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Adaptive Regularization Minimization Algorithms with Non-Smooth Norms and Euclidean Curvature

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Abstract

A regularization algorithm (AR1pGN) for unconstrained nonlinear minimization is considered, which uses a model consisting of a Taylor expansion of arbitrary degree and regularization term involving a possibly non-smooth norm. It is shown that the non-smoothness of the norm does not affect the $\mathcal{O}(\epsilon_1^{-(p+1)/p})$ upper bound on evaluation complexity for finding first-order ϵ_1-approximate minimizers using p derivatives, and that this result does not hinge on the equivalence of norms in $\mathbb{R}^n$. It is also shown that, if $p = 2$, the bound of $\mathcal{O}(\epsilon_2^{-3})$ evaluations for finding second-order ϵ_2-approximate minimizers still holds for a variant of AR1pGN named AR2GN, despite the possibly non-smooth nature of the regularization term. Moreover, the adaptation of the existing theory for handling the non-smoothness results in an interesting modification of the subproblem termination rules, leading to an even more compact complexity analysis. In particular, it is shown when the Newton's step is acceptable for an adaptive regularization method. The approximate minimization of quadratic polynomials regularized with non-smooth norms is then discussed, and a new approximate second-order necessary optimality condition is derived for this case. An specialized algorithm is then proposed to enforce the first- and second-order conditions that are strong enough to ensure the existence of a suitable step in AR1pGN (when $p = 2$) and in AR2GN, and its iteration complexity is analyzed.

Keywords: nonlinear optimization, adaptive regularization, evaluation complexity, non-smooth norms, second-order minimizers.

Note: This paper is a close variant of [23] where the theory presented there is adapted to the use of Euclidean curvature for second-order optimality conditions, instead of the (potentially hard to compute) most negative curvature in a non-smooth norm. We have chosen to keep the complete presentation to preserve its self-contained character and because the necessary modifications to [23] are scattered throughout the text.

1 Introduction

This paper is concerned with the derivation of upper bounds on the evaluation complexity of adaptive regularization algorithms for the solution of the unconstrained nonconvex optimization problem

$$\min_{x\in\mathbb{R}^n} f(x). \tag{1.1}$$

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This research area has been remarkably active in recent years (see, for instance, [24, 30, 6, 8, 10, 3, 20, 4, 5, 28, 21, 2, 1, 13]). Adaptive regularization algorithms, the class of methods considered here, compute steps from one iterate to the next by building and (often approximately) minimizing a model consisting of a truncated Taylor expansion of f, which is then "regularized" by adding a suitable power of the norm of the putative step. Several authors have considered various smooth norms for this regularization term [30, 8, 18, 5, 16, 15], showing that, under suitable assumptions, the resulting adaptive regularization method must find a first-order ϵ_1-approximate minimizer for problem (1.1) (that is an iterate x_k such $\|\nabla_x^1 f(x_k)\| \leq \epsilon_1$) in at most $\mathcal{O}(\epsilon_1^{-(p+1)/p})$ evaluations of the objective function and its derivatives. In addition, second-order variants of this algorithm are bound to find a second-order ϵ_2-approximate minimizer (that is an iterate x_k such the smallest eigenvalue of $\nabla_x^2 f(x_k)$ exceeds $-\epsilon_2$) in at most $\mathcal{O}(\epsilon_2^{-(p+1)/(p-1)})$ such evaluations. The detailed algorithms considered in these contributions all depend on the central tenet that the regularized model (whose approximate minimization yields the step from one iterate to the next) is smooth, and thus that this approximate minimization can be carried out using algorithms for smooth functions and can be terminated using approximate optimality conditions for smooth problems. We show in this paper that the same evaluation complexity bounds still holds for first-order approximate minimizers in the case where non-smooth norms (such as ℓ_1 or ℓ_∞) are considered, provided the algorithm is suitably modified. We also show that, when $p = 2$, the evaluation complexity bound in $\mathcal{O}(\epsilon_2^{-3})$ is also maintained in the same context for a variant of the algorithm. Unsurprisingly, both results require redefining the termination conditions for model minimization. As it turns out, the resulting modifications of the standard adaptive regularization method are extremely simple and their use in the complexity theory results in a remarlably compact formulation.

One may argue that, since all norms are equivalent in finite dimensional spaces, the stated complexity bound can be derived for any norm from known results in Euclidean norm [5, 12]. While this is true if one focuses on the order in ϵ_1 and ϵ_2 only, this ignores the influence of the norm equivalence constants, whose size can be significant when n, the dimension of the problem, grows. For instance the equivalence constant between the Euclidean and infinity norm is proportional to the square root of the problem's dimension. Thus obtaining a given accuracy on the gradient norm in the infinity norm by simply applying the norm equivalence principle may require $n^{(p+1)/2p}$ times more evaluations of the objective function and its derivatives than in the Euclidean one. The approach presented here attempts to avoid this potentially problematic increase in cost.

Of course, for the new algorithms to be practical, one needs to show that the model minimization subproblems are solvable by implementable methods. Focusing again on the case where $p = 2$ and the model is a regularized quadratic, we derive a specialized second-order necessary optimality condition for the approximate minimization of such non-smooth functions. We then propose a new algorithm which is able to achieve first- and second-order approximate optimality for this problem and evaluate its iteration complexity. We finally discuss relaxed variants of the new algorithms that are sufficient for solving the subproblems of interest in algorithms for general functions, as well as their iteration complexity.

Our exposition is organized as follows. We present the problem and the first-order algorithm in Section 2 and derive its evaluation complexity theory in Section 3. Section 4 discusses the new approximate second-order necessary condition for global minimizers and establishes the upper bound on evaluation complexity for an adapted variant of the algo-

rithm. A method for approximately minimizing regularized quadratics (enough for solving the subproblems arising in Sections 2 and 4) is then presented and analyzed in Section 5. Finally, a brief conclusion is stated in Section 6.

2 An first-order adaptive regularization in general norms

We consider the unconstrained nonlinear optimization problem (1.1) where f is a (potentially nonconvex) p times continuoulsy differentiable function from $\mathbb{R}^n$ to $\mathbb{R}$, for some integer $p \geq 1$. We define

$$T_{f,p}(x,s) \stackrel{\rm def}{=} f(x) + \sum_{\ell=1}^{p} \frac{1}{\ell!} \nabla_x^\ell f(x)[s]^\ell,$$

the p-th order Taylor expansion of f at x, where the notation $\nabla_x^\ell f(x)[s]^\ell$ denotes the symmetric ℓ-dimensional tensor $\nabla_x^\ell f(x)$ applied on ℓ copies of the vector s.

As outlined in the introduction, adaptive regularization methods are iterative schemes that compute a step form an iterate x_k by constructing a regularized model $m_k(s)$ of $f(x_k+s)$ as

$$m_k(s) \stackrel{\rm def}{=} T_{f,p}(x_k,s) + \frac{\sigma_k}{(p+1)!}\|s\|_r^{p+1}, \tag{2.1}$$

where the p-th order Taylor series is "regularized" by adding the term $\sigma_k\|s\|_r^{p+1}/(p+1)!$ (σ_k is known as the "regularization parameter") and where we allow $\|\cdot\|_r$ to be a general possibly non-smooth norm. This implies that $\|\cdot\|_r$ is convex and Lipschitz continuous with global Lipschitz constant equal to one. Given the $\|\cdot\|_r$ norm and defining

$$\|S_j\|_{r,j} = \max_{\|s\|_r=1} |S_j[s]^j| \tag{2.2}$$

to be the norm of the j-dimensional symmetric (for $j>1$) tensor S_j induced by $\|\cdot\|_r$, we are now interested in finding, for some prespecified accuracy requirement $\epsilon_1 \in (0,1]$, an ϵ_1-approximate first-order critical point, that is a point x_{ϵ_1} such that $\|\nabla_x^1 f(x_\epsilon)\|_{r,1} \leq \epsilon_1$. Note that, because of (2.2), $\|\cdot\|_{r,1}$ is the dual norm of $\|\cdot\|_r$.)

The "regularization term" in (2.1) guarantees that $m_k(s)$ is bounded below and thus makes the procedure of finding a step s_k by (approximately) minimizing $m_k(s)$ well-defined. However, at variance with the usual setting for adaptive regularization methods, *the model $m_k(s)$ may no longer be smooth.* Once the step is computed, the value of the objective function at the trial point x_k+s_k is then computed. If the decrease in f from x_k to x_k+s_k is comparable to that predicted by the second-order Taylor series, the trial point is accepted as the new iterate and the regularization parameter is (possibly) reduced. If this is not the case, the trial point is rejected and the regularization parameter increased. The resulting algorithm is formally stated as the AR1pGN algorithm on the following page.

While the AR1pGN algorithm follows the main lines of existing adaptive regularization methods (see [8, 5] for example), we immediately note that the test (2.6) differs from the test $\|\nabla_s^1 m_s(s_k)\|_2 \leq \theta_1\|s_k\|_2^p$ which is used so far in the literature. Indeed, our framework no longer guarantees that $\nabla_s^1 m_s(s)$ exists, due to the possible lack of smoothness of the regularization term. Note however that, if $\|\cdot\|_r$ is differentiable everywhere except at the origin, then

$$\nabla_s^1 T_{f,p}(x_k,s_k) + \frac{\sigma_k}{p!}\|s\|_r^p \nabla_s^1 \|s\|_r = 0$$

Algorithm 2.1: First-Order Adaptive Regularization with General Norm (AR1pGN)

Step 0: Initialization. An initial point $x_0 \in \mathbb{R}^n$, a regularization parameter σ_0 and a desired final gradient accuracy $\epsilon_1 \in (0,1]$ are given. The constants η_1, η_2, γ_1, γ_2, γ_3, θ_1 and $\sigma_{\min}$ are also given such that

$$\sigma_{\min} \in (0, \sigma_0], \quad 0 < \eta_1 \leq \eta_2 < 1, \quad \theta_1 > 1 \quad \text{and} \quad 0 < \gamma_1 < 1 < \gamma_2 < \gamma_3. \tag{2.3}$$

Compute $f(x_0)$ and set $k = 0$.

Step 1: Check for termination. Terminate with $x_{\epsilon_1} = x_k$ if

$$\|\nabla_x^1 f(x_k)\|_{r,1} \leq \epsilon_1. \tag{2.4}$$

Step 2: Step calculation. Compute a step s_k which sufficiently reduces the model m_k in the sense that

$$m_k(s_k) \leq m_k(0) \tag{2.5}$$

and

$$\|\nabla_s^1 T_{f,p}(x_k, s_k)\|_{r,1} \leq \theta_1 \frac{\sigma_k}{p!} \|s\|_r^p. \tag{2.6}$$

Step 3: Acceptance of the trial point. Compute $f(x_k + s_k)$ and define

$$\rho_k = \frac{f(x_k) - f(x_k + s_k)}{T_{f,p}(x_k, 0) - T_{f,p}(x_k, s_k)}. \tag{2.7}$$

If $\rho_k \geq \eta_1$, then define $x_{k+1} = x_k + s_k$; otherwise define $x_{k+1} = x_k$.

Step 4: Regularization parameter update. Set

$$\sigma_{k+1} \in \begin{cases} [\max(\sigma_{\min}, \gamma_1 \sigma_k), \sigma_k] & \text{if } \rho_k \geq \eta_2, \\ [\sigma_k, \gamma_2 \sigma_k] & \text{if } \rho_k \in [\eta_1, \eta_2), \\ [\gamma_2 \sigma_k, \gamma_3 \sigma_k] & \text{if } \rho_k < \eta_1. \end{cases} \tag{2.8}$$

Increment k by one and go to Step 1.

at a nonzero first-order point of $m_k(s)$, and (2.6) holds at such a point since $\|s\|_r$ is Lipschitz continuous with unit Lipschitz constant, and thus $\|\nabla_s^1\|s\|_r\| \leq 1$. The condition (2.6) is therefore weaker than a more usual condition of the form $\|\nabla_s^1 m_k(s_k)\| \leq \theta_1\|s_k\|^p$.

Remarkably, and at variance with other adaptive regularization methods, the AR1*p*GN algorithm allows the Newton step $s_k = -\nabla_x^2 f(x_k)^{-1}\nabla_x^1 f(x_k)$ when $p = 2$ and the Hessian $\nabla_x^2 f(x_k)$ is positive definite, *provided the regularized model has not increased, that is provided (2.5) holds*. Indeed this step automatically ensures (2.6) since then $\nabla_s^1 T_{f,2}(x_k, s_k) = 0$. The condition (2.5) however avoids situations where the model decrease $m_k(0) - m_k(s_k)$ is tiny but $\|s_k\|_r$ is large, which is exactly what happens in the example of [11] showing convergence of Newton's method to a first-order ϵ_1-approximate minimizer in $\mathcal{O}(\epsilon_1^{-2})$ evaluations.

We also note that we could use an iteration-dependent $\theta_{1,k}$ in (2.6), provided it is bounded below by one and strictly bounded above by a constant. We have ignored this possibility for the sake of simplicity.

Having modified the requirements on the step, we now need to verify that the new conditions (2.5) and (2.6) are compatible. We start by deriving an expression for the subdifferential $\partial(\|\cdot\|_r^{p+1})(s)$.

Lemma 2.1 We have that

$$\partial(\|\cdot\|_r)(s) = \{v \in \mathbb{R}^n \mid v^T s = \|s\|_r \quad \text{and} \quad \|v\|_{r,1} = 1\} \tag{2.9}$$

and

$$\partial_C(\|\cdot\|_r^{p+1})(s) = \partial(\|\cdot\|_r^{p+1})(s) = (p+1)\|s\|_r^p\,\partial(\|\cdot\|_r)(s) \tag{2.10}$$

where ∂_C denotes the Clarke subdifferential.

Proof. The identity (2.9) is standard (see [25, Example 3.1] for instance). By composition of the norm with the increasing convex differentiable function $\phi(t) = t^{p+1}$ (on $\mathbb{R}^+$), we obtain from [25, Theorem 4.3.1] that

$$\partial(\|\cdot\|_r^{p+1})(s) = \{\alpha s \in \mathbb{R}^n s \mid \alpha \in \phi'(\|s\|_r) \text{ and } s \in \partial(\|\cdot\|_r)(s)\}.$$

which is the second equality in (2.10). Since $\|\cdot\|_r^{p+1}$ is also Lipschitz continuous, it is Clarke regular and thus the Clarke subdifferential and the standard one coincide (see [17, Proposition 4.3]), giving the first equality in (2.10). □

This allows us to derive the following characterization of a minimizer of m_k.

Lemma 2.2 Let s_k^* be a local minimizer of m_k. Then

$$\|\nabla_s^1 T_{f,p}(x_k, s_k^*)\|_{r,1} = \tfrac{1}{2}\sigma\|s_k^*\|_r^p. \tag{2.11}$$

Proof. Since m_k is Lipschitz continuous, the Clarke criticality of s_k^* implies that

$$0 \in \partial_C m_k(s_k^*) = \left\{\nabla_x^1 T_{f,p}(x_k, s_k^*)\right\} + \frac{\sigma_k}{(p+1)!}\partial_C(\|s_k^*\|_r^{p+1}), \tag{2.12}$$

where we have used the property of the Clarke subdifferential of the sum of two locally Lipschitz functions [17, Exercice 1.4] and the fact that, since $T_{f,p}(x_k, s)$ is continuously differentiable as a function of s, $\partial_C T_{f,p}(x, .)(s) = \{\nabla_s^1 T_{f,p}(x_k, s)\}$. Using now (2.10), we deduce from this identity and (2.12) that there exists a vector $\xi \in \partial(\|\cdot\|_r)(s_k^*)$ such that

$$\nabla_s^1 T_{f,p}(x_k, s_k^*) = -\frac{\sigma_k}{p!}\|s_k^*\|_r^p\ \xi. \tag{2.13}$$

Moreover, (2.9) implies that $\|\xi\|_{r,1} = 1$. Taking norms in this relation gives (2.11). □

The (scalar) necessary condition (2.11) is clearly weaker that the (vector) identity (2.12), but is nevertheless sufficient to derive the following crucial result.

Corollary 2.3 A step satisfying both (2.5) and (2.6) always exists.

Proof. From

$$m_k(s) \geq \frac{\sigma_k}{(p+1)!}\|s\|_r^{p+1} - |f(x)| - \sum_{\ell=1}^{p}\frac{1}{\ell!}\|\nabla_x^\ell f(x_k)\|_{r,\ell}\|s\|_r^\ell$$

we obtain $\lim_{\|s\|_r\to+\infty} m_k(s) = +\infty$ which, together with the continuity of $m_k(s)$, implies that m_k admits at least one minimizer s_k^* over $\mathbb{R}^n$, satisfying $m_k(s_k^*) \leq m_k(0)$. Applying Lemma 2.2 then gives that (2.6) holds at s_k^* for any $\theta_1 \geq 1$. □

An important comment is in order at this point. Because the Clarke subdifferential of the norm is not necessarily continuous in our context, it may seem at first sight that obtaining a step satisfying the conditions (2.5) and (2.6) may require the computation of an exact minimizer s_k^* of the model, which is potentially costly. Fortunately, this fear is unfounded because both the left- and the right-hand sides of (2.6) are continuous functions of s and the inequality therefore also holds in a neighbourhood of s_k^* provided $\theta_1 > 1$. Any convergent minimization algorithm (such as those proposed, for instance, in [19, 29, 27, 9, 14, 22] or, more generally, in [26], or in Section 5) applied on the model is therefore bound to produce a suitable step s_k in a finite number of iterations.

Following well-established practice, we now define

$$\mathcal{S} \stackrel{\rm def}{=} \{k \geq 0 \mid x_{k+1} = x_k + s_k\} = \{k \geq 0 \mid \rho_k \geq \eta_1\} \text{ and } \mathcal{S}_k \stackrel{\rm def}{=} \mathcal{S} \cap \{1, \ldots, k\},$$

the set of indeces of "successful iterations", and the set of indeces of successful iterations up to iteration k, respectively. We also recall a well-known result bounding the total number of iterations of an adpative regularization method in terms of the number of successful ones.

Lemma 2.4 Suppose that the AR1pGN algorithm is used and that $\sigma_k \leq \sigma_{\max}$ for some $\sigma_{\max} > 0$. Then

$$k \leq |\mathcal{S}_k| \left(1 + \frac{|\log \gamma_1|}{\log \gamma_2}\right) + \frac{1}{\log \gamma_2} \log \left(\frac{\sigma_{\max}}{\sigma_0}\right). \tag{2.14}$$

Proof. See [5, Theorem 2.4]. □

3 Evaluation complexity for the AR1pGN algorithm

Before discussing our analysis of evaluation complexity, we first formalize our assumptions on problem (1.1).

AS.1 f is p times differentiable and its p-th derivative $\nabla_x^p f(x)$ is is globally Lipschitz continuous in the $\|\cdot\|_r$ and $\|\cdot\|_2$ norms, that is there exists $L_{r,p}, L_{2,p} \geq 0$ such that

$$\|\nabla_x^p f(x) - \nabla_x^p f(y)\|_{r,p} \leq L_{r,p}\|x - y\|_r \quad \text{for all} \quad x, y \in \mathbb{R}^n, \tag{3.1}$$

where the $\|\cdot\|_r$ norm in the left-hand side is defined by (2.2), and

$$\|\nabla_x^p f(x) - \nabla_x^p f(y)\|_2 \leq L_{2,p}\|x - y\|_2 \quad \text{for all} \quad x, y \in \mathbb{R}^n. \tag{3.2}$$

AS.2 There exists a constant $f_{\rm low}$ such that $f(x) \geq f_{\rm low}$ for all $x \in \mathbb{R}^n$.

Assumption AS.2 ensures that problem (1.1) is well-defined. Assumption AS.1 recasts the usual context for the analysis of complexity of adaptive regularization methods in the context of the general norms, Note that assuming both (3.1) and (3.2) in AS.1 is important to avoid large dimension-dependent "norm-equivalence" constants in our final evaluation complexity bounds. Thus, for the bounds to be meaningful, we implicitly assume that $L_{2,p}$ and $L_{r,p}$ do not vary too much in size. AS.1 yields the well-known Lipschitz error bounds.

Lemma 3.1 Suppose that AS.1 holds and that $k \in \mathcal{S}$. Then

$$|f(x_{k+1}) - T_{f,p}(x_k, s_k))| \leq \frac{L_{r,p}}{(p+1)!}\|s_k\|_r^{p+1}, \tag{3.3}$$

$$\|\nabla_x^1 f(x_{k+1}) - \nabla_s^1 T_{f,p}(x_k, s_k)\|_{r,1} \leq \frac{L_{r,p}}{p!}\|s_k\|_r^p \tag{3.4}$$

and

$$\|\nabla_x^2 f(x_{k+1}) - \nabla_s^2 T_{f,p}(x_k, s_k)\|_2 \leq \frac{L_{2,p}}{(p-1)!}\|s_k\|_2^{p-1}. \tag{3.5}$$

Proof. The proof of (3.3) and (3.4) is a direct extension of [13, Lemma 2.1] with $\beta = 1$ that now uses (3.1) instead of (3.2) and exploits (2.2). It is given in the appendix of [23]. The inequality (3.5) immediately results from (3.2) and [13, Lemma 2.1]. □

From now on, the analysis in this section follows that presented in [5] quite closely. We first state a simple lower bound on the decrease of the Taylor expansion.

Lemma 3.2

$$\Delta T_{f,p}(x_k, s_k) \stackrel{\rm def}{=} T_{f,p}(x_k, 0) - T_{f,p}(x_k, s) \geq \frac{\sigma_k}{(p+1)!}\|s_k\|_r^{p+1}. \tag{3.6}$$

Proof. Direct from (2.5) and (2.1). □

We next derive an upper bound on the regularization parameter.

Lemma 3.3 Suppose that AS.1 holds. Then, for all $k \geq 0$,

$$\sigma_k \leq \sigma_{\max} \stackrel{\rm def}{=} \gamma_3 \max\left[\sigma_0, \frac{L_{r,p}}{(1-\eta_2)}\right]. \tag{3.7}$$

Proof. See [5, Lemma 2.2]. Using (2.7), (3.3), and (3.6), we obtain that

$$|\rho_k - 1| \leq \frac{(p+1)!|f(x_k+s_k) - T_{f,p}(x_k, s_k)|}{\sigma_k\|s_k\|_r^{p+1}} \leq \frac{L_{r,p}}{\sigma_k}.$$

Thus, if $\sigma_k \geq L_{r,p}/(1-\eta_2)$, then $\rho_k \geq \eta_2$, iteration k is successful and (2.8) implies that $\sigma_{k+1} \leq \sigma_k$. The mechanism of the algorithm then guarantees that (3.7) holds. □

The next lemma remains in the spirit of [5, Lemma 2.3], but now takes the new condition (2.6) into account, avoiding any reference to the model's derivative and resulting in a simpler proof.

Lemma 3.4 Suppose that AS.1 holds and that $k \in \mathcal{S}$ before termination. Then

$$\|s_k\|_r^p \geq \frac{p!}{L_{r,p} + \theta_1\sigma_{\max}}\,\epsilon_1. \tag{3.8}$$

Proof. Successively using the fact that termination does not occur at iteration k, the triangle inequality, (3.4) for $j = 1$, condition (2.6) and (3.7), we deduce that

$$\begin{aligned}
\epsilon_1 &< \|\nabla_x^1 f(x_{k+1})\|_{r,1} \\
&\leq \|\nabla_x^1 f(x_{k+1}) - \nabla_x^1 T_{f,p}(x_k, s_k)\|_{r,1} + \|\nabla_x^1 T_{f,p}(x_k, s_k)\|_{r,1} \\
&\leq \frac{L_{r,p}}{p!}\|s_k\|_r^p + \theta_1\frac{\sigma_k}{p!}\|s_k\|_r^p.
\end{aligned}$$

This in turn directly implies (3.8). □

We may now resort to the classical "telescoping sum" argument to obtain the desired complexity result.

Theorem 3.5 Suppose that AS.1–AS.2 hold. Then the AR1pGN algorithm requires at most

$$\frac{(p+1)!}{\eta_1\sigma_{\min}}\left(\frac{L_{r,p}+\theta_1\sigma_{\max}}{p!}\right)^{\frac{p+1}{p}}\frac{f(x_0)-f_{\low}}{\epsilon_1^{\frac{p+1}{p}}}$$

successful iterations and evaluations of $\{\nabla_x^i f\}_{i=1,2}$ and at most

$$\frac{(p+1)!}{\eta_1\sigma_{\min}}\left(\frac{L_{r,p}+\theta_1\sigma_{\max}}{p!}\right)^{\frac{p+1}{p}}\frac{f(x_0)-f_{\low}}{\epsilon_1^{\frac{p+1}{p}}}\left(1+\frac{|\log\gamma_1|}{\log\gamma_2}\right)+\frac{1}{\log\gamma_2}\log\left(\frac{\sigma_{\max}}{\sigma_0}\right)$$

evaluations of f to produce a vector $x_{\epsilon_1}\in\mathbb{R}^n$ such that $\|\nabla_x^1 f(x_{\epsilon_1})\|_{r,1}\leq\epsilon_1$.

Proof. Let k be the index of an iteration before termination. Then, using AS.2, the definition of successful iterations, (3.6) and (3.8),

$$|\mathcal{S}_k|\leq\frac{(p+1)!}{\eta_1\sigma_{\min}}\left(\frac{L_{r,p}+\theta_1\sigma_{\max}}{p!}\right)^{-\frac{p+1}{p}}\frac{f(x_0)-f_{\low}}{\epsilon_1^{\frac{p+1}{p}}}$$

for any k before termination, and the first conclusion follows since the derivatives are only evaluated once per successful iteration. Applying now Lemma 2.4 gives the second conclusion. □

4 Approximate second-order minimizers for $p=2$

We now turn the second-order case and from now on, limit our analysis to the case where $p=2$. We are thus interested in finding an approximate second-order minimizer, that is, for our present purposes, an iterate x_k such that

$$\lambda_{\min}[\nabla_x^2 f(x_k)]\geq-\epsilon_2, \tag{4.1}$$

In what follows, we assume that, for a symmetric H, we can compute $\lambda_{\min}[H]$ and a an associated vector u such that $\|u\|_2=1$. This is at variance with the approach taken in [23], where (4.1) is replaced by the condition that

$$\min_{\|v\|_r=1}\nabla_x^2 f(x_k)[v]^2\geq-\epsilon_2,$$

a much more difficult task for which approximations are often necessary.

We now establish a second-order necessary condition for a global minimizer of a regularized quadratic m. As a first step, we derive a lower bound on the model decrease that can be obtained along a direction of sufficient negative curvature.

Lemma 4.1 Let $\phi(s) = f_0 + \langle g, s\rangle + \frac{1}{2}\langle Hs, s\rangle$ be a quadratic polynomial in $s \in \mathbb{R}^n$, and $m(s) = \phi(s) + \frac{1}{6}\sigma\|s\|_r^3$, where $\sigma > 0$ is a constant and $\|\cdot\|_r$ is a general norm. Consider $s \neq 0$ and u an eigenvector corresponding to $\lambda_{\min}[H]$ with $\|u\|_2 = 1$ and whose sign is chosen to ensure that $\langle g + Hs, u\rangle \leq 0$. Also assume that $\lambda_{\min}[H] + \sigma\|s\|_r < 0$. Then there exists an $\alpha > 0$ such that

$$m(s) - m(s + \alpha\|s\|_r u) \geq \frac{3(\lambda_{\min}[H] + \sigma\|s\|_r)}{4\sigma^2}\left[\psi(s)\sigma^2\|s\|_r^2 - \frac{3}{4}(\lambda_{\min}[H] + \sigma\|s\|_r)^2\right], \tag{4.2}$$

where

$$\psi(s) \stackrel{\rm def}{=} \max\left[0, 1 + 2\frac{\langle g + Hs, u\rangle}{\sigma\|s\|_r^2}\right]. \tag{4.3}$$

Proof. Setting $d = \|s\|_r u$, we have that, for $\alpha > 0$,

$$\begin{aligned} m_k(s+\alpha d) &= m_k(s) + \alpha\langle g + Hs, d\rangle + \tfrac{1}{2}\alpha^2\langle Hd, d\rangle + \tfrac{1}{6}\sigma\|s + \alpha d\|_r^3 - \tfrac{1}{6}\sigma\|s\|_r^3 \\ &\leq m_k(s) + \tfrac{1}{2}\alpha\sigma\|s\|_r^3\left(2\frac{\langle g + Hs, u\rangle}{\sigma\|s\|_r^2}\right) + \tfrac{1}{2}\alpha^2\lambda_{\min}[H]\|s\|_r^2 + \tfrac{1}{6}\sigma\|s + \alpha s\|_r^3 - \tfrac{1}{6}\sigma\|s\|_r^3, \end{aligned}$$

where we have used the fact that $\|d\|_r = \|s\|_r$ implies the inequality $\|s+\alpha d\|_r^3 \leq \|s+\alpha s\|_r^3$. Moreover

$$\|s + \alpha s\|_r^3 - \|s\|_r^3 = \left[(1+\alpha)^3 - 1\right]\|s\|_r^3 = \alpha\left[3 + 3\alpha + \alpha^2\right]\|s\|_r^3,$$

and hence, using (4.3),

$$m_k(s + \alpha d) \leq m_k(s) + \tfrac{1}{6}\sigma\|s\|_r^3\left(3\alpha\psi(s) + 3\alpha^2 + \alpha^3\right) + \tfrac{1}{2}\alpha^2\lambda_{\min}[H]\|s\|_r^2. \tag{4.4}$$

This in turn yields that, for $\alpha > 0$,

$$m(s) - m(s+\alpha d) \geq -\frac{\alpha\|s\|_r^2}{2}\left[\frac{\sigma\|s\|_r}{3}\alpha^2 + (\lambda_{\min}[H] + \sigma\|s\|_r)\alpha + \sigma\|s\|_r\psi(s)\right] \stackrel{\rm def}{=} -\frac{\alpha\|s\|_r^2}{2}q_0(\alpha).$$

Now $q_0 = a\alpha^2 + b\alpha + c$ is a convex quadratic in α which admits a minimum for $\alpha = -b/(2a)$ of value $q(-b/(2a)) = c - b^2/(4a)$. Since $b = \lambda_{\min}[H] + \sigma\|s\|_r < 0$,

$$m(s) - m(s + \alpha d) \geq \left(\frac{3(\lambda_{\min}[H] + \sigma\|s\|_r)}{2\sigma\|s\|_r}\right)\frac{\|s\|_r^2}{2}\left[\sigma\|s\|_r\psi(s) - \frac{3(\lambda_{\min}[H] + \sigma\|s\|_r)^2}{4\sigma\|s\|_r}\right],$$

for $\alpha = -b/(2a) > 0$, which implies (4.2). □

This leads to the following necessary optimality condition.

Theorem 4.2 Let $\phi(s) = f_0 + \langle g, s\rangle + \frac{1}{2}\langle Hs, s\rangle$ be a quadratic polynomial in $s \in \mathbb{R}^n$, and assume that s_* is a global minimizer of $m(s) = \phi(s) + \frac{1}{6}\sigma\|s\|_r^3$, where $\sigma > 0$ is a constant and $\|\cdot\|_r$ is a general norm. Let u be an eigenvector corresponding to $\lambda_{\min}[H]$ such that $\|u\|_2 = 1$ and whose sign is chosen to ensure that $\langle g + Hs_*, u\rangle \leq 0$. Then

$$\lambda_{\min}[H] + \omega(s_*)\sigma\|s_*\|_r \geq 0, \tag{4.5}$$

where

$$\omega(s) \stackrel{\rm def}{=} \begin{cases} 1 + \dfrac{2\sqrt{\psi(s)}}{\sqrt{3}} \leq 1 + \dfrac{2}{\sqrt{3}} \stackrel{\rm def}{=} \kappa_\omega & \text{if } s \neq 0, \\ 1 & \text{otherwise,} \end{cases} \tag{4.6}$$

and $\psi(s)$ is given by (4.3).

Proof. When H is positive-semidefinite, (4.5) follows trivially. Assume now that H admits at least one negative eigenvalue. Suppose first that $s_* \neq 0$. If $\lambda_{\min}[H]+\sigma\|s_*\|_r \geq 0$, (4.5) trivially follows. Suppose thus that $\lambda_{\min}[H] + \sigma\|s_*\|_r < 0$. Then (4.2) implies that there exists an $\alpha > 0$ such that $m(s_* + \alpha\|s_*\|_r u) < m(s_*)$ (which is impossible), unless

$$\psi(s_*)\sigma^2\|s_*\|_r^2 > \frac{3}{4}(\lambda_{\min}[H] + \sigma\|s_*\|_r)^2.$$

If $\psi(s_*) = 0$, this cannot happen. Otherwise, this last inequality requires that

$$\sqrt{\psi(s_*)}\sigma\|s_*\|_r > \sqrt{\frac{3}{4}}\left|\lambda_{\min}[H] + \sigma\|s_*\|_r\right| > -\frac{\sqrt{3}}{2}\Big(\lambda_{\min}[H] + \sigma\|s_*\|_r\Big),$$

which, given (4.6), yields (4.5).

Suppose now that $s_* = 0$ and that $\lambda_{\min}[H] < 0$. It is then easy to verify that, if the sign of u is chosen to ensure that $\langle g, u\rangle \leq 0$ and

$$\alpha \in \left[0, -\frac{3\lambda_{\min}[H]}{2\sigma}\right],$$

then

$$m(\alpha u) = f_0 + \alpha\langle g, u\rangle + \tfrac{1}{2}\alpha^2\lambda_{\min}[H] + \tfrac{1}{6}\alpha^3\sigma \leq f_0 + \frac{9\lambda_{\min}[H]^3}{16\sigma^2} = m(0) + \frac{9\lambda_{\min}[H]^3}{16\sigma^2} < m(0), \tag{4.7}$$

which again contradicts the assumption that $s_* = 0$ is a global minimum of m. Thus $\lambda_{\min}[H] \geq 0$ and (4.5) also holds. □

It is remarkable that this lemma provides a "second-order" necessary condition for a global minimizer of quadratic polynomial regularized with a cubic term in a possibly non-smooth norm, despite the first and second derivatives of this objective function failing to exist.

It is interesting to pause at this point to stress that the necessary first- and second-order conditions (2.11) and (4.5), while sufficient for our purposes as we will see, are *merely necessary*, and by no means sufficient to guarantee a local minimizer. This is illustrated in

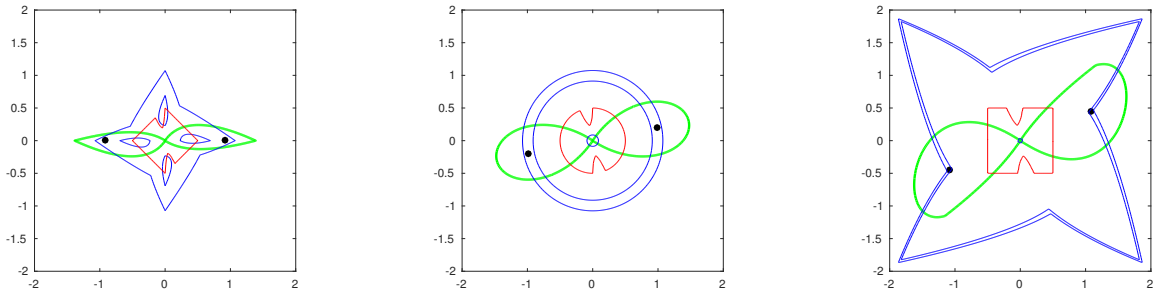

Figure 4.1: Admissible regions for the ℓ_1- (left), ℓ_2- (middle) and ℓ_∞- (right) norms

Figure 4.1 on the current page. In this figure, a two-dimensional model is constructed with a zero gradient at the origin and an indefinite Hessian[(1)], and with the regularization parameter σ is chosen equal to 6. The left picture corresponds to the choice $\|\cdot\|_r = \|\cdot\|_1$, the central one to $\|\cdot\|_r = \|\cdot\|_2$ and the right one to $\|\cdot\|_r = \|\cdot\|_\infty$, all other parameters being identical. In each case, the region of the plane where (2.5) holds is the interior of the two green lobes and the regions were the deviation from (2.11) is bounded by $0.01\,(\frac{1}{2})\sigma\|s\|_r^2$ are shown in blue, the first[(2)] being the small region surrounding the origin (where the gradient is zero) and the second the zone between the two blue concentric curves. Finally, the region where the deviation from (4.5) does not exceed $0.1\,\sigma\|s\|_r$ is the exterior of the region around the origin delineated in red. Note that, when it exists, the region around the origin which is admissible for (2.11) alone is excluded for (4.5). Thus in all cases, the admissible regions for (2.5), (2.11) and (4.5) consist of the regions limited by any of the shown curves and containing the minimizers marked with a black dot. We immediately notice that these regions are relatively large and may extend reasonably far from the minimizers. We also see that the geometry of these regions, while simple for the Euclidean norm, can be quite complicated for other norms.

Our algorithm for finding second-order ϵ_2-approximate minimizers is described on the following page.

As is the case for θ_1 in the AR1*p*GN algorithm, choosing an iteration dependent $\theta_{2,k}$ is possible provided it is strictly bounded below by 1 and bounded above by a constant.

The existence of a suitable step in the AR2GN algorithm directly hinges on Theorem 4.2.

Lemma 4.3 A step satisfying (2.5), (2.6) (for $p = 2$) and (4.10) always exists.

Proof. It follows from Corollary 2.3 and Theorem 4.2 (with $\phi(s) = T_{f,2}(x_k, s)$, $m = m_k$ and $\sigma = \sigma_k$) and the bound $\theta_2 > 1$ that the required conditions are satisfied at every global minimizer of the model m_k. □

As for the first-order case, continuity of $\|s\|_r$ and of $T_{f,2}(x, s)$ with respect to s implies that conditions (2.5), (2.6) and (4.10) also hold in a neighbourhood of a global minimizer whenever

(1) Chosen to be $I - 2uu^T/\langle u, u\rangle$ with $u^T = (5, 1)$.
(2) In the central and left pictures.

Algorithm 4.1: Second-Order Adaptive Regularization with General Norm (AR2GN)

Step 0: Initialization. An initial point $x_0 \in \mathbb{R}^n$, a regularization parameter σ_0 a desired final gradient accuracy $\epsilon, \epsilon_2 \in (0,1]$ and a model degree $p = 2$ are given. The constants η_1, η_2, γ_1, γ_2, γ_3, $\theta_1 > 1$, $\theta_2 > 1$, and $\sigma_{\min}$ are also given such that

$$\sigma_{\min} \in (0, \sigma_0], \;\; 0 < \eta_1 \leq \eta_2 < 1, \;\; \text{and} \;\; 0 < \gamma_1 < 1 < \gamma_2 < \gamma_3. \tag{4.8}$$

Compute $f(x_0)$ and set $k = 0$.

Step 1: Check for termination. Compute $\lambda_{\min}[\nabla_x^2 f(x_k)]$, and terminate with $x_\epsilon = x_k$ if

$$\|\nabla_x^1 f(x_k)\|_{r,1} \leq \epsilon_1 \quad \text{and} \quad \lambda_{\min}[\nabla_x^2 f(x_k)] \geq -\epsilon_2. \tag{4.9}$$

Step 2: Step calculation. Compute a step s_k which sufficiently reduces the model m_k in the sense that (2.5) and (2.6) hold (for $p = 2$) and, additionally,

$$\lambda_{\min}[\nabla_x^2 f(x_k)] + \theta_2 \omega(s_k) \sigma_k \|s_k\|_r \geq 0. \tag{4.10}$$

Step 3: Acceptance of the trial point. Compute $f(x_k + s_k)$ and define ρ_k as in (2.7). If $\rho_k \geq \eta_1$, then define $x_{k+1} = x_k + s_k$; otherwise define $x_{k+1} = x_k$.

Step 4: Regularization parameter update. Set σ_{k+1} according to (2.8), increment k by one and go to Step 1.

$\theta_1 > 1$ and $\theta_2 > 1$. Such a neigbourhood can be reached for instance by using the algorithm discussed in Section 5.

Noting that Lemmas 2.4, 3.1, 3.2 and 3.3 remain valid for the AR2GN algorithm, we now provide a lower bound on the length of the step, which simplifies that of [12, Lemma 3.4].

Lemma 4.4 Suppose that AS.1 holds for $p = 2$ and that $k \in \mathcal{S}$ before termination and such that $\lambda_{\min}[\nabla_x^2 f(x_{k+1})] < -\epsilon_2$. Then

$$\|s_k\|_r \geq \frac{1}{L_{2,p} + \theta_2 \kappa_\omega \sigma_{\max}} \epsilon_2.$$

Proof. Let $k \in \mathcal{S}$ such that $\lambda_{\min}[\nabla_x^2 f(x_{k+1})] < -\epsilon_2$. Since $\min_z[a(z) + b(z)] \geq \min_z a(z) + \min_z b(z)$, we deduce that

$$\begin{aligned}
\lambda_{\min}[\nabla_x^2 f(x_{k+1})] &= \min_{\|d\|_2=1} \langle \nabla_x^2 f(x_{k+1})d, d\rangle \\
&= \min_{\|d\|_2=1} \left[\langle \nabla_x^2 f(x_{k+1})d, d\rangle - \langle \nabla_x^2 f(x_k)d, d\rangle + \langle \nabla_x^2 f(x_k)d, d\rangle\right] \\
&\geq \min_{\|d\|_2=1} \left[\langle \nabla_x^2 f(x_{k+1})d, d\rangle - \langle \nabla_x^2 f(x_k)d, d\rangle\right] + \min_{\|d\|_2=1} \langle \nabla_x^2 f(x_k)d, d\rangle \\
&= \min_{\|d\|_2=1} \langle (\nabla_x^2 f(x_{k+1}) - \nabla_x^2 f(x_k))d, d\rangle + \min_{\|d\|_2=1} \langle \nabla_x^2 f(x_k)d, d\rangle \\
&\geq -\|\nabla_x^2 f(x_{k+1}) - \nabla_x^2 f(x_k)\|_2 + \lambda_{\min}[\nabla_x^2 f(x_k)] \\
&\geq -\left(L_{2,p}\|s_k\|_r + \theta_2 \omega(s_k)\sigma_k \|s_k\|_r\right),
\end{aligned}$$

where we also used (3.5) with $p = 2$, (4.10) and (4.6). The conclusion of the lemma then follows from Lemma 3.3 and the fact that $\lambda_{\min}[\nabla_x^2 f(x_{k+1})] < -\epsilon_2$. □

We conclude our analysis by stating our final evaluation complexity bound for finding second-order ϵ_2-approximate minimizers.

Theorem 4.5 Suppose that AS.1–AS.2 hold for $p = 2$ and let

$$\kappa_{\mathsf{AR2GN}} \stackrel{\rm def}{=} \max\left\{ \left[\tfrac{1}{2}(L_{r,p} + \theta_1\sigma_{\max})\right]^{3/2}, \left[L_{2,p} + \theta_2\kappa_\omega\sigma_{\max}\right]^3, \right\}.$$

Then the AR2GN algorithm requires at most

$$\left(\frac{6\,\kappa_{\mathsf{AR2GN}}}{\eta_1\sigma_{\min}}\right) \frac{f(x_0) - f_{\rm low}}{\min\left[\epsilon_1^{3/2}, \epsilon_2^3\right]} \tag{4.11}$$

successful iterations and evaluations of g and H and at most

$$\left(\frac{6\,\kappa_{\mathsf{AR2GN}}}{\eta_1\sigma_{\min}}\right) \frac{f(x_0) - f_{\rm low}}{\min\left[\epsilon_1^{3/2}, \epsilon_2^3\right]} \left(1 + \frac{|\log\gamma_1|}{\log\gamma_2}\right) + \frac{1}{\log\gamma_2}\log\left(\frac{\sigma_{\max}}{\sigma_0}\right) \tag{4.12}$$

evaluations of f to produce a vector $x_\epsilon \in \mathbb{R}^n$ such that

$$\|\nabla_x^1 f(x_\epsilon)\|_{r,1} \leq \epsilon_1 \quad \text{and} \quad \lambda_{\min}[\nabla_x^2 f(x_\epsilon)] \geq -\epsilon_2.$$

Proof. We prove the upper bounds (4.11) and (4.12) on the number of evaluations requested to produce an iterate x_ϵ at iteration k_ϵ such that

$$\|\nabla_x^1 f(x_\epsilon)\|_{r,1} \leq \epsilon_1 \quad \text{and} \quad \lambda_{\min}[\nabla_x^2 f(x_\epsilon)] \geq -\epsilon_2$$

is identical to that of Theorem 3.5, except that $p = 2$ and the decrease

$$\left(\frac{p!}{L_{r,p} + \theta_1\sigma_{\max}}\right)^{\frac{p+1}{p}} \epsilon_1^{\frac{p+1}{p}}$$

is now replaced, using Lemmas 3.4 and 4.4, by

$$\min\left\{ \left(\frac{2}{L_{r,p} + \theta_1\sigma_{\max}}\right)^{\frac{3}{2}} \epsilon_1^{\frac{3}{2}}, \left(\frac{1}{L_{2,p} + \theta_2\kappa_\omega\sigma_{\max}}\right)^3 \epsilon_2^3 \right\}.$$

yielding the constant κ_{AR2GN}. □

5 An algorithm for approximate minimization of regularized quadratics

The rest of the paper is devoted to the definition and analysis of a method whose purpose is to minimize a model of the form

$$m(s) = f_0 + \langle g, s\rangle + \tfrac{1}{2}\langle Hs, s\rangle + \tfrac{1}{6}\sigma\|s\|_r^3, \tag{5.1}$$

approximately, but enough for the conditions requested in Step 2 of the AR1pGN (for $p = 2$) and AR2GN algorithms to hold for $m = m_k$. We first state a simple technical lemma.

Lemma 5.1 Consider the quadratic polynomial $q(t) = at^2 + bt + c$ with $a \neq 0$ and $c > 0$. Then, for any $\nu > 0$,

$$q(t_*) > \tfrac{1}{2}c \quad \text{for} \quad t_* = \min\left[\frac{c}{\nu + 3|b|}, \frac{1}{3}\sqrt{\frac{c}{|a|}}\right]. \tag{5.2}$$

Proof. We immediately deduce that

$$q(t_*) \geq c - |b|\left(\frac{c}{\nu + 3|b|}\right) - |a|\left(\frac{1}{9}\frac{c}{|a|}\right) \geq c\Big(1 - \tfrac{1}{3} - \tfrac{1}{9}\Big) > \tfrac{1}{2}c.$$

□

The constant ν in (5.2) is introduced to safeguard against $b = 0$ and its value can be chosen for convenience in what follows.

We may start building our specialized method (which we will call the RQMIN algorithm) for minimizing the regularized quadratic (5.1). The algorithm will unsurprisingly be iterative and we will denote its successive iterates by $\{s_k\}_{k\geq 0}$ (the index k refers, for the rest of this section, to RQMIN iterations, and thus $g_k = g + Hs_k$). We will also make the choice to start from the origin, that is $s_0 = 0$. Moreover, we will construct the iterates s_k to ensure that the sequence $\{m(s_k)\}_{k\geq 0}$ is monotonically decreasing from $m(0)$. To motivate the forthcoming detailed description of the algorithm ensuring this property, we now consider the magnitude of the model decrease which can be obtained at a given iterate s_k, if any. We know from Lemma 2.2 that, if s_k were a local minimizer of m, then

$$\|g_k\|_{r,1} = \tfrac{1}{2}\|s_k\|_r^2. \tag{5.3}$$

This is the condition that the RQMIN algorithm will strive to achieve. If (5.3) fails, we will now show that taking a step from s_k along a well chosen direction d_k does produce a model decrease

$$\Delta m(\alpha) \stackrel{\text{def}}{=} m(s_k) - m(s_k + \alpha d_k) = -\alpha\langle g_k, d_k\rangle - \tfrac{1}{2}\alpha^2\langle Hd_k, d_k\rangle - \tfrac{1}{6}\sigma\|s_k + \alpha d_k\|_r^3 + \tfrac{1}{6}\sigma\|s_k\|_r^3 \tag{5.4}$$

which is suitably large. We start by analyzing the case where the step is too short (in view of (5.3)), in which case a generalized "Cauchy point" will provide adequate descent.

Lemma 5.2 Let $s_k \in \mathbb{R}^n$ such that $m(s_k) \leq m(0)$ and

$$\|g_k\|_{r,1} \geq \tfrac{1}{2}\sigma\|s_k\|_r^2. \tag{5.5}$$

Then

$$m(s_k) - m(s_k^C) \geq \tfrac{1}{2}\min\left[\frac{\left|\|g_k\|_{r,1} - \frac{1}{2}\sigma\|s_k\|_r^2\right|^2}{1+\frac{3}{2}\left(\|H\|_{r,2}+\sigma\|s_k\|_r\right)}, \frac{\left|\|g_k\|_{r,1} - \frac{1}{2}\sigma\|s_k\|_r^2\right|^{\frac{3}{2}}}{3\sqrt{\sigma}}\right], \tag{5.6}$$

where $g_k = g + Hs_k$ and

$$s_k^C = s_k + \alpha_k^C d_k, \quad \text{with} \quad d_k = \arg\min_{\|v\|_r=1}\langle g_k, v\rangle \quad \text{and} \quad \alpha_k^C = \arg\min_{\alpha>0} m(s_k+\alpha d_k). \tag{5.7}$$

Proof. If $\|g_k\|_{r,1} = \frac{1}{2}\sigma\|s_k\|_r^2$, the definition of s_k^C implies that $m(s_k) - m(s_k^C) \geq 0$ and (5.6) trivially follows. Suppose therefore that the inequality in (5.5) is strict, and consider the unidimensional minimization of $m(s_k+\alpha d_k)$ as a function of the scalar α, giving (5.4). Suppose first that $s_k = 0$ and thus that $g_k = g$. Then

$$\Delta m(\alpha) = \alpha\, q_0(\alpha) \quad \text{where} \quad q_0(\alpha) = \|g_k\|_{r,1} - \tfrac{1}{2}\alpha\langle Hd_k, d_k\rangle - \tfrac{1}{6}\sigma\alpha^2.$$

We have that $q_0(0) = \|g_k\|_{r,1} > 0$ and $q_0(\alpha)$ is a concave quadratic. Hence the equation $q_0(\alpha) = 0$ has a positive real root and we may apply Lemma 5.1 with $\nu = 1$ to deduce that

$$q_0(\alpha_0) > \tfrac{1}{2}\|g_k\|_{r,1} \quad \text{where} \quad \alpha_0 = \min\left[\frac{\|g_k\|_{r,1}}{1+\frac{3}{2}|\langle Hd_k,d_k\rangle|}, \frac{1}{3}\sqrt{\frac{\|g_k\|_{r,1}}{\frac{1}{6}\sigma}}\right],$$

and thus that

$$\Delta m(\alpha_0) \geq \alpha_0\, q_0(\alpha_0) > \tfrac{1}{2}\|g_k\|_{r,1}\min\left[\frac{\|g_k\|_{r,1}}{1+\frac{3}{2}\|H\|_{r,2}}, \frac{1}{3}\sqrt{\frac{\|g_k\|_{r,1}}{\sigma}}\right]. \tag{5.8}$$

Suppose now that $\|s_k\|_r > 0$ and define $v_k = \|s_k\|_r d_k$. Then, because $\|v_k\|_r = \|s_k\|_r$, we have that $\|s_k+\alpha s_k\|_r \geq \|s_k+\alpha v_k\|_r$ and hence, from (5.4),

$$\Delta m(\alpha) \geq \alpha\|g_k\|_{r,1}\|s_k\|_r - \tfrac{1}{2}\alpha^2\langle Hv_k, v_k\rangle - \tfrac{1}{6}\sigma\|s_k+\alpha s_k\|_r^3 + \tfrac{1}{6}\sigma\|s_k\|_r^3. \tag{5.9}$$

Observe now that

$$\|s_k+\alpha s_k\|_r^3 - \|s_k\|_r^3 = [(1+\alpha)^3 - 1]\|s_k\|_r^3 = \alpha(3+3\alpha+\alpha^2)\|s_k\|_r^3,$$

and thus (5.9) becomes

$$\Delta m(\alpha) \geq \alpha\left[\|g_k\|_{r,1}\|s_k\|_r - \tfrac{1}{2}\alpha\langle Hv_k, v_k\rangle - \tfrac{1}{6}\sigma(3+3\alpha+\alpha^2)\|s_k\|_r^3\right] \stackrel{\rm def}{=} \alpha\, q_1(\alpha), \tag{5.10}$$

where

$$q_1(\alpha) = (\|g_k\|_{r,1}\|s_k\|_r - \beta) - \alpha\left(\tfrac{1}{2}\langle Hv_k, v_k\rangle + \beta\right) - \alpha^2\left(\tfrac{1}{3}\beta\right),$$

where we have defined $\beta \stackrel{\rm def}{=} \frac{1}{2}\sigma\|s_k\|_r^3$. Note that the constant term is positive because we have assumed that $\|g_k\|_{r,1} > \frac{1}{2}\sigma\|s_k\|_r^2$. Moreover, $q_1(\alpha)$ is concave. As above, this implies that the equation $q_1(\alpha) = 0$ has a positive real root and we may then apply Lemma 5.1 with $\nu = \frac{1}{2}\|s_k\|_r^2$ to deduce that

$$q_1(\alpha_1) > \frac{1}{2}\left(\|g_k\|_{r,1}\|s_k\|_r - \beta\right)$$

where

$$\alpha_1 \stackrel{\rm def}{=} \min\left[\frac{\|g_k\|_{r,1}\|s_k\|_r - \beta}{\frac{1}{2}\|s_k\|_r^2 + 3\,|\frac{1}{2}\langle Hv_k, v_k\rangle + \beta|}, \frac{\sqrt{\|g_k\|_{r,1}\|s_k\|_r - \beta}}{3\sqrt{\frac{1}{3}\beta}}\right],$$

and hence, from (5.10),

$$\begin{aligned}\Delta m(\alpha_1) &\geq \alpha_1\, q(\alpha_1)\\ &\geq \frac{1}{2}\left(\|g_k\|_{r,1}\|s_k\|_r - \beta\right)\min\left[\frac{\|g_k\|_{r,1}\|s_k\|_r - \beta}{\frac{1}{2}\|s_k\|_r^2 + 3\,|\frac{1}{2}\langle Hv_k, v_k\rangle + \beta|}, \frac{\sqrt{\|g_k\|_{r,1}\|s_k\|_r - \beta}}{3\sqrt{\frac{1}{3}\beta}}\right]\\ &= \frac{1}{2}\min\left[\frac{(\|g_k\|_{r,1}\|s_k\|_r - \beta)^2}{\frac{1}{2}\|s_k\|_r^2 + 3|\frac{1}{2}\langle Hv_k, v_k\rangle + \beta|}, \frac{(\|g_k\|_{r,1}\|s_k\|_r - \beta)^{\frac{3}{2}}}{3\sqrt{\frac{1}{3}\beta}}\right].\end{aligned}$$

Using the identity $\|v_k\|_r = \|s_k\|_r$ and substituting the definition of β, this finally gives that

$$\Delta m(\alpha_1) \geq \frac{1}{2}\min\left[\frac{(\|g_k\|_{r,1} - \frac{1}{2}\sigma\|s_k\|_r^2)^2}{1 + \frac{3}{2}(\|H\|_{r,2} + \sigma\|s_k\|_r)}, \frac{(\|g_k\|_{r,1} - \frac{1}{2}\sigma\|s_k\|_r^2)^{\frac{3}{2}}}{3\sqrt{\sigma}}\right]. \tag{5.11}$$

Combining now (5.8) and (5.11) gives (5.6). □

As is standard in Cauchy point approaches, the step d_k in (5.7) is made in the direction of the steepest descent for the *unregularized* quadratic, that is *ignoring the regularization term*.

We now consider the alternative to (5.5), which, as (5.3) indicates, means that the step s_k is too large. It therefore makes sense to consider moving back from s_k towards the origin.

Lemma 5.3 Let $s_k \in \mathbb{R}^n$ such that $m(s_k) \leq m(0)$ and

$$\|g_k\|_{r,1} < \tfrac{1}{2}\sigma\|s_k\|_r^2. \tag{5.12}$$

Then

$$m(s_k) - m(s_k^R) \geq \tfrac{1}{2}\min\left[\frac{\left|\|g_k\|_{r,1} - \frac{1}{2}\sigma\|s_k\|_r^2\right|^2}{1 + \frac{3}{2}\left(\|H\|_{r,2} + \sigma\|s_k\|_r\right)}, \frac{\left|\|g_k\|_{r,1} - \frac{1}{2}\sigma\|s_k\|_r^2\right|^{\frac{3}{2}}}{3\sqrt{\sigma}}\right], \tag{5.13}$$

where $g_k = g + Hs_k$ and

$$s_k^R = (1 - \alpha_k^R)s_k \;\text{ with }\; \alpha_k^R = \arg\min_{\alpha>0} m(s_k - \alpha s_k). \tag{5.14}$$

Proof. Note that (5.12) implies that $s_k \neq 0$. Then, from (5.4) with $d_k = -s_k$,

$$\Delta m(\alpha) \geq \alpha\langle g_k, s_k\rangle - \tfrac{1}{2}\alpha^2\langle Hs_k, s_k\rangle - \tfrac{1}{6}\sigma\|s_k - \alpha s_k\|_r^3 + \tfrac{1}{6}\sigma\|s_k\|_r^3. \tag{5.15}$$

Now

$$\|s_k - \alpha s_k\|_r^3 - \|s_k\|_r^3 = [(1-\alpha)^3 - 1]\|s_k\|_r^3 = -\alpha(3 - 3\alpha + \alpha^2)\|s_k\|_r^3,$$

so that, from (5.15),

$$\Delta m(\alpha) \geq \alpha\left[\langle g_k, s_k\rangle - \tfrac{1}{2}\alpha\langle Hs_k, s_k\rangle + \tfrac{1}{6}\sigma(3 - 3\alpha + \alpha^2)\|s_k\|_r^3\right] \stackrel{\rm def}{=} \alpha\, q_2(\alpha), \tag{5.16}$$

where

$$q_2(\alpha) = (\langle g_k, s_k\rangle + \beta) + \alpha\left(-\tfrac{1}{2}\langle Hs_k, s_k\rangle - \beta\right) + \alpha^2\left(\tfrac{1}{3}\beta\right).$$

Observe now that, because $s_k \neq 0$ and, since we have assumed that $m(s_k) \leq m(0)$, we have that $q_2(1) = \Delta m(1) \leq 0$. Moreover, the Cauchy-Schwarz inequality yields that

$$|\langle g_k, s_k\rangle| \leq \|g_k\|_{r,1}\|s_k\|_r < \tfrac{1}{2}\sigma\|s_k\|_r^3 = \beta$$

and hence

$$q_2(0) = \langle g_k, s_k\rangle + \beta > 0.$$

This in turn implies the existence of a real root of $q_2(\alpha)$ in $(0, 1]$, and we may then again apply Lemma 5.1 with $\nu = \frac{1}{2}\|s_k\|_r^2$ to deduce that

$$q_2(\alpha_2) > \tfrac{1}{2}\big(\langle g_k, s_k\rangle + \beta\big) \tag{5.17}$$

where

$$\alpha_2 = \min\left[\frac{\langle g_k, s_k\rangle + \beta}{\frac{1}{2}\|s_k\|_r^2 + 3\,|-\frac{1}{2}\langle Hs_k, s_k\rangle + \beta|}, \frac{1}{3}\sqrt{\frac{\langle g_k, s_k\rangle + \beta}{\frac{1}{3}\beta}}\right]. \tag{5.18}$$

Moreover

$$\langle g_k, s\rangle + \beta \geq -\|g_k\|_{r,1}\|s\|_r + \tfrac{1}{2}\sigma\|s\|_r^3 > 0.$$

Combining this bound with (5.16), (5.18) and (5.17), we obtain that

$$\Delta m(\alpha_2) \geq \tfrac{1}{2}\min\left[\frac{(\frac{1}{2}\sigma\|s_k\|_r^2 - \|g_k\|_{r,1})^2}{1 + \frac{3}{2}\left(\|H\|_{r,2} + \sigma\|s_k\|_r\right)}, \frac{(\frac{1}{2}\sigma\|s_k\|_r^2 - \|g_k\|_{r,1})^{\frac{3}{2}}}{3\sqrt{\sigma}}\right], \tag{5.19}$$

which yields (5.13). □

Remarkably, (5.6) and (5.13) give identical lower bounds for the model decrease. Lemmas 5.2 and 5.3 generalize [7, Lemma 2.1] to the case where $s_k \neq 0$ and general norms are allowed.

We may now complete the analysis of what can happen at iterate s_k (of the still unspecified RQMIN algrithm) if the second-order necessary condition of Theorem (4.2) fails. We first state an easy lemma giving lower and upper bounds on the step s_k, dependent on the "most negative curvature" of the quadratic given by (4.1).

Lemma 5.4 Suppose that, for some s_k and some $\beta \geq 0$,

$$m(s_0) - m(s_k) \geq \beta. \tag{5.20}$$

Then,

$$\|s_k\|_r \leq \frac{\frac{1}{2}\|H\|_{r,2} + \sqrt{\|H\|_{r,2}^2 + \frac{2}{3}\sigma\|g\|_{r,1}}}{\frac{1}{3}\sigma} \stackrel{\rm def}{=} \kappa_{s,\rm upp}, \tag{5.21}$$

and, if $\beta > 0$,

$$\|s_k\|_r \geq \begin{cases} \dfrac{\sqrt{\|g\|_{r,1}^2 + 2\beta|\lambda_r[H]|} - \|g\|_{r,1}}{|\lambda_r[H]|} & \text{if } \lambda_r[H] < 0 \\ \dfrac{\beta}{\|g\|_{r,1}} & \text{otherwise.} \end{cases} \tag{5.22}$$

Proof. Since $s_0 = 0$ and $m(s_k) \geq m(0) + \langle g, s_k\rangle + \frac{1}{2}\langle Hs_k, s_k\rangle$, (5.20) implies that

$$-\|g\|_{r,1}\|s_k\|_r + \tfrac{1}{2}\min\left[0, \lambda_r[H]\right]\|s_k\|_r^2 \leq \langle g, s_k\rangle + \tfrac{1}{2}\langle Hs_k, s_k\rangle \leq m(s_k) - m(0) \leq -\beta,$$

which gives (5.22). Observe now that (5.20) implies that

$$\tfrac{1}{6}\sigma\|s_k\|_r^3 \leq |\langle g, s_k\rangle| + \tfrac{1}{2}|\langle Hs_k, s_k\rangle| \leq \|g\|_{r,1}\|s_k\|_r + \tfrac{1}{2}\|H\|_{r,2}\|s_k\|_r^2,$$

which yields (5.21). □

Armed with this result, we now derive the model decrease when negative curvature is present.

Lemma 5.5 Suppose that u is an eigenvector associated with $\lambda_{\min}[H]$ with $\|u\|_2 = 1$ and that the sequence $\{m(s_k)\}_{k\geq 0}$ is non-increasing. For $k \geq 0$, define

$$s_k^E = s_k + \alpha_k^E u_k \;\text{ where }\; u_k = -\text{sign}\big(\langle g_k, u\rangle\big)u \;\text{ and }\; \alpha_k^E = \arg\min_{\alpha>0} m(s_k + \alpha u_k). \tag{5.23}$$

Then

$$m(s_0) - m(s_1) \geq m(s_0) - m(s_0^E) \geq \frac{9|\lambda_{\min}[H|^3}{16\sigma^2} \tag{5.24}$$

and there exists a constant κ_s such that, for $k \geq 1$,

$$\|s_k\|_r \geq \kappa_s. \tag{5.25}$$

Moreover, if

$$\lambda_{\min}[H] + \sigma\omega(s_k)\|s_k\|_r < 0 \tag{5.26}$$

at iteration $k \geq 1$, then one has that

$$m(s_k) - m(s_k^E) \geq \frac{9\|s_k\|_r^2}{16\sigma^2}\left|\lambda_{\min}[H] + \sigma\omega(s_k)\|s_k\|_r\right|^3. \tag{5.27}$$

Proof. The first inequality in (5.24) results from (5.31) and the second is a direct consequence of the proof of Theorem 4.2 (see (4.7)). The existence of κ_s such that (5.25) holds for $k \geq 1$ then follows from Lemma 5.4 with $\beta = \frac{9}{16}|\lambda_{\min}[H]|^3/\sigma^2$ and our assumption that $\{m(s_k)\}$ is non-increasing. We now prove (5.27). From (5.26), we have that

$$\lambda_{\min}[H] + \sigma\omega(s_k)\|s_k\|_r = -\mu\sigma\|s_k\|_r \tag{5.28}$$

for some $\mu > 0$. But (4.6) implies that $0 > \lambda_{\min}[H] + \sigma\|s_k\|_r = -|\lambda_{\min}[H] + \sigma\|s_k\|_r|$, and thus, from (4.6),

$$|\lambda_{\min}[H] + \sigma\|s_k\|_r| = \frac{2\sqrt{\psi(s_k)}}{\sqrt{3}}\sigma\|s_k\|_r + \mu\sigma\|s_k\|_r, \tag{5.29}$$

from which we obtain that

$$\big(\lambda_{min}[H] + \sigma\|s_k\|_r\big)^2 = \frac{4}{3}\psi(s_k)\sigma^2\|s_k\|_r^2 + \sigma^2\|s_k\|_r^2\left(\mu^2 + \frac{4\sqrt{\psi(s_k)}}{\sqrt{3}}\,\mu\right).$$

Substituting this inequality in (4.2), then gives that there exists an $\alpha > 0$ such that

$$\begin{aligned} m(s_k) - m(s_k + \alpha\|s_k\|_r u) &\geq \frac{3(\lambda_{\min}[H] + \sigma\|s_k\|_r)}{4\sigma^2}\left[-\frac{3}{4}\sigma^2\|s_k\|_r^2\left(\mu^2 + \frac{4\sqrt{\psi(s_k)}}{\sqrt{3}}\,\mu\right)\right] \\ &= \frac{9}{16}\Big|\lambda_{\min}[H] + \sigma\|s_k\|_r\Big|\|s_k\|_r^2\left(\mu^2 + \frac{4\sqrt{\psi(s_k)}}{\sqrt{3}}\,\mu\right). \end{aligned}$$

But (5.29) implies that $\big|\lambda_{\min}[H] + \sigma\|s_k\|_r\big| \geq \mu\sigma\|s_k\|_r$, and thus

$$m(s_k) - m(s_k + \alpha\|s_k\|_r u) \geq \frac{9\sigma\|s_k\|_r^3}{16}\left(\mu^3 + \frac{4\sqrt{\psi(s_k)}}{\sqrt{3}}\,\mu^2\right) \geq \frac{9\sigma\|s_k\|_r^3}{16}\mu^3.$$

The inequality (5.27) then follows from (5.28) and (5.23). □

We now have all ingredients to describe the RQMIN algorithm. It hinges on (5.2), (5.3) and (5.5) and proceeds by successive one-dimensional minimizations of m along the directions s_k^C or s_k^R (depending on the sign of $\|g_k|_{r,1} - \frac{1}{2}\sigma\|s\|_r^2$) and, if needed, s_k^E. It is formally stated on the next page.

Note that the mechanism of the algorithm, which proceeds by successive unidimensional minimizations, guarantees that the sequence $\{m(s_k)\}$ is monotonically decreasing, as announced.

Having established, in Lemma 5.2, 5.3 and 5.5, lower bounds on the decrease in m for all steps produced by the RQMIN algorithm, we are now ready to state its iteration complexity[(3)].

[(3)] At variance with its evaluation complexity, which would be irrelevant here since evaluating $m(s)$ as many times as necessary does not require evaluating f_0, g and H more than once (when the algorithm is called).

Algorithm 5.1: An algorithm for minimization of a regularized quadratic (RQMIN)

The value f_0, gradient g and Hessian H of the quadratic at $s = 0$ are given, as well as a regularization parameter σ and accuracy requests $\epsilon_1 > 0$ and $\theta_2 > 1$.

Step 0: Initialization If unavailable, compute $\lambda_{\min}[H]$ and and associated eignevector u with $\|u\|_2 = 1$. Set $k = 0$, $s_0 = 0$ and $g_0 = g$.

Step 1: Check for termination. Terminate if

$$\left| \|g_k\|_{r,1} - \frac{\sigma}{2}\|s_k\|_r^2 \right| \leq \epsilon_1 \quad \text{and} \quad \lambda_{\min}[H] + \theta_2 \omega(s_k)\sigma\|s_k\|_r \geq 0. \tag{5.30}$$

Step 2: Negative gradient step. If $\|g_k\|_{r,1} > \frac{\sigma}{2}\|s_k\|_r^2$, compute s_k^C according to (5.7), set $m_{k,1} = m(s_k^C)$ and go to Step 4.

Step 3: Retraction step. If $\|g_k\|_{r,1} < \frac{\sigma}{2}\|s_k\|_r^2$, compute s_k^C according to (5.14) and set $m_{k,1} = m(s_k^R)$.

Step 4: Eigenvalue step. If $\lambda_{\min}[H] + \theta_2\omega(s_k)\sigma\|s_k\|_r < 0$, compute s_k^E according to (5.23) and set $m_{k,2} = m(s_k^E)$. Else, set $m_{k,2} = m(s_k)$.

Step 5: Move. Set

$$s_{k+1} = \begin{cases} s_k^C & \text{if } m_{k,1} \leq m_{k,2}, \\ s_k^E & \text{otherwise,} \end{cases} \quad \text{and} \quad g_{k+1} = g_k + H(s_{k+1} - s_k). \tag{5.31}$$

Increment k by one and got to Step 1.

Theorem 5.6 Given $\epsilon_1 > 0$ and $\theta_2 > 1$, there exist a constant $\kappa_{\mathsf{RQMIN}} > 0$ independent of k such that the RQMIN algorithm requires at most

$$\kappa_{\mathsf{RQMIN}} \max\left[\epsilon_1^{-2}, \epsilon_1^{-\frac{3}{2}}, (\theta_2 - 1)^{-3}\right] \tag{5.32}$$

iterations to produce an iterate s_k such that

$$\left|\|g_k\|_{r,1} - \tfrac{1}{2}\sigma\|s_k\|_r^2\right| \leq \epsilon_1 \;\text{ and }\; \lambda_{\min}[H] + \theta_2\omega(s_k)\sigma\|s_k\|_r \geq 0. \tag{5.33}$$

Proof. If the RQMIN algorithm terminates at $k = 0$, then the bound (5.32) is trivially satisfied. Assume therefore that termination does not occur at s_0. We therefore have that, for $k \geq 1$ before termination,

$$\text{either } \;\left|\|g_k\|_{r,1} - \tfrac{1}{2}\theta_1\sigma\|s_k\|_r^2\right| > \epsilon_1 \;\text{ or }\; \lambda_{\min}[H] + \theta_2\omega(s_k)\sigma\|s_k\|_r < 0.$$

Let us define $\mathcal{N} \stackrel{\rm def}{=} \{k \geq 0 \mid s_k = s_k^E\}$ and note that, by construction, this set is non-empty only if $\lambda_{\min}[H] < 0$. We then obtain from $g_0 = g$, (5.31), (5.6) and (5.24) that

$$m(s_0) - m(s_1) \geq \begin{cases} \tfrac{1}{2}\max\left\{\min\left[\dfrac{\|g\|_{r,1}^2}{1+\frac{3}{2}\|H\|_{r,2}}, \dfrac{\|g\|_{r,1}^{\frac{3}{2}}}{3\sqrt{\sigma}}\right], \dfrac{9|\lambda_{\min}[H]|^3}{16\sigma^2}\right\} & \text{if} \quad 0 \in \mathcal{N}, \\ \tfrac{1}{2}\min\left[\dfrac{\|g\|_{r,1}^2}{1+\frac{3}{2}\|H\|_{r,2}}, \dfrac{\|g\|_{r,1}^{\frac{3}{2}}}{3\sqrt{\sigma}},\right] & \text{otherwise.} \end{cases} \tag{5.34}$$

Observe now that the second part of (5.33) cannot hold as long as

$$-\big(\lambda_{\min}[H] + \omega(s_k)\sigma\|s_k\|_r\big) > (\theta_2 - 1)\omega(s_*)\sigma\|s_k\|_r.$$

Hence (5.27) give that, for $k \geq 1$ and $k \in \mathcal{N}$ before termination,

$$m(s_k) - m(s_k^E) \geq \frac{9\|s_k\|_r}{16\sigma^2}\left[(\theta_2 - 1)\omega(s_*)\right]^3. \tag{5.35}$$

Because Lemma 5.4 with β chosen as the relevant right-hand side in (5.34) guarantees the existence of $\kappa_{s,\rm low} > 0$ such that $\|s_k\|_r \geq \kappa_{s,\rm low}$ for all $k \geq 1$, and, because $\omega(s_k) \geq 1$, (5.35) ensures that, before termination and for $k \in \mathcal{N}$, $k \geq 1$,

$$m(s_k) - m(s_k^E) \geq \frac{9\kappa_{s,\rm low}^2}{16\sigma^2}(\theta_2 - 1)^3 \stackrel{\rm def}{=} \frac{9\kappa_{s,\rm low}^2}{16\sigma^2}\epsilon_2^3.$$

Using this together with (5.31) and (5.6) gives that for $k \geq 1$ before termination,

$$m(s_k) - m(s_{k+1}) \geq \begin{cases} \tfrac{1}{2}\max\left\{\min\left[\dfrac{\epsilon_1^2}{1+\frac{3}{2}\left(\|H\|_{r,2}+\sigma\|s_k\|_r\right)}, \dfrac{\epsilon_1^{\frac{3}{2}}}{3\sqrt{\sigma}}\right], \dfrac{9\kappa_{s,\rm low}^2}{16^2}\right\} & \text{if } \; k \in \mathcal{N}, \\ \min\left[\dfrac{\epsilon_1^2}{1+\frac{3}{2}\left(\|H\|_{r,2}+\sigma\|s_k\|_r\right)}, \dfrac{\epsilon_1^{\frac{3}{2}}}{3\sqrt{\sigma}}\right] & \text{otherwise,} \end{cases}$$

and therefore, using (5.21),

$$m(s_k) - m(s_{k+1}) \geq \tfrac{1}{2} \min \left[\frac{1}{1 + \frac{3}{2}\left(\|H\|_{r,2} + \sigma\kappa_{s,\rm upp}\right)}, \frac{1}{3\sqrt{\sigma}}, \frac{9\kappa_{s,\rm low}^2}{16\sigma^2} \right] \min \left[\epsilon_1^2, \epsilon_1^{\frac{3}{2}}, \epsilon_2^3 \right]$$
$$\stackrel{\rm def}{=} \kappa_* \min \left[\epsilon_1^2, \epsilon_1^{\frac{3}{2}}, \epsilon_2^3 \right]. \tag{5.36}$$

We now observe that the definition of $m(s)$ and (5.21) together imply that

$$m(s) \geq m(0) - \|g\|_{r,1}\kappa_{s,\rm upp} - \tfrac{1}{2}\|H\|_{r,2}\kappa_{s,\rm upp}^2 \stackrel{\rm def}{=} m_{\rm low}.$$

Therefore (5.36) implies that the number of iterations required by the RQMIN algorithm to produce an iterate such that (5.33) holds cannot exceed

$$\frac{m(0) - \beta - m_{\rm low}}{\kappa_* \min \left[\epsilon_1^2, \epsilon_1^{\frac{3}{2}}, \epsilon_2^3 \right]}$$

which is (5.32) with $\kappa_{\rm RQMIN} = (m(0) - \beta - m_{\rm low})/\kappa_*$. □

We now consider applying the RQMIN algorithm to find a step s_k in Step 2 of the AR2GN method[(4)]. This latter methods requires the conditions (2.5), (2.6) and (4.10) to hold. We immediately note that (2.5) automatically holds because of the monotonically decreasing nature of the values of m in the RQMIN algorithm. Moreover, (4.10) and the second part of (5.30) are identical. However, the first part of (5.30) is too strong, because it imposes a two-sided inequality on $\|g_k\|_{r,1} - \frac{1}{2}\sigma_k\|s_k\|_r$ while (2.6) only requests

$$\|g_k\|_{r,1} - \tfrac{1}{2}\sigma_k\|s_k\|_r^2 \leq \epsilon_{1s} \stackrel{\rm def}{=} \tfrac{1}{2}(\theta_1 - 1)\sigma_k\|s_k\|_r^2 \tag{5.37}$$

but allows for $\|g_k\|_{r,1} - \frac{1}{2}\sigma_k\|s_k\|_r^2$ to be negative. In Figure 4.1, this amounts to removing the outer blue curve, thus enlarging the admissible regions containing the minimizers. A modified variant of the RQMIN algorithm is therefore suitable if our only objective is to satisfy (2.5), (2.6) and (4.10). This variant, which we call the RQMIN1 algorithm, differs from RQMIN in that

1. the first part of (5.30) is replaced by requiring that (5.37) holds,
2. Step 3 of RQMIN is skipped (as there is no need to correct for negative $\|g_k\|_{r,1} - \frac{1}{2}\sigma_k\|s_k\|_r^2$).

In addition, because (5.37) is weaker that the first part of (5.30), termination of the RQMIN1 algorithm cannot happen later than that what would happen if applying the RQMIN algorithm with $\epsilon_1 = \epsilon_{1s}$. This allows us to derive the following upper bound on the number of iterations of the RQMIN1 algorithm that are necessary to compute a step s_k in Step 2 of AR2GN.

Corollary 5.7 Given $\theta_1 > 1$ and $\theta_2 > 1$, there exist a constant $\kappa_{\rm RQMIN1} > 0$ independent of k such that the RQMIN1 algorithm requires at most

$$\kappa_{\rm RQMIN1} \max \left[(\theta_1 - 1)^{-2}, (\theta_1 - 1)^{-\frac{3}{2}}, (\theta_2 - 1)^{-3} \right] \tag{5.38}$$

iterations to produce an iterate s_k such that (2.5), (2.6) and (4.10) hold.

(4) With $f(x_k) = f_0$, $\nabla_x^1 f(x_k) = g$, $\nabla_x^2 f(x_k) = H$, $\sigma_k = \sigma$ and $x_k = 0$.

Proof. The desired result immediately follows by noting that, by virtue of (2.8), (5.37) and the definition of $\kappa_{s,\rm low}$ in the proof of Theorem 5.6,

$$\epsilon_{1s} \geq \tfrac{1}{2}(\theta_1 - 1)\sigma\kappa_{s,\rm low}^2.$$

The bound (5.38) then follows with

$$\kappa_{\sf RQMIN1} \stackrel{\rm def}{=} \kappa_{\sf RQMIN} \min\left[(\tfrac{1}{2}\sigma\kappa_{s,\rm low}^2)^2, (\tfrac{1}{2}\sigma\kappa_{s,\rm low}^2)^{\frac{3}{2}}\right].$$

□

The reader may have wondered why we did consider the RQMIN method and its two-sided condition at all, since its one-sided version RQMIN1 is sufficient for the purpose of computing a step in AR2GN. Our motivation for RQMIN is that it is likely to achieve a larger model decrease, hopefully reducing the number of iterations needed by AR2GN to terminate.

But the story does not finish here. As we have alluded to in Section 2, an even simpler variant of the RQMIN algorithm can be used to compute s_k in Step 2 of the AR1pGN algorithm when $p = 2$. Since the only requirements on s_k are then (2.5) and (2.6), we may define the RQMIN2 algorithm as a variant of RQMIN where

1. the whole of (5.30) is replaced by requiring that (5.37) holds,
2. Step 3 and Step 4 of RQMIN are skipped and the first part of (5.31) replaced by $s_{k+1} = s_k^C$.

Removing all bounds related to the second-order condition in Theorem 5.6, we then obtain the following iteration bound for the RQMIN2 algorithm (as needed in Step 2 of AR1pGN with $p = 2$).

Corollary 5.8 Given $\theta_1 > 1$, there exist a constant $\kappa_{\sf RQMIN2} > 0$ independent of k such that the RQMIN2 algorithm requires at most

$$\kappa_{\sf RQMIN2} \max\left[(\theta_1 - 1)^{-2}, (\theta_1 - 1)^{-\frac{3}{2}}\right] \tag{5.39}$$

iterations to produce an iterate s_k such that (2.5), and (2.6) hold.

Note that the RQMIN2 algorithm reduces to a standard first-order method (in the $\|\cdot\|_r$ norm), but applied to the quadratic alone, instead of to the complete regularized model.

6 Discussion

We have shown in this variant of [23] that using Euclidean curvature in second-order necessary optimality conditions (see (4.1)) leads to results that are entirely similar to those obtained when attempting to enforce the condition $\min_{\|v\|_r=1}\langle Hv, v\rangle > \epsilon_2$ used in this reference.

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