n$ are the branches of reduced tree-braid-tree diagram of $x_{m-\ell_1}$ then the branches of $(T_+(g_2), br_2, T_-(g_2))$ are $uz_j \to 1^{\ell_1} q_j$, $j = 1, \ldots, n$ as well as all the branches $a_k \to b_k$ of $(T_+(g_1), br_1, T_-(g_1))$, other than $u \to 1^{\ell_1}$.

Let v be such that $1^{\ell_1} \to v$ is a branch of the tree-braid-tree diagram $(T_-(g_1), br_h, T_+(g_1))$ of h. Then $uz_j \to vq_j$, $j = 1, \ldots, n$ are all branch of reduced tree-braid-tree diagram of $g_2 h$. Indeed, by the proof of Lemma 4.6, $T_-(g_2)$ is constructed by attaching the range-tree $T_-(x_{m-\ell_1})$ of reduced tree-braid-tree diagram of $x_{m-\ell_1}$ at the end of branch 1^{ℓ_1} of the $T_-(g_1)$, and so $T_-(g_1)$ is a rooted subtree of $T_-(g_2)$. Then to multiply g_2 by h, we replace $(T_-(g_1), br_h, T_+(g_1))$ by an equivalent tree-braid-tree diagram $(T_+(h), br'_h, T_-(h))$ where $T_+(h) = T_-(g_2)$. By construction, $1^{\ell_1} q_j \to vq_j$ are branches of $(T_+(h), br'_h, T_-(h))$. Hence,

$$(T_+(g_2), br_2, T_-(g_2)) \cdot (T_+(h), br'_h, T_-(h)) = (T_+(g_2), br_2 br'_h, T_-(h))$$

has branches $uz_j \to vq_j$, $j = 1, \ldots, n$. We recall that $z_j \to q_j$, $j = 1, \ldots, n$ are branches of a reduced tree-braid-tree diagram. Hence, we can not reduce the carets formed by the $uz_j \to vq_j$ of the tree-braid-tree diagram $(T_+(g_2), br_2 br'_h, T_-(h))$.

Since $\ell_1 \le N(g_1)$, we have

$$\begin{aligned} N(g_3) = N(g_2 h) &\ge N(x_{m-\ell_1}) \\ &= m - \ell_1 + 2 \\ &\ge m - N(g_1) + 2 \\ &= M(N(g_1)+1) + 1 - N(g_1) + 2 \\ &= (M-1)N(g_1) + M + 3, \end{aligned}$$

as required.

For part (3), let $r = \ell_0(T_+(g_1))$. We note that $r \le N(g_1)$ holds by the same reason as $\ell_1 \le N(g_1)$. Let s be a binary word such that $0^r \to s$ is a branch of $(T_+(g_1), br_1, T_-(g_1))$ of g_1. By the definition of br_h, $s \to 0^r$ is a branch of the diagram $(T_-(g_1), br_h, T_+(g_1))$ of h. Recall that $u \to 1^{\ell_1}$ is a branch of $(T_+(g_1), br_1, T_-(g_1))$ of g_1. We consider two cases: (a) $u \not\equiv 0^r$, and (b) $u \equiv 0^r$.

In case (a), $0^r \to s$ is a branch of $(T_+(g_2), br_2, T_-(g_2))$. Indeed, every branch of $(T_+(g_1), br_1, T_-(g_1))$ of g_1, other than $u \to 1^{\ell_1}$, is also a branch of $(T_+(g_2), br_2, T_-(g_2))$. Then since $(T_+(g_2), br_2, T_-(g_2))$ has the branch $0^r \to s$ and $(T_+(g_1), br_h, T_+(g_1))$ of h has

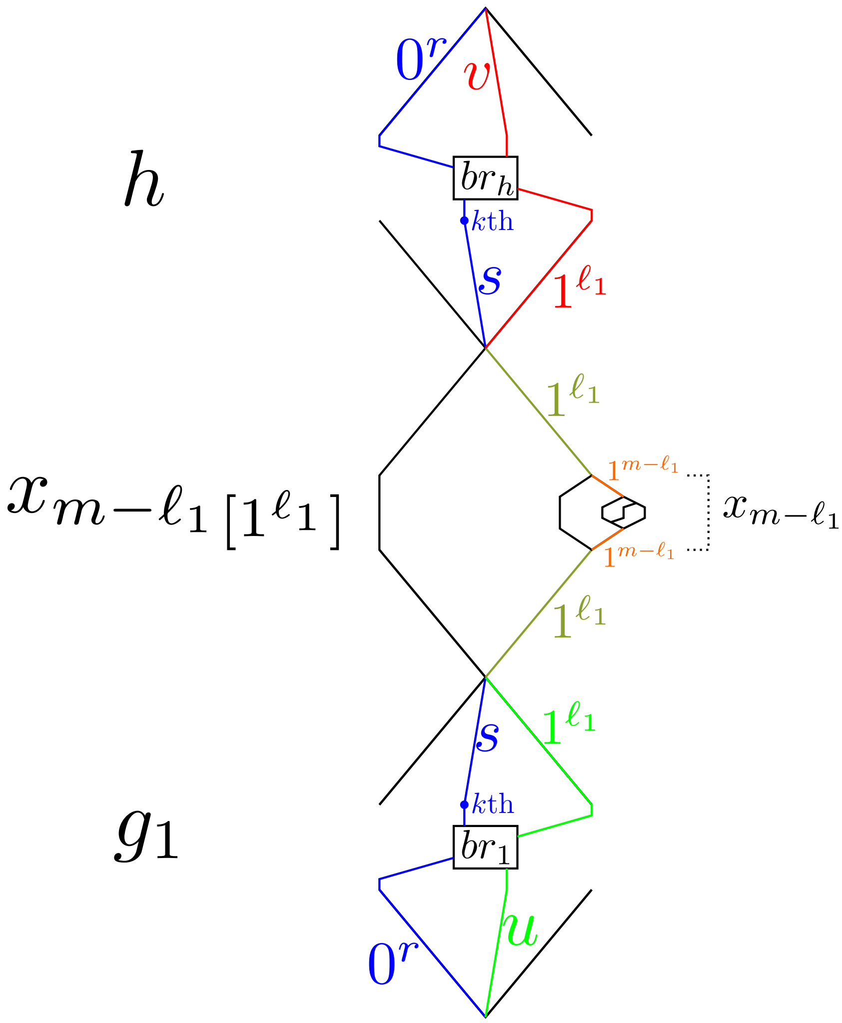

FIGURE 20. A rough sketch of a tree-braid-tree diagram of the product $g_1 x_{m-\ell_1[1^{\ell_1}]} h$ with only the important branches

the branch $s \to 0^r$, by adding some minimal number of caret if necessary, the tree-braid-tree diagram of the product $g_2 h$ has the branch $0^r \to 0^r$. Since this diagram might be not reduced, $\ell_0(g_3) \leq r$ holds. Hence, by $r \leq N(g_1)$ holds, we have $\ell_0(g_3) \leq N(g_1)$ and $0^{\ell_0(g_3)} \to 0^{\ell_0(g_3)}$ is reduced tree-braid-tree diagram of $g_2 h = g_3$, as required. We illustrate a sketch of the product $g_2 h$ in Figure 20.

In case (b), $u \equiv 0^r$, so $1^{\ell_1} \equiv s$. Since the diagram $(T_-(g_1), br_h, T_+(g_1))$ of h has the branches $1^{\ell_1} \to v$ and $s \to 0^r$, we have $v \equiv 0^r \equiv u$. Since the reduced tree-braid-tree diagram of $g_2 h = g_3$ has the branches $uz_j \to vq_j$, this diagram has the branches $0^r z_j \to 0^r q_j$. Since $m - \ell_1 > 0$ holds, $x_{m-\ell}$ has a branch $0 \to 0$. Hence, reduced tree-braid-tree diagram of g_3 has a branch $0^{r+1} \to 0^{r+1}$, as required. □

SUBPATH 4. *We fix an integer $Q \geq 12M/C_1^2$ and let*

$$w_4 \equiv x_0^{Q|g|} x_1^{-1} x_0^{-Q|g|+1}.$$

We also let $g_4 = g_3 w_4$. *We note that* w_4 *is a word representing the terminal vertex of Proposition* 4.8.

Part (1) of the following lemma corresponds to [15, Lemma 2.11].

LEMMA 4.12. *The following assertions hold.*

(1) *For every prefix* w' *of* w_4 *we have*

$$N(g_3 w') \geq N(g_3) + \frac{1}{2}\|w'\| - 2N(g_1) - 1.$$

(2) *As elements in* BV, g_3 *and* w_4 *commute.*

PROOF. To prove part (1), let $\ell = \ell_0(g_3)$. We first consider prefixes w' of w_4 which are positive power of x_0. We note that by Lemma 4.11 (2), 4.9 and 4.11 (3), we have

$$\begin{aligned} N(g_3) &\geq (M-1)N(g_1) + M + 3 \\ &\geq N(g_1) + N(g_1) + M + 3 \\ &\geq N(g) + \ell + M + 2 \\ &\geq 3 + \ell - 1. \end{aligned}$$

Hence, we can apply Corollary 4.3 (1) to g_3. Again we note that $\ell - 1 \leq N(g_1)$ holds by Lemma 4.11 (3). Then we have

$$\begin{aligned} N(g_3 w') = N(g_3 x_0^i) &\geq N(g_3) + i - 2(\ell - 1) \\ &\geq N(g_3) + i - 2N(g_1) \\ &= N(g_3) + \|w'\| - 2N(g_1) \\ &\geq N(g_3) + \frac{1}{2}\|w'\| - 2N(g_1). \end{aligned} \tag{4.5}$$

Thus, to finish the proof of part (1), it suffices to show that for every prefix w' of w_4 which contains the letter x_1^{-1}, we have

$$N(g_3 w') \geq N(g_3 x_0^{Q|g|}) - 1. \tag{4.6}$$

Indeed, in that case, by inequality (4.5),

$$\begin{aligned} N(g_3 w') &\geq N(g_3 x_0^{Q|g|}) - 1 \\ &\geq N(g_3) + Q|g| - 2N(g_1) - 1 \\ &= N(g_3) + \frac{1}{2}\|w_4\| - 2N(g_1) - 1 \\ &\geq N(g_3) + \frac{1}{2}\|w'\| - 2N(g_1) - 1, \end{aligned}$$

as desired. To show inequality (4.6), we first consider the following prefix

$$p \equiv x_0^{Q|g|} x_1^{-1} x_0^{-1} \equiv x_0^{Q|g|-2} \cdot x_0^2 x_1^{-1} x_0^{-1}.$$

Since $C_1 Q \geq 1200$ and $N(g) \geq 3$ hold, by Theorem 4.1, we note that we have

$$\frac{1}{C_1}|g| - Q|g| \leq \frac{|g|}{C_1} - \frac{1200|g|}{C_1} = \frac{-1199}{C_1}|g| \leq -1199N(g) < -5.$$

By Lemma 4.11 (3), 4.9 and Theorem 4.1, we have

$$\ell \leq N(g_1) + 1 \leq N(g) + 3 \leq \frac{1}{C_1}|g| + 3 < Q|g| - 2.$$

Since $N(g_3) \geq 3 + \ell - 1$ holds, by Lemma 4.2 (1) and (2) (if necessary, apply them repeatedly), we have $\ell_0(g_3 x_0^{Q|g|-2}) = 1$. In other words, the range-tree of the reduced tree-braid-tree diagram of $g_3 x_0^{Q|g|-2}$ has a branch 0. By applying Lemma 4.2 (1) to $g_3 x_0^{Q|g|-2}$ twice, we have

$$N(g_3 x_0^{Q|g|}) = N(g_3 x_0^{Q|g|-2} \cdot x_0^2) = N(g_3 x_0^{Q|g|-2} \cdot x_0) + 1 = N(g_3 x_0^{Q|g|-2}) + 2.$$

Since $x_0^2 x_1^{-1} x_0^{-1} = x_{0[0]}$ and $\ell_0(g_3 x_0^{Q|g|-2}) = 1$ hold, by Lemma 4.6, we have

$$\begin{aligned} N(g_3 p) &= N(g_3 x_0^{Q|g|-2} \cdot x_0^2 x_1^{-1} x_0^{-1}) \\ &= N(g_3 x_0^{Q|g|-2} \cdot x_{0[0]}) \\ &= N(g_3 x_0^{Q|g|-2}) + N(x_0) \\ &= N(g_3 x_0^{Q|g|}) - 2 + 2 \\ &= N(g_3 x_0^{Q|g|}), \end{aligned}$$

so inequality (4.6) holds for the prefix p. We also note that by Lemma 4.6, $\ell_0(g_3 p) \neq 1$ and $N(g_3 p) \geq 3$.

To finish the proof, it remains to prove that inequality (4.6) holds for the prefix (a) $w' \equiv x_0^{Q|g|} x_1^{-1}$ and prefixes of the form (b) $w' \equiv x_0^{Q|g|} x_1^{-1} x_0^{-i}$, for $1 < i \leq Q|g| - 1$. For the case (a), we note that $\ell_0(g_3 p) \neq 1$ and $g_3 w' = g_3 p x_0$. Hence, by applying Lemma 4.2 (2) to $g_3 p$, we have

$$N(g_3 w') = N(g_3 p x_0) \geq N(g_3 p) - 1 = N(g_3 x_0^{Q|g|}) - 1,$$

as required. Finally, we note again that $\ell_0(g_3 p) \neq 1$ and prefixes of the form (b) can be written as $w' \equiv p x_0^{-(i-1)}$. Hence, by Corollary 4.5 (2) we have

$$N(g_3 w') = N(g_3 p x_0^{-(i-1)}) \geq N(g_3 p) = N(g_3 x_0^{Q|g|}),$$

as required.

For part (2), we first note that reduced tree-braid-tree diagram of g_3 has a "same length" branch $0^\ell \to 0^\ell$. By calculating $x_0^i (x_0^2 x_1^{-1} x_0^{-1}) x_0^{-i}$, for $i = 1, 2, \ldots, Q|g| - 2$ inductively, we get tree-braid-tree diagram of w_4 as Figure 21. Since $\ell - 1 < Q|g| - 2$ holds, we can calculate $w_4 g_3$ and $g_3 w_4$ as Figure 22 and Figure 23, respectively. Hence, we have $w_4 g_3 = g_3 w_4$, as required. □

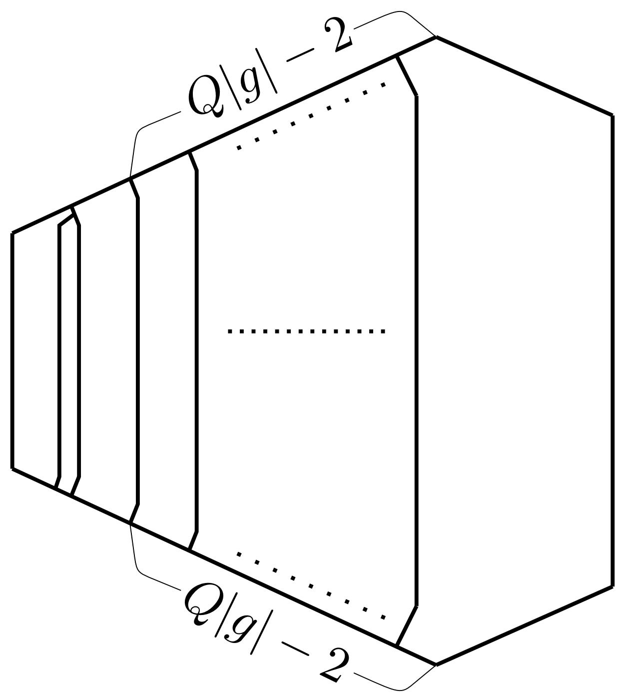

FIGURE 21. A tree-braid-tree diagram of w_4

SUBPATH 5. *Let w_5 be a minimal word in the alphabet X such that $w_5 = g_3^{-1}$ in BV. Let $g_5 = g_4 w_5$.*

It follows from Lemma 4.12 (2) that

$$gw = gw_1w_2w_3w_4w_5 = g_5 = g_3w_4g_3^{-1} = w_4.$$

Hence, $gw = x_0^{Q|g|}x_1^{-1}x_0^{-Q|g|+1}$ for $w \equiv w_1w_2w_3w_4w_5$, as required.

It remains to prove that one can choose constants δ, D (independently of g), so that path w satisfies the conditions in the Proposition 4.8. First, by definitions of subpaths, we have the following.

$$\begin{aligned}
\|w\| &\leq \|w_1w_2w_3\| + \|w_4\| + \|w_5\| \\
&= \|w_1w_2w_3\| + 2Q|g| + |g_3| \\
&= \|w_1w_2w_3\| + 2Q|g| + |gw_1w_2w_3| \\
&\leq \|w_1w_2w_3\| + 2Q|g| + |g| + \|w_1w_2w_3\| \\
&= 2\|w_1w_2w_3\| + (2Q+1)|g| \\
&\leq 2\|w_1w_2w_3\| + 3Q|g|.
\end{aligned}$$

Furthermore, we have a upper bound of $\|w_1w_2w_3\|$ as follows.

$$\begin{aligned}
\|w_1w_2w_3\| &\leq \|w_1\| + \|w_2\| + \|w_3\| \\
&\leq 4 + 2M(N(g_1)+1) + 1 + 14N(g_1)
\end{aligned}$$

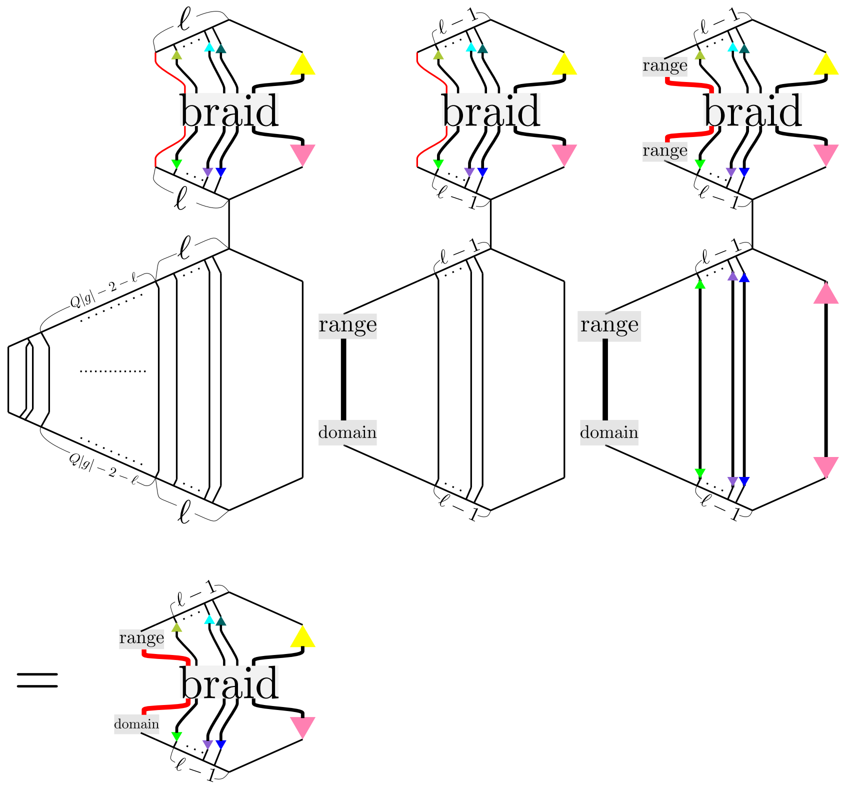

FIGURE 22. Calculation of $w_4 g_3$

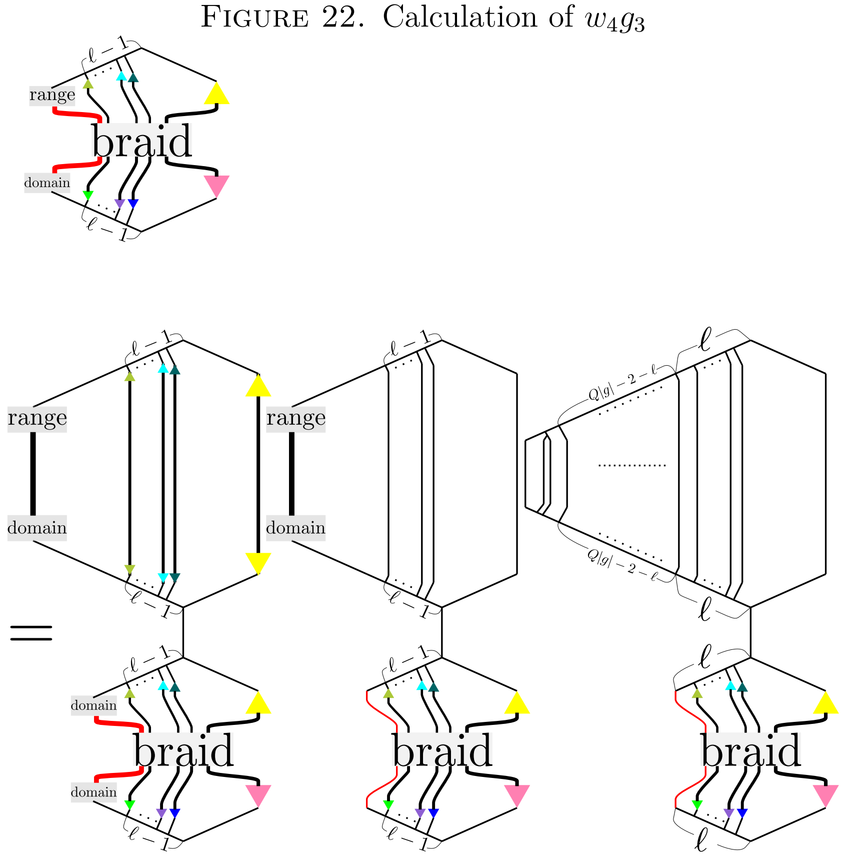

FIGURE 23. Calculation of $g_3 w_4$

$$
\begin{aligned}
&= 2MN(g_1) + 14N(g_1) + 5 + 2M \\
&\leq 2M(N(g)+2) + 14(N(g)+2) + 5 + 2M \\
&= 2MN(g) + 14N(g) + 33 + 6M \\
&< 2MN(g) + 14N(g) + 33N(g) + 2M \times 3 \\
&= 2MN(g) + 47N(g) + 2M \times 3 \\
&\leq 2MN(g) + MN(g) + 2MN(g) \\
&= 5MN(g) \qquad (4.7) \\
&\leq \frac{5M}{C_1}|g|, \qquad (3.7)'
\end{aligned}
$$

where these inequalities follow from the definition of the subpaths, Lemma 4.11 (1), 4.9, the definition of M and Theorem 4.1. Therefore, we have $\|w\| \leq D|g|$ where $D = 10M/C_1 + 3Q$, as required. Now, let $\delta = C_1/10M$. The following lemma corresponds to [15, Lemma 2.12] and completes the proof of Proposition 4.8.

Lemma 4.13. *Let w' be a prefix of w. Then $|gw'| > \delta|g|$.*

Proof. First, we note that by Lemma 4.11 (2),

$$N(g_3) \geq (M-1)N(g_1) + M + 3.$$

Then for each prefix $\tilde{w} \leq w_4$ we have by Lemma 4.12,

$$
\begin{aligned}
N(g_3\tilde{w}) &\geq N(g_3) + \frac{1}{2}\|\tilde{w}\| - 2N(g_1) - 1 \\
&\geq \frac{1}{2}\|\tilde{w}\| + (M-3)N(g_1) + M + 2 \\
&> \frac{1}{2}\|\tilde{w}\|. \qquad (4.8)
\end{aligned}
$$

We separate the proof into two cases depending on the length of g.

Case (1): $|g| < 10MN(g)$.

It follows from Lemma 4.9 and 4.10 (1) that for every prefix $w' \leq w_1w_2$, we have $N(gw') \geq N(g)$. Then, by applying Theorem 4.1 to gw', we have

$$|gw'| \geq C_1N(gw') \geq C_1N(g) > \frac{C_1}{10M}|g| = \delta|g|,$$

as required. Next, we consider a prefix $w' \leq w_3$. By Theorem 4.1, Lemma 4.10 (2) and $M \geq 100/C_1$,

$$|g_2| \geq C_1N(g_2) \geq C_1MN(g_1) \geq 100N(g_1).$$

Since we already know that $\|w_3\| \leq 14N(g)$ (Lemma 4.11 (1)), $N(g_1) \geq N(g)$ (Lemma 4.9) and $N(g) > |g|/10M$ (assumption of case (1)) hold, we have

$$|g_2w'| \geq |g_2| - \|w'\|$$

$$
\begin{aligned}
&\geq |g_2| - \|w_3\| \\
&\geq 100N(g_1) - 14N(g_1) \\
&= 86N(g_1) \\
&\geq 86N(g) \\
&> \frac{86}{10M}|g| \\
&> \frac{C_1}{10M}|g| = \delta|g|,
\end{aligned}
$$

as required. Now, let w' be a prefix of w_4. By Theorem 4.1,Lemma 4.12 (1), 4.11 (2), 4.9, and assumption of case (1), we have

$$
\begin{aligned}
|g_3w'| &\geq C_1N(g_3w') \\
&\geq C_1(N(g_3) - 2N(g_1) - 1) \\
&\geq C_1((M-1)N(g_1) + M + 3 - 2N(g_1) - 1) \\
&= C_1((M-3)N(g_1) + M + 2) \\
&> C_1N(g) \\
&> \frac{C_1}{10M}|g| = \delta|g|,
\end{aligned}
$$

as required. Finally, we consider a prefix $w' \leq w_5$. Since $\|w_1w_2w_3\| \leq (5M/C_1)|g|$ (inequality (3.7)$'$) and $Q \geq 12M/C_1^2$, we have

$$
\begin{aligned}
\|w_5\| = |g_3| = |gw_1w_2w_3| &\leq |g| + \|w_1w_2w_3\| \\
&\leq |g| + \frac{5M}{C_1}|g| \\
&< \frac{M}{C_1}|g| + \frac{5M}{C_1}|g| \\
&= \frac{6M}{C_1}|g| \\
&\leq \frac{C_1Q}{2}|g|.
\end{aligned}
$$

By Theorem 4.1, inequality (4.8) for $\tilde{w} = w_4$ (then $\|w_4\| = 2Q|g|$) and the definition of Q, we have

$$
\begin{aligned}
|g_4w'| \geq |g_4| - \|w'\| \geq |g_4| - \|w_5\| & \\
&> C_1N(g_4) - \frac{C_1Q}{2}|g| \\
&\geq C_1Q|g| - \frac{C_1Q}{2}|g| \\
&= \frac{C_1Q}{2}|g|
\end{aligned}
$$

$$> \frac{C_1}{10M}|g| = \delta|g|,$$

as required. Hence, the lemma holds in case (1).

Case (2): $|g| \geq 10MN(g)$.

Since $\|w_1w_2w_3\| \leq 5MN(g)$ (inequality (4.7)) and $|g|/2 \geq 5MN(g)$ (assumption of case (2)), for any prefix $w' \leq w_1w_2w_3$ we have

$$\begin{aligned}|gw'| &\geq |g| - \|w'\| \\ &\geq |g| - \|w_1w_2w_3\| \\ &\geq |g| - 5MN(g) \\ &\geq |g| - \frac{1}{2}|g| \\ &= \frac{1}{2}|g| \\ &> \frac{C_1}{10M}|g| = \delta|g|,\end{aligned}$$

as required. In particular, we note that $|g_3| \geq |g|/2$ where $g_3 = gw_1w_2w_3$. Let $w' \leq w_4$. If $\|w'\|\leq|g|/5$, then we have

$$|g_3w'| \geq |g_3| - \|w'\| \geq \frac{1}{2}|g| - \frac{1}{5}|g| = \frac{3}{10}|g| > \frac{C_1}{10M}|g| = \delta|g|,$$

as desired. Hence, we can assume that $\|w'\| > |g|/5$. In that case, by Theorem 4.1 and inequality (4.8) we have

$$|g_3w'| \geq C_1N(g_3w') > \frac{C_1}{2}\|w'\| > \frac{C_1}{10}|g| > \frac{C_1}{10M}|g| = \delta|g|,$$

as required. To finish the proof, we note that by Theorem 4.1 and inequality (4.8) for $\tilde{w} = w_4$ (then $\|w_4\| = 2Q|g|$),

$$|g_4| = |g_3w_4| \geq C_1N(g_3w_4) > \frac{C_1}{2}\|w_4\| = C_1Q|g|.$$

By inequality (4.7) and assumption of case (2), we also note that we have

$$\begin{aligned}\|w_5\| = |g_3| &= |gw_1w_2w_3| \\ &\leq |g| + \|w_1w_2w_3\| \\ &\leq |g| + 5MN(g) \\ &\leq |g| + \frac{1}{2}|g| \\ &= \frac{3}{2}|g|.\end{aligned}$$

Hence, since $Q \geq 12M/C_1^2$, we have

$$|g_4w'| \geq |g_4| - \|w'\| \geq |g_4| - \|w_5\|$$

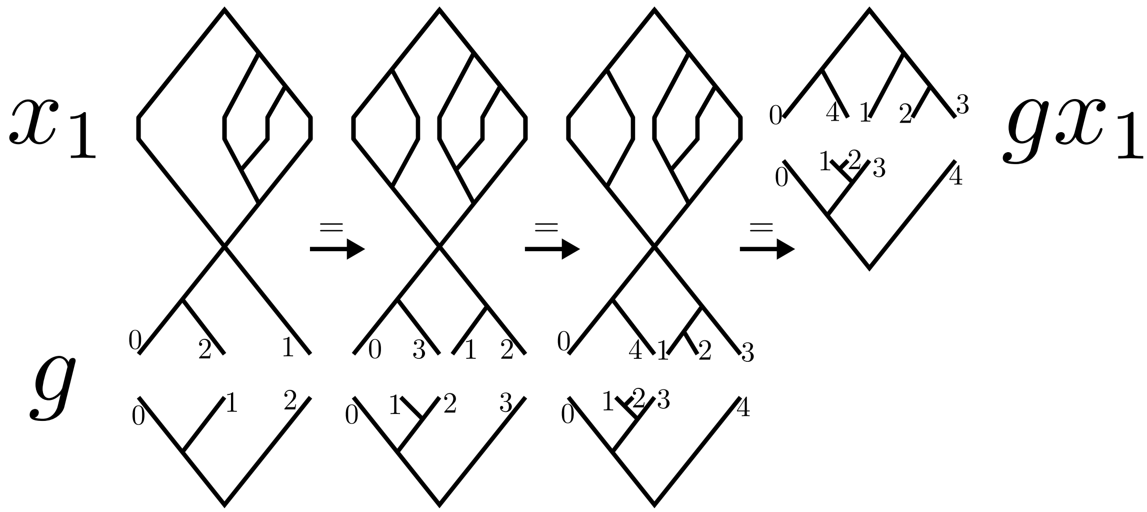

FIGURE 24. Calculating example of gx_1

$$
\begin{aligned}
&> C_1 Q|g| - \frac{3}{2}|g| \\
&= (C_1 Q - \frac{3}{2})|g| \\
&> |g| \\
&> \frac{C_1}{10M}|g| = \delta|g|,
\end{aligned}
$$

as required. □

□

By multiplying x_1 on the right to an element having one or two carets, we get the element of BV satisfying the assumption of Proposition 4.8.

LEMMA 4.14. *Let $g \in BV$ be such that $N(g) \leq 2$. Then $N(gx_1) \geq 3$.*

PROOF. By regarding each braid as just a permutation, it can be shown by finite number of direct calculations. Indeed, each tree-braid-tree diagram is reduced if there exists no strands pair such that they have a same parent. Hence, if it is reduced when considering g as the element of V, then it is also reduced in BV. For example, see Figure 24. The endpoints of each strand are represented by the same number, with a blank representing some braid. □

LEMMA 4.15. *Let $g \in BV$. Then*

$$|g| - 1 \leq |gx_1| \leq |g| + 1.$$

PROOF. The first inequality follows from $|g| \leq |gx_1| + |x_1^{-1}| = |gx_1| + 1$. The second inequality follows from $|gx_1| \leq |g| + |x_1| = |g| + 1$. □

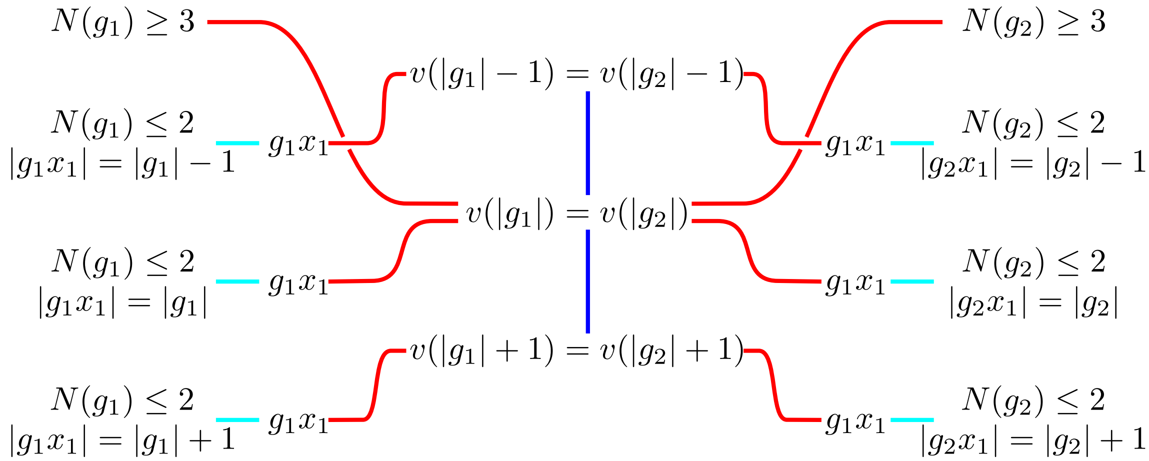

FIGURE 25. The path connecting g_1 and g_2 where $|g_1|, |g_2| \geq 2$

The following proposition immediately implies that braided Thompson group has liner divergence, completing the proof of Theorem 1.1. See Figure 25 for the overview of the path. The idea of paths (corresponding "normal" blue paths) in the following proposition comes from [15, Theorem 2.13].

PROPOSITION 4.16. *There exist constants δ_{BV} and $D_{BV} > 0$ such that the following holds. Let $g \in BV$ be an element with $|g| \geq 2$. Then there exists a path of length at most $D_{BV}|g|$ in the Cayley graph $\Gamma = \mathrm{Cay}(BV, X)$ which avoids a $\delta_{BV}|g|$-neighborhood of the identity and which has initial vertex g and terminal vertex $x_0^{Q|g|}x_1^{-1}x_0^{-Q|g|+1}$.*

In other words, there exists a word w_{BV} in the alphabet X such that $\|w\| < D_{BV}|g|$; for any prefix w' of w_{BV}, we have $|gw'| > \delta_{BV}|g|$ and such that

$$gw_{BV} = x_0^{Q|g|}x_1^{-1}x_0^{-Q|g|+1}.$$

PROOF. For each natural number $k > 0$, let

$$v(k) = x_0^{Qk}x_1^{-1}x_0^{-Qk+1}.$$

First, if $N(g) \geq 3$ then the proposition follows from Proposition 4.8. Hence, we can assume that $N(g) \leq 2$. Then by Lemma 4.15, $|gx_1| = |g| - 1$, $|g|$ or $|g| + 1$ and by Lemma 4.14, $N(gx_1) \geq 3$. Let

$$D_{BV} = 2D + 4Q + 1$$
$$\delta_{BV} = \min\{\frac{1}{2}\delta, \frac{1}{2}C_1Q\}.$$

In the following, we will use Proposition 4.8 to construct the path connecting gx_1 and $v(|g|)$. We consider three cases depending on the length of gx_1.

Case (1): $|gx_1| = |g| - 1$.

By Proposition 4.8, there exists a path of length at most $D(|g| - 1)$ which avoids a $\delta(|g| - 1)$-neighborhood of identity and which has initial vertex gx_1 and terminal vertex

$v(|g|-1)$. Since $|g| \geq 2$, we have $\delta(|g|-1) \geq (\delta/2)|g|$. Hence this path avoids a $(\delta/2)|g|$-neighborhood of identity. Thus, we construct a path connecting $v(|g|-1)$ and $v(|g|)$. Let

$$p(|g|-1) \equiv x_0^{Q(|g|-1)-1} x_1 x_0^Q x_1^{-1} x_0^{-Q|g|+1}.$$

It is clear that $p(|g|-1)$ labels a path from $v(|g|-1)$ to $v(|g|)$ and the length of $p(|g|-1)$ is at most $2Q|g|$. In the following, we prove that for any prefix p' of $p(|g|-1)$, we have $|v(|g|-1)p'| > C_1 Q|g|$. Indeed, it is easy to see that the positive part of the normal form of the element $v(|g|-1)p'$ is x_0^i for $i \geq Q(|g|-1)$. Hence, by [8, Theorem 3], we have

$$N(v(|g|-1)p') \geq N(x_0^i) \geq Q(|g|-1)+1 > Q(|g|-1),$$

where we note that this theorem claims only the relationship between the number of carets of elements in F and their exponents, so it can be applied to BV. Hence, by Theorem 4.1, we have

$$|v(|g|-1)p'| \geq C_1 N(v(|g|-1)p') > C_1 Q(|g|-1).$$

Since $|g| \geq 2$, we have $C_1 Q(|g|-1) \geq (C_1 Q/2)|g|$, as required.

Case (2): $|gx_1| = |g|$.

By Proposition 4.8, all assertions follow.

Case (3): $|gx_1| = |g|+1$.

By Proposition 4.8, there exists a path of length at most $D(|g|+1)$ which avoids a $\delta(|g|+1)$-neighborhood of identity and which has initial vertex gx_1 and terminal vertex $v(|g|+1)$. Since $|g| \geq 2$, we have $D(|g|+1) \leq 2D|g|$. Thus, we construct a path connecting $v(|g|)$ and $v(|g|+1)$. Let

$$p(|g|) \equiv x_0^{Q(|g|)-1} x_1 x_0^Q x_1^{-1} x_0^{-Q(|g|+1)+1}.$$

It is clear that $p(|g|)$ labels a path from $v(|g|)$ to $v(|g|+1)$ and the length of $p(|g|)$ is at most $2Q(|g|+1)$. Since $|g| \geq 2$, we have $2Q(|g|+1) \leq 4Q|g|$. By the almost same argument as case (1), we have

$$|v(|g|)p'| \geq C_1 N(v(|g|)p') \geq C_1 N(x_0^i) > C_1 Q|g|,$$

for any prefix $p' \leq p(|g|)$ and corresponding $i \geq Q|g|$, as required. □

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