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Valuing Switching Options with the Moving-Boundary Method

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Abstract

Switching options can be deployed in various complex switching problems such as tolling agreements and the offshoring-backshoring problem. Closed form solutions to valuing switching options are not only hard, but also computationally intensive when solving numerically. We develop a new computational method to value switching options based on the moving boundary method. We show how the free boundary problem arising from switching options can be converted into a sequence of fixed boundary problems. We formulate the problem, and solve the optimal switching problem in two regimes over a finite time horizon. We establish the theoretical guarantees for this method (maximum principles, uniqueness and convergence). We demonstrate this with a numerical example and show the sensitivity of the solution with respect to problem parameters.
Keywords: switching options, moving boundary, optimal policy

1. Introduction

Managerial problems typically boil down to answering the following question: "What action should be taken, and when?". Such decision problems can be characterized as optimal-switching problems, a larger class of problems that encompass optimal-stopping problems. Optimal-switching problems are best explained in the context of optimal-stopping problems. An optimal-stopping problem involves the determination of a single optimal stopping time to take a single action. For example, a typical optimal-stopping problem is the problem of exercising an American option. The decision maker, the option holder in this instance, is presented with a single action, exercising the option, and must decide when to take this action. The optimal time to exercise the option is known as the optimal stopping time. Optimal-switching problems, on the other hand, present the decision-maker with a multitude of actions. The decision maker needs to determine when to take an action (switching times) and select an associated action from the available set of actions. Switching options can be defined as a sequence of transactions where the exercise of the option creates one or more additional options. These problems are very commonly observed, such as in tolling agreements, where one has to decide when to turn the power generator on and off, and in the offshoring-reshoring problem where a firm has to decide when to onshore and offshore its production. In the tolling agreement example,

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when the manager decides to turn on the power generation, she receives a new option to turn it off, and so on. Given the frequency with which these decision-making problems occur in practice, and the impact that some of these decisions could have, managers would benefit by knowing the switching times and the associated optimal actions. Such decision rules are obtained by solving optimal-switching problems. Solving these optimal-switching problems is akin to valuing switching options, a variant of the American option described above. The optimal value of the switching option is the discounted expected net benefit that the firm can accrue.

The problem of valuing switching options gives rise to a free-boundary problem characterized by a system of Quasi-Variational inequalities (QVIs). Solving this system of QVIs yields both the optimal value functions and the optimal control (i.e., a policy that dictates when a switch should be made and to which state). Explicit solutions exist for certain infinite horizon problems, such as those shown by Brekke and Oksendal (1994), Duckworth and Zervos (2001) and Yushkevich (2001), but place severe restrictions on the state dynamics and other problem parameters. For most of the problems, an analytical or closed-form solution rarely exists thus necessitating the need for numerical solutions. Several methods are outlined in Broadie and Detemple (2004) in their application to pricing American options. Non-PDE based methods such as the Monte-Carlo simulation approach are also used to solve these problems. However, these numerical solutions are harder to obtain due to high computational resource needs, and exacerbated by the multiple curses of dimensionality.

In this paper, we show a numerical scheme that leverages upon the 'moving-boundary' framework developed by Muthuraman (2008) for a two-regime switching problem. We focus our attention on the two-regime switching problem, because while a manager may theoretically be faced with selecting one action from a set of two or more actions, practically, a substantial number of managerial decisions can be cast as real options with two possible actions. For example, Brekke and Oksendal (1994) and Brennan and Schwartz (1985) consider natural resource extraction problems where an operator decides when to open or close a field/mine for resource extraction, incurring fixed opening and closing costs. Dixit (1989) considers the problem of when to invest in an asset, and when to abandon it. Carmona and Ludkovski (2008) and Deng and Xia (2006) focus on tolling agreements where an operator decides when a power generating plant that has been leased should be operated as the spread between the input price (of natural gas) and the output price (of electricity) fluctuates. The valuation of energy storage facilities is also based on two decisions facing an operator, to withdraw or inject the commodity as seen in Carmona and Ludkovski (2010). All of these problems are quite difficult to solve, and require extensive computational efforts. The moving boundary method in Chockalingam and Muthuraman (2011) is shown to be accurate as well as demonstrably faster than other existing methods in solving similar problems.

We first formulate a 2-regime switching problem that can be solved using an extension of the earlier moving boundary method developed by Chockalingam and Muthuraman (2011). Our contributions in this research are two-fold, (i) we develop this moving boundary computational method to solve a switching problem in two regimes over a finite time horizon, and (ii) we provide the necessary theoretical guarantees (maximum principles, uniqueness and convergence) for our proposed computational method. Additionally, we provide some numerical insights into the behavior of our method for a specific case.

The rest of the paper is organized as follows. In Section 2, we briefly review the current literature. The model formulation is presented in Section 3. In Section 4, we develop our generalization of the earlier Moving-Boundary method, which is used to solve optimal switching problems. In Section 5, we provide some numerical studies. And finally in Section 6 we introduce two applications of switching options from the real world that could be solved using our methodology, and make concluding remarks.

2. Literature Review

Stochastic control problems are problems where a controller attempts to control a system governed by an evolving stochastic process with the aim of optimizing an objective functional. Due to the costs involved in controlling and making changes to the underlying system, the controller needs an optimal control policy that will dictate the actions necessary to optimize the controller's objective functional. The cost structure determines the type of stochastic control problem facing the controller. There are several types of stochastic control problems such as optimal stopping problems, singular control problems, and impulse control problems. The more classical setup of these problems is continuous control, where a continuous control action is possible between successive impulses (see Bensoussan and Lions (1975)). The theory of optimal stopping is concerned with the problem of choosing a time to take a particular action- like exercising an American option, in order to maximize an expected reward or minimize an expected cost. Singular stochastic control problems are those problems in which the system controller can change the state of the system instantaneously by paying just the proportional cost, while in impulse control problems there is a fixed cost associated with each state change. Hence, when the control has proportional cost, the problem is called singular control wherein the optimal control only makes infinitesimal changes in states, but when the cost contains fixed component as well, the problem is called impulse control, and it is usually optimal for the controller to bring about non-infinitesimal changes to the state (there can be an impulse in singular control problems at the outset, to reach the boundary if the state is in the interior of the action region). Singular control problems have been extensively studied in literature, see (Merton 1992, Karatzas and Shreve 1998). When positive fixed control costs are present, the problem is called impulse control. The classical method to solve this problem has been shown in the fundamental work of Bensoussan and Lions (1984). In Constantinides and Richard (1978), the authors show an optimal policy for an infinite horizon discounted cost cash management problem using impulse control. They obtain analytic expressions for the optimal policy for a special case. A special case of the impulse control problem described above is considered in Sulem (1986) where a classic stochastic inventory control problem is solved. Recent work by Bensoussan and Chevalier-Roignant (2019) consider strategic capacity expansions in a continuous-time setting with fixed costs by employing the impulse control methodology. Guo and Tomecek (2008) establish a theoretical connection between singular control problems and optimal switching problems. Many of these impulse control problems are solved using verification theorems by assuming a priori the smooth fit property through the action and continuation regions, although verification theorems without assuming the smooth fit property also exist. These regularity properties and smooth fit properties are shown in (Guo and Wu 2009).

Using dynamic programming arguments, stochastic control problems typically involve solving a system of differential equations, either ordinary or partial. The domains over which these systems of equations are to be solved are unknown a priori, therefore resulting in free-boundary problems. Closed-form solutions rarely exist for such problems. Numerical solutions are then needed to determine the optimal value functions and policies. The Moving-Boundary method is a numerical scheme that can be used to solve a certain class of such free-boundary problems. The method transforms the free-boundary problem into a sequence of fixed-boundary problems that are easier to solve.

Kumar and Muthuraman (2004) applied Moving-Boundary method for the first time to solve singular control problems. They combine finite element methods that numerically solve partial differential equations with a policy update procedure based on the principle of smooth pasting to iteratively solve Hamilton-Jacobi-Bellman equations associated with the stochastic control problem. Following this, Muthuraman and Kumar (2006) provide a computational study of the problem of how to optimally allocate wealth among multiple stocks and a bank account to maximize the infinite horizon discounted utility of consumption. They consider the situation where the transfer of wealth from one asset to another involves transaction costs that are proportional to the amount of wealth transferred. Their model allows for correlation between the price processes, which in turn gives rise to interesting hedging strategies. This results in a stochastic control problem with both drift-rate and singular controls, which can be recast as a free boundary problem in partial differential equations. Adapting the finite element method and using an iterative procedure that converts the free boundary problem into a sequence of fixed boundary problems, Kumar and Muthuraman (2004) provide an efficient numerical method for solving this problem. Muthuraman (2008) develops the variant of their method for optimal stopping problems. Interestingly, at that time he showed that the Moving-Boundary idea also worked for optimal stopping problems, in particular for pricing American options. Apart from providing an efficient methodology to solve the free boundary problem in American option pricing, he extends the Moving-Boundary approach to solve an optimal stopping problem, thereby taking a modest step in making the Moving-Boundary method more general. Chockalingam and Muthuraman (2011) apply the Moving-Boundary method to price American options under stochastic volatility. First, they develop a transformation procedure to compute the optimal-exercise policy and option price and provide theoretical guarantees for convergence. Second, using this computational tool, they explore a variety of questions that seek insights into the dependence of option prices, exercise policies, implied volatilities on the market price of volatility risk and correlation between the asset and stochastic volatility. They compare the speed and accuracy of the procedure against existing methods such as the PSOR method outlined by Cryer (1971) and others shown in Ikonen and Toivanen (2008). The authors in Chockalingam and Muthuraman (2011, 2010,

2015) show that the moving boundary method is faster than these methods while being just as accurate. Feng and Muthuraman (2010) extend the Moving-Boundary method in stochastic impulse control problems. They develop a methodology that converts the free-boundary problem into a sequence of fixed boundary problems. They show that the arising sequence has monotonically improving solutions and that the sequence converges. Provided the converged solution is continuous and once differentiable (C^1), they show that it is the optimal solution and that the optimal policy takes the form of a control band policy which has a simple and intuitive representation. They also provide an ϵ-optimality result that provides an upper bound on the error when the sequence is terminated after convergence to within a tolerance. There are several differences in their approach as compared to ours. They work in an infinite horizon setting and the controller is able to change the state of the system, and the policy is independent of time. In our work, we focus on a finite horizon setting where the optimal policy is independent of the state but it is time dependent.

Switching options are another kind of stochastic control problems which have found diverse applications in operations management and finance such as Tolling agreements. Ludkovski (2005) defines a tolling agreement as any temporary contract between the permanent owner and another agent that allows the latter to claim ownership and management of the output, subject to pre-specified exercise rules. The holder of the tolling agreement can gain the benefits or suffer the losses depending on commodity prices and operational costs. Limited operational flexibility is also a feature of tolling agreements such that the manager may not be allowed to scale production up or down in a way that is most advantageous. This is where optimal switching is of particular interest in that it addresses the question when it is optimal for an investor to enter, or exit the market or adjust the investment. The pioneering work of Brekke and Oksendal (1994) considers the problem of finding the optimal sequence of opening and closing times of a multi-activity production process, given the costs of opening, running, and closing the activities and assuming that the state of the economic system is a stochastic process. They formulated the problem as an extended impulse control problem over an infinite time horizon and solved using stochastic calculus. Duckworth and Zervos (2001) address the problem of determining in an optimal way the sequence of times at which a firm can enter or exit an economic activity. They consider an investment model which involves production scheduling as well as a sequence of entry and exit decisions over an infinite time horizon. The pricing of an investment using this model gives rise to a stochastic impulse control problem that they explicitly solve. They do not assume additional costs associated with scaling production up or down, as long as this does not involve a complete shut-down. They assume the profit function is upper semi-continuous however, and impose further constraints to justify their analysis. Ly Vath and Pham (2007) consider the problem of determining the optimal sequence of stopping times for a diffusion process subject to regime switching decisions. They use a viscosity solution approach combined with the smooth-fit property, and explicitly solve the problem in the two-regime case when the state process is Geometric Brownian in nature. The results of their analysis take several qualitatively different forms, depending on model parameter values. They focus on two cases: one when the underlying dynamics are unaffected by the switching regime, but the profit functions are different; and another when the profit functions are identical, but the underlying dynamics depend on the switching regime. Djehiche et al. (2009) consider a finite time horizon and two production regimes. They use backward stochastic differential equations and Snell envelopes to solve completely the starting and stopping problem when the dynamics of the system are a general adapted stochastic process. Ludkovski (2005) allow multiple production regimes. However, the number of allowed switches is limited. They propose a new method of numerical solution based on Monte Carlo regressions. The scheme uses dynamic programming to simultaneously approximate the optimal switching times along all the simulated paths.

3. The Switching Model

Consider an operational process that can run under 2 different regimes or states - 'ON' (or state 1) and 'OFF' (or state 0). A firm has to run this process switching between these regimes over a finite time horizon $0 < T < \infty$. The output of this process is a product that sells in the market for the price X_t at time t. Our model setup is similar to that in Ludkovski (2005), Carmona and Ludkovski (2008), and Djehiche et al. (2009). Let the dynamics of this price be an Ito diffusion process represented by

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \tag{1}$$

where W_t is a one-dimensional Brownian motion. The model primitives $\mu(\cdot)$ and $\sigma(\cdot)$ are locally Lipschitz, implying that the solution to Eq 1 exists and is unique. Let $h(t, X_t, m_t)$ denote the instantaneous profit generated at time t by the operational process in the regime m_t and selling the output at the price X_t. This profit function is assumed to be continuous and locally Lipschitz. However, in practice h is usually increasing in X and profit is higher in the 'ON' state for higher values of X compared to the 'OFF' state, i.e. it is better to operate the process in the ON state at high values of X and in the OFF state for low values of X. The profit function could also be set to represent the utility of the firm that consumes the revenue by running this process. Let $K_{ij} > 0$ denote the cost of switching from regime i to j, $i \neq j$ and $\{i, j\} \in \{0, 1\}$. Let $(\tau_n)_{n\geq 1}$ be a sequence of non-decreasing stopping times such that only a finite number will occur in any bounded interval with probability one. A management policy ν is defined as

$$\nu = (\tau_1, \zeta_1; \tau_2, \zeta_2; \cdots ; \tau_k, \zeta_k; \cdots) \tag{2}$$

where the action $(\zeta_n)_{n\geq 1}$ taking values in $\{0, 1\}$ represents the regime the process is switched to, i.e. at τ_n for $n \geq 1$, the process is switched from its current state ζ_{n-1} to ζ_n. The regime in which the operational process starts (ζ_0) is deterministic. For a given policy the expected profit over the time horizon $[0, T]$ is

$$\mathcal{J}_{x,a}(\nu) = \mathbb{E}\left[\int_0^T h(s, X_s, m_s)ds - \sum_n K_{\zeta_{n-1},\zeta_n}(\tau_n)\right], \tag{3}$$

where we set $X(0) = x$ and $\zeta_0 = a$. The objective is to maximize the expected profit over the time horizon $[0, T]$ and we thus arrive at the following stochastic control problem:

$$V(x) = \sup_\nu \mathcal{J}_{x,a}(\nu) \tag{4}$$

The value function properties and verification theorems are shown by Ludkovski (2005), Carmona and Ludkovski (2010), Djehiche et al. (2009). The first two articles assume continuity of the value function, and this gap is bridged in (Djehiche et al. 2009). Let the operator $\mathcal{L}$ (infinitesimal generator of X_t) be defined as

$$\mathcal{L} = \frac{\partial}{\partial t} + \mu(t,x)\frac{\partial}{\partial x} + \sigma^2(t,x)\frac{\partial}{\partial x^2} \tag{5}$$

Without the continuity assumption (that the value function is C^2), we could use the weak version of the Ito's Lemma (see Revuz and Yor (2013)) which allows the differentiability of the function to be somewhat relaxed to being C^1 at certain coordinates. In our case, these will be the boundaries. We would then assume that the value function is at least C^1 at the boundaries and C^2 elsewhere.

Since the continuity has been shown in Djehiche et al. (2009), we proceed assuming this to be true. Using the dynamic programming principle and Ito's Lemma, we can arrive at the quasi-variational inequality (QVI) form of the problem. At a given point in time t when the system exists in regime i and $X(t) = x$, we have

$$\mathcal{L}V + h(t,x,i) \leq 0 \tag{6}$$

$$V(t,x,i) \geq V(t,x,j) - K_{ij} \quad (i \neq j) \tag{7}$$

where only one of the two inequalities needs to be strict at any point in time for a given regime. This tells us whether it is optimal to switch or stay in the current regime. If Eq (7) is strict, we stay in the current regime, and when Eq (6) is strict, we switch. Solving the QVI gives us the value function which is used to easily obtain the optimal policy.

Thus, we have a set of QVI's for each regime; a PDE for the continuation region for that regime and another for switching the operating regime. In the continuation region, the firm should continue in its current regime. The other tells the firm when it should change its operating regime. To characterize these regions, we can start by conjecturing the shape of the optimal policy. The optimal exercise policy can be represented by the continuous boundaries $u(t)$ and $d(t)$, associated with the regimes 1 and 0 (on and off) respectively. The continuation region for regime 1 is the area above and below $u(t)$ up to but not including the boundary $d(t)$.

$$C_1 = \Big\{(t,x) \in [0,T] \times [0,\infty);\ x > d(t)\Big\} \tag{8}$$

$$C_0 = \Big\{(t,x) \in [0,T] \times [0,\infty);\ x < u(t)\Big\} \tag{9}$$

These conjectured boundaries help us visualize the optimal policy. Let us say the process is in regime 1 and $x \in C_1$. As long as x stays in the continuation region C_1, the firm should continue in regime 1. However, once x hits or crosses the boundary $d(t)$, the firm should switch to the 'OFF' regime, i.e. regime 0. Figure 1 shows the shape of this (conjectured) policy schematically. Closer to the start of the time period the boundaries will be closer and will increase towards maturity T. As we move towards the T, the cost of switching is higher than any profits from switching which widens the inaction region.

When the firm is in regime 1 and the market price of the product $X(t)$ is high, the probability that the price will drop to or below the lower boundary $d(t)$ decreases. Hence, the firm will never switch from

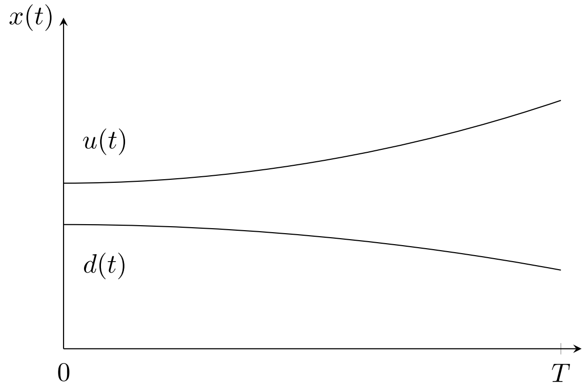


Figure 1: (Conjectured) shape of the optimal policy

this regime to the other. Consequently, the change in the value function with respect to the price is zero, $\lim_{x\to\infty} \frac{\delta V(t,x,1)}{\delta x} = 0$. A similar condition exists in regime 0 when the price $X(t)$ is too low, since the probability it will rise to or above $u(t)$ decreases. Hence, $\lim_{x\to 0} \frac{\delta V(t,x,0)}{\delta x} = 0$. We also know that the firm will end the process at time T, so $V(T,x,i) = 0, \forall i \in [0,1]$. Thus, we can say that $V(t,x,1)$ and $V(t,x,0)$ are the unique solutions to the free boundary problems:

$$\begin{aligned} &\mathcal{L}V(t,x,1) + h(t,x,1) \leq 0, \quad \text{in } C_1 \\ &V(t,x,1) \geq V(t,x,0) - K_{10} \\ &V(T,x,1) = 0 \\ &\lim_{x\to\infty} \frac{\delta V(t,x,1)}{\delta x} = 0 \end{aligned} \tag{10}$$

$$\begin{aligned} &\mathcal{L}V(t,x,0) + h(t,x,0) \leq 0, \quad \text{in } C_0 \\ &V(t,x,0) \geq V(t,x,1) - K_{01} \\ &V(T,x,0) = 0 \\ &\lim_{x\to 0} \frac{\delta V(t,x,0)}{\delta x} = 0 \end{aligned} \tag{11}$$

Note that Problems 10 and 11 are not equivalent to Equations 6 and 7.

4. Method

In this section, we propose a methodology to solve the free boundary problem described in Section 3. This methodology involves converting the free boundary problem into a sequence of fixed boundary problems which converges to the optimal solution. We also show and prove the necessary theoretical guarantees.

To begin our search for the optimal policy, we start with an initial guess boundary and compute its associated value function. This initial guess is chosen with a large inaction region (or the intersection of the two continuation regions C_1 and C_0, see Fig 2), such that the lower boundary d is very close to the x-axis and the upper boundary u is far from the x-axis. Each iteration generates a new boundary that monotonically decreases this inaction region - the lower boundary moves upwards and the upper boundary moves downwards. Since the space is closed, convergence is guaranteed with an improved speed to convergence.

For any policy (d^n, u^n) and its associated value function V^n, the following conditions guarantee that the current inaction region is a superset of the optimal inaction region (this is because they indicate that the current inaction region is too conservative, that switching is beneficial and should take place earlier).

$$V^n(t, d^n+, 1) - V^n(t, d^n, 1) < V^n(t, d^n+, 0) - V^n(t, d^n, 1) - K_{10} \tag{12}$$

$$V^n(t, u^n+, 0) - V^n(t, u^n, 0) < V^n(t, u^n+, 1) - V^n(t, u^n, 0) - K_{01} \tag{13}$$

We start with the initial guess policy (d^0, u^0). If this does not satisfy the above conditions, we need to start with smaller d^0 and/or a larger u^0. After solving the fixed boundary problem to compute V^0, we begin our iteration with $n = 0$, (d^n, u^n) and V^n. We define d^{n+1} by

$$\begin{aligned} d^{n+1}(t) = \quad & \sup \big\{ x \in (d^n(t), \infty) \big| \\ & \quad V^n(t, x_0+, 1) - V^n(t, x_0, 1) \leq V^n(t, x_0+, 0) - V^n(t, x_0, 1) - K_{10} \\ & \quad \forall\ x_0 \in [d^n(t), x)] \big\} \end{aligned} \tag{14}$$

and u^{n+1} by

$$\begin{aligned} u^{n+1}(t) = \quad & \inf \big\{ x \in (0, u^n(t)) \big| \\ & \quad V^n(t, x_0+, 0) - V^n(t, x_0, 0) \leq V^n(t, x_0+, 1) - V^n(t, x_0, 0) - K_{01} \\ & \quad \forall\ x_0 \in (x, u^n(t)] \big\} \end{aligned} \tag{15}$$

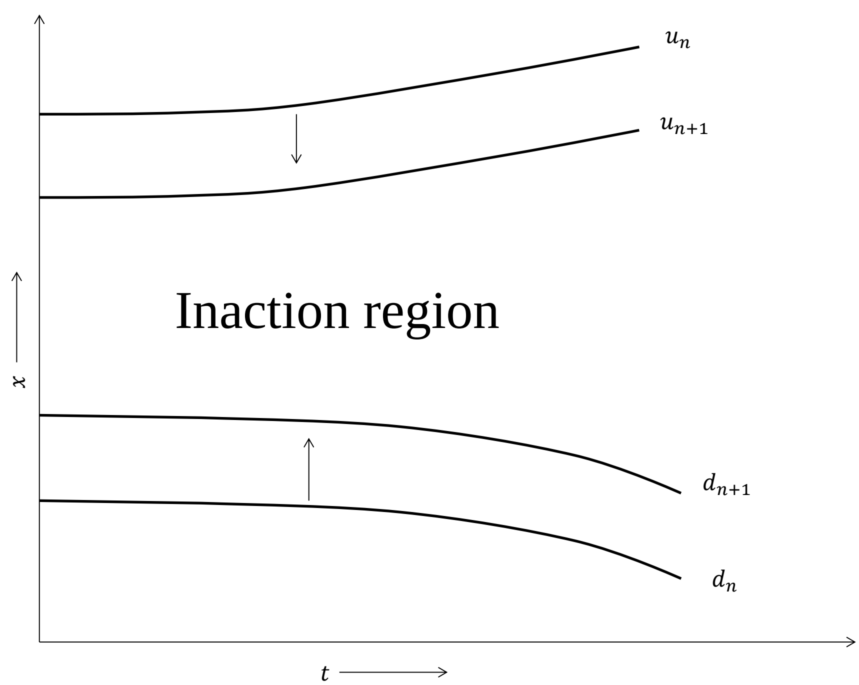


Figure 2: Moving Boundary

These policy updates are devised with the following intuition. The update for d looks for the largest x such that switching to the 'OFF' regime increases the value more than continuing in the 'ON' regime. This is done while maintaining the inaction region a superset of the optimal inaction region and leaving us with the necessary superset condition Eq 12 to iterate further. Similarly, the update for u looks for the smallest x such that switching to the 'ON' regime yields a higher value than continuing in the 'OFF' regime. Again, the new inaction region is a superset of the optimal inaction region leaving us with the necessary superset condition Eq 13 to iterate further (see Fig 2).

Using the updated policy (d^{n+1}, u^{n+1}), we solve the fixed boundary problems 10 and 11 to obtain the associated value function V^{n+1}. To show convergence, we need to show the following results. First, we need to show that the updated policy is better, i.e. show that $V^{n+1} \geq V^n$. Next, we need to ensure that the superset conditions Eqs 12 and 13 hold true for both V^n and V^{n+1} to allow for repetitive improvements. Since the space is closed, convergence is guaranteed, but we still need to show that the converged policy is the optimal policy.

The rest of this section establishes the theoretical guarantees for the method. Lemmas 1 and 2 establish the maximum principles, i.e. show that the optima lie on the boundaries. These two Lemmas correspond to the two continuation regions of the optimal policy, and the difference between them lies in the limit (i.e. as $x \to \infty$ or $x \to 0$) in Problems 16 and 17. Using these, we show in Theorem 3 that the solutions to the Problems 10 and 11 are unique. In Theorem 4 we show that the optimal condition can be violated by superset conditions, and in Theorem 5 we show that the value function improves at each iteration as well as show the convergence of the method.

Lemma 1. *For a given function $G(t)$, a continuous $c(t) > 0$ and a $T' \in (0, \infty)$ let us define the region $C_{T'} = \{(t, x) \in (0, T') \times \mathbb{R}_+ \mid x > c(t)\}$. Let ∂O represent the boundary $(0, x)$ for $x \in (c(0), \infty)$ and let ∂C represent the boundary $(t, c(t))$ for $t \in [0, T')$. Let f be the solution of*

$$\begin{aligned}
&\mathcal{L}f = 0, \;\; \textit{in } C_{T'} \\
&f(t, c(t)) = G(t), \;\; t \in [0, T') \\
&f(T', x) = 0 \\
&\lim_{x \to \infty} f_x(t, x) = 0
\end{aligned} \tag{16}$$

where $\mathcal{L}f = f_t + \mu(t, x) f_x + 0.5\sigma^2(t, x) f_{xx}$ with the subscripts representing the partial derivatives. If $G(t) > 0$ $\forall t$ then the maxima of f are attained only on the boundary ∂C and the minimum value of f is 0 and is attained on (T', x). If $G(t) = 0$ $\forall t$ then $f(t, x) = 0$ for all $(t, x) \in C_{T'}$.

Proof. We show that the maxima of f is attained only on the boundary ∂C. To do so we rule out any other possibilities. Let's say that an internal maxima exists attained at some (t', x'). When we have $G(t) > 0$, then this maxima must be greater than $f(t, x)$ for all (t, x), meaning that $f(t', x') \geq \max_{t \in [0, T')} G(t) > 0$. From the required conditions for maxima, we have $f_x = 0$, $f_t = 0$, and $f_{xx} < 0$, which on substitution in $\mathcal{L}f = 0$ yields a contradiction. Next say that the maxima of f is attained at $(0, x') \in \partial O$. Then it must be true that $f(0, x') \geq \max_{t \in [0, T')} G(t) > 0$ and with $f_x = 0$, $f_t \leq 0$, and $f_{xx} < 0$, which on substitution in $\mathcal{L}f = 0$ yields a contradiction. Hence the maxima is attained on ∂C.

Similar arguments and reasoning as above will yield that the minima of f is 0 and is attained on (T', x). When $G(t) = 0$ for all t, the above arguments directly show that $f(t, x) = 0$ for all $(t, x) \in C_{T'}$. ■

Lemma 2. *For a given function $G(t)$, a continuous $c(t) > 0$ and a $T' \in (0, \infty)$ let us define the region $C_{T'} = \{(t, x \in (0, T') \times \mathbb{R}_+ \mid x < c(t)\}$ and let ∂O represent the boundary $(0, x)$ for $x \in (c(0), \infty)$ and let ∂C represent the boundary $(t, c(t))$ for $t \in [0, T')$. Let f be the solution of*

$$\begin{aligned} &\mathcal{L}f = 0, \ \textit{in } C_{T'} \\ &f(t, c(t)) = G(t), \ t \in [0, T') \\ &f(T', x) = 0 \\ &\lim_{x \to 0} f_x(t, x) = 0 \end{aligned} \tag{17}$$

If $G(t) > 0 \ \forall t$ then the maxima of f is attained only on the boundary ∂C and the minimum value of f is 0 and is attained on (T', x). If $G(t) = 0$ for all t then $f(t, x) = 0$ for all $(t, x) \in C_{T'}$.

Proof. Similar to Lemma 1. ■

The following theorem shows that the solutions to Eq (10) and Eq (11) are unique.

Theorem 3. *The solutions to the initial value problems*

$$\begin{aligned} &\mathcal{L}V^n(t, x, 1) + h(t, x, 1) = 0, \ \textit{in } C_1^n \\ &V^n(t, u(t), 1) = V^n(t, u(t), 0) - K_{10}, \quad t \in [0, T) \\ &V^n(T, x, 1) = 0 \\ &\lim_{x \to \infty} V_x(t, x, 1) = 0 \end{aligned}$$

and

$$\begin{aligned} &\mathcal{L}V^n(t, x, 0) + h(t, x, 0) = 0, \ \textit{in } C_0^n \\ &V^n(t, d(t), 0) = V^n(t, d(t), 1) - K_{01}, \quad t \in [0, T) \\ &V^n(T, x, 0) = 0 \\ &\lim_{x \to 0} V_x(t, x, 0) = 0 \end{aligned}$$

are unique.

Proof. Suppose not, and that there exist two solutions f_a and f_b for $V^n(t, x, 1)$. Then $\bar{V} = f_a - f_b$ solves

$$\begin{aligned} &\mathcal{L}\bar{V} = 0, \ \text{in } C_1 \\ &\bar{V}(t, u(t), 1) = 0, \quad t \in [0, T') \\ &\bar{V}(T', x, 1) = 0 \\ &\lim_{x \to \infty} \bar{V}_x(t, x, 1) = 0 \end{aligned}$$

Now from Lemma 1, $\bar{V}$ is uniformly zero in C_1 and the results follow. One can easily show the same for $V^n(t, x, 0)$. ■

The following theorem shows that the optimal condition can be violated by superset conditions. This is very important for the initial policy, because the algorithm should start from a boundary at which the superset conditions Eq (12) and Eq (13) are violated.

Theorem 4. *If $d^0(t)- < d(t)$ for all $t \in (0,T)$, then*

$$V^0(t, d^0(t)+, 1) - V^0(t, d^0(t), 1) < V(t, d^0(t)+, 0) - V^0(t, d^0(t), 1) - K_{10}$$

and if $u^0(t)- > u(t)$ for all $t \in (0,T)$, then

$$V^0(t, u^0(t)+, 0) - V^0(t, u^0(t), 0) < V(t, u^0(t)+, 1) - V^0(t, u^0(t), 1) - K_{01}$$

Proof. Consider the region $C_1^0 \backslash C_1 = \{(t,x) : d^0(t) < x < d(t)\}$. This is the region in which it is strictly optimal to switch but the policy d^0 chooses not to. Therefore for any x and $x+$ in this region

$$V^0(t, x, 1) < V(t, x, 1) = V(t, x, 0) - K_{10}$$

and

$$V^0(t, x+, 1) < V(t, x+, 1) = V(t, x+, 0) - K_{10}$$

means that at any point in $C^0_{ON} \backslash C_{ON}$, we have

$$V^0(t, x+, 1) - V^0(t, x, 1) < V(t, x+, 0) - V^0(t, x, 1) - K_{10}$$
$$V^0(t, d^0(t)+, 1) - V^0(t, d^0(t), 1) < V(t, d^0(t)+, 0) - V^0(t, d^0(t), 1) - K_{10}$$

giving us our results. The same argument follows to show the condition for u. ■

Finally, the following two theorems show that the value improves at each iteration, and that the method converges to the optimal boundaries, i.e. we show that the value function is monotonically increasing in each iteration and eventually converges to its maximum.

Theorem 5. *If $V^n(t, x, 1)$ is the solution to the initial value problem*

$$\begin{aligned}
&\mathcal{L}V^n(t,x,1) + h(t,x,1) = 0, \ \text{in } C_1^n \\
&V^n(t, d^n(t)-, 1) = V(t, d^n(t)-, 0) - K_{10}, \quad t \in [0,T) \\
&V^n(T, x, 1) = 0 \\
&\lim_{x\to\infty} V^n_x(t, x, 1) = 0
\end{aligned}$$

and

$$V^n(t, d^n(t)+, 1) - V^n(t, d^n(t), 1) < V(t, d^n(t)+, 0) - V^n(t, d^n(t), 1) - K_{10}$$

for all $t \in (0,T)$ then $d^{n+1}-$ from Eq 14 is well defined. Moreover, $V^{n+1}(t,x,1)$ (the solution to initial value problem with boundary $d^{n+1}-$) is such that $V^{n+1}(t,x,1) > V^n(t,x,1)$ and

$$V^{n+1}(t, d^{n+1}(t)+, 1) - V^{n+1}(t, d^{n+1}(t), 1) < V(t, d^{n+1}(t)+, 0) - V^{n+1}(t, d^{n+1}(t), 1) - K_{10} \quad \forall t \in (0,T).$$

Proof. Since $V^n(t, d^n(t)+, 1) - V^n(t, d^n(t), 1) < V(t, d^n(t)+, 0) - V^n(t, d^n(t), 1) - K_{10}$ and $\lim_{x\to\infty} V^n_x(t,x,1) = 0$, we have from continuity of $V^n(t,x,1)$ in C_1^n, the existence of $d^{n+1}-$. Moreover since $V^n(t, x+, 1) - V^n(t, x, 1) < V(t, x+, 0) - V^n(t, x, 1) - K_{10}$ for $x \in (d^n-, d^{n+1}(t)-)$, then the following condition holds:

$$V^n(t, d^{n+1}(t)-, 1) - V^n(t, d^n(t)-, 1) < V(t, d^{n+1} - (t), 0) - V^n(t, d^n(t)-, 1) - K_{10}$$

This implies that

$$V^n(t, d^{n+1}(t)-, 1) < V(t, d^{n+1} - (t), 0) - K_{10}$$

$$V^n(t, d^{n+1}(t)-, 1) < V^{n+1}(t, d^{n+1}(t)-, 1)$$

Now consider the function $P(t, x, 1) = V^{n+1}(t, x, 1) - V^n(t, x, 1)$ in the region C_1^{n+1}. We have

$$\mathcal{L}P(t, x, 1) = 0$$

$$P(t, d^{n+1}(t)-, 1) > 0$$

$$P(T, x, 1) = 0$$

$$\lim_{x\to\infty} P_x(t, x, 1) = 0$$

Now directly from Lemma 1, $P(t, x, 1)$ is positive in C_1^{n+1}.

Next we need to show that

$$V^{n+1}(t, d^{n+1}(t)+, 1) - V^{n+1}(t, d^{n+1}(t), 1) < V(t, d^{n+1}(t)+, 0) - V^{n+1}(t, d^{n+1}(t), 1) - K_{10}$$

To do this, it is sufficient to show that $P_x(t, x, 1) < 0 \quad \forall t \in (0, T)$. Suppose not, and say $P_x(t_0, d^{n+1}(t_0), 1) \geq 0$ for some t_0. Now since $P(t_0, d^{n+1}(t_0), 1) > 0$ and $\lim_{x\to\infty} P(t_0, x, 1) = 0$ this implies that a maxima along x is attained in the interior of C_1^{n+1} contradicting Lemma 1.

Hence $P_x(t_0, d^{n+1}(t_0), 1) < 0$, i.e. $V_x^{n+1}(t, d^{n+1}(t), 1) < V_x(t, d^{n+1}(t), 1)$ implies that

$$V^{n+1}(t, d^{n+1}(t)+, 1) - V^{n+1}(t, d^{n+1}(t), 1) < V^n(t, d^{n+1}(t)+, 1) - V^n(t, d^{n+1}(t), 1)$$

and since

$$V^n(t, d^{n+1}(t)+, 1) - V^n(t, d^{n+1}(t), 1) = V(t, d^{n+1}(t)+, 0) - V^{n+1}(t, d^{n+1}(t), 1) - K_{10}$$

we conclude that

$$V^{n+1}(t, d^{n+1}(t)+, 1) - V^{n+1}(t, d^{n+1}(t), 1) < V(t, d^{n+1}(t)+, 0) - V^{n+1}(t, d^{n+1}(t), 1) - K_{10}$$

Convergence is achieved when the boundary update d^{n+1} from Eq 14 is the same as d^n. ■

Theorem 6. *If $V^n(t, x, 0)$is the solution to the initial value problem*

$$\begin{aligned}
&\mathcal{L}V^n(t, x, 0) + h(t, x, 0) = 0, \text{ in } C_{OFF}^n \\
&V^n(t, u^n(t)+, 0) = V(t, u^n(t)+, 1) - K_{01}, \ t \in [0, T) \\
&V^n(T, x, 1) = 0 \\
&\lim_{x\to 0} V_x^n(t, x, 1) = 0
\end{aligned}$$

and

$$V^n(t, u^n(t), 0) - V^n(t, u^n(t)-, 0) < V(t, u^n(t), 1) - V^n(t, u^n(t)-, 0) - K_{01}$$

for all $t \in (0, T)$ then $u^{n+1}+$ from Eq 14 is well defined. Moreover, $V^{n+1}(t, x, 0)$ (the solution to initial value problem with boundary $u^{n+1}+$) is such that

$$V^{n+1}(t, x, 0) > V^n(t, x, 0)$$

and

$$V^{n+1}(t, u^{n+1}(t), 0) - V^{n+1}(t, u^{n+1}(t)-, 0) < V(t, u^{n+1}(t), 1) - V^{n+1}(t, u^{n+1}(t)-, 0) - K_{01} \quad \forall t \in (0, T)$$

Proof. Similar to Theorem 5. ■

5. Numerical Studies

We now illustrate our Moving-Boundary methodology with a numerical example. Before we move on to show the example, we make a brief note about the implementation. In order to solve the fixed boundary problems there are some standard methods. In line with Chockalingam and Muthuraman (2011), we use the finite difference method in this example. As with other numerical methods, we need to choose a finite domain and impose boundary conditions on the finite boundaries. As our time horizon is finite, there is no problem to truncate the time axis; however the truncation of the axis for X_t leads to an approximation, since the truncation is imposed at a finite boundary rather than at infinity. To reduce this problem, we choose an X which is sufficiently large.

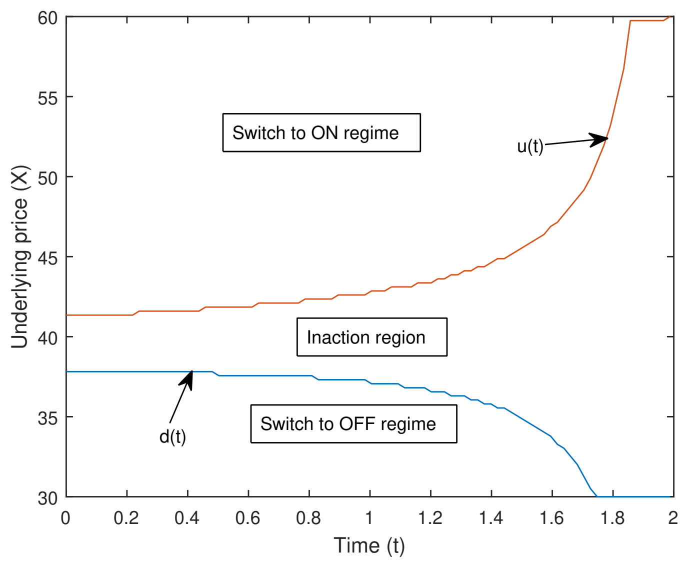


Figure 3: Optimal Policy

In this numerical example, we model X_t to follow a simple one-dimensional Brownian Motion driving process $dX_t = 0.5d_t + 1.4dW_t$; where $X_0 = 30$ and the time horizon is $T = 2$. We have two regimes with continuous reward rates of $h(x, t, 0) = 0$ and $h(x, t, 1) = x - 42$, and the switching cost between them is $K_{01} = K_{10} = 10$.

Our aim in this numerical study is to demonstrate how the iterative procedure presented above computes the optimal policy and value function. the Figure 3, the area above the orange curve $u(t)$ is the set of prices where the firm should switch on its production. Conversely, the area below the blue curve $d(t)$ encompasses all the prices where the firm should switch off its production. The continuation region is the region between $d(t)$ and $u(t)$. In the continuation region the regime should not change, irrespective of whether the firm's current regime is ON or OFF. We see that towards the end of the time horizon the operating regime does not change as fast as the beginning of the period, because the remaining time may not be sufficient to compensate the switching costs. Closer to T the boundaries $u(t)$ and $d(t)$ will flatten out mimicking a situation similar

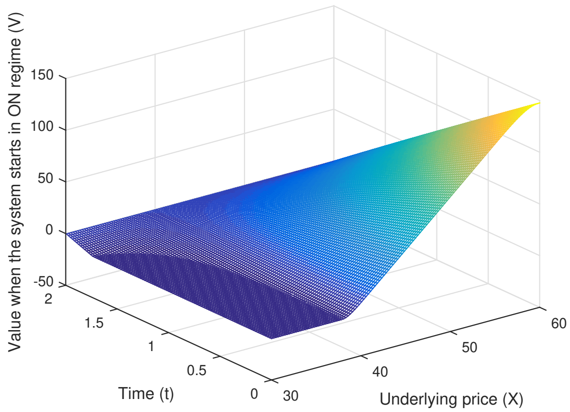


Figure 4: Value of the switching option when the firm is in ON state

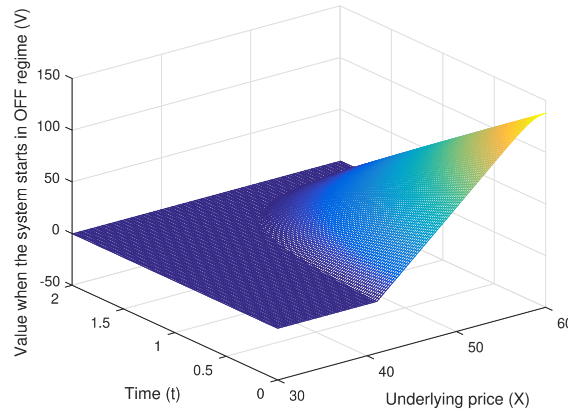


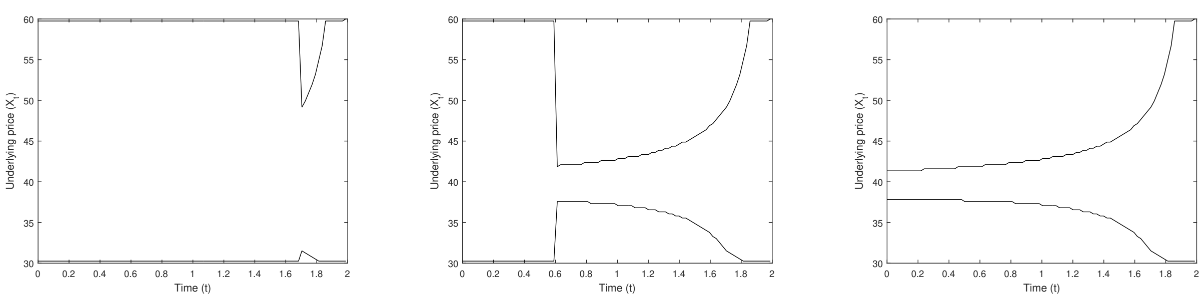
Figure 5: Value of the switching option when the firm is in OFF state

to an infinite horizon problem where these boundaries will be constant.

Figure 4 shows the value function when the firm starts in ON state $V(x, t, 1)$, and Figure 5 shows the value function when the firm starts in OFF state $V(x, t, 0)$. We see that in both figures that the value function is sufficiently smooth. The smoothness of the value function is shown in Carmona and Ludkovski (2008) and in Djehiche et al. (2009). In our case, since we use a numerical method we can say that the value function is smooth in the continuation region, and at least C^1 at the boundary.

5.1. Convergence in Different Iterations

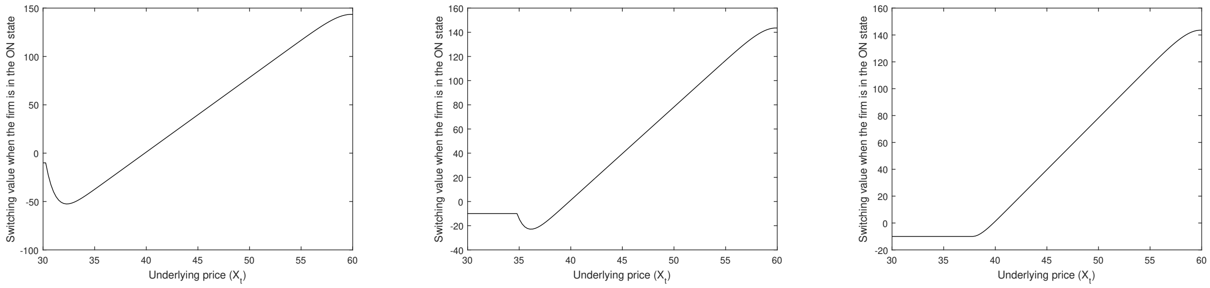
In Section 4 we have presented the theoretical guarantees for this method. As we mentioned there, our Moving-Boundary approach converts the free boundary problem into a sequence of fixed boundary problems which can be solved numerically. This sequence improves the value function and converges to an optimal policy where the value is maximized. Now we show some iterations to get a better understanding of how the policy and value functions change in different iterations.



(a) First iteration (b) Second iteration (c) Last iteration

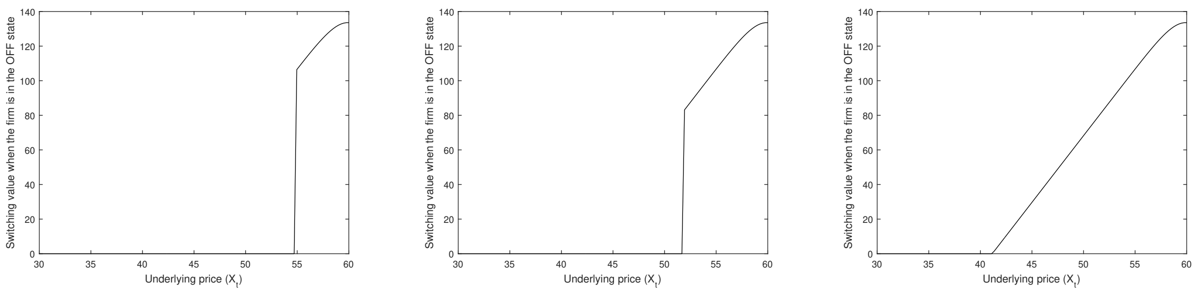
Figure 6: Policy convergence in different iterations (convergence achieved at iteration 5)

Figure 6 plots the policy in three different iterations. Figure 6(a) is the first iteration after the initial policy. The initial policy should hold the superset condition where the inaction region is large enough to start



(a) First iteration (b) Second iteration (c) Last iteration

Figure 7: ON state value convergence (convergence achieved at iteration 5)



(a) First iteration (b) Second iteration (c) Last iteration

Figure 8: OFF state value convergence (convergence achieved at iteration 5)

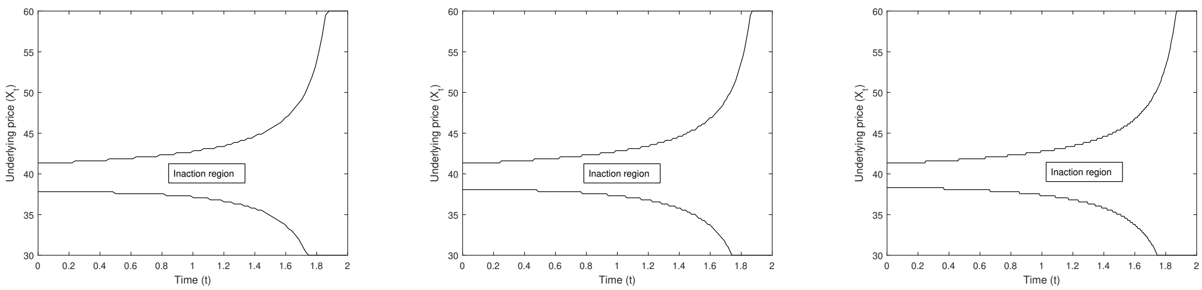
the Moving-Boundary algorithm. As we see from Figure 6(a) to Figure 6(c), the inaction region becomes smaller, and the policy converges to the optimal.

Figure 7 plots the ON state value function convergence in the different iterations. Figure 7(a) shows the value function associated with the initial policy. As we see in the last iteration (Figure 7(c)), the value function is sufficiently smooth. Figure 8 plots the OFF state value function at the different iterations. Convergence is achieved at iteration 5 for this example (for the policy and value functions). As mentioned before, the continuity of the value functions is quite a technical challenge. We refer the reader to Djehiche et al. (2009) for the discussion on continuity. In our case with the numerical method, we can say that the value function is smooth in the continuation region, and at least C^1 at the boundary.

5.2. Effect of volatility and switching costs on the Optimal Policy

Volatility is an important risk measure which can affect the optimal policy. We thus seek to investigate what this effect looks like. Figure 9 plots the optimal policy for different levels of volatility. The inaction region becomes smaller as volatility increases, meaning that the firm might need to switch more frequently when the volatility is relatively high. However, note that this change is very small. As the level of volatility may be difficult to estimate, the robustness of the optimal policy with regard to different levels of volatility is of significant managerial value.

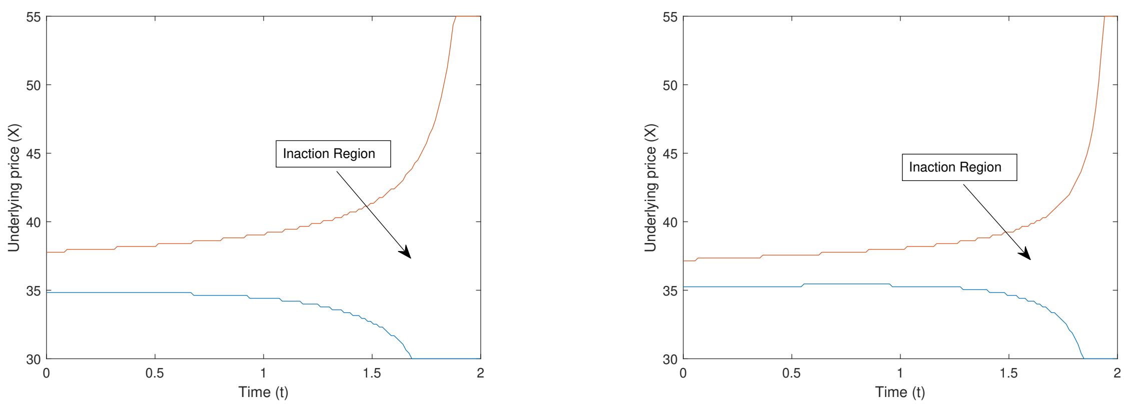
The reason why volatility does not have such a large effect is because the size of the inaction region is primarily governed by the switching cost. Figure 10 shows how decreasing the switching cost from 10 to 5



(a) $\sigma = 0.5$ (b) $\sigma = 1$ (c) $\sigma = 1.5$

Figure 9: Influence of volatility on optimal policy

affects the optimal policy. We see that the inaction region decreases substantially. When it is cheaper to switch between states, the boundaries $u(t)$ and $d(t)$ get closer. This is an important insight for managers operating switching processes - that changes in switching costs have greater influence on the optimal policies than the changes in the underlying asset volatility.



(a) $K_{01} = K_{10} = 10, \sigma = 1.4$ (b) $K_{01} = K_{10} = 5, \sigma = 1.4$

Figure 10: Influence of switching costs on the optimal policy

6. Discussion and Conclusion

Project costs in the energy industry tend to be high, a typical natural gas power plant or an oil refinery costs hundreds of millions of dollars and may take three to five years to build (Ludkovski 2005). Due to the capital intensive nature of the industry a few large companies own these power plants and lease them to operators. The operators, therefore, circumvent the capital intensive side of the business. The lease agreements between the operator and the owner are termed as tolling agreements. The key decision facing the lessee is how and when the power plant should be operated, taking into consideration the spread between the input and output prices. Limited operational flexibility in tolling agreements prevents scaling the production up or down in all proportions of capacity. Hence, the only choices for the manager are either to enter the market with full capacity or to exit the market completely, and shut the plant down. The firm procures gas at price G_t, using it to generate power that it sells at the price P_t. The revenue from running the plant is then given by the spread $X_t = (P_t - G_t)$ which pays the difference between the market price of the power and the market price of gas needed to produce the power. Due to limited operational flexibility in tolling agreements, managers have the option just either to run the plant (mode ON) or to shut it down (mode OFF). Managers frequently face such switching problems. Another example is in the offshore dual sourcing problem, in which firms have located their production plants in two different geographical locations and consequently switch between the locations as time passes. Such a strategy can create value for firms by decreasing the cost of production, but relocation incurs substantial costs. This can be modeled as an optimal switching problem where the firm can choose between two production plants. It is problems like these that form the managerial decision background for switching options.

In this paper, we formulate a switching problem with two regimes, and develop a numerical scheme that leverages upon the 'moving-boundary' framework developed by Muthuraman (2008) for this switching problem. Switching problems are quite difficult to solve, and require extensive computational efforts.

Our novel computational method is based on the Moving-Boundary method that can solve the optimal switching problem in two regimes over a finite time horizon. The idea is to convert a free boundary problem in to a sequence of fixed boundary problems (as was first developed for singular control problems in Kumar and Muthuraman (2004)), and then extended it to optimal stopping and impulse control. The optimal switching problem can be thought of as sequence of optimal stopping problems and has complicating features, making an extension of the Moving-Boundary method to tackle such problems non-trivial. In this paper, we have demonstrated that the idea of converting to a sequence of fixed boundary problems is still possible for free boundary problems arising from switching problems; the specific method, however, is different.

In conclusion, we formulate the 2-regime switching problem and contribute to the existing literature in two ways: (i) we extend the moving boundary computational method developed by Chockalingam and Muthuraman (2011) to solve a switching problem in two regimes over a finite time horizon, and (ii) we provide the necessary theoretical guarantees (maximum principles, uniqueness and convergence) for our proposed computational method.

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