The Exact Number of Nonnegative Integer Solutions for a Linear Diophantine Inequality

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Abstract—**In this paper, we present a simple and fast method for counting the number of nonnegative integer solutions to the equality $a_1x_1+a_2x_2+\ldots+a_rx_r = n$ where $a_1, a_2, ..., a_r$ and n are positive integers. As an application, we use the method for finding the number of solutions of a Diophantine inequality.**

Keywords: Counting, Nonnegative integer solutions, Diophantine inequality.

1 Introduction

Counting techniques play an important role in computing probabilities in random experiments of throwing dice, or classical occupancy problems. As a result, they have come to form a major part of the mathematics curriculum in many statistical publications. First we will consider some important applications of counting techniques.

Ross [3] showed that the number of ways for placing n identical objects into the r distinct cells is equivalent to the number of nonnegative integer solutions to the equation

$$x_1 + x_2 + \ldots + x_r = n \ \ (\text{with} \ \ x_i \geq 0, \ \ i = 1, \ldots, r). \qquad (1)$$

He also showed that the number of positive integers solutions of (1) is $\binom{n-1}{r-1}$. The number of nonnegative integer solutions of (1), subject to the constraint $x_i \geq b_i$ for $i = 1, \ldots, r$ is $\binom{n+r-(b_1+b_2+\ldots+b_r)-1}{r-1}$. Letting $x_i = y_i + b_i$ for each i yields the equation

$$y_1 + \ldots + y_r = n - (b_1 + b_2 + \ldots + b_r), \qquad (2)$$

to be solved in nonnegative integers. The number of such solutions where $x_i \leq b_i$ $(i = 1, \ldots, r)$ can be obtained using the inclusion/exclusion principle (see, for example, Rosen et al. [1]). For the latter situation, Murty [4] obtained a simple method of counting the favoured number of solutions. One generalization of (1) is the number of nonnegative integer solutions of the following equation,

$$a_1x_1 + a_2x_2 + \ldots + a_rx_r = n. \qquad (3)$$

Equation (3) is well-known as a Linear Diophantine Equation. As is discussed above for the simple case, it is possible to obtain the number solutions of equation (1) with some bounds on x_i's from (1) without any bounds on x_i's. It has been shown that the number solutions of (3) by some bounds on x_i's can be expressed as a function of the number solutions of (3) without any bounds on x_i's (Eisenbeis et al. [5]). Therefore, it is enough to restrict our effort to determine the number solutions of (3) without any bounds on x_i's. Given positive integers $a_1, a_2, \ldots, a_r$ that are relatively prime, it is well-known that for all sufficiently large n the equation (3) has a solution with nonnegative integers x_i (Tripathi [2]). The generating function of equation (3) has the form

$$\varphi(t) = [(1 - t^{a_1})(1 - t^{a_2}) \ldots (1 - t^{a_r})]^{-1},$$

and the number of non-negative integer solutions $J(n)$ of equation (2) is given by the formula:

$$J(n) = \frac{1}{n!}\varphi^n(0). \qquad (4)$$

Calculation of $J(n)$ is difficult in most situations. Antimirov and Matvejevs, in [6] have discussed several possible methods methods for its calculation. Eisenbeis et al.(1992) [5] presented fast methods for computing the exact or approximate number of solutions. In summary, there are two main problem for finding the number of nonnegative integer solution solutions of (3); the present methods, owing to the difficulty of the problem, are complicated, time consuming, and encounter difficulties when one wishes to extract a list of such solutions. These issues motivated us to obtain a simple method for finding the number of nonnegative integer solutions of (3) and provide a list of the obtained solutions.

2 New Method

Among the two problems considered, i.e., computing the number solutions and generating the solutions, the first one is by far the most complex. Therefore, it is vital

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to simplify the problem as much as possible in order to obtain efficient computation. Let us first consider $a_i = 1$ for $i = 2, ..., r$ in (3). In this case, we must find the number of nonnegative integer solutions for

$$a_1x_1 + x_2 + \ldots + x_r = n. \tag{5}$$

For solving (5), we can give the possible values of x_1 and reform (5) to form (1). Therefore,

$$\sum_{w_1=0}^{[n/a_1]} \binom{n - a_1w_1 + r - 2}{r - 2} \tag{6}$$

is the number of nonnegative integer solutions for equation (5), where $[u]$ is the integer part of u and r is a positive integer and $r > 2$. If $r = 2$ we must use $\sum_{w_1=0}^{[n/a_1]} I(a_2, w_1)$ as the number of nonnegative integer solutions, where

$$I(a_2, w_1) = \begin{cases} 1 & a_2|n - a_1w_1 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Now, let $a_i = 1$ for $i = 3, ..., r$. In this case, we must find the number of nonnegative integer solutions for

$$a_1x_1 + a_2x_2 + x_3 + \ldots + x_r = n. \tag{8}$$

For solving (8), we can give the possible values of x_1, x_2 and reform (8) to form (1). Therefore,

$$\sum_{w_1=0}^{[n/a_1]} \sum_{w_2=0}^{[(n-a_1w_1)/a_2]} \binom{n - a_1w_1 - a_2w_2 + r - 3}{r - 3} \tag{9}$$

is the number of nonnegative integer solutions for this equation. It should be noted that, the formula is true when r is a positive integer and $r > 3$. However, if $r = 3$ we use $\sum_{w_1=0}^{[n/a_1]} \sum_{w_1=0}^{[(n-a_1w_1)/a_2]} I(a_3, w_1, w_2)$ as the number of nonnegative integer solutions, where

$$I(a_3, w_1, w_2) = \begin{cases} 1 & a_3|n - a_1w_1 - a_2w_2 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Continuing the procedure, we can get the following formula for the number of nonnegative integer solutions of (3).

$$s(a_1, \ldots, a_r; n) :=$$
$$\sum_{w_1=0}^{[n/a_1]} \sum_{w_2=0}^{[(n-a_1w_1)/a_2]} \cdots \sum_{w_{r-1}=0}^{[(n-a_1w_1-\ldots-a_{r-2}w_{r-2})/a_{r-1}]} I(a_r; w_1, \ldots, w_{r-1}) \tag{11}$$

where

$$I(a_r; w_1, \ldots, w_{r-1}) = \begin{cases} 1 & a_r|n - a_1w_1 - ...a_{r-1}w_{r-1} \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Note also that if $a_i = 1$ for all i, then $s(a_1, \ldots, a_r; n)$ is equal to $\binom{n+r-1}{r-1}$, since

$$\begin{aligned} s(a_1, \ldots, a_r; n) &= \sum_{w_1=0}^{n} \sum_{w_2=0}^{n-w_1} \cdots \sum_{w_{r-1}=0}^{n-w_1-\ldots-w_{r-2}} 1 \\ &= \sum_{w_1=0}^{n} \sum_{w_2=0}^{n-w_1} \cdots \sum_{w_{r-2}=0}^{n-w_1-\ldots-w_{r-3}} \binom{n - w_1 - \ldots - w_{r-2} + 1}{1} \\ &= \sum_{w_1=0}^{n} \sum_{w_2=0}^{n-w_1} \cdots \sum_{w_{r-2}=0}^{n+1-w_1-\ldots-w_{r-3}-1} \binom{1 + w_{r-2}}{1}. \end{aligned} \tag{13}$$

Now equality is obtained using the fact that $\sum_{k=0}^{n-m} \binom{m+k}{m} = \binom{n+1}{m+1}$.

3 An application

There are many problems which can be solved using the proposed algorithm. As a useful example, we use the algorithm for solving the Diophantine inequality

$$a_1x_1 + \ldots + a_rx_r \leq n. \tag{14}$$

Let us now briefly consider the characteristic of the Diophantine Inequality (for more information see, for example, [16][17][18][19]). The main statement of the aforementioned theorem is in the language of lattices in number theory. That is for any convex set in the r-dimensional Euclidean space $\mathbb{R}^r$ symmetric with respect to the origin, and with volume greater than 2^r, must contain a lattice point other than that of the origin. In the language of linear forms the problem is restated as

$$\sum_{j=1}^{r} a_{ij}x_j = L_i(X), \ \ 1 \leq i \leq r, \tag{15}$$

with real coefficients a_{ij} such that $\det(a_{ij}) \neq 0$, supposing that there exist r positive real numbers b_i, $i = 1 \ldots r$ with $\prod_{i=1}^{r} b_i \geq \det(a_{ij})$. Then there exists an integer vector C such that $L_i(C) \leq b_i$, $1 \leq i \leq r$, thus implying that a solution *exists* for the above equations and indeed the implied inequality. The paper by Cheema, [13] suggests techniques similar to the programming of this research in its working, and indeed uses Minkowski's theorem to state that, where $||\cdot||$ denotes the distance of a number from its nearest integer, that there always exists a nonzero integer-vector solution $X = (x_1, \ldots, x_r)$ to the inequalities:

$$||L_j(X)|| \leq C, \ (1 \leq j \leq r). \tag{16}$$

Another practical application of the discussed problem is that of the "Knapsack" model, encountered in many areas with a cleat explanation offered in [14]; "the question of how to fill a knapsack of limited weight capacity with different items which best meet the needs of one's trip". Beged-Dov [14], first introduced bounds on the number, N, of solutions to $\sum_{i=1}^{r} a_i x_i \leq n$ with the a_i's all being natural-valued, as

$$\frac{n^r}{r!\prod_{i=1}^{r} a_i} \leq N \leq \frac{(n + a_1 + \ldots + a_r)^r}{r!\prod_{i=1}^{r} a_i}. \tag{17}$$

These bounds were obtained in the following way. Denote the rectangular box $B(y_1, \ldots, y_r)$ as the set of points $Y = (y_1, \ldots, y_r)$ such that

$$a_i x_i \leq y_i \leq (x_i + 1)a_i \ \text{ for } \ i = 1, \ldots, r \tag{18}$$

which has r-dimensional volume $\prod_{i=1}^{r} a_i$. Secondly, define the pyramid $P(n)$ with volume $\frac{n^r}{r!}$, which denotes the set of points satisfying $y_i \geq 0$ for $i = 1, \ldots, r$ and $\sum_{i=1}^{r} y_i \leq n$. The bounds are obtained as a consequence of the fact that each point x_i as defined above belongs to a unique B, which is the one with $x_i = \left[\frac{y_i}{a_i}\right]$, and if that x_i lies in the pyramid $P(n)$, then it necessarily obeys the linear diophantine inequality in question. So the union of the N boxes contains $P(n)$. This somewhat simple topological argument allows the derivation of the above bounds. To add weight to Beged-Dov's argument in [14], some experimental results are calculated using an algorithm which could be considered to be an early precursor to the results of this paper. The tendency of the upper and lower bounds of the number of solutions to the linear Diophantine inequality to become close with increased number of variables and right hand side is also touched upon.

Padberg and Lambe sought to respectively improve upon Beged-Dov's bounds. In the latter case an approximate number of solutions was eventually sought and found in [7]. Padberg [12] considered the following lower bound

$$\frac{(n+1)^r}{r!\sum_{i=1}^{r} a_i} \leq N \tag{19}$$

Very soon after [14] was submitted, Padberg took its result in [12] and sharpened Beged-Dov's result to the following inequality:

$$\max\left(\frac{(n+1)^r}{r!\prod_{i=1}^{r} a_i}, \ \binom{r+a^*}{r}\right) \leq N$$

$$\text{and} \tag{20}$$

$$N \leq \min\left(\frac{(n+\sum_{j=1}^{r} a_j)^r}{r!\prod_{i=1}^{r} a_i}, \ \binom{r+a^{**}}{r}\right).$$

Here a^* and a^{**} are integers satisfying $a^* \leq \frac{n}{a_j}$ and $a^{**} \geq \left[\frac{n}{a_j}\right]$ for all $j = 1, \ldots, r$. The initial adjustment to the original result is made by definition of the new pyramid $P(n+\delta)$, whence

$$\sum_{j=1}^{r} x_{ij} \leq n+\delta, \ x_{ij} \geq 0 \text{ for } j = 1, \ldots r, \ 0 \leq \delta < 1, \tag{21}$$

Then as above, taking a vector $\xi \in P(n+\delta)$, then summing over each element of the vector we have (since $[x] \leq x, \ \forall x > 0$):

$$\sum_{j=1}^{r} x_j \left[\frac{\xi_j}{x_j}\right] \leq \sum_{j=1}^{r} x_j \left(\frac{\xi_j}{x_j}\right) \leq n+\delta. \tag{22}$$

The lower bound $\frac{(n+\delta)^r}{r!\prod_{j=1}^{r} a_j} \leq N$ is obtained with the substitution of $P(n+\delta)$ with $P(n)$ in the previous proof, which is then sharpened by taking the limit of this bound as $\delta \to 1$. The result is improved further by making the above substitution for a^* and a^{**} above, noting that

$$\sum_{j=1}^{r} x_j \leq \sum_{j=1}^{r} x_j \frac{a_j}{a_{\min}} \leq \frac{n}{a_{\min}}, \tag{23}$$

to obtain the bounds stated above.

The paper of Padberg also introduces the formula for the number of possible partitions explored in this paper, and quotes that another proof is mentioned in the book [15]. Lambe in his paper [11] of 1974 introduced bounds which in most cases were better still, than what had been previously discussed:

$$\binom{n+r}{r}\prod_{i=1}^{r}\frac{1}{a_i} \leq N \leq \binom{n+r\overline{a}}{r}\prod_{i=1}^{r}\frac{1}{a_i}, \tag{24}$$

Table 1: Comparison between current methods and the new algorithm.

$\{a_i\}$	n	New (Exact)	(17)		(20)		(24)		(19)
			lower	upper	lower	upper	lower	upper	lower
2, 3, 5	10	20	6	44	10	56	10	38	8
2, 3, 5	50	947	695	1200	737	1200	781	1140	737
2, 3, 5	100	6518	5556	7394	5724	73946	5896	7194	5724
2, 3, 5	200	48202	44444	51450	45115	51450	45791	50717	45115
1, 1, 10	12	97	29	230	37	230	46	202	37
2, 3, 4, 4, 5	11	53	3	356	21	252	9	247	21
2, 4, 4, 4, 5	11	41	2	316	21	252	7	223	21
2, 3, 4, 4, 5, 7	15	162	5	1693	28	1693	16	1142	7
2, 3, 4, 5, 6, 7	20	364	18	2970	28	2970	46	2130	24
2, 3, 5, 7, 9, 11, 13	50	8872	574	73412	659	73412	978	59228	659

where $\overline{a} = \sum_{i=1}^{r} \frac{a_i}{r}$. His new bounds were also able to show that the ratio of upper to lower bounds tends to unity as r and n grow large. To attain the lower bound, the inequalities

$$\sum_{i=1}^{g-1} a_i y_i + \sum_{i=g}^{r} y_i \leq n, \tag{25}$$

with the y_i's all integers and $g \in \{1, \ldots r+1\}$ are considered. The proof requires - where P_i denotes the number of *feasible* (that is, nonnegative) solutions to (25) - the proving of

$$P_g \leq a_g P_{g+1}, \text{ for } g = 1, \ldots, r. \tag{26}$$

The proof of the upper bound is achieved using the inequalities

$$\sum_{i=1}^{g-1} a_i y_i + \sum_{i=g}^{r} y_i \leq n + \sum_{i=g}^{r} (x_i - 1), \tag{27}$$

and requires the assertion - where Q_i denotes the number of feasible solutions to (25) and (27) - that

$$Q_g \geq a_g Q_{g+1}, \text{ for } g = 1, \ldots, r. \tag{28}$$

Both are achieved in similar fashion.

As mentioned above, Lambe in [7], discovered upper and lower bounds for this number. However, the algorithm proposed here is able to compute the exact number of solutions. To do this, we convert (14) to (3) by adding an extra nonnegative integer variable x_r to (14). Then we need to solve $a_1 x_1 + \ldots + a_{r-1} x_{r-1} + x_r = n$ and using the algorithm the number of nonnegative integer solutions to (14) is:

$$s(a_1, \ldots, a_{r-1}, 1; n) = \sum_{w_1=0}^{[n/a_1]} \sum_{w_2=0}^{[(n-a_1x_1)/a_2]} \cdots \sum_{w_{r-1}=0}^{[(n-a_1x_1-\ldots-a_{r-2}x_{r-2})/a_{r-1}]} 1. \tag{29}$$

It should be noted that in the reduced form of inequality we have $a_r = 1$. Therefore $I(a_r; w_1, \ldots, w_{r-1}) = 1$ for all $w_1, \ldots, w_{r-1}$. Let us first consider an simple example. Suppose we are interested in finding the number of nonnegative solutions to

$$10x_1 + x_2 + x_3 \leq 12. \tag{30}$$

The lower and upper bounds on the number of solutions to this inequality, 4 and 455 respectively, are obtained from the algorithm of (20), whilst we know the exact number of solution is 97. It can be seen easily that these bounds represent a wide deviation from the actual number of solutions. Let us now use the proposed algorithm for solving (30). As we mentioned above, first we need to reform (30) to $10x_1 + x_2 + x_3 + x_4 = 12$. Thus, the solution is as follows

$$\sum_{w_1=0}^{[12/10]} \sum_{w_2=0}^{[(12-10w_1)/1]} \sum_{w_3=0}^{[(12-10w_1-w_2)/1]} 1 = \\ = \sum_{w_2=0}^{12} \sum_{w_3=0}^{12-w_2} 1 + \sum_{w_2=0}^{2} \sum_{w_3=0}^{2-w_2} 1 = 97. \tag{31}$$

Table 1 shows the resulting lower and upper bounds given for the number of solutions to the inequality with coefficients a_i and relevant n. The third column shows the exact number of solutions given by the method of this note.

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