Implementing **Dempster-Shafer Theory for Property Similarity in Conceptual Spaces Modeling**

Jeremy R. Chapman,* John L. Crassidis,† James Llinas,‡ Barry Smith§ David Kasmier,¶
University at Buffalo, The State University of New York, Amherst, NY, 14260-4400

Alexander P. Cox ‖
CUBRC Inc., Buffalo, NY 14225

Previous work has shown that the Complex Conceptual Spaces − Single Observation mathematical framework is a useful tool for event characterization. This mathematical framework is developed on the basis of Conceptual Spaces and uses integer linear programming to find the needed similarity values. The work of this paper is focused primarily on space event characterization. In particular, the focus is on the ranking of threats for malicious space events such as a kinetic kill. To make the Conceptual Spaces framework work, the similarity values between the contents of observations on the one hand and the properties of the entities observed on the other needs to be found. This paper shows how to exploit Dempster-Shafer theory to implement a statistical approach for finding these similarities values. This approach will allow a user to identify the uncertainty involved in similarity value data, which can later be propagated through the developed mathematical model in order for the user to know the overall uncertainty in the observation-to-concept mappings needed for space event characterization.

I. Introduction

The primary purpose of this paper is to further explore data fusion of *hard* and *soft* data for space event characterization. This work is a continuation of work presented in [1–3]. The present paper implements the Conceptual Spaces technology to mimic human decision-making that is involved in assessing which Red spacecraft is most capable of executing a kinetic kill in relation to a pre-specified Blue spacecraft. When humans make decisions they use an abundant amount of potentially helpful information including measurable data, contextual data, human intuition, etc. The goal is to incorporate as many such sources as possible into a process of automated decision-making.

For space events the relevant information includes observation data (i.e. photometer data and position data) derived from sensors, such as radar and telescopes. Measurable information of these types are examples of hard data. Other information about a spacecraft used in decision making that is not derived directly from sensor measurements but instead from human perception, judgment and analysis, are commonly referred to as soft data. This type of data is often in textual form. Some examples of soft data include, but are not limited to, social and political aspects of the relationship between Red and Blue spacecraft countries of ownership as well as information regarding the spacecraft themselves (for example, their component systems and thruster types). Human derived information is not easily quantified, and its uncertainty is also difficult to quantify. This paper presents a method of utilizing Conceptual Spaces to show how both types of data can be exploited to establish the relative threat posed by a given Red to a given Blue spacecraft.

Fusion of hard and soft data into a single framework permits for a better understanding of the situation at hand, allowing for a more acceptable anticipatory decision-making tool. The primary method for data fusion presented in this paper is based on the principles of Conceptual Spaces model of human cognition [4]. The underlying mathematical model used to represent Conceptual Spaces developed by Holender implements linear integer programming to develop similarity values between the contents of observations and the properties of the entities observed [5, 6].

This paper begins with an overview of Conceptual Spaces. Then it introduces the mathematical framework of Complex Conceptual Spaces − Single Observation and shows how this framework can be utilized to find similarity

*Graduate Student, Department of Mechanical & Aerospace Engineering. Email: jeremych@buffalo.edu, Student Member AIAA.

†SUNY Distinguished Professor and Samuel P. Capen Chair Professor, Department of Mechanical & Aerospace Engineering. Email: johnc@buffalo.edu. Fellow AIAA.

‡Emeritus Research Professor, Department of Industrial & Systems Engineering. Email: llinas@buffalo.edu.

§SUNY Distinguished Professor and Julian Park Chair of Philosophy, Department of Philosophy. Email: phismith@buffalo.edu.

¶Postdoctoral Researcher, Department of Philosophy. Email: djkasmie@buffalo.edu.

‖Ontologist, Data Science and Information Fusion Group. Email: alexander.cox@cubrc.org.

values between observations and their associated concepts. It also shows how uncertainty in the similarity values for properties can have an adverse affect on the overall results. Afterwards, Holender's method for solving single domain Conceptual Spaces is presented and an example of its use is provided. An alternative method to solving single domain Conceptual Spaces (i.e. properties) is explored using Dempster-Shafer theory to develop the similarities between the properties and the observations. The results from each of the two methods are compared.

II. Conceptual Spaces

Conceptual Spaces is a cognitive model developed to represent how the human mind perceives concepts. Originally developed by Peter Gärdenfors as a method for combining traditional cognitive models [4], which are of two primary sorts, namely *symbolic* and *associative*. Symbolic models take the form of a working computer program and can be thought of as a Turing machine [7, 8]. This type of model is based on the properties of Newell and Simon's physical symbol system hypothesis, which claims that human thinking is a kind of symbol manipulation [9]. The second type of cognitive model is the associative model, and it rests on the idea that the human mind perceives concepts on the basis of associations. For example, if one hears certain words, such as "eggs," "bacon," and "juice" they will likely associate them with the concept breakfast [10].

The idea behind Conceptual Spaces is that one can create a geometric structure to represent the concepts used in human thinking. Gärdenfors looked for a way to represent associations between observations and concepts that could be represented in a systematic way and used in a symbolic model in order to come up with an assessment of the similarity between observations and concepts. To see what this involves, a formal definition of the underlying geometrical structure used by Conceptual Spaces and of the associated terminology is provided.

A. World

To apply Conceptual Spaces to the underlying "world" that is being explored needs to be understood. Throughout different languages and even within a single language there are often cases where a single word has multiple different meanings. Take for example, the word 'orbit'; if the world of outer space is being considered it is understood that this word is relating to the path of one body around another body and that the path is subject in every case to physical laws. In the world of anatomy, however, the word 'orbit' is used to refer to a certain cavity in the skull of a vertebrate, namely the eye socket. Understanding the domain that is being worked on is important also for the identification of properties. Properties are analogous to adjectives (e.g. strong, fast, athletic) in the way that they describe a feature withing a signal domain [4]. In different worlds, properties can also take on different definitions. Consider the property 'large' as applied, first, in the world of birds, and secondly in the world of emus. In the world of birds, emus are considered large; in the world of emus there might be a subspecies of emus that are considered small, but birds of this type would still be considered large within the world of birds.

To understand the definitions and properties of a particular world it helps to have an appropriate set of ontologies as well as associated domain knowledge on the part of experts. The world that is being dealt with throughout this paper involves space events. To make sure all the terms and definitions are consistent the Space Domain Ontologies (SDO) illustrated in Fig. 1 is followed. The top-level ontology used here is the Basic Formal Ontology (BFO) ISO/IEC 21838 [11], which is a small, very general ontology that is designed for use in supporting information retrieval, analysis, and integration in scientific as well as other domains. Extending from BFO are the mid-level Common Core Ontologies (CCO), which together are a candidate standard mid-level ontology under INCITS 573-2. Together, they form a suite of 12 ontologies (outlined in blue in Fig. 1 [12, 13]).

Underneath the CCO, domain ontologies extending CCO come into play. In particular this paper deals with the Space Domain Ontology, which is itself a suite of 5 sub-ontologies relevant to the space domain, including the Space Object and the Space Event Ontologies [14, 15]. As noted, additional domain ontologies in Fig. 1 are tangential to but also sometimes supportive of the Space Domain Ontologies.

The SDO currently contains more than 800 classes that represent entities of a wide range of different types. Space object data is aligned with classes and relations in the SDO and stored in a dynamically updated Resource Description Framework (RDF) triple store. The RDF can be queried to support Space Situational Awareness (SSA) and also to support the needs of the spacecraft, ground operators and other stakeholders. For each type of entity in the domain the ontology provides a definition, a classification and relational links to other entities. The SDO ontologies taken as a whole represent the reality and complexity of the entities in the space domain.

Once ontologically structured, data is available for any number of applications. The fundamental idea is that for data to be unambiguous across all applications, it must be properly represented using terms from the ontology. For this paper

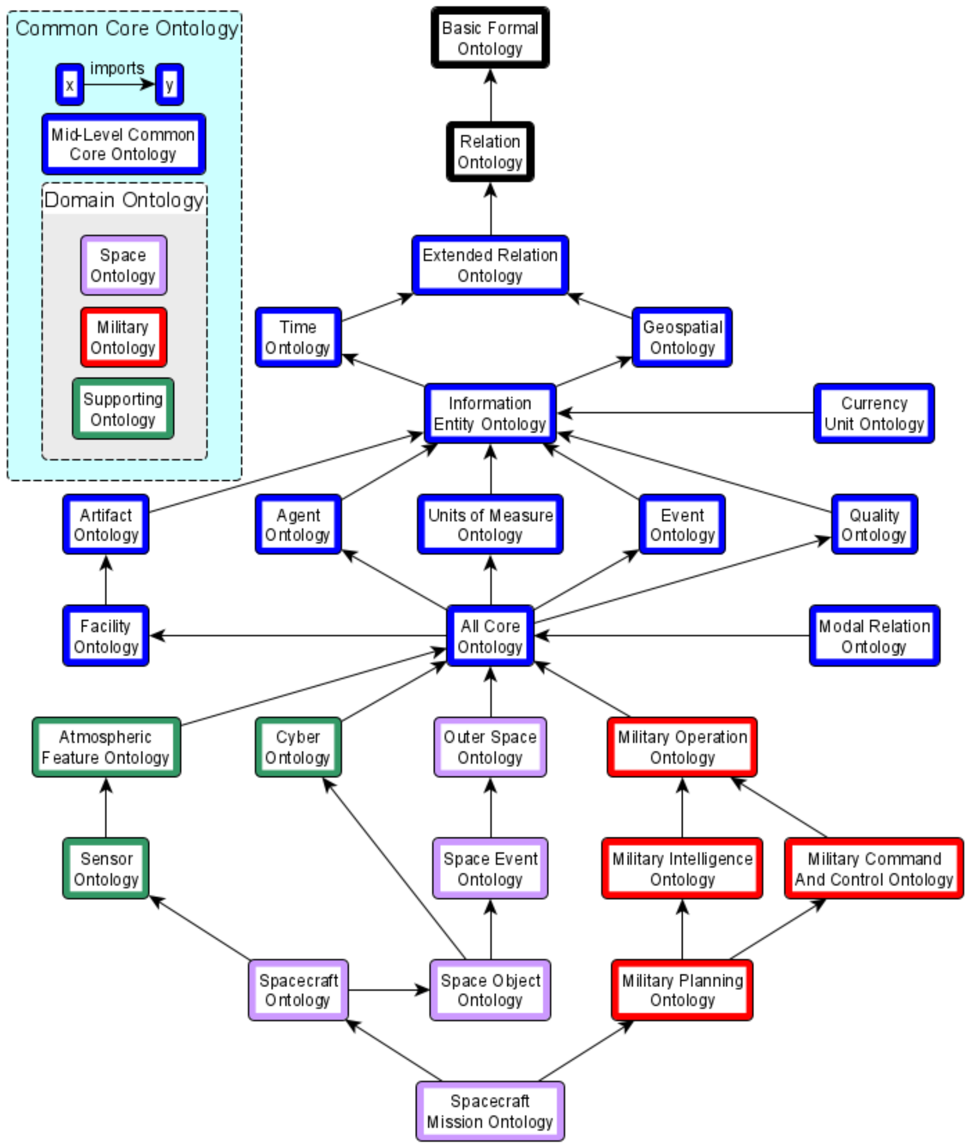


Fig. 1 Hub and Spokes Import Diagram for the Space Domain Ontologies

the primary application of the Conceptual Spaces model is created for space event characterization. The SDO will be used to extend the Conceptual Spaces framework in two ways: first, by extending the framework to include concepts outside the sphere of human cognition upon which Gärdenfors focused his attentions; and second, by ontologically structuring the data used in the Conceptual Spaces modeling process.

The goal of ontologies is to reduce the effort required to turn collected data into information accessible to, and supportive of, effective and timely decision making. Of concern here are the data relevant to threat rank assessments for Red and Blue spacecraft pairs in the space operating environment. The aim is to allow, as much as possible, the inclusion of any available sorts of data usable by analysts to establish the presence, likelihood and severity of a threat. The application of the Conceptual Spaces model to a body of heterogeneous yet ontologically structured data is intended to produce an automated form of threat analysis and ranking. The result is designed to serve as a time-critical aide to analysts and decision makers responsible for SSA.

One of the challenges analysts face is the amount of data available and the heterogeneous nature of the data collected. Data provided to analysts often comes from different kinds of sources, and pertains to different and sometimes seemingly unrelated entities, processes and features. An ontology in some ways works like a map of a piece of territory placing each of these data elements into a unique location. It defines each type of entity, structures those types vertically into a hierarchy of classes and subclasses, and provides further relations that link these types together laterally. The ontology contains definitions in both a human readable and a machine readable format.

Multi-sourced data poses terminological difficulties. In addition to a single word having multiple meanings, another common difficulty is maintaining the distinction between a piece of information and what that information is about. Consider the term 'mission' as in 'spacecraft mission.' 'Mission' can indicate 1. an objective to be achieved (for example, to monitor weather), 2. the plan for achieving this objective, and 3. the process of carrying out this plan. In the space domain realizing a mission might be something that a spacecraft is engaged in continuously over a given period of time, for example, of monitoring the weather. The objective is achieved by the process, and while humans easily adjust their understanding between these different meanings, machines cannot. An ontology structures the terms and definitions in a machine readable way that allows machines to unambiguously process information about mission plans and objectives without conflating them with processes prescribed by those plans.

There are several benefits to ontologically structured data. First, both analysts and machines can be assured that they are talking about the same things. Second, the ontology provides a way to automate inferences and thus reasoning over large quantities of data. Simple inferences, such as from the fact that a spacecraft has a thruster of type T it can be concluded that it uses a propellant of type P, are automatically derived by running reasoners over the ontologically enhanced data. Consequently, sophisticated queries can be formulated to call and use data as desired, including looking for correlations and interesting connections among the data. This provides analysts opportunities for reasoning which go beyond single human capabilities. Third, ontologies make it possible for data of heterogeneous types-for example, relating to the quantity of available fuel, the velocity of a spacecraft at a certain time, the type and number of thrusters a spacecraft possesses, the owner of the spacecraft, the history of its station-keeping maneuvers, and so forth, to all be linked together in a logical and searchable way within a single framework.

B. Quality Dimensions

At the basis of Conceptual Spaces are *quality dimensions*; for example, hue, saturation or brightness in the color domain or pitch, timbre and loudness in the domain of musical tones, as depicted in Fig. 2a. Quality dimensions are any observable parameters for which a (quantitative or qualitative) notion of distance is defined. Quality dimensions denote features by which concepts and observations can be compared. The fundamental role of quality dimensions is to build up the domains that are needed to represent what the Conceptual Spaces model calls a *concept*, but for this paper it can be understood to be a type or kind.

Quality dimensions are not always necessary for Conceptual Spaces. In the work presented in [16], it is explained that sometime the quality dimensions are unknown, as for example in the shape or friendship domains. However, even without a formal understanding of the quality dimensions that make up a particular domain, there is still an understanding of similarity for certain concepts. For example, with the shape domain it is known that a cube is more similar to a right rectangular prism than it is to a sphere.

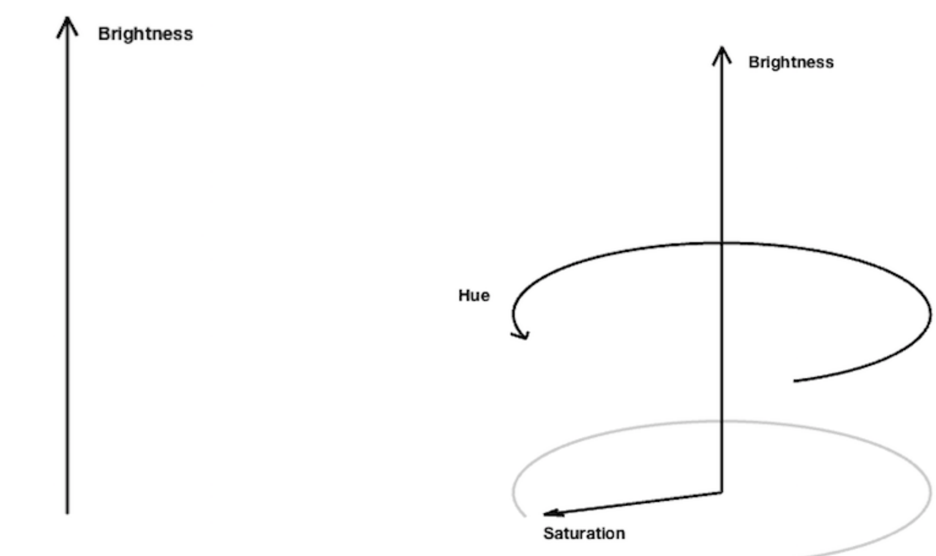
A collection of quality dimensions is said to be 'integral' when one quality cannot be assigned a value without assigning a value to all the others [17–19]. Quality dimensions that are not integral are called 'separable', as for example in the case of pitch and hue. An example of integral quality dimensions are pitch, timbre and loudness. If someone is measuring the sound of something their measurements would need all three quality dimensions to define the sound.

On the other hand an example of separable quality dimensions would be hue and pitch; that is, we can measure pitch without measuring hue.

C. Domains

Built upon quality dimensions are *domains*, where a domain is defined as a set of integral quality dimensions (i.e. of quality dimensions that cannot be independently assigned a value). For example, combining the three quality dimensions of hue, brightness and saturation would generate the domain of color as shown in Fig. 2b. When attempting to identify a color, the color would be unidentified unless the values for hue, saturation and brightness are known. However, to identify a color they would not need to know anything about its pitch, timbre or loudness, as these are quality dimensions of the sounds domain and play no role in helping us to identify the color. The quality dimensions within any single domain are integral, and this means that a quality dimension cannot bear a measurable value without the other quality dimensions also bearing some measurable value. The fundamental reason for decomposing a concept into domains is the assumption that the concept can be assigned certain properties within the domain that are independent of other properties.

One interesting feature about domains is that sometimes there are options when choosing which quality dimensions are necessary to represent the features being highlighted. Sometimes the underlying quality dimensions for a particular domain can be interchangeable. Take for example the color domain: it can either be represented by hue, brightness and saturation or these quality dimensions can be converted to red, blue and green (using the rbg scale). In some cases the quality dimensions for a domain may be not be known.



(a) Dimension (b) Domain

Fig. 2 Characteristics Lying within a Single Domain of a Conceptual Spaces Model

D. Properties

After the domains are built, convex regions of a single domain can start to be carved out to represent what are called *properties* of the given domain. In the color domain, for example, where the color blue is being examined, blue would be represented by a certain set of values from each of the quality dimensions within the color domain. It is well understood that there will be a range of alternative convex regions within the color space that could represent a particular color. In Fig. 3 the colors on the left side represent blue, which could be bounded by constraints on the hue, brightness and saturation dimensions. One can relax or constrict the constraints on the property to make it more or less specific. Some may say that true blue is only represented by a single point in the color domain, while others would allow for a range of lighter to darker blues to all be classified as blue.

A property has to be something that different individual objects can have in common. If two objects have the same particular property then in some respect they are similar to each other. Properties will differ from world to world. In this paper the world that is being examined is the world of the space domain.

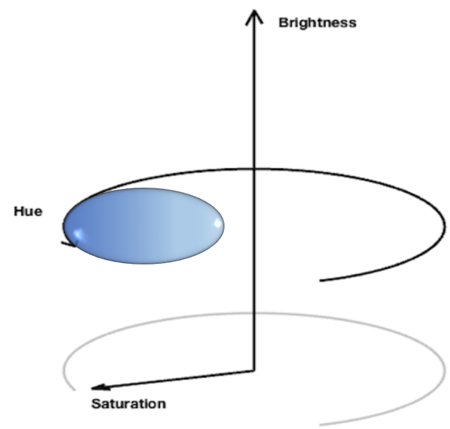


Fig. 3 Properties within the Color Domain

E. Concept

Alongside properties, concepts play a central role through the entire Conceptual Spaces model. A property is a type of concept that has only one domain, where concepts in general may span multiple domains, and they can in principle span all domains of a single Conceptual Spaces model. If there are multiple concepts that are being represented in a Conceptual Spaces model, then not all of them are required to have a property in all of the identified domains. However, if a domain in Conceptual Spaces model is such that no concept in the model has data to identify a property from that domain, then that domain is not relevant to the model and it can be deleted. Concepts are generally represented as nouns, since they describe a specific "Person," "Place," or "Thing." Work has been done to show that in addition to nouns, verbs (designating events) can also be represented as concepts in a Conceptual Spaces model [20].

A property is a subset of a concept; it is essentially the most basic type of concept. It is a concept with only one domain. Concepts can be used to build more sophisticated concepts. Take for example the concept of a dress suit. The dress suit could be broken down into the three domains: pants, shirt and shoes. Properties within these three domains can be thought of as concepts. These concepts can be broken down even further in order to identify properties that would represent the category of each domain. For example in the domain of pants, properties of pants might be: dress pants, jeans, sweats and yoga; these properties can now be treated as concepts, and a Conceptual Spaces model for these concepts can be constructed. Domains within the Conceptual Spaces model for the concept of pants might include material and color. From this example, jeans would have the property of denim for the material domain and blue for the color domain, while dress paints would have properties such as cotton, viscose or silk for material, and black, tan or grey for color. In theory, this breakdown of concepts can be continued until you achieve the most fine-grained features for your overall concept.

Concepts are unique from properties in that they span several domains. This is closer to how a human would identify objects, events and qualities in the world, as knowing only one property such as color generally is not enough to know what is being described. The distinction between properties and concepts is not used in traditional symbolic or associationist representation models. This distinction provides an added benefit to the Conceptual Spaces approach. Concepts can have associations between multiple domains; such links between domains are known as *cross-domain property associations* and are useful for building concepts.

F. Object

The final term in Conceptual Spaces that needs to be defined is *object*. An object is essentially just a point within the Conceptual Spaces model. It should have defined property values for each of the properties from one or more of the domains. Each object should also be associated with one or more observations that contain measurable features for the object being observed.Object in the Conceptual Spaces jargon comprehends both physical things and events. Each can be a target of observations. An object can be thought of as noun-like if the associated concept is a noun (that is a physical thing such as a spacecraft or rocket body), or it can be thought of as verb-like if it is an event that is extended in time (such as a satellite maneuver or spacecraft shadowing). If a Conceptual Spaces model is built correctly, then for each object there should be a concept with which the object is being compared.

III. Complex Conceptual Spaces − Single Observations

Now that we have an understanding of how Conceptual Spaces work a mathematical model for identifying similarities between concepts and observations can be outlined. It should be noted that Gärdenfors' work on Conceptual Spaces lacked any mathematical grounding beyond the notion of distance. Several mathematical models have however been developed to represent Conceptual Spaces since Gärdenfors original work, and some of these models are shown in [5, 6, 21–24]. Conceptual Spaces gives a geometric representation of how humans understand concepts. [5, 6] use a Conceptual Spaces − Mathematical Programming Hybrid model in order to describe the underlying mathematics. Mathematical programming, here, is a form of *optimization*, which is a means by which one minimizes or maximizes a function over a set of constraints. A set of constraints, also known as a constraint set, represents potential solutions of a problem that are either feasible or not feasible. Take for example the vehicle routing problem of determining "the optimal set of routes for a fleet of vehicles to traverse in order to deliver to a given set of customers?", where constraints that need to be considered might include the range of the vehicle, the driver's work day, that the driver needs periodic breaks, and so on. The *constraint set* for a given problem generates a region within the space of possible solutions consisting of those solutions that satisfy the problem's constraints, forming what is known as the *feasible set*. The functions to be minimized or maximized are known as the *objective functions*, and they can be optimized over the feasible set to find the best solution.

There are many types of optimization problems that can be implemented to solve different problems. Some of the most common types of optimization problems are known as linear programming problems [25]. The general form of linear programming is to minimize a linear objective function subject to linear constraints. Linear programs are problems that can be expressed in conical form as:

$$\begin{array}{ll} \text{Find a Vector} & \mathbf{x} \\ \text{That Maximizes} & \mathbf{c}^T\mathbf{x} \\ \text{Subject to} & A\mathbf{x} \leq \mathbf{b} \\ \text{and} & \mathbf{x} > \mathbf{0} \\ \text{with} & \mathbf{x} \in S \end{array}$$

where **x** is a vector of unknown variables (also known as the decision variables); **c** is the cost vector; **b** is a vector determined by the multiplication of the constraint matrix with the decision variables; A is the constraint matrix; and S is the set of decision variables. In general the constraint matrix does not need to be square; in other words the problem is underdetermined, and thus the problem is not easily solved. A more special form of linear programming is integer linear programming, in which all of the variables are restricted to take on integer values. Only some of the variables are restricted to take on integer values then this problem becomes a mixed integer linear programming problem [26].

The model focused on primarily throughout this paper is the complex Conceptual Spaces − Single Observation Model developed in [5, 6], which involves solving a linear integer optimization problem, in order to find the similarity values between observations and concepts. To set up the optimization problem we must first define a set of property combinations, for domain i and properties j, that can be observed together to represent a specific concept:

$$F = \left[\left(i^1, j^1\right),\ \left(i^2, j^2\right), \ldots,\ (i^n, j^n)\right] \tag{1}$$

For example, in the concept "Apple":

$$F_{apple} = \{(\text{Green, Sour, Smooth}),\ (\text{Red, Sweet, Smooth}),\ (\text{Brown, Bitter, Rotten}), \ldots\}$$

where the subscript on F represents the concept. The optimization problem is setup to find the best F that maximizes the cost function

$$\max \sum_i \sum_j s_{ij} x_{ij} \tag{2}$$

or to find the similarities between an observed object and the concept. Here s_{ij} is the similarity between the observation and the concept of property j from domain i. In integer programming x_{ij} represents the decision variable for property j from domain i. Not every decision variable needs to be taken into account, which leads to

$$x_{ij} = \begin{cases} 1, \text{ if property } j \text{ from domain } i \text{ is considered} \\ 0, \text{ otherwise} \end{cases}$$

In addition to this, a constraint set consisting of inequality and equality constraints needs to be formulated. Together these two types of constraints form the feasible regions for the concept being described, over which objective function is to be optimized. Constraints must be set up that represent the allowable pairs. For the equality constraints there can only be one decision variable from each domain that exists within the concept, stated mathematically as

$$\sum_{j=1}^{n_i} x_{ij} = \begin{cases} 1 \ \forall P^i \neq \emptyset \text{ (Domain } i \text{ exists within the concept)} \\ 0 \ \forall P^i = \emptyset \text{ (Domain } i \text{ does not exist within the concept)} \end{cases}$$

The equality set only allows for a single property to be chosen from each domain. The rationale for this is that two properties cannot overlap within a single domain, so only one property can exist for each observation. From this as a starting point, we can continue to construct the concept by building the inequality constraints or cross-property constraints that represent it. This can be done as in:

$$x_{ij} + x_{i'j'} \leqslant 1 \ \forall \{(i, \ j), (i', \ j')\} \in F$$

Since integer programming is being used, the decision variable can hold only integer values, i.e.

$$x_{ij} = 0 \text{ or } 1 \ \forall \ i, \ j$$

Once the optimization problem is formulated the goal is to maximize the objective function, shown in Eq. (2). The result from solving the linear optimization problem will be the objective value of the cost function. This value can be normalized by dividing by the number of domains within the concept. Once normalized, the results will be the percent similarity between the observation and the concept.

Now that there is an understanding of the mathematical framework, the effects of adding noise to the similarity values between the properties and observations can be explored. It is evident that no matter how these similarity values are derived, there is going to be some uncertainty in these values. This uncertainty will propagate through the Conceptual Spaces model, and result in uncertainties in the correlations of concepts and observations.

IV. Uncertainty Quantification

The effects of uncertainty can be examined by developing an example problem and adding random Gaussian noise into the similarity values of the properties in order to see when the associations start to break down. First a hypothetical Conceptual Spaces model will be developed that will have four concepts denoted A, B, C and D. For this model there will be five arbitrarily defined domains with the following set of arbitrarily defined number of quality dimensions in each domain {4, 3, 5, 3, 2}. Now that the domains and quality dimensions have been defined, the constraint sets shown in Table 1 can be used to represent the four concepts.

We can now proceed to solving the optimization problem subject to the cost function from Eq. (2). First, simulated observations can be generated by using the similarity values found in Table 2. These similarity values are meant to represent the four concepts. That is, observation A is meant to represent concept A, and so on for B, C and D. The normalized similarities between the four observations and the four concepts before any noise is added to the model can be shown in Table 3. The observation that represents the concept has the highest similarity value for that concept, which makes sense and indicates the observations are correctly identified as observations of their respective concepts.

A Monte-Carlo simulation is now performed with increasing variance on the similarity values, s_{ij}, between the properties and observations. The variance is increased from zero to twenty. Concept A only is used and note is taken of the number of times the algorithm is correct. These results are then plotted, as shown in Fig. 4. Once the variance passes a certain threshold, the Conceptual Spaces start to break down and are no longer reliable. Thus, knowing the input variance on the similarity values for the properties is important. The variances of hard data, such as the errors in

Table 1 Constraint Sets

Concept A	Concept B	Concept C	Concept D
$x_{11} = 1$	$x_{23} = 1$	$x_{12} = 1$	$x_{14} = 1$
$x_{21} = 1$	$x_{33} = 1$	$x_{22} = 1$	$x_{21} = 1$
$x_{32} = 1$	$x_{42} = 1$	$x_{43} = 1$	$x_{32} = 1$
$x_{41} = 1$	$x_{51} = 1$	$x_{52} = 1$	$x_{42} = 1$
$x_{11} + x_{22} \leq 1$	$x_{14} + x_{21} \leq 1$	$x_{12} + x_{42} \leq 1$	
$x_{13} + x_{22} \leq 1$	$x_{13} + x_{22} \leq 1$	$x_{12} + x_{32} \leq 1$	
$x_{14} + x_{21} \leq 1$		$x_{14} + x_{42} \leq 1$	
$x_{22} + x_{31} \leq 1$		$x_{14} + x_{22} \leq 1$	
$x_{11} + x_{32} \leq 1$			
$x_{13} + x_{32} \leq 1$			

position measurements, are easy to quantify. However, the variance of soft data, such as geopolitical tension or other data that comes from subjective reasoning, may be difficult to quantify. This uncertainty quantification is what leads to the use of Dempster-Shafer theory for similarity identification.

Table 2 Initial Observation Values

Observation A	Observation B	Observation C	Observation D
$s_{12} = 1$	$s_{23} = 1$	$s_{12} = 1$	$s_{14} = 1$
$s_{21} = 1$	$s_{33} = 1$	$s_{22} = 1$	$s_{23} = 1$
$s_{32} = 1$	$s_{42} = 1$	$s_{43} = 1$	$s_{32} = 1$
$s_{41} = 1$	$s_{51} = 1$	$s_{52} = 1$	$s_{41} = 1$

Table 3 Simulated Results with No Noise Added to Property Similarity Values

	Concept A	Concept B	Concept C	Concept D
Observation A	1	0	0.25	0.25
Observation B	0	1	0.25	0.25
Observation C	0.25	0	1	0
Observation D	0.25	0.5	0	1

V. Single Domain Conceptual Spaces

To find similarity values for a concept within a single domain (i.e. properties), Holender developed what he called Single Domain Conceptual Spaces. We will now show how Single Domain Conceptual Spaces are derived, in order to compare them to the Dempster-Shafer method. To begin the derivation, some basic notation for the representation of terminology in Conceptual Spaces must first be understood. Note that some of the notation used for Single Domain Conceptual Spaces differs from the notation used in Complex Conceptual Spaces – Single Observation. As in all Conceptual Space models, the starting point should be to define the world which has a set number of concepts denoted as

$$W = \text{World of interest composed of } m \text{ concepts}$$

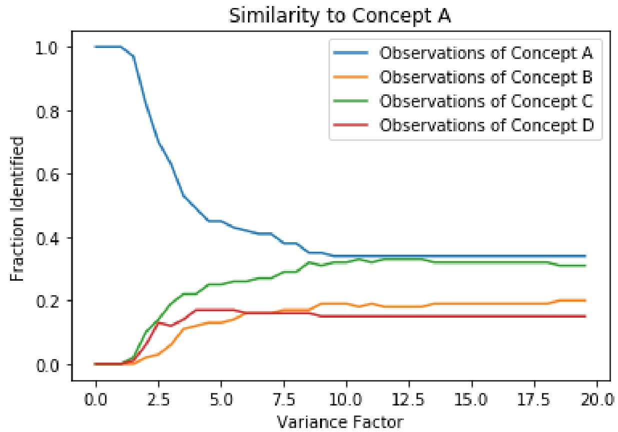


Fig. 4 Simulated Results with Increasing Noise Added to Property Similarity values

Within that world, there will be a library of m defined concepts with each concept denoted as

$$C^q = \text{Set of } m \text{ Concepts } q \text{ for } (q = 1, ..., m)$$

These concepts will be from a single domain, D. Within the world observations, O, of n objects, O can be made such that

$$O_o = \text{Set of } n \text{ Objects } o \text{ for } (o = 1, ..., n)$$

The observation is given by

$$s_{k,o} = \text{Observation } k \text{ of object } o$$

These can be be used to find the similarities, shown by

$$\mu^q_{o,k} = \text{Probability that concept } q \text{ is object } o \text{ under observation } k$$

$$\text{for } k = 1, ..., n_o, \forall\, q,\ o \text{ with } \sum_{q=1}^{m} \mu^q_{ok} = 1 \text{ for } k = 1, ..., n, \forall\, j$$

Finally the probabilistic matrix for each object can be constructed as

$$U_o = \begin{bmatrix} \mu_{o,1} \\ \mu_{o,2} \\ \vdots \\ \mu_{o,n_o} \end{bmatrix} = \begin{bmatrix} \mu^1_{o,1} & \cdots & \mu^m_{o,1} \\ \vdots & \ddots & \vdots \\ \mu_{o,n_o} & \cdots & \mu^m_{o,n_o} \end{bmatrix} = \text{Probabilistic Matrix for object } o\ \forall\, o$$

After the observations are made, four different methods are available in Single Domain Conceptual Spaces, to find the distance between the observation and the concept, known as the normalized distance. These four methods are shown here:

Average Normalized Distance

$$d^q_o = \frac{\sum_{k=1}^{n_o} \mu^q_{o,k}}{\sum_{h=1}^{m} \sum_{k=1}^{n_o} \mu^h_{o,k}} \quad \forall\, o, q \tag{3a}$$

Combined Mass Normalized Distance

$$d^q_o = \frac{\amalg_{k=1}^{n_o} \mu^q_{o,k}}{\sum_{h=1}^{m} \amalg_{k=1}^{n_o} \mu^h_{o,k}} \quad \forall\, o, q \tag{3b}$$

Minimum Mean Normalized Distance

$$d_o^q = \frac{\min\limits_{1 \le k \le n_o} \mu_{o,k}^q}{\sum_{h=1}^m \min\limits_{1 \le k \le n_o} \mu_{o,k}^h} \quad \forall\, o, q \tag{3c}$$

Maximum Mean Normalized Distance

$$d_o^q = \frac{\max\limits_{1 \le k \le n_o} \mu_{o,k}^q}{\sum_{h=1}^m \max\limits_{1 \le k \le n_o} \mu_{o,k}^h} \quad \forall\, o, q \tag{3d}$$

Now a simple example problem is shown. Say there is a space domain world with three identified concepts:

$$W = \{Satellite,\ Aircraft,\ Rocket\ Body\}$$
$$C^1 = Satellite,\ C^2 = Aircraft,\ C^3 = Rocket\ Body$$

Next, say two distinct objects have been identified, O_1 and O_2. For the first object three and for the second object four different observations are made. These observations will be made by experts who are not completely certain about the classification of what they are seeing, resulting in different experts having different beliefs in what concept the observation represents. The first and second object can be represented by the following values:

Object 1

$$\mu_{11}^1 = 0.8 \quad \mu_{11}^2 = 0.1 \quad \mu_{11}^3 = 0.1$$
$$\mu_{12}^1 = 0.7 \quad \mu_{12}^2 = 0.2 \quad \mu_{12}^3 = 0.1$$
$$\mu_{13}^1 = 0.6 \quad \mu_{13}^2 = 0.2 \quad \mu_{13}^3 = 0.2$$

Probabilistic Matrix for object 1

$$U_1 = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.7 & 0.2 & 0.1 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}$$

Object 2

$$\mu_{21}^1 = 0.0 \quad \mu_{21}^2 = 0.7 \quad \mu_{21}^3 = 0.3$$
$$\mu_{22}^1 = 0.1 \quad \mu_{22}^2 = 0.8 \quad \mu_{22}^3 = 0.1$$
$$\mu_{23}^1 = 0.1 \quad \mu_{23}^2 = 0.6 \quad \mu_{23}^3 = 0.3$$
$$\mu_{24}^1 = 0.2 \quad \mu_{24}^2 = 0.8 \quad \mu_{24}^3 = 0.0$$

Probabilistic Matrix for object 2

$$U_2 = \begin{bmatrix} 0.0 & 0.7 & 0.3 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0.2 & 0.8 & 0.0 \end{bmatrix}$$

Now the four different normalized distance from each object to each corresponding concept can be calculated using Eqs. (3a) through (3d), To get the following results:

Average Normalized Distance

$$d_1^1 = 0.7 \quad d_1^2 = 0.1667 \quad d_1^3 = 0.1333$$
$$d_2^1 = 0.1 \quad d_2^2 = 0.7250 \quad d_2^3 = 0.1750$$

Combined Mass Normalized Distance

$$d_1^1 = 0.9825 \quad d_1^2 = 0.0117 \quad d_1^3 = 0.0058$$
$$d_2^1 = 0 \quad d_2^2 = 1 \quad d_2^3 = 0$$

Minimum Mean Normalized Distance

$$d_1^1 = 0.75 \quad d_1^2 = 0.125 \quad d_1^3 = 0.125$$
$$d_2^1 = 0 \quad d_2^2 = 1 \quad d_2^3 = 0$$

Maximum Mean Normalized Distance

$$d_1^1 = 0.6667 \quad d_1^2 = 0.1667 \quad d_1^3 = 0.1667$$
$$d_2^1 = 0.1538 \quad d_2^2 = 0.6154 \quad d_2^3 = 0.2308$$

The highest similarity from each of the four different normalized distances for object 1 is to the $Satellite$. While the highest similarity from each of the four different normalized distances for object 2 is to the $Aircraft$. This is a clear indication that object 1 is most likely a satellite while object 2 is most likely an aircraft. However, the uncertainty in these measurements still remains unknown.

VI. Dempster-Shafer Theory

Dempster-Shafer theory is a general framework for reasoning with uncertainty, as well as connections to other frameworks, such as probability, possibility and imprecise probability theory. First proposed by Arthur P. Dempster [27], and later further developed as a framework for modeling epistemic uncertainty by his student Glenn Shafer [28], the resultant theory allows for the combining of evidence from different sources. It also allows for arriving at a degree of belief between the properties of an observation and the properties of a concept. For degree of belief (often referred to as its 'mass') the definition in subjective probability varies significantly. The first definition, which is used throughout this paper for degree of belief come from Ramsey and de Finetti. They define degrees of belief "in terms of what a person prefers, or would choose if given the option" [29–31]. De Finetti states that degree of belief or mass in a proposition of A is the odds at which you would regard a bet on A that pays \$1. An example, based on what you would be willing to bet for proposition A, is to say you are willing to bet \$0.75 then you are placing the odds of A occurring at 75%.

First, start by assuming that there is a collection of all the entities that one wishes to consider in a given situation. This collection of all entities will be represented by X. The power set can now be generated as

$$2^X \tag{4}$$

This power set is a set of all subsets including the empty set and the set itself. The reason the power set is base two is because, for every set in the power set there exists a complement set that includes all the elements of the universal set that are not present in the given set. For example, if the set only includes subsets $\{a\}$ and $\{b\}$ then the power set would be

$$2^X = \{\emptyset,\ \{a\},\ \{b\},\ X\} \tag{5}$$

Each subset is defined as a hypothesis and can receive a belief value between the bounds of zero and one. In the original from this we can infer:

$$m : 2^X \to [0,\ 1]$$

The assignment of beliefs is known as the Basic Probability Assignment (BPA) or Basic Belief Assignment. (Determining the BPA is still an issue for Dempster-Shafer Theory, though some methods are described in [32–35].) The BPA has two properties. The first is that the mass of the empty set is equal to zero, thus

$$m(\emptyset) = 0$$

The second is that the sum of the masses of all the members of the power set must add up to total 1, leading to

$$\sum_{A \in 2^X} m(A) = 1 \tag{6}$$

Intuitively, these statements make sense. The first statement simply states that one value from the set must be true, thus the probability of the statement falling under the empty set has to be zero. Say for example we have the power set to identify the color of red. The set would than be $\{\emptyset$, {red}, {not red}, {both red and not red}$\}$. Clearly the empty set needs to be equivalent to zero because we cannot be neither red nor not red. The same holds true for all entities, which is why the mass of the empty set is always equivalent to zero. The second simply states that all of the masses combined must sum up to one, thus indicating that all of the possible outcomes need to be represented by the set. Because the set includes the set and the complimentary set, the masses must add to one.

Next, the upper and lower bounds of a probability interval can be defined. In Dempster-Shafer theory, the lower bound of the probability of an event A occurring is defined by the amount of the belief in A, which is determined by the sum of the masses of all subsets of the hypothesis set. It is the amount of belief that directly supports either the given hypothesis or a more specific one. Belief is a measure of the strength of the evidence of the proposition. It ranges from zero, indicating no evidence, to one, indicating absolute certainty. According to Dempster-Shafer theory, the upper bound of the probability is defined by the plausibility of A. Plausibility is obtained by the sum of the masses of all sets whose intersection with the hypothesis is not empty.

This is mathematically expressed as:

$$bel(A) \ \leq\ P(A) \ \leq\ pl(A) \tag{7}$$

The belief for a set can be defined as the sum of all the masses from the subset of the set of interest, leading to

$$bel\,(A) = \sum_{B|B\cap A\subseteq\emptyset} m\,(B) \tag{8}$$

On the other hand the plausibility is the sum of all the masses of the sets B that intersects the set of interest A:

$$pl\,(A) = \sum_{B\ |\ B\cap A\neq\emptyset} m\,(B) \tag{9}$$

Summation of Eqs. (8) and (9) would result in one since everything is either enclosed in or not enclosed in set A. Rearranging results in the following:

$$pl\,(A) + bel\,(A') = 1 \rightarrow pl\,(A) = 1 - bel\,(A') \tag{10}$$

Conversely, the masses, $m\,(A)$, can be found with the inverse function, given by

$$m\,(A) = \sum_{B|B\subseteq A} (-1)^{|A-B|}\, bel\,(B) \tag{11}$$

where in Eq. (11) $|A - B|$ denotes the difference of the cardinalities, or number of elements, between the two sets.

To help illustrate the fact that belief is the lower bound and plausibility is the upper bound an example will be illustrated. If someone is trying to figure out if a satellite operators in a ground station are working or not. The hypothesis table shown in Table 4 can be set up. From the example, the masses all sum to 1 (i.e. $0.4 + 0.2 + 0.4 = 1$). In addition to this, it can be seen that the plausibility is equivalent to $1 - Bel(p')$ (i.e. $Pl(\text{Working}) = 1 - Bel(\text{Not Working}) = 1 - 0.2 = 0.8$).

Table 4 Probability Table Satellite Operators Working in Ground Station

Hypothesis	Mass	Belief	Plausibility
Null (neither working nor not working)	0	0	0
Working	0.4	0.4	0.8
Not working	0.2	0.2	0.6
Either (working or not working)	0.4	1.0	1.0

Dempster-Shafer theory is often used as a method for sensor data fusion. First, one must obtain a degree of belief for a question from subjective probability for multiple sensors. Degree of belief from subjective probability incorporates individuals' personal beliefs and intuitions, and its value is derived from domain experts in a particular field making predictions about the belief of an observation to a hypothesis. Next, implement Dempster's rule of combination from [36], given by

$$\begin{aligned} &m_{1,\,2}\,(\emptyset) = 0 \\ &m_{1,\,2}\,(A) = (m_1 \oplus m_2)\,(A) = \frac{1}{1 - \sum_{B\in C=\emptyset} m_1(B)m_2(C)} \sum_{B\in C=A\neq\emptyset} m_1\,(B)\,m_2\,(C) \end{aligned} \tag{12}$$

to combine degrees of belief from multiple sensors.

Often more than two sensors need to be combined at a given time. In the example from this paper the first object has three sensor readings while the second object has four sensor readings. Thus, we can infer that a method for handling multiple sensors is needed. Methods for combining three sensors together are discussed in [37, 38]. A visual representation of this method is shown in Fig. 5, the formula for doing so is given by:

$$m_{1,2,3}(D) = \frac{\sum_{A\in B\in C=D} m_1(A)m_2(B)m_3(C)}{1-K} \tag{13}$$

Where the degree of conflict for multiple sensor fusion can be expressed by

$$K = \sum_{A\in B\in C=\emptyset} m_1(A)m_2(B)m_3(C) \tag{14}$$

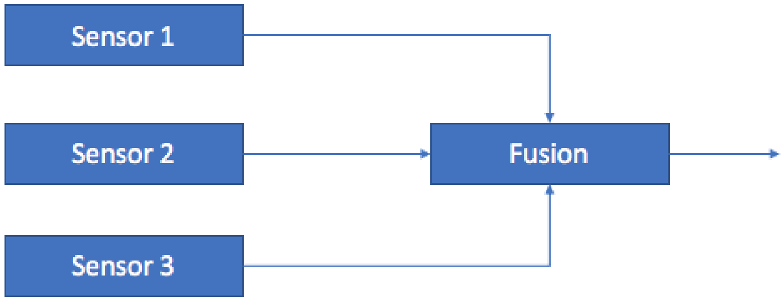


Fig. 5 Three Sensor Fusion

Alternatively, Challa and Koks show in [37] that Eq. (13) can be written as follows:

$$m_{1,2,3}(D) = \frac{\sum_{A \in B \in C = D} m_1(A) m_2(B) m_3(C)}{1 - \sum_{A \in B \in C = \emptyset} m_1(A) m_2(B) m_3(C)} = \frac{\sum_{A \in B \in C = D} m_1(A) m_2(B) m_3(C)}{\sum_{A \in B \in C \neq \emptyset} m_1(A) m_2(B) m_3(C)}$$

This method only allows for three sensor (i.e. three sources of information) fusion.

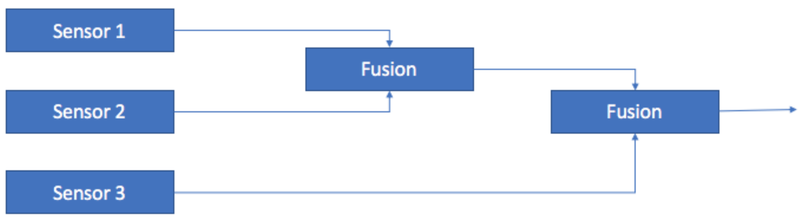


Fig. 6 Three Sensor Fusion in Steps

Often more than three sources of information are available, so a method for handling more than three sensors is required. A technique of this sort is given by Challa and Koks in [37], and can be represented by Fig. 6. This method is performed by taking the results of fusing the data from sensors 1 and 2, then using the two-sensor fusion technique to fuse this result with the data from a third sensor. The fused previously found results act, in effect, as a pseudosensor taking the place of a single sensor. When using this method that it is no longer possible to obtain the degree of conflict between the three sensors. Instead, the degree of conflict between a pseudosensor and the third sensor is obtained. This method can be repeated for fusion of data from more and more sensors.

VII. Comparing Results

We can now see that the Single Domain Conceptual Spaces model is essentially just another method for sensor fusion using Dempster's rule of combination. The overall goal is to achieve a higher similarity to the actual concept that the observation represents. From the results above it can be shown that, depending on the method used results vary greatly in the normalized distance between the four different techniques. For object one, the lowest normalized distance measurement is calculated to be 66% while the highest normalized distance is calculated 98.25%. For object two, these numbers ranged between 61.54% up to 100%.

Using the example of the $Satellite$, $Aircraft$ and $Rocket\ Body$ from Section V , by Dempster's rule of combination the following results can be calculated:

$$d_1^1 = 0.9333 \quad d_1^2 = 0.0069 \quad d_1^3 = 0.0031$$
$$d_2^1 = 0 \quad d_2^2 = 0.9825 \quad d_2^3 = 0$$

For object one, there is a 93.33% confidence of the object being concept one. This outperforms the results from the average normalized distance, the minimum mean normalized distance and the maximum mean normalized distance.

While for object two, there is a 98.25% confidence of the object being concept two. This outperforms the results from the average normalized distance and the maximum mean normalized distance.

In addition to this, the similarity between object one to $Aircraft$ and $Rocket\ Body$ should be zero, since it is known that this object is neither of these concepts. For these two similarity values, Dempster's rule of combination is closer to the truth than all four methods, with the single exception of the combined mean normalized distance, which performs better for observation 1 with the concept of a $Rocket\ Body$. For object 2, the similarity values between the $Satellite$ and the $Rocket\ Body$ should be exactly zero. Dempster's rule of combination yields the correct results not only for these values, but also for the combined mean normalized distance and the minimum mean normalized distance.

Another interesting feature of using Dempster's rule of combination is that the degree of conflict for object one can be tracked. This is because there are only three sensor sources being observed for this object. Thus, the degree of conflict can be calculated to be 0.1880 or in other words 18.8% conflict between measurements. As shown in section IV, this degree of conflict can be represented as noise in property similarity values. It would be wise to track degree of conflict between multiple sensors to see if one sensor has an abnormally high degree of conflict relative to other sensors. If this is so, it may be an outlier and should be considered for removal from the analysis.

VIII. Conclusion

In this paper a new method for solving Single Domain Conceptual Spaces is presented. This method involves using Dempster-Shafer theory for fusing together information from multiple sources of data. This method provides more accurate results when compared to normalized distance methods. It also allows for the tracking of degree of conflict which can be used to quantify variance that is normally difficult to quantify, such as with soft data. It is shown that, as variance in the property values increases, the results of the similarity between the observation and the overall concept will decrease. Thus, an accurate tracking of this variance is important for an understanding of accuracy of the overall results.

Acknowledgments

Our research team would like to thank Dr. Erik Blasch, Air Force Office of Scientific Research, for continuous support and funding of this project. Our group would also like to show our gratitude to Ms. Carolyn Sheaff of the Air Force Research Laboratory Information Directorate, and her colleagues for their continuous technical support on this project.

References

[1] Chapman, J. R., Kasmier, D., Limbaugh, D., Gagnon, S., Crassidis, J. L., Llinas, J., Smith, B., and Cox, A. P., "Conceptual Space Modeling for Space Event Characterization," *23rd International Conference on Information Fusion (FUSION)*, Rustenburg, South Africa, 2020, pp. 1–8. doi:10.23919/FUSION45008.2020.9190163.

[2] Chapman, J. R., Crassidis, J. L., Kasmier, D., Limbaugh, D., Gagnon, S., Llinas, J., Smith, B., and Cox, A. P., "Conceptual Spaces for Space Event Characterization via Hard and Soft Data Fusion," *AIAA SciTech 2021 Forum*, AIAA, 2021, pp. 1–18. doi:10.2514/6.2021-1163.

[3] Chapman, J. R., Kasmier, D., Crassidis, J. L., Llinas, J., Smith, B., and Cox, A. P., "An Introduction to Hard and Soft Data Fusion via Conceptual Spaces Modeling for Space Event Characterization," *National Symposium on Sensor & Data Fusion Committee (NSSDF)*, Military Sensing Symposia (MSS), 2021.

[4] Gärdenfors, P., *Conceptual Spaces: The Geometry of Thought*, MIT Press, Cambridge, MA, 2000. doi:10.7551/mitpress/2076.001.0001.

[5] Holender, M., Sudit, M., Nagi, R., and Rickard, J., "Information Fusion Using Conceptual Spaces: Mathematical Programming Models and Methods," *10th International Conference on Information Fusion (FUSION)*, Quebec City, Quebec, 2007, pp. 1–8. doi:10.1109/ICIF.2007.4408111.

[6] Holender, M. N., "Conceptual Spaces as a Modeling System for Information Fusion," Ph.D. thesis, University at Buffalo, State University of New York, 2008.

[7] Lewis, R., "Cognitive Modeling, Symbolic," *The MIT Encyclopedia of the Cognitive Sciences*, Vol. 132, edited by R. Wilson and F. Keil, MIT Press, Cambridge, MA, 1999, pp. 140–142. doi:10.7551/mitpress/1888.001.0001.

[8] Newman, M. H. A., *Alan Mathison Turing, 1912-1954*, Vol. 1, Royal Society, 1955. doi:10.1098/rsbm.1955.0019.

[9] Newell, A., and Simon, H. A., "Computer Science as Empirical Inquiry: Symbols and Search," *Communications of the ACM*, Vol. 19, No. 3, 1976, pp. 113–126. doi:10.1145/1283920.1283930.

[10] Carlston, D. E., *Associative Network*, SAGE Publications, Inc., Thousand Oaks, California, 2007. doi:10.4135/9781412956253.

[11] International Standards Organization, "Information Technology – Top-Level Ontologies (TLO) – Part 2: Basic Formal Ontology (BFO)," Standard, International Organization for Standardization, Geneva, CH, Oct. 2019.

[12] CUBRC, "Common Core Ontologies," `https://github.com/CommonCoreOntology/CommonCoreOntologies`, 2017-2020.

[13] CUBRC, "An Overview of the Common Core Ontologies 1.1," `https://github.com/CommonCoreOntology/CommonCoreOntologies/blob/master/documentationAnOverviewoftheCommonCoreOntologies1.1.docx`, 2019.

[14] Cox, A. P., Nebelecky, C. K., Rudnicki, R., Tagliaferri, W. A., Crassidis, J. L., and Smith, B., "The Space Object Ontology," *19th International Conference on Information Fusion (FUSION)*, Heidelberg, Germany, 2016, pp. 146–153.

[15] Cox, A. P., Nebelecky, C. K., Rudnicki, R., Tagliaferri, W. A., Crassidis, J. L., and Smith, B., "The Space Domain Ontologies," *National Symposium on Sensor & Data Fusion Committee (NSSDF)*, Military Sensing Symposia (MSS), 2021.

[16] Bechberger, L., and Kühnberger, K., "Towards Grounding Conceptual Spaces in Neural Representations," *CoRR*, Vol. abs/1706.04825, 2017.

[17] Gärdenfors, P., "Conceptual Spaces as a Framework for Knowledge Representation," *Mind and Matter*, Vol. 2, 2004, pp. 9–27. doi:10.1016/j.isci.2020.101429.

[18] Garner, W. R., *The Processing of Information and Structure*, 1st ed., Erlbaum, Potomac, MD, 1974. doi:10.4324/9781315802862.

[19] Melara, R. D., "The Concept of Perceptual Similarity: From Psychophysics to Cognitive Psychology," *Psychophysical Approaches to Cognition*, Advances in Psychology, Vol. 92, edited by D. Algom, North-Holland, 1992, pp. 303–388. URL `https://www.sciencedirect.com/science/article/pii/S0166411508617823`, doi:10.1016/S0166-4115(08)61782-3.

[20] Warglien, M., Gärdenfors, P., and Westera, M., "Event Structure, Conceptual Spaces and the Semantics of Verbs," *Theoretical Linguistics*, Vol. 38, 2012, pp. 159–193. doi:10.1515/tl-2012-0010.

[21] Aisbett, J., and Gibbon, G., "A General Formulation of Conceptual Spaces as a Meso Level Representation," *Artificial Intelligence*, Vol. 133, No. 1-2, 2001. doi:10.1016/s0004-3702(01)00144-8.

[22] Raubal, M., "Formalizing Conceptual Spaces," *Third International Conference on Formal Ontology in Information Systems*, 2004, pp. 153–164. doi:10.1007/978-3-319-67190-1-5.

[23] Rickard, J. T., and Yager, R. R., "Hypercube Graph Representations and Fuzzy Measures of Graph Properties," *IEEE Transactions on Fuzzy Systems*, Vol. 15, No. 6, 2007, pp. 1278–1293.

[24] McConky, K. T., "Applications of Location Similarity Measures and Conceptual Spaces to Event coreference and Classification," Ph.D. thesis, University at Buffalo, State University of New York, 2013.

[25] Schrijver, A., *Theory of Linear and Integer Programming*, John Wiley & Sons, Inc., USA, 1986.

[26] Nemhauser, G. L., and Wolsey, L. A., *Integer and Combinatorial Optimization*, Wiley-Interscience, USA, 1988.

[27] Dempster, A. P., "Upper and Lower Probabilities Induced by a Multivalued Mapping," *The Annals of Mathematical Statistics*, Vol. 38, No. 2, 1967, pp. 325–339. doi:10.1214/aoms/1177698950.

[28] Shafer, G., *A Mathematical Theory of Evidence*, Princeton University Press, Princeton, N.J., 1976.

[29] Jonathan, W., "Varieties of Bayesianism," *Inductive Logic*, Handbook of the History of Logic, Vol. 10, edited by D. M. Gabbay, S. Hartmann, and J. Woods, North-Holland, 2011, pp. 477–551. doi:10.1016/B978-0-444-52936-7.50013-6.

[30] Barlow, R. E., *Introduction to de Finetti (1937) Foresight: Its Logical Laws, Its Subjective Sources*, chapter and pages, pp. 127–133. doi:10.1007/978-1-4612-0919-5-9.

[31] Ramsey, F. P., "The Foundations of Mathematics," *Proceedings of the London Mathematical Society*, Vol. s2-25, No. 1, 1926, pp. 338–384. doi:10.1112/plms/s2-25.1.338.

[32] Bloch, I., "Some Aspects of Dempster-Shafer Evidence Theory for Classification of Multi-Modality Medical Images Taking Partial Volume Effect Into Account," *Pattern Recognit. Lett.*, Vol. 17, 1996, pp. 905–919.

[33] Yager, R. R., "A Class of Fuzzy Measures Generated From a Dempster-Shafer Belief Structure," *Int. J. Intell. Syst.*, Vol. 14, 1999, pp. 1239–1247. doi:10.1002/(SICI)1098-111X(199912)14:12<1239::AID-INT5>3.0.CO2-G.

[34] Zhu, Y., Bentabet, L., Dupuis, O., Kaftandjian, V., Babot, D., and Rombaut, M., "Automatic Determination of Mass Functions in Dempster-Shafer Theory Using Fuzzy C-Means and Spatial Neighbourhood Information for Image Segmentation," *Optical Engineering*, Vol. 41, 2002, pp. 760–770. doi:10.1117/1.1457458.

[35] Jiang, W., Duanmu, D., Fan, X., and Deng, Y., "A New Method to Determine Basic Probability Assignment Under Fuzzy Environment," *2012 International Conference on Systems and Informatics (ICSAI2012)*, 2012, pp. 758–762. doi:10.1109/ICSAI.2012.6223121.

[36] Jøsang, A., and Pope, S., "Dempster's Rule as Seen by Little Colored Balls," *Computational Intelligence*, Vol. 28, 2012, pp. 453–474. doi:10.1111/j.1467-8640.2012.00421.x.

[37] Challa, S., and Koks, D., "Bayesian and Dempster-Shafer Fusion," *Sãdhãna*, Vol. 29, 2004, pp. 145–174. doi:10.1007/BF02703729.

[38] Li, J., Luo, S., and Jin, J. S., "Sensor Data Fusion for Accurate Cloud Presence Prediction Using Dempster-Shafer Evidence Theory," *Sensors*, Vol. 10, No. 10, 2010, pp. 9384–9396. doi:10.3390/s101009384.