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Worldline approach to noncommutative field theory

R Bonezzi[1], O Corradini[2], S A Franchino Viñas[3,4] and P A G Pisani[3]

[1] Dipartimento di Fisica, Università di Bologna and INFN, Sezione di Bologna, Via Irnerio 46, I-40126 Bologna, Italy
[2] Centro de Estudios en Física y Matemáticas Básicas y Aplicadas, Universidad Autónoma de Chiapas, Tuxtla Gutiérrez, Mexico
[3] IFLP-CONICET/ Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, CC 67 (1900), La Plata, Argentina
[4] Dipartimento di Fisica, Università di Roma 'La Sapienza' and INFN, Sezione di Roma1, Piazzale Aldo Moro 2, I-00185 Roma, Italy

E-mail: bonezzi@bo.infn.it, olindo.corradini@unach.mx, safranchino@yahoo.com.ar and pisani@fisica.unlp.edu.ar

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Abstract

The study of the heat-trace expansion in non-commutative field theory has shown the existence of Moyal non-local Seeley–DeWitt coefficients which are related to the UV/IR mixing and manifest, in some cases, the non-renormalizability of the theory. We show that these models can be studied in a worldline approach implemented in phase space and arrive at a master formula for the n-point contribution to the heat-trace expansion. This formulation could be useful in understanding some open problems in this area, as the heat-trace expansion for the non-commutative torus or the introduction of renormalizing terms in the action, as well as for generalizations to other non-local operators.

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1. Introduction

Non-commutative field theory [1, 2] involves two concepts which could be considered as fundamental ingredients of a theory of quantum gravity: non-locality and a minimal length scale. As a consequence, the theory presents interesting renormalization properties, such as the so-called UV/IR mixing [3].

Renormalization in quantum field theory can be implemented with heat-kernel techniques [4], since the one-loop counterterms that regularize the high energy divergencies of the effective action can be obtained in terms of the Seeley–DeWitt (SDW) coefficients, which are determined by the asymptotic expansion, for small values of the proper time, of the heat trace of a relevant operator. Such an operator defines the spectrum of quantum fluctuations and is obtained as the second-order correction of the classical action in a background field expansion.

In a commutative field theory, this operator of quantum fluctuations is, in general, a (local) differential operator whose SDW coefficients have been extensively studied; in particular, the worldline formalism (WF) has proved an efficient technique for the computation of these coefficients [5].

On the other hand, in a non-commutative field theory the operator of quantum fluctuations is, in general, a non-local operator. Lately it has been shown that the SDW coefficients corresponding to these types of non-local operators have peculiar contributions—which are related to the non-planar diagrams of the UV/IR mixing—and are linked to the (non)renormalizability of the theory (see the short review [6]).

Let us also mention that in the context of the spectral action principle [7] the SDW coefficients for models defined on non-commutative spaces have an essential role in the determination of the corresponding classical actions.

In this paper, we use WF techniques to obtain a systematic description of the SDW coefficients of non-local operators relevant for the quantization of non-commutative self-interacting scalar fields on Moyal Euclidean spacetime. In order to do that we implement the WF in phase space and we derive a master formula (equation (4.2)) that can be applied to different settings. We consider this formula to be a step toward a more systematic understanding of the heat-trace expansion of non-local operators as well as a potentially useful approach to some open problems in this topic, as those one encounters for example on the NC torus [8]. Further applications to other non-local operators could also be considered.

Although the main motivation of this paper is to develop a new tool for performing one-loop calculations in non-commutative field theories, the present technique also yields a path integral representation of non-commutative quantum mechanical transition amplitudes which, to the best of our knowledge, has not been used in previous works on quantum mechanics in Moyal spaces. Previous path integral expressions were derived in [9, 10] based on the symplectic form that defines non-commutativity, whereas our worldline formulation is suitable for the particular representation of the non-commutative algebra given by the Moyal product. Other path integral representations of transition amplitudes have been derived in [11, 12] in the coherent states approach to non-commutative quantum mechanics (in [12] coherent states are defined in the Hilbert space consisting of a certain class of Hilbert–Schmidt operators). Such formulations are connected to the representation of non-commutativity given by the Voros product. There exist applications of path integrals to quantum mechanics in the Moyal plane [13, 14] where the Hamiltonians are quadratic in momenta and coordinates so that they can be reduced to analogue models in the commutative plane. Let us also mention the application of path integrals to the Aharonov–Bohm problem in the Moyal plane given in [15], which is performed to leading order in the parameter that defines non-commutativity.

1.1. Effective action in Euclidean Moyal spacetime

A field theory on Euclidean Moyal spacetime can be formulated in terms of the non-local Moyal product usually defined as

$$(\phi \star \psi)(x) := \mathrm{e}^{\mathrm{i}\,\partial^{\phi}\Theta\partial^{\psi}}\,\phi(x)\psi(x). \tag{1.1}$$

The conditions under which the exponential in this expression is well defined are studied in [16]. The scalar functions ϕ and ψ depend on $x \in \mathbb{R}^d$; ∂^{ϕ} and ∂^{ψ} denote their gradients, respectively. The expression $\partial^{\phi}\Theta\partial^{\psi}$ represents $\Theta^{ab}\partial^{\phi}_{x^a}\partial^{\psi}_{x^b}$, where Θ^{ab} are the components of an antisymmetric matrix Θ independent of x.

With respect to this $\star$-product, the coordinates do not commute:

$$[x^a, x^b]_\star := x^a \star x^b - x^b \star x^a = 2\mathrm{i}\,\Theta^{ab}. \tag{1.2}$$

Thus, the matrix Θ characterizes the non-commutativity of the base space. Throughout this paper, we will consider a possibly degenerate matrix Θ. Assuming Θ of rank $2b$ we split $\mathbb{R}^d$ into a commutative $\mathbb{R}^c$ and a non-commutative $\mathbb{R}^{2b}$, with $d = c + 2b$, by choosing coordinates $x = (\tilde{x}, \hat{x})$ with commuting $\tilde{x} \in \mathbb{R}^c$ and non-commuting $\hat{x} \in \mathbb{R}^{2b}$. Consequently, the matrix Θ can be written as

$$\Theta = \mathbf{0}_c \oplus \Xi\,, \tag{1.3}$$

where $\mathbf{0}_c$ is the null matrix in $\mathbb{R}^c$ and Ξ is a non-degenerate antisymmetric matrix in $\mathbb{R}^{2b}$. In terms of the Fourier transform

$$\tilde{\phi}(p) = \int \frac{\mathrm{d}x}{(2\pi)^d}\, \mathrm{e}^{-\mathrm{i}px}\phi(x) \tag{1.4}$$

the Moyal product reads

$$\widetilde{\phi \star \psi}(p) = \int \mathrm{d}q\, \mathrm{e}^{-\mathrm{i}p\Theta q}\, \tilde{\phi}(p-q)\tilde{\psi}(q). \tag{1.5}$$

A simple model in non-commutative field theory in Euclidean Moyal spacetime is given by a scalar field with a $\star$ - cubic self-interaction, described by the Langrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{3!}\phi_\star^3, \tag{1.6}$$

where $\phi_\star^3 := \phi \star \phi \star \phi$.

In the ordinary commutative case (i.e. $\Theta = 0$) the one-loop effective action Γ_C can be represented as

$$\Gamma_C = \tfrac{1}{2}\log \mathrm{Det}\{-\partial^2 + m^2 + \lambda\,\phi(x)\}, \tag{1.7}$$

where the Schrödinger differential operator between brackets, which determines the spectrum of quantum fluctuations, arises from the second functional derivative of the action with respect to the background field ϕ. As we have already mentioned, the regularization of the effective action can be implemented in terms of the heat trace of this Schrödinger operator.

The spectral theory of Schrödinger operators of the type $-\partial^2 + V(x)$ on $\mathbb{R}^d$ shows that for regular potentials $V(x)$ the heat trace admits the following asymptotic expansion in powers of the proper time $\beta > 0$ [17]:

$$\mathrm{Tr}(f(x)\cdot \mathrm{e}^{-\beta\{-\partial^2+V(x)\}}) \sim \frac{1}{(4\pi\beta)^{d/2}}\sum_{n=0}^{\infty} a_n\beta^n. \tag{1.8}$$

In this expression, we have introduced a smearing function $f(x)$. The coefficients a_n are called SDW coefficients and are given by integrals on $\mathbb{R}^d$ of products of the smearing function $f(x)$, the potential $V(x)$ and derivatives thereof (the reader can find some proof of this statement in subsection 4.1).

In the non-commutative case ($\Theta \neq 0$), the one-loop effective action Γ_{NC} corresponding to the Lagrangian (1.6) instead results in[5]

$$\Gamma_{\mathrm{NC}} = \frac{1}{2}\log\mathrm{Det}\left\{-\partial^2 + m^2 + \frac{\lambda}{2}L(\phi) + \frac{\lambda}{2}R(\phi)\right\}, \tag{1.9}$$

where $L(\phi)$ is an operator whose action on a function $\psi(x)$ is defined as $L(\phi)\psi(x) := (\phi\star\psi)(x)$ whereas $R(\phi)\psi(x) := (\psi\star\phi)(x)$. These non-local operators represent respectively the left- and right-Moyal multiplication by ϕ and can also be expressed as

$$\begin{aligned} L(\phi)\psi(x) &= \phi(x + \mathrm{i}\Theta\partial)\psi(x), \\ R(\phi)\psi(x) &= \phi(x - \mathrm{i}\Theta\partial)\psi(x), \end{aligned} \tag{1.10}$$

where $x \pm \mathrm{i}\Theta\partial$ have components $x^a \pm \mathrm{i}\Theta^{ab}\partial_b$.

[5] A computation of Γ_{NC} in terms of Feynman diagrams can be found in [3].

Thus, in order to regularize the non-commutative effective action Γ_{NC} we must study the heat trace of the non-local operator:

$$-\partial^2 + m^2 + \frac{\lambda}{2}\phi(x+\mathrm{i}\Theta\partial) + \frac{\lambda}{2}\phi(x-\mathrm{i}\Theta\partial). \tag{1.11}$$

Note that in the case of a quartic self-interaction $\phi_\star^4$, the operator corresponding to the second functional derivative of the action contains the following terms: $L(\phi_\star^2)$, $R(\phi_\star^2)$, $L(\phi)R(\phi)$.

It has been shown that the heat trace of an operator of the type

$$-\partial^2 + L(l_1(x)) + R(r_1(x)) + L(l_2(x))R(r_2(x)) \tag{1.12}$$

also admits an asymptotic expansion in powers of the proper time β [18] . However, the SDW coefficients are not given, in general, by integrals on $\mathbb{R}^d$ of local expressions depending on the smearing function and the potential functions $l_i(x)$, $r_i(x)$.

Nevertheless, if the operator (1.12) contains only a left-Moyal product (i.e. $r_1(x) = r_2(x) = 0$) and the smearing function acts by left-Moyal multiplication, then the SDW coefficients can be obtained from the commutative ones by replacing every commutative pointwise product by the non-commutative Moyal product[6] [19, 20]. The same holds if the operator contains only a right-Moyal product (i.e. $l_1(x) = l_2(x) = 0$) and the smearing function acts by right-Moyal multiplication. Therefore, when there is no mixing between left- and right-Moyal multiplications, the SDW coefficients are still integrals of 'local products' (but in a Moyal sense) of the smearing function, the potential functions and their derivatives; we will refer to these as 'Moyal local' coefficients. In consequence, they are likely to be introduced as counterterms in the Lagrangian.

In contrast, for the general case given by expression (1.12) some SDW coefficients are not even Moyal local, i.e. they cannot be written as integrals of Moyal products of the potential functions and the smearing function. In some field theories, these coefficients manifest the non-renormalizability of the corresponding effective action[7] [21].

In this paper, we develop a worldline approach to the computation of the SDW coefficients of operator (1.12). For simplicity, we will consider the case $l_1(x) = r_1(x) = 0$ since the mixing term $L(l_2)R(r_2)$ will suffice to show the appearance of the (Moyal) non-local coefficients. In order to do that, we write a representation of the heat kernel in terms of a path integral in phase space; this is done in section 2. This path integral is solved in section 3, where we compute the corresponding generating functional. Our main result, the master formula for the non-commutative heat trace, is given in section 4, where we also show how it works for some particular settings and confirm existing results [6]. In section 5, we show a simple application of formula (4.2) to non-commutative field theroy: we consider a scalar field with a quartic self-interaction in Euclidean non-commutative spacetime to show how (Moyal) non-local SDW coefficients manifest two different phenomena that could lead to non-renormalizability: the so-called UV/IR mixing and the existence of more than one commuting direction in the non-commutative spacetime. Finally, in section 6 we draw our conclusions.

2. Path integral in phase space

Let us consider an operator H in Euclidean spacetime which contains a non-local potential mixing left- and right-Moyal multiplications

$$H = -\partial^2 + L(l)\,R(r), \tag{2.1}$$

[6] As we will see, to avoid ordering ambiguities, one should consider a commutative operator with matrix-valued coefficients.

[7] Let us mention that in some non-commutative models in emergent gravity a low-energy regime is to be considered and this UV/IR mixing problem is avoided [22].

where $l(x)$ and $r(x)$ are functions of $x \in \mathbb{R}^d$. Note that, due to the associativity of Moyal product, $L(l)$ and $R(r)$ commute with each other. In accordance with representation (1.10), we write

$$H = p^2 + l(x - \Theta p)\, r(x + \Theta p), \tag{2.2}$$

where $p = -\mathrm{i}\partial$. Hamiltonian operator (2.2) has a fixed ordering given by replacing coordinate x with operator $x - \Theta p$ in l, and with operator $x + \Theta p$ in r. In order to obtain the path integral representation of the transition amplitude associated with (2.2), it is convenient to *rearrange* the Hamiltonian operator in a different form, for example the Weyl-ordered form. A phase-space operator $A(x, p)$ is written in Weyl-ordered form when it is arranged in such a way that $A(x, p) = A_S(x, p) + \Delta A \equiv A_W(x, p)$, where $A_S(x, p)$ involves symmetric products of xs and ps and ΔA includes all terms resulting from eventual commutators between xs and ps, necessary to rearrange $A(x, p)$ in its symmetric form; for example $xp = (xp)_S + \frac{1}{2}[x, p] = (xp)_S + \frac{\mathrm{i}\hbar}{2} \equiv (xp)_W$ (for details see, e.g., appendices B and C in [5]). Above, since the phase-space operator $l(x-\Theta p)\, r(x+\Theta p)$ mixes coordinates and momenta it is not, *a priori*, written in symmetrized form: in other words $l(x - \Theta p)\, r(x + \Theta p) = (l(x - \Theta p)\, r(x + \Theta p))_S + \Delta V$. However, one can show by Taylor expanding the functions l and r, that the operator (2.2) only involves symmetric products; in other words, products of xs and ps in (2.2) can be cast in their symmetrized form (for example $(xp)_S = \frac{1}{2}(xp + px)$ and $(x^2p)_S = \frac{1}{3}(x^2p + xpx + px^2) = \frac{1}{2}(x^2p + px^2)$) without introducing extra terms, i.e. $\Delta V = 0$. Let us, for example, consider the product of the linear contributions in the Taylor expansions of l and r: it involves the product $x_-^a x_+^b \equiv (x^a - \Theta^{ac} p_c)(x^b + \Theta^{bd} p_d)$ whose Weyl ordering reads

$$\begin{aligned} x_-^a x_+^b &= (x^a x^b)_S - \Theta^{ac}\Theta^{bd}(p_c p_d)_S - \Theta^{ac} p_c x^b + \Theta^{bd} x^a p_d \\ &= (x_-^a x_+^b)_S - \tfrac{1}{2}\Theta^{ac}[p_c, x^b] + \tfrac{1}{2}\Theta^{bd}[x^a, p_d] = (x_-^a x_+^b)_S \end{aligned} \tag{2.3}$$

thanks to the antisymmetry of the Θ^{ab} symbol. It is thus easy to convince oneself that the latter antisymmetry, along with the commutativity property $[x_-^a, x_+^b] = 0$ and the total symmetry of the coefficients of Taylor expansions of functions $l(x)$ and $r(x)$, allows us to easily prove that all the contributions to the product of Taylor expansions of operators $l(x-\Theta p)$ and $r(x+\Theta p)$ are symmetric.

Therefore—using the midpoint rule [23, 5]—one can write the following path integral representation for the heat kernel of operator (2.2):

$$\begin{aligned} \langle x + z|\, \mathrm{e}^{-\beta H} |x\rangle = \int \mathcal{D}x(t)\mathcal{D}p(t) \exp\left(-\int_0^\beta \mathrm{d}t\, \left\{p^2(t) - \mathrm{i}p(t)\dot{x}(t)\right\}\right) \exp\Bigg(-\int_0^\beta \mathrm{d}t\, l(x(t) \\ - \Theta p(t))\, r(x(t) + \Theta p(t))\Bigg), \end{aligned} \tag{2.4}$$

where $\beta > 0$ and $x(t)$, $p(t)$ represent trajectories in phase space $\mathbb{R}^{2d}$. The path integral is performed on trajectories $x(t)$ that satisfy the boundary conditions $x(0) = x$ and $x(\beta) = x + z$ and on trajectories $p(t)$ that do not satisfy any boundary condition.

It is convenient to replace the integral on the trajectories $x(t)$ by an integral on perturbations $q(t)$ about the free classical path $x_{\mathrm{cl}}(t) = zt/\beta + x$. We also make the following rescaling: $t \to \beta t$, $q \to \sqrt{\beta} q$, $p \to p/\sqrt{\beta}$. Expression (2.4) then takes the form

$$\begin{aligned} \langle x + z|\, \mathrm{e}^{-\beta H} |x\rangle &= \beta^{-d/2} \int \mathcal{D}q\mathcal{D}p \exp\left(-\int_0^1 \mathrm{d}t\, \{p^2 - \mathrm{i}p\dot{q}\}\right) \exp\left(\mathrm{i}\frac{z}{\sqrt{\beta}}\int_0^1 \mathrm{d}t\, p\right) \\ &\quad \times \exp\left(-\beta\int_0^1 \mathrm{d}t\, l(x + tz + \sqrt{\beta}q - \Theta p/\sqrt{\beta})\, r(x + tz + \sqrt{\beta}q + \Theta p/\sqrt{\beta})\right), \end{aligned} \tag{2.5}$$

where the perturbations q in configuration space satisfy $q(0) = q(1) = 0$.

If we define the mean value of a functional $f[q(t), p(t)]$ as

$$\langle f[q(t), p(t)]\rangle := \frac{\int \mathcal{D}q\mathcal{D}p \exp\left(-\int_0^1 \mathrm{d}t\,\{p^2 - \mathrm{i}p\dot{q}\}\right) f[q(t), p(t)]}{\int \mathcal{D}q\mathcal{D}p \exp\left(-\int_0^1 \mathrm{d}t\,\{p^2 - \mathrm{i}p\dot{q}\}\right)} \tag{2.6}$$

then the transition amplitude (2.5) reads

$$\langle x+z|\,\mathrm{e}^{-\beta H}|x\rangle = \frac{1}{(4\pi\beta)^{d/2}}\Bigg\langle \exp\left(\mathrm{i}\frac{z}{\sqrt{\beta}}\int_0^1 \mathrm{d}t\, p\right) \times \exp\left(-\beta\int_0^1 \mathrm{d}t\, l(x+tz+\sqrt{\beta}q-\Theta p/\sqrt{\beta})\, r(x+tz+\sqrt{\beta}q+\Theta p/\sqrt{\beta})\right)\Bigg\rangle. \tag{2.7}$$

Next, we make a small β expansion of the second exponential and a Taylor expansion of l and r around x

$$\begin{aligned}\langle x+z|\,\mathrm{e}^{-\beta H}|x\rangle = {}& \frac{1}{(4\pi\beta)^{d/2}}\sum_{n=0}^{\infty}\frac{(-\beta)^n}{n!}\int_0^1 \mathrm{d}t_1\cdots\int_0^1 \mathrm{d}t_n \\ &\times\Bigg\langle \exp\left(\mathrm{i}\frac{z}{\sqrt{\beta}}\int_0^1 \mathrm{d}t\, p\right)\exp\Bigg(\sum_{i=1}^{n}[t_i z+\sqrt{\beta}q(t_i)-\Theta p(t_i)/\sqrt{\beta}]\partial_i^l \\ &+[t_i z+\sqrt{\beta}q(t_i)+\Theta p(t_i)/\sqrt{\beta}]\partial_i^r\Bigg)\Bigg\rangle l(x_1)\cdots l(x_n)\, r(x_1)\cdots r(x_n)|_x,\end{aligned} \tag{2.8}$$

where $\partial_i^l, \partial_i^r$ are the gradients[8] of $l(x_i), r(x_i)$. As indicated, all x_i must be evaluated at x after performing these derivatives. Expression (2.8) can be written as

$$\begin{aligned}\langle x+z|\,\mathrm{e}^{-\beta H}|x\rangle = {}& \frac{1}{(4\pi\beta)^{d/2}}\sum_{n=0}^{\infty}\frac{(-\beta)^n}{n!}\int_0^1 \mathrm{d}t_1\cdots\int_0^1 \mathrm{d}t_n \\ &\times\exp\left(\sum_{i=1}^{n} t_i z(\partial_i^l+\partial_i^r)\right)\left\langle \exp\left(\int_0^1 \mathrm{d}t\,(p\,k_n+q\,j_n)\right)\right\rangle l(x_1)\cdots l(x_n)\, r(x_1)\cdots r(x_n)\Bigg|_x\end{aligned} \tag{2.9}$$

if we define the sources $k_n(t), j_n(t)$ as

$$\begin{aligned}k_n(t) &= \frac{\mathrm{i}z}{\sqrt{\beta}}+\frac{\Theta}{\sqrt{\beta}}\sum_{i=1}^{n}\delta(t-t_i)(\partial_i^l-\partial_i^r),\\ j_n(t) &= \sqrt{\beta}\sum_{i=1}^{n}\delta(t-t_i)(\partial_i^l+\partial_i^r).\end{aligned} \tag{2.10}$$

In the following section, we will compute the expectation value $\left\langle \exp\left(\int_0^1 \mathrm{d}t\,(p\,k+q\,j)\right)\right\rangle$ for arbitrary sources $k(t), j(t)$.

3. The generating functional in phase space

Let us compute the generating functional

$$\begin{aligned}Z[k,j] := \left\langle \exp\left(\int_0^1 \mathrm{d}t\,(pk+qj)\right)\right\rangle &= \frac{\int \mathcal{D}q\mathcal{D}p\exp\left(-\int_0^1 \mathrm{d}t\,(p^2-\mathrm{i}p\dot{q})\right)\exp\left(\int_0^1 \mathrm{d}t\,(pk+qj)\right)}{\int \mathcal{D}q\mathcal{D}p\exp\left(-\int_0^1 \mathrm{d}t\,(p^2-\mathrm{i}p\dot{q})\right)} \\ &= \frac{\int \mathcal{D}P\,\exp\left(-\frac{1}{2}\int_0^1 \mathrm{d}t\, P^t A P+\int_0^1 \mathrm{d}t\, P^t K\right)}{\int \mathcal{D}P\,\exp\left(-\int_0^1 \mathrm{d}t\, P^t A P\right)}\end{aligned} \tag{3.1}$$

[8] Note that the subindex i does not denote spacetime components but refers to the ith spacetime point x_i.

for arbitrary sources $k(t)$, $j(t)$. In this last expression, we have defined the vectors

$$P := \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} \quad K := \begin{pmatrix} k(t) \\ j(t) \end{pmatrix} \tag{3.2}$$

and the operator

$$A := \begin{pmatrix} 2 & -\mathrm{i}\partial_t \\ \mathrm{i}\partial_t & 0 \end{pmatrix}. \tag{3.3}$$

Completing squares and inverting the operator A—taking into account the boundary conditions $q(0) = q(1) = 0$—we obtain the generating functional in phase space:

$$Z[k, j] = \exp\left(\frac{1}{2}\int_0^1 \mathrm{d}t\, K^t A^{-1} K\right). \tag{3.4}$$

The kernel of the operator A^{-1} is given by

$$A^{-1}(t, t') = \begin{pmatrix} \frac{1}{2} & \frac{\mathrm{i}}{2}[h(t, t') + f(t, t')] \\ \frac{\mathrm{i}}{2}[h(t, t') - f(t, t')] & 2g(t, t') \end{pmatrix}, \tag{3.5}$$

where

$$\begin{aligned} h(t, t') &:= 1 - t - t', \\ f(t, t') &:= t - t' - \epsilon(t - t'), \\ g(t, t') &:= t(1 - t')H(t' - t) + t'(1 - t)H(t - t'). \end{aligned} \tag{3.6}$$

The sign function $\epsilon(\cdot)$ is ± 1 if its argument is positive or negative, respectively; $H(\cdot)$ represents the Heaviside function.

To obtain the expectation value of expression (2.9) we replace the sources given by equations (2.10) into expression (3.4):

$$\left\langle \exp\left(\int_0^1 \mathrm{d}t\, (p\, k_n + q\, j_n)\right)\right\rangle = \exp\left(-\frac{z^2}{4\beta} + \frac{\mathrm{i}z}{2\beta}\Theta\sum_{i=1}^n (\partial_i^l - \partial_i^r)\right) \mathrm{e}^{\triangle_n}, \tag{3.7}$$

where the operator $\triangle_n$ is defined as

$$\begin{aligned} \triangle_n := \sum_{i,j=1}^n \Bigg[&\beta g(t_i, t_j)\big(\partial_i^l + \partial_i^r\big)\big(\partial_j^l + \partial_j^r\big) - \frac{1}{4\beta}\big(\partial_i^l - \partial_i^r\big)\Theta^2\big(\partial_j^l - \partial_j^r\big) \\ &- \frac{\mathrm{i}}{2} f(t_i, t_j)\big(\partial_i^l \Theta \partial_j^l - \partial_i^r \Theta \partial_j^r\big) - \mathrm{i}h(t_i, t_j)\partial_i^l \Theta \partial_j^r \Bigg]. \end{aligned} \tag{3.8}$$

4. The non-commutative heat kernel

The smeared heat trace of the non-local operator H, defined in equation (2.2), can be written as

$$\mathrm{Tr}(f(x) \bar{\star}\, \mathrm{e}^{-\beta H}) = \int_{\mathbb{R}^d} \mathrm{d}x\, \langle x + z|\, \mathrm{e}^{-\beta H} |x\rangle|_{z=-\mathrm{i}\bar{\Theta}\partial^f} f(x), \tag{4.1}$$

where $\bar{\star}$ represents a Moyal product as defined in (1.1) but in terms of another antisymmetric matrix $\bar{\Theta}$ which, in principle, differs from Θ. Expression (4.1) can be easily obtained by inserting in the lhs a spectral decomposition of unity in terms of position eigenstates and using that $\exp\big(-\mathrm{i}\bar{\Theta}\partial^f\partial\big)\langle x| = \langle x - \mathrm{i}\bar{\Theta}\partial^f|$.

The matrix $\bar{\Theta}$ allows us to consider at the same time the cases where the smearing function f acts by commutative pointwise multiplication ($\bar{\Theta} = 0$), by left-Moyal multiplication ($\bar{\Theta} = \Theta$) or right-Moyal multiplication ($\bar{\Theta} = -\Theta$). As indicated, the variable z must be replaced by the operator $-\mathrm{i}\bar{\Theta}\partial^f$, where ∂^f is the gradient acting on the smearing function f only. The heat kernel $\langle x+z|\mathrm{e}^{-\beta H}|x\rangle$ is obtained by replacing equation (3.7) into expression (2.9). Inserting the result into (4.1), we obtain our master formula

$$\mathrm{Tr}\left(f(x)\,\bar{\star}\,\mathrm{e}^{-\beta H}\right) = \frac{1}{(4\pi\beta)^{d/2}}\sum_{n=0}^{\infty}(-\beta)^n\int_{\mathbb{R}^d}\mathrm{d}x\, f(x)\exp\left(\frac{1}{4\beta}\sum_{i,j=1}^{n}D_iD_j\right)$$
$$\times\int_0^1\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\cdots\int_0^{t_{n-1}}\mathrm{d}t_n\,\mathrm{e}^{\mathrm{i}\triangle_n^{\mathrm{NC}}+\beta\triangle_n^C}l(x_1)\cdots l(x_n)\,r(x_1)\cdots r(x_n)\Big|_x, \tag{4.2}$$

where we have defined the following differential operators:

$$\triangle_n^C := \sum_{i,j=1}^{n} g(t_i,t_j)(\partial_i^l+\partial_i^r)(\partial_j^l+\partial_j^r),$$
$$D_i := (\Theta-\bar{\Theta})\partial_i^l - (\Theta+\bar{\Theta})\partial_i^r,$$
$$\triangle_n^{\mathrm{NC}} := \sum_{i<j=1}^{n}\left\{\left[\partial_i^l\Theta\partial_j^l-\partial_i^r\Theta\partial_j^r\right]-(1-t_i-t_j)(\partial_i^l\Theta\partial_j^r-\partial_i^r\Theta\partial_j^l)\right.$$
$$\left.+(t_i-t_j)\left[\partial_i^l(-\Theta+\bar{\Theta})\partial_j^l+\partial_i^r(\Theta+\bar{\Theta})\partial_j^r+\partial_i^l\bar{\Theta}\partial_j^r+\partial_i^r\bar{\Theta}\partial_j^l\right]\right\}$$
$$-\sum_{i=1}^{n}(1-2t_i)\partial_i^l\Theta\partial_i^r. \tag{4.3}$$

In the derivation of (4.2), we have used the symmetry of the integrand with respect to permutations of the variables t_i and we have integrated by parts to make the replacement $\partial^f\to-\sum_{i=1}^n(\partial_i^l+\partial_i^r)$.

A few remarks are now in order. As we will see next, the SDW coefficients for the commutative case are fully determined by the action of the operator $\triangle_n^C$ since $\triangle_n^{\mathrm{NC}}$ and D_i vanish for $\Theta=\bar{\Theta}=0$. On the other hand, if $r(x)\equiv 1$ (or $l(x)\equiv 1$) and the smearing function acts by left- (respectively right-) Moyal multiplication, then D_i vanishes and the only non-vanishing term in $\triangle_n^{\mathrm{NC}}$ is the first one $\partial_i^l\Theta\partial_j^l$ (respectively $-\partial_i^r\Theta\partial_j^r$) which replaces any pointwise product by a left- (respectively right-) Moyal product. Let us also mention that the heat trace with no smearing function corresponds to $f(x)\equiv 1$ and $\bar{\Theta}=0$.

Finally, we will show that the $1/\beta$ coefficient of $\sum_{i,j}D_iD_j$ in expression (4.2) is responsible for Moyal non-local SDW coefficients, which can be shown to correspond to contributions of non-planar diagrams, leading to the well-known UV/IR mixing.

In the rest of this section, we will apply master formula (4.2) to these different settings in order to describe the SDW coefficients.

4.1. Commutative limit

First of all, we apply formula (4.2) to the heat trace in the commutative case for future comparison with the subsequent non-commutative expressions. For $\bar{\Theta}=\Theta=0$, it is sufficient to consider the case $r(x)\equiv 1$. As already mentioned, the operators D_i and $\triangle_n^{\mathrm{NC}}$ vanish so that formula (4.2) reads

$$\mathrm{Tr}\big(f(x)\,\cdot\,\mathrm{e}^{-\beta H}\big) = \frac{1}{(4\pi\beta)^{d/2}}\sum_{n=0}^{\infty}(-\beta)^n\int_{\mathbb{R}^d}\mathrm{d}x\,f(x)$$
$$\times\int_0^1\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\cdots\int_0^{t_{n-1}}\mathrm{d}t_n\exp\left(\beta\sum_{i,j=1}^{n}g(t_i,t_j)\partial_i\partial_j\right)\,l(x_1)\cdots l(x_n)\Big|_x. \tag{4.4}$$

This expression shows that the SDW coefficients a_n are integrals of products of the smearing function, the potential and derivatives thereof (see equation (1.8)).

For later use, in expression (4.4) we keep the time ordering explicit in the multi-worldline integral so that (4.4) gives the correct SDW coefficients even if the potential $l(x)$ is a matrix potential. In such a case, the product of two adjacent $l(x)$s has to be understood as a spatially pointwise matricial product.

4.2. Pointwise multiplication by a smearing function

In this subsection, we consider the non-commutative operator (2.2) but for the case in which the smearing function $f(x)$ acts by pointwise multiplication, i.e. $\bar{\Theta} = 0$. We will show that some SDW coefficients are Moyal non-local [19].

For $\bar{\Theta} = 0$, formula (4.2) reads

$$\operatorname{Tr}\big(f(x)\cdot \mathrm{e}^{-\beta H}\big) = \frac{1}{(4\pi\beta)^{d/2}}\sum_{n=0}^{\infty}(-\beta)^n\int_{\mathbb{R}^d}\mathrm{d}x\, f(x) \\ \times\int_0^1 \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\cdots\int_0^{t_{n-1}}\mathrm{d}t_n\, \mathrm{e}^{\triangle_n} l(x_1)\cdots l(x_n)\, r(x_1)\cdots r(x_n)\Big|_x . \tag{4.5}$$

For the purposes of this section, it will suffice to consider a potential which contains only a left-Moyal product so we will consider the case $r(x)\equiv 1$. Thus, the leading terms in (4.5) read

$$\operatorname{Tr}\big(f(x)\cdot \mathrm{e}^{-\beta H}\big) = \frac{1}{(4\pi\beta)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}x\, f(x)\left(1-\beta\exp\left(-\frac{1}{4\beta}\partial^l\Theta^2\partial^l\right)l(x)+\cdots\right). \tag{4.6}$$

The first term coincides with the leading term—the volume contribution—in the commutative case, whereas the second one can be written as (see the discussion below equation (1.2))

$$-\frac{\beta}{(4\pi\beta)^{d/2}}\int_{\mathbb{R}^c}\mathrm{d}\tilde{x}\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{x}\, f(\tilde{x},\hat{x})\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{y}\frac{\beta^b}{\pi^b\det\Xi}\exp\big(\beta(\hat{x}-\hat{y})\Xi^{-2}(\hat{x}-\hat{y})\big)\; l(\tilde{x},\hat{y}) \\ \sim -\frac{1}{(4\pi\beta)^{d/2}}\cdot\frac{\beta^{b+1}}{\pi^b\det\Xi}\int_{\mathbb{R}^c}\mathrm{d}\tilde{x}\left(\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{x}\, f(\tilde{x},\hat{x})\right)\left(\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{y}\, l(\tilde{x},\hat{y})\right)+\cdots . \tag{4.7}$$

As can be seen from this expression, the presence of a smearing function that acts by pointwise multiplication yields a contribution to the SDW coefficient a_{b+1} which is non-local in the generalized Moyal sense. Note that this Moyal non-local coefficient was obtained when considering the case $r(x)\equiv 1$, i.e. even when the potential in H does not mix left- and right-Moyal actions.

4.3. No mixing between left- and right-Moyal multiplications

We will now consider the case in which the heat trace involves only left-Moyal multiplication ($r(x)\equiv 1$ and $\bar{\Theta}=\Theta$) or only right-Moyal multiplication ($l(x)\equiv 1$ and $\bar{\Theta}=-\Theta$). Under any of these alternative assumptions the operators D_i vanish, whereas $\triangle_n^{\mathrm{NC}}$ takes the form

$$\triangle_n^{\mathrm{NC}} := \pm\sum_{i<j=1}^{n}\partial_i\Theta\partial_j, \tag{4.8}$$

where the upper (lower) sign corresponds to the case where only left- (right-) Moyal multiplication is considered; the derivatives ∂_i act consequently on $l(x_i)$ ($r(x_i)$). Therefore, according to formula (4.2), the action of the operator $\mathrm{e}^{\mathrm{i}\triangle_n^{\mathrm{NC}}}$ is the only effect of non-commutativity on the SDW coefficients. Note that the action of this operator defines Moyal multiplication (see equation (1.1)), so that the heat trace (for the left-Moyal case) reduces to

$$\mathrm{Tr}\big(f(x)\star\mathrm{e}^{-\beta H}\big)=\frac{1}{(4\pi\beta)^{d/2}}\sum_{n=0}^{\infty}(-\beta)^n\int_{\mathbb{R}^d}\mathrm{d}x\, f(x)\\ \times\int_0^1\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\cdots\int_0^{t_{n-1}}\mathrm{d}t_n\,\exp\left(\beta\sum_{i,j=1}^{n}g(t_i,t_j)\partial_i\partial_j\right)\, l(x_1)\star\cdots\star l(x_n)\Bigg|_x . \tag{4.9}$$

In this case, we conclude that the non-commutative SDW coefficients can be obtained from the commutative coefficients for a matrix potential $l(x)$ (cf (4.4)), by replacing every spatially pointwise matrix product of adjacent potentials with left- (right-) Moyal products. For example, up to β^3 one obtains

$$\mathrm{Tr}\left(f(x)\star\mathrm{e}^{-\beta H}\right)=\frac{1}{(4\pi\beta)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}x\, f(x)\\ \times\left[1-\beta l(x)+\beta^2\left(\frac{1}{2}l_\star^2(x)-\frac{1}{6}\partial^2 l(x)\right)+\beta^3\left(-\frac{1}{60}\partial^4 l(x)\right.\right.\\ \left.\left.+\frac{1}{12}\left(\partial^2 l\star l(x)+l\star\partial^2 l(x)+\partial l\star\partial l(x)\right)-\frac{1}{3!}l_\star^3(x)\right)+\cdots\right]. \tag{4.10}$$

In summary, when there is only left- (right-) Moyal multiplication the SDW coefficients are Moyal local [19, 20].

4.4. Mixing between left- and right-Moyal multiplications

In this last subsection, we will consider the presence of both functions $l(x)$ and $r(x)$ in the potential of the operator (2.2). The calculation goes along the same line followed in subsection 4.2 and we will arrive to a similar conclusion, namely the existence of Moyal non-local SDW coefficients. For simplicity, we will consider $f(x)\equiv 1$. Formula (4.2) then reads

$$\mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right)=\frac{1}{(4\pi\beta)^{d/2}}\sum_{n=0}^{\infty}(-\beta)^n\int_{\mathbb{R}^d}\mathrm{d}x\\ \times\int_0^1\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\cdots\int_0^{t_{n-1}}\mathrm{d}t_n\,\mathrm{e}^{\triangle_n}l(x_1)\cdots l(x_n)\,r(x_1)\cdots r(x_n)\Big|_x . \tag{4.11}$$

Let us consider the leading terms in expansion (4.11) corresponding to an increasing number n of insertions of $l(x)$ and $r(x)$.

For $n=0$, we obtain the commutative volume contribution. For $n=1$, we obtain the leading contribution

$$-\frac{\beta}{(4\pi\beta)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}x\, r(x)\exp\left(-\frac{1}{\beta}\partial^l\Theta^2\partial^l\right)l(x), \tag{4.12}$$

where upon integration by parts we have replaced $\partial^r\to-\partial^l$. Expression (4.12) can be obtained from the subleading term of (4.6) by identifying $r(x)$ with $f(x)$. Proceeding analogously, we obtain for (4.12)

$$-\frac{1}{(4\pi\beta)^{d/2}}\frac{\beta^{b+1}}{(4\pi)^b\det\Xi}\int_{\mathbb{R}^c}\mathrm{d}\tilde{x}\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{x}\, r(\tilde{x},\hat{x})\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{y}\exp\left(\frac{\beta}{4}(\hat{x}-\hat{y})\Xi^{-2}(\hat{x}-\hat{y})\right)\, l(\tilde{x},\hat{y})$$

$$\sim -\frac{1}{(4\pi\beta)^{d/2}}\cdot\frac{\beta^{b+1}}{(4\pi)^b\det\Xi}\int_{\mathbb{R}^c}\mathrm{d}\tilde{x}\left\{\left(\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{x}\,r(\tilde{x},\hat{x})\right)\left(\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{y}\,l(\tilde{x},\hat{y})\right)\right.$$
$$+\frac{\beta}{4}(\Xi^{-2})_{ij}\left[\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{x}\,\hat{x}^i\hat{x}^j r(\tilde{x},\hat{x})\cdot\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{y}\,l(\tilde{x},\hat{y})\right.$$
$$+\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{x}\,r(\tilde{x},\hat{x})\cdot\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{y}\,\hat{y}^i\hat{y}^j l(\tilde{x},\hat{y})$$
$$\left.\left.-2\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{x}\,\hat{x}^i r(\tilde{x},\hat{x})\int_{\mathbb{R}^{2b}}\mathrm{d}\hat{y}\,\hat{y}^j l(\tilde{x},\hat{y})\right]+\cdots\right\}. \tag{4.13}$$

In this expression, we have explicitly written the splitting of the coordinates $x\in\mathbb{R}^d$ into commuting $\tilde{x}\in\mathbb{R}^c$ and non-commuting $\hat{x}\in\mathbb{R}^{2b}$ coordinates (see the discussion below equation (1.2)). Expansion (4.13) shows the Moyal non-local contributions, which are linear in the product $r(x)l(y)$, to the coefficients a_{b+1} and a_{b+2} [18] that could lead to the non-renormalizability of the corresponding theory [21].

4.5. The non-commutative torus

As our last example, we will consider the operator (2.2) on the d-dimensional non-commutative torus T^d, as defined in [8]. The coordinates $x=(\tilde{x},\hat{x})$ on T^d can be split into commuting $\tilde{x}\in T^c$ and non-commuting $\hat{x}\in T^{2b}$ components, with $d=c+2b$, as discussed after equation (1.2). We also define $0\leqslant x_i\leqslant L_i$.

The heat kernel $\langle y|\,\mathrm{e}^{-\beta H}|x\rangle_{T^d}$ subject to the corresponding periodic boundary conditions can be obtained as the sum of infinitely many transition amplitudes $\langle y+t_k|\mathrm{e}^{-\beta H}|x\rangle_{\mathbb{R}^d}$ computed on the whole space $\mathbb{R}^d$, where $t_k=(L_1k_1,\ldots,L_dk_d)$ and $k=(k_1,\ldots,k_d)\in\mathbb{Z}^d$:

$$\langle y|\mathrm{e}^{-\beta H}|x\rangle_{T^d}=\sum_{k\in\mathbb{Z}^d}\langle y+t_k|\,\mathrm{e}^{-\beta H}|x\rangle_{\mathbb{R}^d}. \tag{4.14}$$

The functions l, r in the operator H of the rhs of this equation are the periodic extensions to the whole space $\mathbb{R}^d$ of the functions l, r in the operator H of the lhs.

Correspondingly, the heat trace on the torus can be computed as

$$\mathrm{Tr}(\mathrm{e}^{-\beta H})=\int_{T^d}\mathrm{d}x\,\langle x|\,\mathrm{e}^{-\beta H}|x\rangle_{T^d}=\int_{T^d}\mathrm{d}x\sum_{k\in\mathbb{Z}^d}\langle x+t_k|\,\mathrm{e}^{-\beta H}|x\rangle_{\mathbb{R}^d}. \tag{4.15}$$

The transition amplitudes on the whole $\mathbb{R}^d$ were already computed and can be read from equations (2.9) and (3.7). The first contribution, corresponding to $n=0$ in equation (2.9), gives the volume term

$$\frac{1}{(4\pi\beta)^{d/2}}\prod_{i=1}^{d}L_i\sum_{k\in\mathbb{Z}}\exp\left(-\frac{L_i^2}{4\beta}k^2\right)\sim\frac{1}{(4\pi\beta)^{d/2}}L_1,\ldots,L_d. \tag{4.16}$$

Note that the only contribution to the asymptotic expansion for small β comes from the term corresponding to $k=0$.

On the other hand, the term corresponding to $n=1$ in equation (2.9) reads

$$-\frac{\beta}{(4\pi\beta)^{d/2}}\sum_{k\in\mathbb{Z}^d}\int_{T^d}\mathrm{d}x\,r(x)\exp\left(-\frac{1}{\beta}\left(-\mathrm{i}\Theta\partial+\frac{1}{2}t_k\right)^2\right)l(x), \tag{4.17}$$

where we have used the periodicity of $l(x)$ and $r(x)$ to replace ∂^r by $-\partial^l$. This contribution has been analyzed in [8] for the case in which the matrix elements of Θ satisfy a Diophantine condition.

5. Application to $\lambda\,\phi_\star^4$

In this section, we will apply our formula (4.2) to a simple non-commutative model. In particular, we will compute some SDW coefficients to study the one-loop corrections to the propagator of a real scalar field with a quartic self-interaction defined on Euclidean spacetime $\mathbb{R}^d$. We therefore consider the following action:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi_\star^4, \tag{5.1}$$

where $\phi_\star^4 := \phi\star\phi\star\phi\star\phi$. The one-loop effective action Γ can be written as [18]

$$\begin{aligned}\Gamma &= \frac{1}{2}\log\mathrm{Det}\left\{-\partial^2 + m^2 + \frac{\lambda}{3}\,[L(\phi_\star^2) + R(\phi_\star^2) + L(\phi)R(\phi)]\right\}\\ &= -\frac{1}{2}\int_{\Lambda^{-2}}^{\infty}\frac{\mathrm{d}\beta}{\beta}\,\mathrm{e}^{-\beta m^2}\,\mathrm{Tr}\exp\left(-\beta\left\{-\partial^2 + \frac{\lambda}{3}\,\left[L(\phi_\star^2) + R(\phi_\star^2) + L(\phi)R(\phi)\right]\right\}\right),\end{aligned} \tag{5.2}$$

where we have used the Schwinger proper time approach to represent the functional determinant and we have introduced an UV-cutoff Λ.

Since the propagator is obtained from the quadratic terms in the effective action, we only need to consider the terms in expression (5.2) which are linear in λ. Therefore, we use formula (4.2) to compute the contributions of the three terms $L(\phi_\star^2)$, $R(\phi_\star^2)$ and $L(\phi)R(\phi)$ separately. By replacing $f(x)\equiv 1$, $\bar{\Theta}=0$ and $r(x)\equiv 1$ in the term corresponding to $n=1$ in equation (4.2) we obtain the contribution of $L(\phi_\star^2)$ to the effective action. Analogously, the contribution corresponding to $R(\phi_\star^2)$ is obtained by replacing instead $l(x)\equiv 1$. Both contributions are equal and the sum of them reads

$$\int_{\Lambda^{-2}}^{\infty}\mathrm{d}\beta\,\mathrm{e}^{-\beta m^2}\,\frac{1}{(4\pi\beta)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}x\,\frac{\lambda}{3}\phi_\star^2(x) = \frac{\lambda}{3}\frac{m^{d-2}}{(2\pi)^{d/2}}\,\Gamma(1-d/2, m^2/\Lambda^2)\int_{\mathbb{R}^d}\phi^2, \tag{5.3}$$

where $\Gamma(\cdot,\cdot)$ represents the incomplete gamma function [24]. In the limit $\Lambda\to\infty$, this contribution diverges as $\log\Lambda$, for $d=2$, and as Λ^{d-2}, for $d>2$. This divergence is eliminated by a mass redefinition; in this way, the dependence of the mass with the cutoff Λ is determined. Note that result (5.3)—which corresponds to the contribution of one-loop planar diagrams—does not depend on the non-commutativity parameters and holds also for the commutative case.

The remaining contribution, corresponding to the term $L(\phi)R(\phi)$, is obtained by replacing $f(x)\equiv 1$, $\bar{\Theta}=0$ and $\partial^l = -\partial^r$ in the term corresponding to $n=1$ in equation (4.2); the result reads

$$\begin{aligned}&\frac{1}{2}\int_{\Lambda^{-2}}^{\infty}\mathrm{d}\beta\,\mathrm{e}^{-\beta m^2}\,\frac{1}{(4\pi\beta)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}x\,\frac{\lambda}{3}\phi(x)\exp\left(-\frac{1}{\beta}\partial\Theta^2\partial\right)\phi(x)\\ &\qquad = \frac{\lambda}{6}(2\pi)^{3d/2}m^{d-2}\int\mathrm{d}^c\tilde{p}\,\mathrm{d}^{2b}\hat{p}\,\tilde{\phi}^*(\tilde{p},\hat{p})\tilde{\phi}(\tilde{p},\hat{p})\cdot\Sigma_{\mathrm{NP}}(\hat{p}),\end{aligned} \tag{5.4}$$

where

$$\Sigma_{\mathrm{NP}}(\hat{p}) := \int_0^{\infty}\frac{\mathrm{d}\beta}{\beta^{d/2}}\exp\left(-\beta - \frac{m^2}{\beta}|\Xi\hat{p}|^2\right) = 2(m|\Xi\hat{p}|)^{1-d/2}K_{d/2-1}(2m|\Xi\hat{p}|), \tag{5.5}$$

with $K_{d/2-1}(\cdot)$ being the modified Bessel function. In equation (5.4), we have made use of the splitting defined in equation (1.3) to separate spacetime $\mathbb{R}^d$ into $\mathbb{R}^c$, which is described by commuting coordinates, and $\mathbb{R}^{2b}$, where non-commutativity is defined by the non-degenerate matrix Ξ. We have also written this contribution to the effective action in terms of the Fourier transform $\tilde{\phi}(\tilde{p},\hat{p})$ of the field, where $\tilde{p}\in\mathbb{R}^c$ and $\hat{p}\in\mathbb{R}^{2b}$. Note also that in expression (5.5) the term depending on Ξ makes the integral convergent at $\beta\to 0$ and, in consequence, the cutoff Λ has been removed. In other words, non-commutativity regularizes UV-divergence at the one-loop level. It can be shown that expression (5.4) corresponds to the contribution of one-loop non-planar diagrams.

Since $\Sigma_{\mathrm{NP}}(\hat{p}) \sim |\hat{p}|^{-d+2}$, for small $|\hat{p}|$ and $d > 2$, the integrand in (5.4) grows as $|\hat{p}|^{1-c}$ for $|\hat{p}| \to 0$, where $c = d - 2b$ is the number of commuting coordinates. Therefore, this contribution to the one-loop effective action is divergent if the number of commuting coordinates is greater or equal than 2. The same result can be obtained from expression (4.13), whose leading term reads

$$-\frac{1}{(4\pi\beta)^{d/2}} \cdot \frac{\beta^{b+1}}{(4\pi)^b \det \Xi} \int_{\mathbb{R}^c} \mathrm{d}\tilde{x} \left(\int_{\mathbb{R}^{2b}} \mathrm{d}\hat{x}\, \phi(\tilde{x}, \hat{x}) \right) \left(\int_{\mathbb{R}^{2b}} \mathrm{d}\hat{y}\, \phi(\tilde{x}, \hat{y}) \right). \quad (5.6)$$

If this contribution is inserted in the last line of expression (5.2) one can see that the integrand behaves as $\beta^{-c/2}$ for small β and then the integral diverges as $\Lambda \to \infty$ if $c \geqslant 2$.

As is well known, the S-matrix may present pathological behaviors if time is a non-commutative coordinate [25–27]. This suggests that, if the number of space-like coordinates is odd, then at least two coordinates should be commutative, i.e. $c \geqslant 2$ in our Wick-rotated Euclidean scenario. In this case, as we have seen, some one-loop non-planar diagrams generate divergencies that cannot be regularized by a redefinition of the parameters of the theory. In consequence, as was shown by Gayral *et al* [21], the four-dimensional model defined by the Lagrangian (5.1) does not have a well-defined effective action. The renormalizability of such a model by the introduction of a new type of non-local term was proved in [28].

On the other hand, note that the function $\Sigma_{\mathrm{NP}}(\hat{p})$ behaves as

$$\Sigma_{\mathrm{NP}}(\hat{p}) = \left[(d/2-2)!(m|\Xi\hat{p}|)^{2-d} + \frac{2(-1)^{d/2}}{(d/2-1)!} \log\left(m|\Xi\hat{p}|\right) \right] (1 + O(|\hat{p}|^2)), \quad (5.7)$$

for small $|\hat{p}|$ and $d > 2$. The corresponding result for $d = 2$ reads

$$\Sigma_{\mathrm{NP}}(\hat{p}) = -2\left[\log m|\Xi\hat{p}| + \gamma\right](1 + O(|\hat{p}|^2)), \quad (5.8)$$

where γ is Euler's constant. The result of expression (5.7) evaluated at $d = 4$ and $c = 0$ corresponds to the contribution to the effective action computed by Minwalla *et al* [3] by considering one-loop non-planar diagrams.

The divergent behavior at small $\hat{p}$ shown in expressions (5.7) and (5.8) implies that the propagator receives one-loop corrections which are divergent for small values of the momentum in the non-commutative directions. This is the well-known UV/IR mixing, which shows that in some non-commutative theories the integration of internal momenta can generate divergencies at small values of the external momenta, even for massive fields. As a consequence, these non-commutative theories are non-renormalizable.

6. Conclusions

We have determined the phase-space propagator and the phase-space-generating functional and written a path integral formulation for non-local operators which are relevant in field theories on non-commutative spacetimes. In this formulation, we have derived a master formula (cf (4.2)) for the heat-trace expansion which can be applied to different non-commutative settings. In particular, we considered a non-local operator involving the product of left- and right-Moyal multiplications.

We have shown that the natural rescaling $x \to \sqrt{\beta}x$ and $p \to p/\sqrt{\beta}$ in phase space introduces $O(1/\beta)$ differential operators which act on the potentials. These operators are given by D_i, defined in equations (4.3). Note that when the heat operator involves only left-Moyal multiplication ($\partial^r = 0$ and $\bar{\Theta} = \Theta$) or only right-Moyal multiplication ($\partial^l = 0$ and

$\bar{\Theta} = -\Theta$) the operators D_i vanish. However, in the case where both left- and right-Moyal multiplications are present, the operators D_i generate SDW coefficients which are non-local even in the generalized Moyal sense. These non-local SDW coefficients are equivalent to the non-planar contributions in a perturbative calculation of the effective action in terms of Feynman diagrams.

Our phase-space formulation provides a simple derivation of these results and is suitable for further generalizations in non-commutative models. In particular, the introduction of a Grosse–Wulkenhaar term [29, 30] can be straightforwardly implemented in our phase-space approach by replacing the vanishing matrix element of the propagator given by equation (3.3) by a constant matrix element, proportional to the squared frequency of the harmonic oscillator term. Finally, we consider that this formalism could be a useful tool in the study of other models involving more general non-local operators. Research along these lines is currently in progress.

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