

Integrable achiral $D5$ -brane reflections and asymptotic Bethe equations

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Abstract

We study the reflection of magnons from a $D5$ -brane in the framework of the AdS/CFT correspondence. We consider two possible orientations of the $D5$ -brane with respect to the reference vacuum state, namely vacuum states aligned along “vertical” and “horizontal” directions. We show that the reflections are of the achiral type. We also show that the reflection matrices satisfy the boundary Yang-Baxter equations for both orientations. In the horizontal case the reflection matrix can be interpreted in terms of a bulk S-matrix, $S(p, -p)$, and factorizability of boundary scattering therefore follows from that of bulk scattering. Finally, we solve the nested coordinate Bethe ansatz for the system in the vertical case to find the Bethe equations. In the horizontal case, the Bethe equations are of the same form as those for the closed string.

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1 Introduction

Integrable QFTs with boundaries have long been studied because of their applications in many problems in physics. Over the past few years, certain nonlinear σ -models corresponding to strings propagating in AdS spacetimes have been the subject of intense investigations because of their importance for the understanding of the AdS/CFT correspondence. The AdS/CFT correspondence is a duality between certain seemingly different theories [1]. It states, for example, that the spectrum of scale dimensions in the conformal $\mathcal{N} = 4$ super Yang-Mills theory in 4 dimensions coincides with the spectrum of energies of strings propagating in $AdS_5 \times S^5$. For the full range of values of the interacting CFT coupling constant, this spectral problem is believed to be exactly integrable in the planar limit, i.e. in the limit of the gauge group rank $N \rightarrow \infty$. With the use of integrability methods substantial and steady progress has been observed in the resolution of this planar spectral problem (see [2] for

a complete overview). Though most studies deal with periodic boundary conditions, cases with open boundary conditions have also been considered (see [3] and references therein). In some cases, the addition of open boundaries enriches the problem, for example, by making it less supersymmetric and by adding matter in the fundamental representation of the gauge group.

Supersymmetric $D3$, $D5$ and $D7$ -branes are natural candidates to introduce Dirichlet boundary conditions in the nonlinear σ -model for strings on $AdS_5 \times S^5$. An all-loop derivation of the reflection matrix for certain $D3$ -branes was performed in [4], where the authors also showed that the boundary Yang-Baxter equation (bYBE) is satisfied. This was in agreement with explicit weak and strong coupling limits where integrability had been previously observed [5, 6]. $D7$ -branes also provide integrable boundary conditions [7].

However, the integrability of the supersymmetric $D5$ -brane boundary was less clear. On the one hand, 1-loop results on the gauge theory side [8] indicated that the dilatation operator in the scalar sector was an integrable Hamiltonian. On the other hand, the same procedure successfully used to show the classical integrability of the $D3$ and $D7$ -brane boundary conditions [6], could not be used to construct the infinite set of non-local charges for the case of the $D5$ -brane¹. This made it seem questionable whether integrability was present beyond 1-loop, though it by no means ruled it out. In part, it was this uncertainty that motivated [7], where the all-loop reflection matrices for two possible orientations of the $D5$ -brane were obtained. Unfortunately, that paper contained a sign error – originating, as we discuss below, in a graded permutation – which led to the erroneous conclusion that the bYBE was not fulfilled with $D5$ -brane reflection matrices. In the present paper we revise that statement, to find not only that the $D5$ -brane boundary conditions are integrable but also that they have additional interesting features. For instance, for one of the two possible $D5$ -brane orientations, the reflection can be fully understood in terms of a bulk S-matrix of the form $\mathcal{S}(p, -p)$. Another interesting aspect is the achiral nature of the boundary reflection i.e. the incoming left particles become right ones after the reflection and the right particles become left ones [9]. These new features play an important role in constructing a nested Bethe ansatz leading to the Bethe equations to solve the spectral problem with $D5$ -brane boundary conditions.

The structure of this paper is as follows. In section 2 we review the all-loop $D5$ -brane reflection matrices for the two possible relative orientations between the brane and the polarization of the vacuum. For the horizontal vacuum orientation we use a basis for the vector representation different than the one used in [7], for which it becomes evident that the reflection matrices for a right boundary can be understood as a bulk S-matrices of the form $\mathcal{S}(p, -p)$. The factorizability of the bulk S-matrix constitutes a strong hint for the factoriz-

¹For the $D5$ -brane, the charges could be constructed in an $SU(2)$ sub-sector only.

ability of the reflection matrix. We show that bYBE is fulfilled for both relative orientations and indicate what the error was in [7]. In section 3 we formulate a nested Bethe ansatz for the vertical vacuum orientation, and derive the corresponding Bethe equations. We conclude in section 4 with a discussion about our results and possible future directions.

2 Reflection from the $D5$ -brane

We begin by briefly recalling the setup and symmetries of the $D5$ -brane as well as the representations of the matter content living on the brane. The details can be found in [7]. We then present the corresponding reflection matrices, and show that they obey the boundary Yang-Baxter equation.

2.1 Symmetries

The symmetry algebra in the bulk of the scattering theory is $\mathfrak{psu}(2|2) \times \widetilde{\mathfrak{psu}(2|2)} \ltimes \mathbb{R}^3$, consisting of two copies (left and right) of the centrally-extended algebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ with their central charges identified. The generators of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ are the central charges \mathbb{H} , \mathbb{C} and \mathbb{C}^\dagger , two sets of bosonic rotation generators \mathbb{R}_a^b , \mathbb{L}_α^β and two sets of fermionic supersymmetry generators \mathbb{Q}_α^a , \mathbb{G}_a^α . The non-trivial commutation relations are [10]

$$\begin{aligned}
[\mathbb{L}_\alpha^\beta, \mathbb{J}^\gamma] &= \delta_\alpha^\gamma \mathbb{J}^\beta - \frac{1}{2} \delta_\alpha^\beta \mathbb{J}^\gamma, & [\mathbb{L}_\alpha^\beta, \mathbb{J}_\gamma] &= -\delta_\gamma^\beta \mathbb{J}_\alpha + \frac{1}{2} \delta_\alpha^\beta \mathbb{J}_\gamma, \\
[\mathbb{R}_a^b, \mathbb{J}^c] &= \delta_a^c \mathbb{J}^b - \frac{1}{2} \delta_a^b \mathbb{J}^c, & [\mathbb{R}_a^b, \mathbb{J}_c] &= -\delta_c^b \mathbb{J}_a + \frac{1}{2} \delta_a^b \mathbb{J}_c, \\
\{\mathbb{Q}_\alpha^a, \mathbb{Q}_b^\beta\} &= \epsilon_{ab} \epsilon^{\alpha\beta} \mathbb{C}, & \{\mathbb{G}_\alpha^a, \mathbb{G}_\beta^b\} &= \epsilon_{\alpha\beta} \epsilon^{ab} \mathbb{C}^\dagger, \\
\{\mathbb{Q}_\alpha^a, \mathbb{G}_\beta^b\} &= \delta_a^b \mathbb{L}_\beta^\alpha + \delta_\beta^\alpha \mathbb{R}_a^b + \frac{1}{2} \delta_a^b \delta_\beta^\alpha \mathbb{H}, & & (1)
\end{aligned}$$

where $a, b, \dots = 1, 2$ and $\alpha, \beta, \dots = 3, 4$. We use undotted (a, α) and dotted ($\dot{a}, \dot{\alpha}$) indices to distinguish generators of left and right $\mathfrak{psu}(2|2)$.

The kind of $D5$ -brane we consider wraps an $AdS_4 \subset AdS_5$ and a maximal $S^2 \subset S^5$. The AdS_4 part of the brane defines a $2 + 1$ dimensional defect hypersurface of the $3 + 1$ dimensional conformal boundary. The fundamental matter living on the defect hypersurface is a 3d hypermultiplet [11]. The original $\mathfrak{so}(6)$ R-symmetry of $\mathcal{N} = 4$ SYM is broken by the presence of the $D5$ -brane down to $\mathfrak{so}(3)_H \times \mathfrak{so}(3)_V$. We shall fix the bulk vacuum state to be $Z = X^5 + iX^6$ and consider two inequivalent embeddings of the $D5$ -brane into $AdS_5 \times S^5$ in which the maximal $S^2 \subset S^5$ is specified by:

- $X^4 = X^5 = X^6 = 0$, for which the vacuum is “vertical”;

- $X^1 = X^2 = X^3 = 0$, for which the vacuum is “horizontal”.

At the boundary of the scattering theory, only those bulk symmetries that are also symmetries of the $D5$ -brane are preserved. The preserved symmetry algebra is a “diagonal” copy

$$\mathfrak{psu}(2|2)_D \times \mathbb{R}^3 \subset \mathfrak{psu}(2|2) \times \widetilde{\mathfrak{psu}}(2|2) \times \mathbb{R}^3,$$

whose generators, obeying the canonical commutation relations (1), we shall write as $\check{\mathbb{L}}_{\check{\beta}}^{\check{\alpha}}$, $\check{\mathbb{R}}_{\check{b}}^{\check{a}}$, $\check{\mathbb{Q}}_{\check{a}}^{\check{\alpha}}$, $\check{\mathbb{G}}_{\check{\alpha}}^{\check{a}}$ and $\check{\mathbb{H}}$, $\check{\mathbb{C}}$ and $\check{\mathbb{C}}^\dagger$. These generators are given by $\check{\mathbb{H}} = \mathbb{H} + \check{\mathbb{H}}$, $\check{\mathbb{C}} = \mathbb{C} + \kappa^2 \check{\mathbb{C}}$, $\check{\mathbb{C}}^\dagger = \mathbb{C}^\dagger + \kappa^{-2} \check{\mathbb{C}}^\dagger$ and

$$\check{\mathbb{L}}_{\check{\beta}}^{\check{\alpha}} = \mathbb{L}_{\beta}^{\alpha} + \check{\mathbb{L}}_{\check{\beta}}^{\check{\alpha}}, \quad \check{\mathbb{R}}_{\check{b}}^{\check{a}} = \mathbb{R}_b^a + \check{\mathbb{R}}_{\check{b}}^{\check{a}}; \quad (2)$$

$$\check{\mathbb{Q}}_{\check{a}}^{\check{\alpha}} = \mathbb{Q}_a^{\alpha} + \kappa \check{\mathbb{Q}}_{\check{a}}^{\check{\alpha}}, \quad \check{\mathbb{G}}_{\check{\alpha}}^{\check{a}} = \mathbb{G}_{\alpha}^a + \kappa^{-1} \check{\mathbb{G}}_{\check{\alpha}}^{\check{a}}, \quad (3)$$

where the bar above the dotted indices acts as $\bar{3} = \dot{4}$ and $\bar{4} = \dot{3}$. Here the number κ depends on the orientation of the $D5$ -brane [7]:

$$\kappa = \begin{cases} -i & \text{vertical case} \\ -1 & \text{horizontal case.} \end{cases} \quad (4)$$

The preserved R -symmetries $\check{\mathbb{R}}_{\check{b}}^{\check{a}}$ are the generators of $\mathfrak{so}(3)_H$ in the vertical case, and of $\mathfrak{so}(3)_V$ in the horizontal case.

2.2 Bulk representation

We need to determine how the elementary bulk magnons transform with respect to the preserved boundary symmetry algebra $\mathfrak{psu}(2|2)_D \times \mathbb{R}^3$ generated by (2)-(3). Recall that with respect to the bulk symmetry algebra $\mathfrak{psu}(2|2) \times \widetilde{\mathfrak{psu}}(2|2) \times \mathbb{R}^3$, the bulk magnon transforms [10] in the bifundamental representation $(\square_{(a,b,c,d)}, \widetilde{\square}_{(a,b,c,d)})$. The representation labels (a, b, c, d) are the same for both left and right factors, and are conveniently parametrized by²

$$a = \sqrt{\frac{g}{2}} \eta, \quad b = \sqrt{\frac{g}{2}} \frac{i\zeta}{\eta} \left(\frac{x^+}{x^-} - 1 \right), \quad c = -\sqrt{\frac{g}{2}} \frac{\eta}{\zeta x^+}, \quad d = -\sqrt{\frac{g}{2}} \frac{x^+}{i\eta} \left(\frac{x^-}{x^+} - 1 \right), \quad (5)$$

where $\zeta = e^{2i\xi}$ is the magnon phase and unitarity requires $\eta = e^{i\xi} e^{i\frac{\xi}{2}} \sqrt{i(x^- - x^+)}$. The spectral parameters x^\pm are constrained to satisfy the mass-shell condition

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}. \quad (6)$$

²Note that here we use parametrization different from the one used in [7].

The magnon momentum is p where $e^{ip} = x^+/x^-$. We shall sometimes use the alternative notation $\mathcal{V}(p, \zeta)$ for the fundamental representation $\square_{(a,b,c,d)}$.

It follows that, under the action of the diagonal subalgebra $\mathfrak{psu}(2|2)_D \times \mathbb{R}^3$ generated by (2)-(3), the bulk magnon transforms in some tensor product representation $\square_{(a',b',c',d')} \otimes \square_{(\tilde{a},\tilde{b},\tilde{c},\tilde{d})}$, where (a',b',c',d') and $(\tilde{a},\tilde{b},\tilde{c},\tilde{d})$ are representation labels that we must determine. For the left factor we have simply $(a',b',c',d') = (a,b,c,d)$. For the right factor, the choices of defect orientation and gamma matrices made in [7] lead to the relations $\bar{3} = \dot{4}$ and $\bar{4} = \dot{3}$. Because of this and the κ factors appearing in (3), one must change basis in order for the action of the generators (2)-(3) to be the canonical one:

$$(\tilde{\phi}^{\dot{1}}, \tilde{\phi}^{\dot{2}} | \tilde{\psi}^{\dot{3}}, \tilde{\psi}^{\dot{4}}) := (\tilde{\phi}^{\dot{1}}, \tilde{\phi}^{\dot{2}} | \kappa \tilde{\psi}^{\dot{4}}, \kappa \tilde{\psi}^{\dot{3}}). \quad (7)$$

One then finds that $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (a, -\kappa^2 b, -\kappa^{-2} c, d)$. Therefore, with respect to the boundary symmetry algebra $\mathfrak{psu}(2|2)_D \times \mathbb{R}^3$ generated by (2)-(3), the bulk magnon transforms in the representation

$$\square_{(a,b,c,d)} \otimes \square_{(a,-\kappa^2 b,-\kappa^{-2} c,d)}. \quad (8)$$

Representations of $\mathfrak{psu}(2|2)_D \times \mathbb{R}^3$ can also be labelled by the values of the central charges [12]: for the fundamental representation we have $\square_{(a,b,c,d)} \cong \langle 0, 0; H, C, C^\dagger \rangle$ where $H = ad + bc$, $C = ab$, $C^\dagger = cd$. In these terms, the bulk magnon lives in the representation

$$\langle 0, 0; H, C, C^\dagger \rangle \otimes \langle 0, 0; H, -\kappa^2 C, -\kappa^{-2} C^\dagger \rangle \cong \{0, 0, 2H, (1 - \kappa^2)C, (1 - \kappa^{-2})C^\dagger\}. \quad (9)$$

We now consider separately the horizontal and vertical vacua.

Horizontal vacuum. This case corresponds to $\kappa = -1$ in (3). With respect to $\mathfrak{psu}(2|2)_D \times \mathbb{R}^3$, the bulk magnon transforms in the tensor representation

$$\square_{(a,b,c,d)} \otimes \square_{(a,-b,-c,d)} = \mathcal{V}(p, \zeta) \otimes \mathcal{V}(-p, \zeta e^{ip}). \quad (10)$$

One way of interpreting this is as two consecutive magnons with momenta p and $-p$, as depicted in figure 1. The central charges \check{C} and \mathbb{C}^\dagger vanish and

$$\langle 0, 0; H, C, C^\dagger \rangle \otimes \langle 0, 0; H, -C, -C^\dagger \rangle = \{0, 0, 2H, 0, 0\}. \quad (11)$$

Vertical vacuum. This case corresponds to $\kappa = -i$. The bulk magnon transforms in the tensor representation

$$\square_{(a,b,c,d)} \otimes \square_{(a,b,c,d)} = \mathcal{V}(p, \zeta) \otimes \mathcal{V}(p, \zeta), \quad (12)$$

under $\mathfrak{psu}(2|2)_D \times \mathbb{R}^3$. The values of the central charges \check{C} , \check{C}^\dagger and \check{H} are

$$\check{H} = 2H, \quad \check{C} = 2C, \quad \check{C}^\dagger = 2C, \quad (13)$$

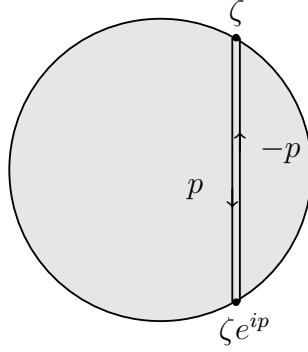


Figure 1: Bulk magnon diagonal representation looks like two consecutive magnons with momentum p and phase ζ and momentum $-p$ and phase ζe^{ip} .

These values satisfy the multiplet splitting condition $\check{H}^2 - \check{C}\check{C}^\dagger = 1$, according to which

$$\{0, 0; 2H, 2C, 2C^\dagger\} = \langle 1, 0; 2H, 2C, 2C^\dagger \rangle \oplus \langle 0, 1; 2H, 2C, 2C^\dagger \rangle = \boxtimes \oplus \boxtimes. \quad (14)$$

Thus, bulk magnons transform in the direct sum of symmetric and antisymmetric short representations of $\mathfrak{psu}(2|2)_D \ltimes \mathbb{R}^3$. In the conventions of [13, 14] these representations are equivalent to two-magnon ($M = 2$) bound-state and mirror bound-state representations with the labels having doubled coupling constant, i.e. g replaced by $2g$. This doubling of the coupling constant dependence of the labels of the diagonal representations explains why the all-loop dispersion relation is still given by $H(p) = \sqrt{1 + 8g^2 \sin^2(\frac{p}{2})}$ in spite of having $M = 2$ reps³. We shall observe a similar duplication for the boundary representation labels in this case.

2.3 Reflection matrix: horizontal case

We consider first the reflection of a bulk magnon from the boundary in the horizontal case. For definiteness, we consider a right boundary. In the horizontal case the boundary is a singlet. Then reflection from a right boundary sends $p \mapsto -p$ and $\zeta \mapsto \zeta$ [4]. Thus the reflection matrix is a map

$$\mathcal{K}^h : \mathcal{V}(p, \zeta) \otimes \mathcal{V}(-p, \zeta e^{ip}) \otimes 1 \rightarrow \mathcal{V}(-p, \zeta) \otimes \mathcal{V}(p, \zeta e^{-ip}) \otimes 1. \quad (15)$$

As noted above, the tensor representation (10) corresponds to two consecutive magnons with momenta p and $-p$. Therefore the K -matrix $\mathcal{K}^h(p, -p)$ intertwines the same representations

³For the $M = 2$ reps we have $\check{H} = 2H = \sqrt{2^2 + 8(2g)^2 \sin^2(\frac{p}{2})}$.

as the bulk S -matrix $\mathcal{S}(p, -p)$, and the two must be equal up to a phase (since this intertwiner is fixed by symmetry, up to a phase). Details of \mathcal{S} and \mathcal{K}^h are given in appendices A and B.

In order to check that the boundary is integrable one has to consider the boundary Yang-Baxter equation (bYBE), which computes the difference between the two possible ways of factorizing the scattering of two incoming magnons off a boundary.

It is convenient to do all the calculations in terms of representations of the preserved boundary symmetry algebra $\mathfrak{psu}(2|2)_D \times \mathbb{R}^3$ generated by (2)-(3). The bYBE represents two incoming bulk magnons reflecting from the boundary:

$$\begin{aligned} \text{bYBE} : \mathcal{V}_L(p_1, \zeta) \otimes \mathcal{V}_R(-p_1, \zeta e^{ip_1}) \otimes \mathcal{V}_L(p_2, \zeta e^{ip_1}) \otimes \mathcal{V}_R(-p_2, \zeta e^{i(p_1+p_2)}) \rightarrow \\ \mathcal{V}_L(-p_1, \zeta) \otimes \mathcal{V}_R(p_1, \zeta e^{-ip_1}) \otimes \mathcal{V}_L(-p_2, \zeta e^{-ip_1}) \otimes \mathcal{V}_R(p_2, \zeta e^{-i(p_1+p_2)}). \end{aligned} \quad (16)$$

Here \mathcal{V}_L (\mathcal{V}_R) are representations of the boundary algebra originating as left (respectively, right) factors of bulk magnons. We must not lose track of this information, because it affects how the representations scatter, as follows.

For the bulk scattering, left (right) states scatter with left (respectively, right) states only. When scattering two left representations we use the standard S -matrix, but when scattering two right representations we must allow for the change of basis, (7), which produces additional signs in the ζ -dependent components:

$$\begin{aligned} \langle \psi^3 \psi^4 | \mathcal{S} | \phi^1 \phi^2 \rangle &= -\langle \psi^3 \psi^4 | \mathcal{S} | \phi^1 \phi^2 \rangle = -a_7 \\ \langle \phi^1 \phi^2 | \mathcal{S} | \psi^3 \psi^4 \rangle &= -\langle \phi^1 \phi^2 | \mathcal{S} | \psi^3 \psi^4 \rangle = -a_8. \end{aligned} \quad (17)$$

Given that a_7 and a_8 depend linearly on the phase, this sign change is just $\zeta \mapsto -\zeta$. Next, to exchange a left state with a right state in the tensor product one must use a graded permutation, which also produces certain minus signs.⁴

The pictorial version of the bYBE is presented in figure 2 and the equation itself is

$$\begin{aligned} &\mathcal{K}_{34}(p_2, \zeta e^{-ip_1}; -p_2, \zeta e^{-i(p_1-p_2)}) P_{23} \mathcal{S}_{34}(-p_2, -\zeta e^{ip_2}; p_1, -\zeta e^{i(p_2-p_1)}) \\ &\quad \times \mathcal{S}_{12}(p_2, \zeta; -p_1, \zeta e^{ip_2}) P_{23} \mathcal{K}_{34}(p_1, \zeta e^{ip_2}; -p_1, \zeta e^{i(p_1+p_2)}) \\ &\quad \times P_{23} \mathcal{S}_{12}(p_1, \zeta; p_2, \zeta e^{ip_1}) \mathcal{S}_{34}(-p_1, -\zeta e^{ip_1}; -p_2, -\zeta e^{i(p_1+p_2)}) P_{23} \\ &- P_{23} \mathcal{S}_{34}(p_2, -\zeta e^{-ip_2}; p_1, -\zeta e^{-i(p_2+p_1)}) \mathcal{S}_{12}(-p_2, \zeta; -p_1, \zeta e^{-ip_2}) P_{23} \\ &\quad \times \mathcal{K}_{34}(p_1, \zeta e^{-p_2}; -p_1, \zeta e^{-i(p_2-p_1)}) P_{23} \mathcal{S}_{12}(p_1, \zeta; -p_2, \zeta e^{ip_1}) \\ &\quad \times \mathcal{S}_{34}(-p_1, -\zeta e^{ip_1}; p_2, -\zeta e^{i(p_1-p_2)}) P_{23} \mathcal{K}_{34}(p_2, \zeta e^{ip_1}; -p_2, \zeta e^{i(p_1+p_2)}) = 0, \end{aligned} \quad (18)$$

where the subscripts 12, 23, 34 indicate the tensor factors on which the operators act, P_{ij} is the graded permutation operator permuting left-right states, \mathcal{S}_{ij}^L and \mathcal{S}_{ij}^R are the left and

⁴This graded permutation was overlooked in the calculations of [7] thus obscuring the integrability of the $D5$ -brane boundary conditions.

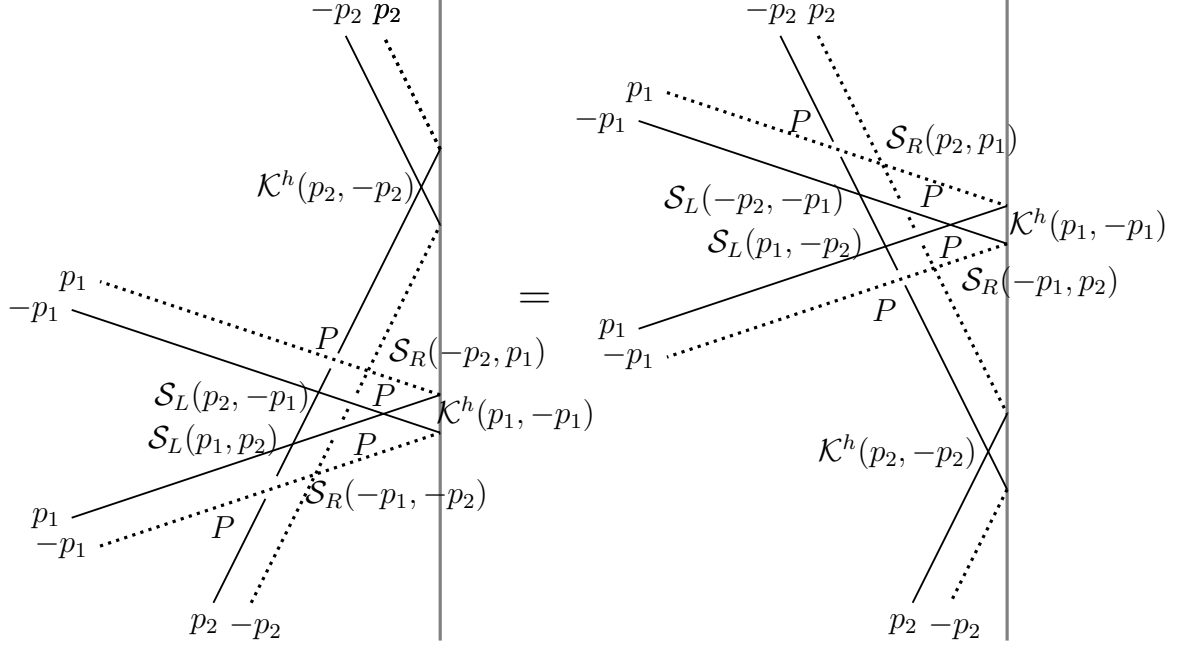


Figure 2: bYBE for the reflection in the horizontal case. Solid lines correspond to the left reps while the dotted lines correspond to right reps.

right bulk S -matrices and \mathcal{K}_{34} is the reflection matrix. We have checked directly that this boundary YBE is satisfied.

Another way to verify that the boundary YBE is satisfied is to note that it may be mapped to a standard *bulk* YBE, as follows. One can verify that whenever a phase-dependent component appears, the extra sign in the right S -matrix is canceled with a minus sign from a graded permutation. Then, using also the relation between \mathcal{K} and the bulk S -matrix, the above equation is equivalent to

$$\begin{aligned}
& \mathcal{S}_{23}(p_2, \zeta e^{-ip_1}; -p_2, \zeta e^{ip_2-ip_1}) \mathcal{S}_{34}(p_1, \zeta e^{ip_2-ip_1}; -p_2, \zeta e^{ip_2}) \mathcal{S}_{12}(p_2, \zeta; -p_1, \zeta e^{ip_2}) \\
& \quad \times \mathcal{S}_{23}(p_1, \zeta e^{ip_2}; -p_1, \zeta e^{ip_1+ip_2}) \mathcal{S}_{34}(-p_2, \zeta e^{ip_1+ip_2}; -p_1, \zeta e^{ip_1}) \mathcal{S}_{12}(p_1, \zeta; p_2, \zeta e^{ip_1}) \\
& - \mathcal{S}_{34}(p_1, \zeta^{-ip_1-ip_2}; p_2, \zeta e^{-ip_2}) \mathcal{S}_{12}(-p_2, \zeta; -p_1, \zeta e^{-ip_2}) \mathcal{S}_{23}(p_1, \zeta e^{-ip_2}; -p_1, \zeta e^{ip_1-ip_2}) \\
& \quad \times \mathcal{S}_{34}(p_2, \zeta e^{ip_1-ip_2}; -p_1, \zeta e^{ip_1}) \mathcal{S}_{12}(p_1, \zeta; -p_2, \zeta e^{ip_1}) \mathcal{S}_{23}(p_2, \zeta e^{ip_1}; -p_2, \zeta e^{ip_1+ip_2}) = 0.
\end{aligned} \tag{19}$$

In this way we have “unfolded” the bYBE into a succession of bulk scattering processes. Consequently, the boundary YBE follows from a particular case of the bulk YBE. The meaning of (19) is represented in figure 3.

From this second, “unfolded”, point of view, the boundary is seen to be “achiral”, meaning

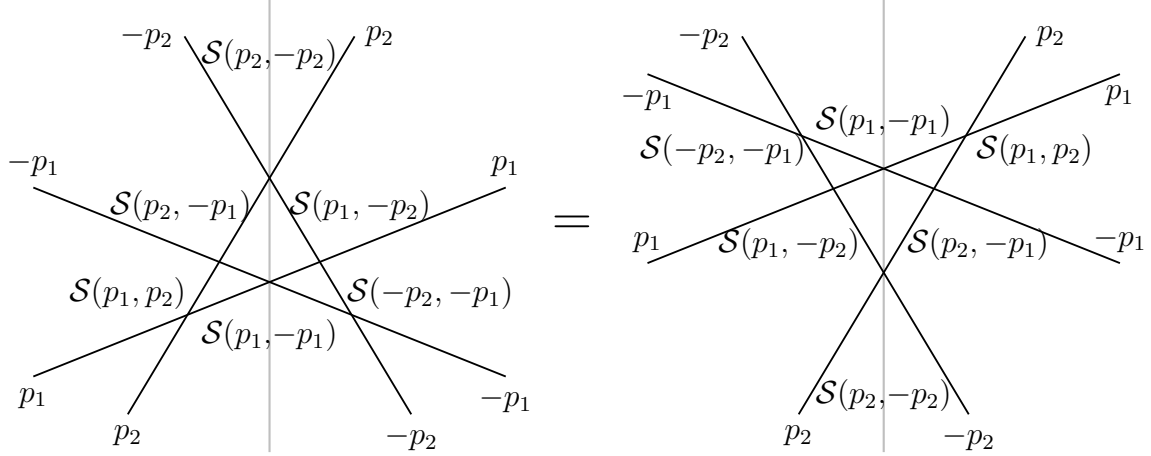


Figure 3: The unfolded bYBE as a 4-particle scattering in the bulk. The vertical line plays no role in the unfolded picture, but is drawn as a reminder that $S(p, -p)$ represents boundary reflections.

that an incoming left state becomes a right one after the reflection and a right one becomes a left.

2.4 Reflection matrix: vertical case

In the vertical case the boundary carries a degree of freedom transforming in a fundamental representation $\square_{(a_B, b_B, c_B, d_B)}$ of $\mathfrak{psu}(2|2)_D \times \mathbb{R}^3$. The representation labels specifying this representation are⁵

$$a_B = \sqrt{g}\eta_B, \quad b_B = -\sqrt{g}\frac{i\zeta}{\eta_B}, \quad c_B = -\sqrt{g}\frac{\eta_B}{\zeta x_B}, \quad d_B = \sqrt{g}\frac{x_B}{i\eta_B}. \quad (20)$$

This representation is related to a radial line segment in the LLM disc picture [4, 15, 16]. The unitarity and mass-shell conditions give

$$|\eta_B|^2 = -ix_B, \quad x_B \equiv \frac{i(1 + \sqrt{1 + 4g^2})}{2g}. \quad (21)$$

Thus, the exact energy of the boundary excitation is

$$\check{H} = \mathcal{D} - J_{56} = \frac{1}{2}\sqrt{1 + 4g^2}. \quad (22)$$

For the boundary degree of freedom, the representation labels (20) and the mass-shell condition (21) are those of the boundary fundamental degree of freedom in the D3-brane case

⁵Note that here we use a different parametrization from the one used in [7].

[4], but with a coupling constant g twice bigger. This doubling of the coupling constant is crucial for integrability to hold and for the exact boundary energy (22) to consistently reproduce 1-loop anomalous dimensions.

As we saw, the elementary bulk magnons transform, under the boundary symmetry algebra, in direct sum of two $M = 2$ bound state representations (symmetric and antisymmetric). Therefore we have the following two scattering processes:

$$\mathcal{K} : \begin{array}{|c|} \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array}, \quad (23)$$

$$\bar{\mathcal{K}} : \begin{array}{|c|} \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array}. \quad (24)$$

As in [7, 17], following [13], the reflection matrices in the symmetric and antisymmetric channels (23) are

$$\mathcal{K}^{Ba} = \sum_{i=1}^{19} k_i^{(S)} \Lambda_i, \quad \bar{\mathcal{K}}^{Ba} = \sum_{i=1}^{19} k_i^{(A)} \bar{\Lambda}_i, \quad (25)$$

where Λ_i are certain differential operators (see appendix C for details), $\bar{\Lambda}_i$ are obtained from Λ_i by exchanging indices $\check{1} \leftrightarrow \check{3}$ and $\check{2} \leftrightarrow \check{4}$, and where $k_i^{(S,A)}$ are the reflection coefficients. In both cases, the symmetry algebra alone fixes all reflection coefficients up to an overall phase. Interestingly, the two channels are related by $k_i^{(A)}(p, x_B) = k_i^{(S)}(-p, x_B)$ [7]. Note that the reflection coefficients do not explicitly depend on g , thus they coincide with the ones found in [17].

It is easy to check that symmetric and antisymmetric reflection matrices \mathcal{K}^{Ba} and $\bar{\mathcal{K}}^{Ba}$ do satisfy bYBE on their own. The bYBE invariance of \mathcal{K}^{Ba} was checked in [17], while for checking the bYBE invariance of $\bar{\mathcal{K}}^{Ba}$ we had to construct an antisymmetric bound state S -matrix $\bar{\mathcal{S}}^{BB}$ which is the mirror-model partner of the ordinary bound state S -matrix \mathcal{S}^{BB} .

For the vertical vacuum case the complete reflection matrix must be some linear combination:

$$\mathcal{K}^v = k_0 \mathcal{K}^{Ba} + \bar{\mathcal{K}}^{Ba}, \quad (26)$$

with k_0 being a function of bulk and boundary representation parameters. The important question is whether there exists any choice of this function, such that the system is integrable, i.e. such that the complete reflection matrix obeys the boundary Yang-Baxter equation. For this purpose one needs to consider the complete bulk 16×16 -dim. S -matrix \mathcal{S}^{AAAA} which may be constructed as a tensor product of two fundamental S -matrices \mathcal{S}^{AA} and $\mathcal{S}^{\bar{A}\bar{A}}$. It is convenient to compute \mathcal{S}^{AAAA} in the basis of (graded) symmetric and antisymmetric states i.e. on the superspace and the mirror-superspace. The complete S -matrix is not block-diagonal in this basis; rather it mixes symmetric and antisymmetric states during the scattering. But it is important to note that it is invariant under the symmetries preserved by the boundary (this is natural as the boundary algebra is a subalgebra of the bulk algebra).

The bYBE for the reflection in this vertical case reads as

$$\begin{aligned} \text{bYBE} : \mathcal{V}_L(p_1, \zeta) \otimes \mathcal{V}_R(p_1, \zeta) \otimes \mathcal{V}_L(p_2, \zeta e^{ip_1}) \otimes \mathcal{V}_R(p_2, \zeta e^{ip_1}) \otimes \mathcal{V}_B(x_B, \zeta e^{i(p_1+p_2)}) \rightarrow \\ \mathcal{V}_L(-p_1, \zeta) \otimes \mathcal{V}_R(-p_1, \zeta) \otimes \mathcal{V}_L(-p_2, \zeta e^{-ip_1}) \otimes \mathcal{V}_R(-p_2, \zeta e^{-ip_1}) \otimes \mathcal{V}_B(x_B, \zeta e^{-i(p_1+p_2)}), \end{aligned} \quad (27)$$

where once again the scattering in the bulk is between left-left and right-right states only, while the permutation of left-right and right-left states produces a graded minus sign. In the contrast to the horizontal case, the right S -matrix is equivalent to the left S -matrix, i.e. it does not acquire an extra minus sign in the ζ -dependent components, since now $-\kappa^2 = +1$. Also, all phases in (27) are increasing from left to right. The graphical interpretation of bYBE is almost the same as for the horizontal case. The difference is that the boundary in this case does not act diagonally but mixes bulk and boundary flavours.

A general matrix element of the bYBE (27) has a complicated structure. We found the particular matrix element

$$\left\langle \phi_1^{\{\bar{3}\bar{4}\}} \otimes \phi_2^{\{\bar{1}\bar{1}\}} \otimes \phi_B^{\bar{1}} \middle| \text{bYBE} \middle| \psi_1^{\{\bar{1}\bar{3}\}} \otimes \phi_2^{\{\bar{1}\bar{1}\}} \otimes \psi_B^{\bar{4}} \right\rangle \quad (28)$$

to be quite tractable and by treating minus signs coming from permuting left and right reps carefully (i.e. $\mathcal{S}^{A_1 \bar{A}_2 A_3 \bar{A}_4} = (-1)^{[\bar{A}_2][A_3]} \mathcal{S}^{A_1 A_3} \otimes \mathcal{S}^{\bar{A}_2 \bar{A}_4}$) we find the required ratio has to be

$$k_0 = -\frac{x^-(x_B - x^-)^2 \eta^2 \eta_B}{x^+(x_B + x^+)^2 \tilde{\eta}^2 \tilde{\eta}_B}. \quad (29)$$

for (28) to vanish. We have then checked that, using this ratio, the reflection matrix K^v (26) satisfies all matrix elements of bYBE (27). Thus we conclude that the reflection in the vertical case is indeed integrable. We also claim that it is an achiral boundary in the same sense as in the horizontal case: at this stage the ‘‘unfolded’’ picture of the reflection is not obvious, but it will become clear when we consider the nested Bethe ansatz.

To end this section we would like to produce a weak coupling consistency check for the ratio k_0 between the reflection of symmetric and antisymmetric components. We shall focus on the right reflection of components $\phi^{\{\bar{1}\bar{1}\}}$ of \boxtimes and $\phi^{[\bar{1}\bar{2}]}$ of \boxminus , both of which are reflected diagonally when the right boundary has the defect field $\phi^{\bar{1}}$. Using the exact expressions for (29) and (69), we expand in powers of g^2 and obtain⁶

$$\frac{K_R^{\phi^{\{\bar{1}\bar{1}\}}}}{K_R^{\phi^{[\bar{1}\bar{2}]}}} = \frac{k_0 k_1^{(S)}}{k_9^{(A)}} = \frac{3 - e^{ip}}{1 - 3e^{ip}} + \mathcal{O}(g^2). \quad (30)$$

The leading order is in exact agreement with the ratio

$$\frac{K_L^Y(-p)}{K_L^{X_V^{\bar{1}}}(-p)} = \frac{3 - e^{ip}}{1 - 3e^{ip}}, \quad (31)$$

⁶We consider η 's in the spin chain basis, in which $\frac{\eta^2 \eta_B}{\tilde{\eta}^2 \tilde{\eta}_B} = 1$.

of the reflection factors obtained in the appendix D from the 1-loop mixing matrix of anomalous dimensions. Therefore, the unique choice of k_0 consistent with the bYBE is also consistent with the available weak coupling results.

3 Coordinate Bethe Ansatz

We now proceed to derive the asymptotic Bethe equations for the vertical case, by means of the nested coordinate Bethe ansatz [18]. Our treatment is similar to that in [19].

3.1 Bethe ansatz

Let us start by considering the scattering problem on the half-line with a right boundary. As in the previous section, we work in terms of representations of the symmetry algebra $\mathfrak{psu}(2|2)_D \ltimes \mathbb{R}^3$ preserved by the boundary. An asymptotic state with N^I elementary bulk magnons of momenta (p_1, \dots, p_{N^I}) and phases $(\zeta_1, \dots, \zeta_{N^I})$ then transforms in the representation (c.f. 12)

$$\left(\mathcal{V}(p_1, \zeta_1) \otimes \mathcal{V}(p_1, \zeta_1)\right) \otimes \cdots \otimes \left(\mathcal{V}(p_{N^I}, \zeta_{N^I}) \otimes \mathcal{V}(p_{N^I}, \zeta_{N^I})\right) \otimes \mathcal{V}_B(\zeta_B) \quad (32)$$

with $2N^I + 1$ tensor factors. We would like to go to an “unfolded” picture of the boundary, as we did for the horizontal case in §2.3, so let us choose to write these tensor factors in a different order, namely

$$\mathcal{V}(p_1, \zeta_1) \otimes \cdots \otimes \mathcal{V}(p_{N^I}, \zeta_{N^I}) \otimes \mathcal{V}_B(\zeta_B) \otimes \mathcal{V}(p_{N^I}, \zeta_{N^I}) \otimes \cdots \otimes \mathcal{V}(p_1, \zeta_1). \quad (33)$$

Doing so introduces minus signs when permuting fermions, which we need to keep careful account of below. We shall write basis vectors as

$$|\chi_1^a \cdots \chi_{N^I}^y \chi_B^z \chi_{N^I}^y \cdots \chi_1^a\rangle, \quad (34)$$

where all indices run over $1 \dots 4$. We no longer decorate indices that originated as right multiplets with dots, nor the boundary ones with checks. After all, these indices all transform canonically under the preserved symmetry algebra. (So, a should strictly be \bar{a} in the notation of section 2.)

We write $\mathcal{S}_{i,i+1}^I$ for the fundamental left or right S -matrix. As noted above, these are identical since $-\kappa^2 = +1$. We write \mathcal{K}^I for the reflection matrix found in (26).

Level II

We start from defining the *level II* vacuum to be the state consisting only of ψ^3

$$|0\rangle^{\text{II}} = |\psi_1^3 \cdots \psi_{N^I}^3 \psi_B^3 \psi_{N^I}^3 \cdots \psi_1^3\rangle, \quad (35)$$

On this state $\mathcal{S}_{i,i+1}^I$ and \mathcal{K}^I act diagonally: $\mathcal{S}_{i,i+1}^I |0\rangle^{\text{II}} = |0\rangle^{\text{II}} \mathcal{S}_{i,i+1}^I$, $\mathcal{K}^I |0\rangle^{\text{II}} = |0\rangle^{\text{II}} K^I$. We normalize the scattering and reflection matrices in such way that

$$S_{i,i+1}^I = -1, \quad \text{and} \quad K^I = +1. \quad (36)$$

Reflection of level II excitations. Next we define *level II* excitations, defined to transform under $\mathcal{S}_{i,i+1}^I$ and \mathcal{K}^I in exactly the same fashion as $|0\rangle^{\text{II}}$ (the *compatibility condition*).

We consider first a single excitation. As usual we make a spin-wave ansatz in which the particle has a “tail” running away behind it. This ansatz is the sum of an “ingoing” and an “outgoing” piece, plus a term in which the excitation has just reached the boundary. So we have, using a pictorial notation,

$$\begin{aligned} |\Psi^a(y)\rangle^{\text{II}} &= \dots + \begin{array}{c} \nearrow \\ | \\ \vdots \\ | \\ \vdots \\ | \\ \vdots \\ | \\ \vdots \end{array} + \begin{array}{c} \nearrow \\ | \\ \vdots \\ | \\ \vdots \\ | \\ \vdots \\ | \\ \vdots \end{array} + \begin{array}{c} \nearrow \\ | \\ \vdots \\ | \\ \vdots \\ | \\ \vdots \\ | \\ \vdots \end{array} + \dots \\ &= \sum_{k=1}^{N^I} |\psi_1^3 \dots \phi_k^a \dots \psi_{N^I}^3 \psi_B^3 \psi_{N^I}^3 \dots \psi_1^3\rangle \prod_{l=1}^{k-1} S^{\text{II},I}(y; x_l) f^{\text{in}}(y; x_k, \eta_k), \\ &\quad + |\psi_1^3 \dots \psi_{N^I}^3 \phi_B^a \psi_{N^I}^3 \dots \psi_1^3\rangle \prod_{l=1}^{N^I} S^{\text{II},I}(y; x_l) f^\tau(y; x_B, \eta_B) \\ &\quad + \sum_{k=1}^{N^I} |\psi_1^3 \dots \psi_{N^I}^3 \psi_B^3 \psi_{N^I}^3 \dots \phi_k^a \dots \psi_1^3\rangle \prod_{l=1}^{N^I} S^{\text{II},I}(y; x_l) K^{\text{II}}(y; x_B) \\ &\quad \times \prod_{l=k+1}^{N^I} S^{\text{I},\text{II}}(x_l; -y) f^{\text{out}}(x_k, \eta_k; -y). \quad (37) \end{aligned}$$

Here (see [10] for the details)

$$\begin{aligned} S^{\text{II},I}(y, x_i) &= -\frac{y - x_i^+}{y - x_i^-}, \quad S^{\text{I},\text{II}}(x_i; y) 1/S^{\text{II},I}(y; x_i) = -\frac{y - x_i^-}{y - x_i^+}, \\ f^{\text{in}}(y; x_i, \eta_i) &= \frac{x_i^+ - x_i^-}{y - x_i^-} \frac{1}{\eta_i}, \quad f^{\text{out}}(x_i, \eta_i; y) = S^{\text{I},\text{II}}(x_i; y) f^{\text{in}}(y; x_i, \eta_i) = \frac{x_i^- - x_i^+}{y - x_i^+} \frac{1}{\eta_i}. \quad (38) \end{aligned}$$



The new unknown functions K^{II} and f^τ are fixed by the compatibility condition for the scattering through the boundary




$$\begin{aligned} \mathcal{K}^I |\Psi^a(y)\rangle^{\text{II}} &= |\Psi^a(y)\rangle_\tau^{\text{II}} K^I = |\Psi^a(y)\rangle_\tau^{\text{II}} \\ \mathcal{K}^I |\Psi^a(y)\rangle^{\text{II}} &= |\Psi^a(y)\rangle_\tau^{\text{II}} K^I = |\Psi^a(y)\rangle_\tau^{\text{II}}, \quad (39) \end{aligned}$$

where τ merely acts by sending $x_{N^I}^\pm \rightarrow -x_{N^I}^\mp$.

We find⁷ the unique solution

$$K^{\text{II}}(y; x_B) = \frac{y - x_B}{y + x_B}, \quad f^\tau(y; x_B, \eta_B) = -\frac{\sqrt{2}x_B}{y + x_B} \frac{1}{\eta_B}. \quad (42)$$

Note that we did not include terms  and  in the ansatz. One could include such terms with some reflection coefficient $\tilde{K}^{\text{II}}(y; x_B)$, but solving the compatibility relation (39) one finds that $\tilde{K}^{\text{II}}(y; x_B) = 0$. In this sense, the scattering from the boundary is indeed achiral: it is a sum of a left excitation with momentum y and a right excitation with momentum $-y$, plus the boundary term.

For clarity in the pictures below, it is useful also to work with the spin-wave ansatz with its “tail” trailing to the right rather than the left – that is, pictorially, $\dots +$  $+$  $+$  $+$ \dots . However, such states are not linearly independent of those of the form (37). Thus, in contrast to the usual open boundaries case, there is only one type of level II excitation.

Scattering of level II particles. The scattering of two *level II* excitations in the bulk works as in the usual open-boundaries case; neglecting the boundary, the *level II* state of

⁷It is sufficient to consider a $N^{\text{I}} = 1$ state, i.e. one left, one right and one boundary site. A useful trick is to consider left-right graded-(anti)symmetric versions of the ansatz (37):

$$\begin{aligned} |\Psi^{\{a\}}(y)\rangle^{\text{II}} &= \mathbb{A} \left(|\phi_1^a \psi_B^3 \psi_1^3\rangle + |\phi_1^3 \psi_B^3 \psi_1^a\rangle \right), \\ |\Psi^{[a]}(y)\rangle^{\text{II}} &= \mathbb{B} \left(|\phi_1^a \psi_B^3 \psi_1^3\rangle - |\phi_1^3 \psi_B^3 \psi_1^a\rangle \right) + \mathbb{C} |\phi_1^3 \psi_B^a \psi_1^3\rangle, \end{aligned} \quad (40)$$

where

$$\begin{aligned} \mathbb{A} &= \begin{array}{c} \nearrow \uparrow \uparrow \\ \vdots \quad \vdots \\ \nearrow \uparrow \uparrow \end{array} + \begin{array}{c} \nearrow \uparrow \uparrow \\ \vdots \quad \vdots \\ \nearrow \uparrow \uparrow \end{array} = \begin{array}{c} \uparrow \uparrow \uparrow \\ \vdots \quad \vdots \\ \uparrow \uparrow \uparrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ \vdots \quad \vdots \\ \uparrow \uparrow \uparrow \end{array} \\ &= f^{\text{in}}(y; x_1, \eta_1) + S^{\text{II,I}}(y; x_1) K^{\text{II}}(y; x_B) f^{\text{out}}(x_1, \eta_1; -y), \\ \mathbb{B} &= \begin{array}{c} \nearrow \uparrow \uparrow \\ \vdots \quad \vdots \\ \nearrow \uparrow \uparrow \end{array} - \begin{array}{c} \nearrow \uparrow \uparrow \\ \vdots \quad \vdots \\ \nearrow \uparrow \uparrow \end{array} = \begin{array}{c} \uparrow \uparrow \uparrow \\ \vdots \quad \vdots \\ \uparrow \uparrow \uparrow \end{array} - \begin{array}{c} \uparrow \uparrow \uparrow \\ \vdots \quad \vdots \\ \uparrow \uparrow \uparrow \end{array} \\ &= f^{\text{in}}(y; x_1, \eta_1) - S^{\text{II,I}}(y; x_1) K^{\text{II}}(y; x_B) f^{\text{out}}(x_1, \eta_1; -y), \\ \mathbb{C} &= \begin{array}{c} \nearrow \uparrow \uparrow \\ \vdots \quad \vdots \\ \nearrow \uparrow \uparrow \end{array} - \begin{array}{c} \nearrow \uparrow \uparrow \\ \vdots \quad \vdots \\ \nearrow \uparrow \uparrow \end{array} = 2 S^{\text{II,I}}(y; x_1) f^\tau(y; x_B, \eta_B). \end{aligned} \quad (41)$$

The compatibility conditions (39) then explicitly become $k_0 k_7^{(\text{S})} \mathbb{A} = \mathbb{A}_\tau$, for the graded-symmetric state and $k_3^{(\text{A})} \mathbb{B} + k_{18}^{(\text{A})} \mathbb{C} = \mathbb{B}_\tau$, $2k_{19}^{(\text{A})} \mathbb{B} + k_5^{(\text{A})} \mathbb{C} = \mathbb{C}_\tau$ for the graded-antisymmetric one.

two particles is in a background consisting of two *level I* sites is

$$\begin{aligned} |\phi^a(y_1)\phi^b(y_2)\rangle_{\text{two site}}^{\text{II}} &= \mathbb{A} |\phi_1^a\phi_2^b\rangle + \mathbb{B} (M |\phi_1^a\phi_2^b\rangle + N |\phi_1^b\phi_2^a\rangle) \\ &\quad + \varepsilon^{ab} (\mathbb{C} |\psi_1^4\psi_2^3\rangle + \mathbb{D} |\psi_1^3\psi_2^4\rangle), \end{aligned} \quad (43)$$

where the shorthands are

$$\begin{aligned} \mathbb{A} &= \begin{array}{c} \nearrow \\ \nearrow \\ \vdots \end{array} = f^{\text{in}}(y_1; x_1, \eta_1) S^{\text{II,I}}(y_2; x_1) f^{\text{in}}(y_2; x_2, \eta_2), \\ \mathbb{B} &= \begin{array}{c} \nearrow \\ \nearrow \\ \vdots \end{array} = S^{\text{II,I}}(y_1; x_1) f^{\text{in}}(y_1; x_2, \eta_2) f^{\text{in}}(y_2; x_1, \eta_1), \\ \mathbb{C} &= \begin{array}{c} \bullet \\ \nearrow \\ \nearrow \\ \vdots \end{array} = f^{\text{in}}(y_1; x_1, \eta_1) f^{\text{in}}(y_2; x_1, \eta_1) f^\sigma(y_1, y_2; x_1, \eta_1, \zeta_1), \\ \mathbb{D} &= \begin{array}{c} \bullet \\ \nearrow \\ \nearrow \\ \vdots \end{array} = S^{\text{II,I}}(y_1; x_1) f^{\text{in}}(y_1; x_1, \eta_1) S^{\text{II,I}}(y_2; x_1) f^{\text{in}}(y_2; x_1, \eta_1) f^\sigma(y_1, y_2; x_2, \eta_2, \zeta_2). \end{aligned} \quad (44)$$

Here [10]

$$f^\sigma(y_1, y_2; x_k, \eta_k, \zeta_k) = \frac{x_k^+ x_k^- - y_1 y_2 \eta_k}{x_k^+ (x_k^+ - x_k^-) \zeta_k} \left(-\frac{\frac{i}{y_1} - \frac{i}{y_2}}{v_1 - v_2 - \frac{2i}{g}} \right), \quad (45)$$

and

$$M(y_1, y_2) = \frac{\frac{2i}{g}}{v_1 - v_2 - \frac{2i}{g}}, \quad N(y_1, y_2) = -\frac{v_1 - v_2}{v_1 - v_2 - \frac{2i}{g}}, \quad (46)$$

where $v_i = y_i + 1/y_i$.

In the presence of a boundary there is only one new type of term needed in the ansatz, corresponding to both particles sitting at the boundary. The full two-particle ansatz has many terms, so for brevity we shall write it out only in the case of a chain with $N^{\text{I}} = 1$ bulk sites. This is sufficient to determine the new coefficient, $f^{\sigma\tau}$.

Thus, consider a state of two impurities propagating on a background of a three-site *level I* chain of left, right and boundary slots. For clarity in the pictures we suppose that the impurities have their ‘‘tails’’ running one to the left and one to the right. The *level II* ansatz is

$$\begin{aligned} |\phi^a(y_1)\phi^b(y_2)\rangle_{\text{three site}}^{\text{II}} &= \mathbb{A} |\phi_1^a\psi_B^3\phi_2^b\rangle + \mathbb{B} |\phi_1^a\psi_B^b\phi_2^3\rangle - \mathbb{C} |\phi_1^3\psi_B^a\phi_2^b\rangle \\ &\quad + M(y_1, -y_2) (\mathbb{D} |\phi_1^a\psi_B^3\phi_2^b\rangle + \mathbb{E} |\phi_1^a\psi_B^b\phi_2^3\rangle - \mathbb{F} |\phi_1^3\psi_B^a\phi_2^b\rangle) \\ &\quad + N(y_1, -y_2) (\mathbb{D} |\phi_1^b\psi_B^3\phi_2^a\rangle + \mathbb{E} |\phi_1^b\psi_B^a\phi_2^3\rangle - \mathbb{F} |\phi_1^3\psi_B^a\phi_2^b\rangle) \\ &\quad + \varepsilon^{ab} (\mathbb{G} |\psi_1^4\psi_B^3\psi_2^3\rangle + \mathbb{H} |\psi_1^3\psi_B^4\psi_2^3\rangle + \mathbb{K} |\psi_1^3\psi_B^3\psi_2^4\rangle), \end{aligned} \quad (47)$$

where the shorthands are

$$\begin{aligned}
\mathbb{A} &= \begin{array}{c} \nearrow \\ | \\ \searrow \end{array} = f^{in}(y_1; x_1, \eta_1) f^{in}(y_2; x_2, \eta_2), \\
\mathbb{B} &= \begin{array}{c} \nearrow \\ | \\ \searrow \end{array} = -f^{in}(y_1; x_1, \eta_1) S^{\text{II,I}}(y_2; x_2) f^\tau(y_2; x_B, \eta_B), \\
\mathbb{C} &= \begin{array}{c} \nearrow \\ | \\ \searrow \end{array} = S^{\text{II,I}}(y_1; x_1) f^\tau(y_1; x_B, \eta_B) f^{in}(y_2; x_2, \eta_2), \\
\mathbb{D} &= \begin{array}{c} \nearrow \\ | \\ \searrow \end{array} = S^{\text{II,I}}(y_1; x_1) K^{\text{II}}(y_1, x_B) f^{out}(x_2, \eta_2; -y_1) \\
&\quad \times S^{\text{II,I}}(y_2; x_2) K^{\text{II,I}}(y_2, x_B) f^{out}(x_1, \eta_1; -y_2), \\
\mathbb{E} &= \begin{array}{c} \nearrow \\ | \\ \searrow \end{array} = S^{\text{II,I}}(y_1; x_1) f^\tau(y_1; x_B, \eta_B) S^{\text{II,I}}(y_2; x_2) K^{\text{II,I}}(y_2, x_B) f^{out}(x_1, \eta_1; -y_2), \\
\mathbb{F} &= \begin{array}{c} \nearrow \\ | \\ \searrow \end{array} = S^{\text{II,I}}(y_1; x_1) K^{\text{II,I}}(y_1, x_B) f^{out}(x_2, \eta_2; -y_1) S^{\text{II,I}}(y_2; x_2) f^\tau(y_2; x_B, \eta_B), \\
\mathbb{G} &= \begin{array}{c} \nearrow \\ | \\ \searrow \end{array} = f^{in}(y_1; x_1, \eta_1) S^{\text{II,I}}(y_2; x_2) K^{\text{II,I}}(y_2, x_B) f^{out}(x_1, \eta_1; -y_2) f^\sigma(y_1, -y_2; x_1, \eta_1, \zeta_1), \\
\mathbb{H} &= \begin{array}{c} \nearrow \\ | \\ \searrow \end{array} = S^{\text{II,I}}(y_1; x_1) f^\tau(y_1; x_B, \eta_B) S^{\text{II,I}}(y_2; x_2) f^\tau(y_2; x_B, \eta_B) f^{\sigma\tau}(y_1, y_2; x_B, \eta_B, \zeta_B), \\
\mathbb{K} &= \begin{array}{c} \nearrow \\ | \\ \searrow \end{array} = S^{\text{II,I}}(y_1; x_1) K^{\text{II,I}}(y_1, x_B) f^{out}(x_2, \eta_2; -y_1) f^{in}(y_2; x_1, \eta_1) f^\sigma(-y_1, y_2; x_2, \eta_2, \zeta_2).
\end{aligned} \tag{48}$$

The minus signs in (47) appear because of the graded permutation of left and right representations, which is not explicitly seen in the unfolded picture, but is revealed by folding the incoming left tail to the right side:

$$- \begin{array}{c} \nearrow \\ | \\ \searrow \end{array} \sim \begin{array}{c} \nearrow \\ | \\ \searrow \end{array}, \quad \text{and} \quad - \begin{array}{c} \nearrow \\ | \\ \searrow \end{array} \sim \begin{array}{c} \nearrow \\ | \\ \searrow \end{array}.$$

The minus sign in \mathbb{B} is already present in the ansatz for a single *level II* particle with its tail running to the right.

The compatibility relation for the case under the consideration is

$$\mathcal{K}^{\text{I}} |\phi^a(y_1) \phi^b(y_2)\rangle^{\text{II}} = |\phi^a(y_1) \phi^b(y_1)\rangle_\tau^{\text{II}} K^{\text{I}} = |\phi^a(y_1) \phi^b(y_2)\rangle_\tau^{\text{II}}, \tag{49}$$

and is trivially satisfied when $a = b$, because the terms with unknown function $f^{\sigma\tau}$ do not appear. It is good to start from this case, as it is a way to check that the ansatz is written consistently and all signs are correct.

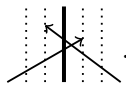
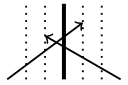
The easiest way to find $f^{\sigma\tau}$ it is to consider the overlap of the consistency condition with the composite excitation $|\psi^4(y_1, y_2)\rangle_{\text{three site}}^{\text{II}}$ leading to a very simple equation

$$k_5^{(\text{A})} (\mathbb{G} + \mathbb{H} - \mathbb{K}) = \mathbb{G}_\tau + \mathbb{H}_\tau - \mathbb{K}_\tau, \tag{50}$$

which may be solved straightforwardly giving a unique solution

$$f^{\sigma\tau}(y_1, y_2, x_B, \eta_B, \zeta_B) = -i \frac{\eta_B^2 \left(\frac{1}{y_1} + \frac{1}{y_2} \right) \left(1 - \frac{y_1 y_2}{x_B^2} \right)}{\zeta_B \left(v_1 + v_2 - \frac{2i}{g} \right)}. \quad (51)$$

Then it is easy to check that all other constraints coming from the compatibility relation are satisfied using this result and the mass-shell relation.

Note that we did not include the following diagram in (47). . It is a valid scattering diagram, but it does not need to be included as it is equivalent to the diagram we have already included, namely . (The *level II* reflection matrix has only one component, namely K^{II} (42), and thus acts only diagonally.)

Level III

The final level of the nesting is very similar to that in [19]. One finds the usual *level III* S-matrices of [10]:

$$S^{\text{III,II}}(w; y) = \frac{w - v + \frac{i}{g}}{w - v - \frac{i}{g}}, \quad S^{\text{III}}(w_1, w_2) = \frac{w_1 - w_2 - \frac{2i}{g}}{w_1 - w_2 + \frac{2i}{g}}. \quad (52)$$

and the *level III* reflection matrix

$$K^{\text{III}}(w) = -1. \quad (53)$$

3.2 Bethe equations

The nested coordinate Bethe ansatz we have presented applies to a semi-infinite system with a right boundary. The picture was unfolded into an infinite system with the right boundary represented in the middle. Adding a left boundary at a distance L , corresponds to closing the infinite line into a circle of length $2L$. Then the Bethe equations are obtained by inserting excitations into the closed spin-chain at any level (I, II or III) and moving them around the circle, scattering them with all other states and with left and right boundaries (see figure 4).

The *level I* chain has $2N^0 + 2$ sites on which to place N^{I} left, N^{I} right, and two boundary excitations. The full revolution of any *level I* excitation around the circle results in a phase factor e^{2ipL} , where L is ‘a half of the circumference of the circle’.

At *level II* there are N^{II} excitations that propagate in the inhomogeneous background of the $2N^{\text{I}} + 2$ *level I* sites. Thus there can be any number $N^{\text{II}} \in [0, 2N^{\text{I}} + 2]$ of *level II* states. The final level, *level III*, is similar, but there are no boundary sites. There can be any number $N^{\text{III}} \in [0, N^{\text{II}}]$ of *level III* excitations.

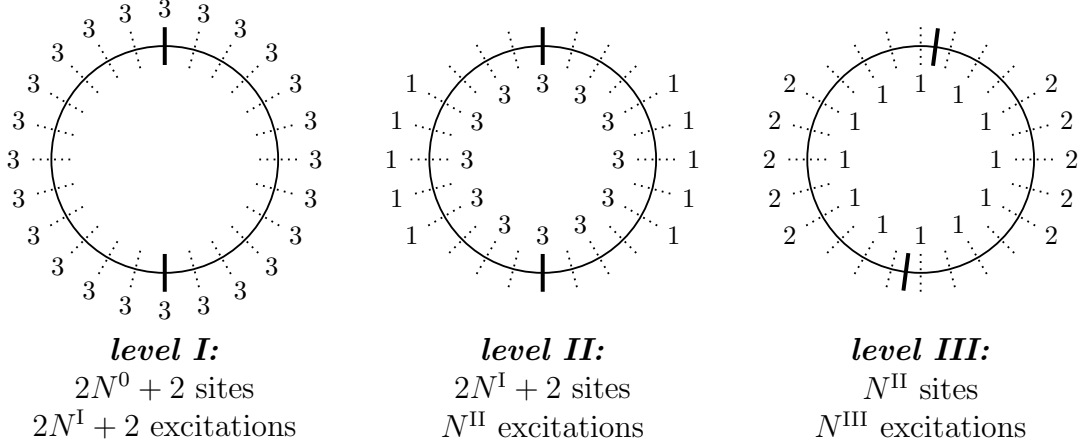


Figure 4: Schematical representation of Bethe equations for all levels. The dotted lines represent bulk sites and the solid lines represent boundary sites. The inside of the circles corresponds to the background states and the outside corresponds to the excitations.

Each excitation of every level has a rapidity x_k^A associated, and for each one of them we obtain a Bethe equation:

$$K_R^A(x_k^A) K_L^A(-x_k^A) \prod_{\substack{B=I \\ (A,k) \neq (B,\ell)}}^{III} \prod_{l=1}^{N^B} S^{A,B}(x_k^A, x_l^B) S^{B,A}(x_l^B, -x_k^A) = \begin{cases} \left(\frac{x_k^+}{x_k^-}\right)^{-2L} & \text{for } A = \text{I} \\ 1 & \text{for } A = \text{II, III.} \end{cases} \quad (54)$$

For *level I* rapidities we use the spectral parameters $x^I = x^\pm$ and for *level II* and *level III* rapidities we use y and w respectively. We can now write the Bethe equations explicitly for each level using (36), (38), (42), (52) and (53) and simplify them with the help of parity symmetry, which ensures that $K_L^A(-x^A) = K_R^A(x^A)$ and $S^{B,A}(x_l^B, -x_k^A) = S^{A,B}(x_k^A, -x_l^B)$. Then, the expressions of the Bethe equations for the $\mathfrak{su}(2|2)^2$ scattering theory with the ‘‘achiral vertical’’ boundary conditions are

$$1 = K_0(x_k^\pm)^2 \left(\frac{x_k^+}{x_k^-}\right)^{2L} \prod_{l \neq k}^{N^I} S_0(x_k^\pm, x_l^\pm)^2 S_0(x_k^\pm, -x_l^\mp)^2 \prod_{l=1}^{N^{II}} \frac{y_l - x_k^-}{y_l - x_k^+} \frac{y_l + x_k^-}{y_l + x_k^+}, \quad (55)$$

$$1 = \left(\frac{y_k - x_B}{y_k + x_B}\right)^2 \prod_{l=1}^{N^I} \frac{y_k - x_l^+}{y_k - x_l^-} \frac{y_k + x_l^-}{y_k + x_l^+} \prod_{l=1}^{N^{III}} \frac{w_l - v_k - \frac{i}{g}}{w_l - v_k + \frac{i}{g}} \frac{w_l + v_k + \frac{i}{g}}{w_l + v_k - \frac{i}{g}}, \quad (56)$$

$$1 = \prod_{l=1}^{N^{II}} \frac{w_k - v_l + \frac{i}{g}}{w_k - v_l - \frac{i}{g}} \frac{w_k + v_l - \frac{i}{g}}{w_k + v_l + \frac{i}{g}} \prod_{l \neq k}^{N^{III}} \frac{w_k - v_l - \frac{2i}{g}}{w_k - v_l + \frac{2i}{g}} \frac{w_k + v_l - \frac{2i}{g}}{w_k + v_l + \frac{2i}{g}}. \quad (57)$$

Note that we added the overall scalar factors $K_0(x_k^\pm)$ and $S_0(x_k^\pm, x_l^\pm)^2$ to the *level I* scattering factors (36), which are not determined by symmetry arguments. The bulk factor

$S_0(x_k^\pm, x_l^\pm)^2$ was found by an educated guess which relied on the crossing symmetry and many sophisticated weak and strong coupling verifications [20, 21, 22]. However, the analogous boundary factor $K_0(x_k^\pm)$ for the $D5$ -brane reflection is not known yet.

Note the similarity between the Bethe equations (55)-(57) and the Bethe equations for the $\mathfrak{su}(2|2)$ scattering theory with a “ $Z = 0$ ” boundary [19]. The only differences are in the dressing phase K_0 , and that the equations lack an index α in *level II* and *level III* rapidities, which distinguishes between left and right excitations. This is because the achiral nature of the reflection, which means that left and right can no longer be distinguished.

The flavour of *level I* excitations is a matter of choice. If we had we chosen them to be ϕ^1 instead of ψ^3 , the ψ^α would have been the *level II* excitations and for the scattering factors of the nested Bethe ansatz we would have obtained:

$$\begin{aligned}
S^I(x_1; x_2) &= \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \sqrt{\frac{x_1^+ x_2^-}{x_1^- x_2^+}}, & K^I(x; x_B) &= -\frac{x^+}{x^-} \left(\frac{x^- - x_B}{x^+ + x_B} \right)^2, \\
K^{II}(y; x_B) &= \frac{y + x_B}{y - x_B}, & S^{II,I}(y; x) &= \frac{y - x^-}{y - x^+}, \\
S^{III,II}(w; v) &= \frac{w - v - \frac{i}{g}}{w - v + \frac{i}{g}}, & S^{III}(w_1; w_2) &= \frac{w_1 - w_2 + \frac{2i}{g}}{w_1 - w_2 - \frac{2i}{g}}.
\end{aligned} \tag{58}$$

Thus the Bethe equations would be of the following form:

$$\begin{aligned}
1 &= K_0(x_k^\pm)^2 \left(\frac{x_k^+}{x_k^-} \right)^{2L} \left(\frac{x_k^+}{x_k^-} \right)^2 \left(\frac{x_k^- - x_B}{x_k^+ + x_B} \right)^4 \\
&\times \prod_{l \neq k}^{N^I} S_0(x_k^\pm, x_l^\pm)^2 S_0(x_k^\pm, -x_l^\mp)^2 \left(\frac{x_k^- - x_l^+}{x_k^+ - x_l^-} \sqrt{\frac{x_k^+ x_l^-}{x_k^- x_l^+}} \right)^2 \left(\frac{x_k^- + x_l^-}{x_k^+ + x_l^+} \sqrt{\frac{x_k^+ x_l^+}{x_k^- x_l^-}} \right)^2 \\
&\times \prod_{l=1}^{N^{II}} \frac{y_l - x_k^+}{y_l - x_k^-} \frac{y_l + x_k^+}{y_l + x_k^-},
\end{aligned} \tag{59}$$

$$1 = \left(\frac{y_k + x_B}{y_k - x_B} \right)^2 \prod_{l=1}^{N^I} \frac{y_k - x_l^-}{y_k - x_l^+} \frac{y_k + x_l^+}{y_k + x_l^-} \prod_{l=1}^{N^{III}} \frac{w_l - v_k + \frac{i}{g}}{w_l - v_k - \frac{i}{g}} \frac{w_l + v_k - \frac{i}{g}}{w_l + v_k + \frac{i}{g}}, \tag{60}$$

$$1 = \prod_{l=1}^{N^{II}} \frac{w_k - v_l - \frac{i}{g}}{w_k - v_l + \frac{i}{g}} \frac{w_k + v_l + \frac{i}{g}}{w_k + v_l - \frac{i}{g}} \prod_{l \neq k}^{N^{III}} \frac{w_k - v_l + \frac{2i}{g}}{w_k - v_l - \frac{2i}{g}} \frac{w_k + w_l + \frac{2i}{g}}{w_k + w_l - \frac{2i}{g}}. \tag{61}$$

As a matter of convention, we take the distance L to be the number of bulk fields N^0 . After having made this choice, the overall factor K_0 could be fixed order by order by comparison with weak and strong coupling results.

Given that the spin-chain length may vary under mixing [10], N^0 is not a good quantum number. As in [19], the Bethe equations can be re-expressed in terms of the charge $J =$

J_{56} . The expression of the conserved charge in terms of the N^A depends on the vacuum orientation. For the vacuum ψ^3 choice this would be $J \sim N^0 - N^{\text{II}}$ while for the vacuum ϕ^1 would be $J \sim N^0 - N^{\text{I}} + N^{\text{II}}$.

Horizontal vacuum. Let us go back and consider the reflection in the horizontal vacuum case. We had shown that this case is equivalent to the scattering in the bulk (as in [10]). The Bethe equations for the boundary scattering in the horizontal case are essentially the same as in the vertical case. For instance, eq. (55)-(57) would remain the same, except for the factor $\frac{y_k - x_B}{y_k + x_B}$ in (56) which would not appear. Of course, the boundary dressing phase would also be different. The analogue of eq. (55)-(57) in the horizontal case would be identical to the Bethe equations of a closed spin chain of length $2L$ with $2N^{\text{I}}$ bulk magnons of momenta $(p_1, \dots, p_{N^{\text{I}}}, -p_{N^{\text{I}}}, \dots, -p_1)$ if the boundary dressing factor in this case was $K_0(p) = S_0(p, -p)$, but that has not been proven.

4 Discussion

In this paper we have considered the reflection of fundamental magnons from certain $D5$ -branes in an $AdS_5 \times S^5$ background. There are two interesting cases of embedding $D5$ -branes that have different vacuum orientations, which we have named “horizontal” and “vertical” vacua. A previous attempt to show the integrability of these configurations [7] was unsuccessful because some crucial minus signs were overlooked. In this paper we have shown that the $D5$ -brane, from the scattering theory point of view, allows integrable boundary conditions and furthermore, by solving Bethe ansatz equations, we have shown that they are of the achiral (chirality-reversing) type.

Interestingly, the reflection from the achiral boundary has an “unfolded” picture where the reflection from the boundary may be considered as a “scattering through the boundary”. In this “unfolded” picture, the scattering in the “horizontal” vacuum open spin-chain is completely equivalent to the scattering in an closed spin-chain. This is so because the boundary is achiral and has no boundary degrees of freedom (is a singlet). Thus, the Bethe equations are essentially the same, up to a possibly different dressing factor.

Some interesting calculations could be done to verify the assertion of all-loop integrability for the considered $D5$ -brane boundary conditions. In the weak coupling limit, one could perform a 2-loop computation to analyze whether the dilatation operator is integrable beyond 1-loop. In the opposite regime, in the strong coupling, one should expect the superstring theory to be classically integrable. In particular, it would be very interesting to understand why the infinite set of non-local charges could be constructed for the $D3$ and the $D7$ case

but failed for the $D5$ in the formalism of [6]. As a separate question, one could also compute the leading finite size corrections, as was done for the $D3$ case in [23].

Although we have analyzed vertical and horizontal cases separately, the integrability of both cases ought to be related. Consider the open spin chain corresponding to our $D5$ -branes. If we knew its exact anomalous dimension Hamiltonian, integrability would depend on the existence of conserved charges only, independently of any vacuum choice. However, while the reflection matrix in the horizontal case necessarily satisfied the bYBE, in the vertical case the boundary symmetry constraints were not enough to fix the reflection matrix to satisfy bYBE. In this regard, an interesting question is whether there is an associated Yangian symmetry that would constrain the coefficient k_0 and the bound-state reflection matrices without the use of the boundary Yang-Baxter equation. Such an “achiral twisted Yangian”, of a similar structure to the Yangian of the $Y = 0$ $D3$ -brane introduced in [24] and explored in [25, 26], will be considered in a forthcoming paper.

Acknowledgments. We thank Juan Martín Maldacena for helpful discussions and for the suggestion to reconsider the integrability of the $D5$ -brane boundary conditions. We also want to thank Niall MacKay for many valuable discussions and for reading the manuscript and Alessandro Torrielli for lots of useful discussions and comments. V.R. and C.A.S.Y. are supported by UK EPSRC grant EP/H000054/1.

A Fundamental S -matrix

The fundamental S -matrix may be neatly defined as a differential operator

$$\mathcal{S}(p_1, p_2) = a_i(p_1, p_2) \Lambda_i, \quad (62)$$

acting on the superspace with the basis $\{\omega_1, \omega_2, \theta_3, \theta_4\}$, where ω_a and θ_α are bosonic and fermionic variables respectively (see [13] for details), and Λ_i are the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ invariant differential operators:

$$\begin{aligned} \Lambda_1 &= \frac{1}{2} (\omega_a^1 \omega_b^2 + \omega_b^1 \omega_a^2) \frac{\partial^2}{\partial \omega_b^2 \partial \omega_a^1}, & \Lambda_6 &= \omega_a^2 \theta_\alpha^1 \frac{\partial^2}{\partial \omega_a^2 \partial \theta_\alpha^1}, \\ \Lambda_2 &= \frac{1}{2} (\omega_a^1 \omega_b^2 - \omega_b^1 \omega_a^2) \frac{\partial^2}{\partial \omega_b^2 \partial \omega_a^1}, & \Lambda_7 &= \epsilon^{ab} \omega_a^1 \omega_b^2 \epsilon_{\alpha\beta} \frac{\partial^2}{\partial \theta_\beta^2 \partial \theta_\alpha^1}, \\ \Lambda_3 &= \frac{1}{2} (\theta_\alpha^1 \theta_\beta^2 + \theta_\beta^1 \theta_\alpha^2) \frac{\partial^2}{\partial \theta_\beta^2 \partial \theta_\alpha^1}, & \Lambda_8 &= \frac{1}{2} \epsilon^{\alpha\beta} \theta_\alpha^1 \theta_\beta^2 \epsilon_{ab} \frac{\partial^2}{\partial \omega_b^2 \partial \omega_a^1}, \\ \Lambda_4 &= \frac{1}{2} (\theta_\alpha^1 \theta_\beta^2 - \theta_\beta^1 \theta_\alpha^2) \frac{\partial^2}{\partial \theta_\beta^2 \partial \theta_\alpha^1}, & \Lambda_9 &= \omega_a^1 \theta_\alpha^2 \frac{\partial^2}{\partial \omega_a^2 \partial \theta_\alpha^1}, \\ \Lambda_5 &= \omega_a^1 \theta_\alpha^2 \frac{\partial^2}{\partial \omega_a^1 \partial \theta_\alpha^2}, & \Lambda_{10} &= \omega_a^2 \theta_\alpha^1 \frac{\partial^2}{\partial \omega_a^1 \partial \theta_\alpha^2}, \end{aligned} \quad (63)$$

The physical S -matrix ($\mathcal{S} : \mathcal{V}_1 \otimes \mathcal{V}_2 \mapsto \mathcal{V}_2 \otimes \mathcal{V}_1$), that we were using in our calculations is acquired by acting with a graded permutation on the S -matrix defined on superspace, $\mathcal{S}_{\text{physical}} := P_{12} \mathcal{S}_{\text{superspace}}$. The coefficients of the fundamental (physical) left S -matrix that we were using in our calculations are:

$$\begin{aligned} a_1 &= -\frac{x_1^- - x_2^+}{x_2^- - x_1^+} \frac{\eta_1 \eta_2}{\tilde{\eta}_1 \tilde{\eta}_2}, & a_6 &= -\frac{x_1^- - x_2^-}{x_2^- - x_1^+} \frac{\eta_2}{\tilde{\eta}_2}, \\ a_2 &= -\frac{x_2^+ - x_1^-}{x_2^- - x_1^+} \left(1 - 2 \frac{1 - 1/x_2^- x_1^+}{1 - 1/x_2^+ x_1^+} \frac{x_2^- - x_1^-}{x_2^+ - x_1^-} \right) \frac{\eta_1 \eta_2}{\tilde{\eta}_1 \tilde{\eta}_2}, & a_7 &= \frac{i\zeta(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(-1 + x_1^- x_2^-)(x_2^- - x_1^+) \tilde{\eta}_1 \tilde{\eta}_2}, \\ a_3 &= -1, & a_8 &= \frac{i(x_1^- - x_2^-) \eta_1 \eta_2}{\zeta(x_2^- - x_1^+)(-1 + x_1^+ x_2^+)}, \\ a_4 &= \left(1 - 2 \frac{1 - 1/x_2^+ x_1^-}{1 - 1/x_2^- x_1^-} \frac{x_2^+ - x_1^+}{x_2^- - x_1^-} \right), & a_9 &= \frac{x_1^- - x_1^+}{-x_2^- + x_1^+} \frac{\eta_2}{\tilde{\eta}_1}, \\ a_5 &= -\frac{x_1^+ - x_2^+}{x_2^- - x_1^+} \frac{\eta_1}{\tilde{\eta}_1}, & a_{10} &= -\frac{x_2^- - x_2^+}{x_2^- - x_1^+} \frac{\eta_1}{\tilde{\eta}_2}. \end{aligned} \quad (64)$$

B Reflection matrix \mathcal{K}^h

The K -matrix for the reflection in the horizontal vacuum case may be defined on the superspace as

$$\mathcal{K}^h(p, -p) = k_i(p) \Lambda_i, \quad (65)$$

where Λ_i are the same as in (63). The reflection coefficients are:

$$k_i(p) = a_i(p, -p). \quad (66)$$

C Reflection matrices \mathcal{K}^{Ba} and $\overline{\mathcal{K}}^{Ba}$

The supersymmetric reflection K -matrix \mathcal{K}^{Ba} describing the reflection of the two-magnon bound states in the bulk from the fundamental states on the boundary may be defined as a differential operator

$$\mathcal{K}^{Ba}(p_1, x_B) = k_i^{(S)}(p_1, x_B) \Lambda_i \quad (67)$$

acting on the superspace, where Λ_i are

$$\begin{aligned} \Lambda_1 &= \frac{1}{6} (\omega_a^1 \omega_b^1 \omega_c^2 + \omega_c^1 \omega_b^1 \omega_a^2 + \omega_a^1 \omega_c^1 \omega_b^2) \frac{\partial^3}{\partial \omega_c^2 \partial \omega_b^1 \partial \omega_a^1}, & \Lambda_{10} &= \frac{1}{2} \epsilon^{kl} \omega_c^1 \omega_k^1 \omega_l^2 \epsilon_{\alpha\beta} \frac{\partial^3}{\partial \omega_c^2 \partial \theta_\beta^1 \partial \theta_\alpha^1}, \\ \Lambda_2 &= \frac{1}{6} (\epsilon_{bc} \omega_a^1 + \epsilon_{ac} \omega_b^1) \epsilon^{kl} \omega_k^1 \omega_l^2 \frac{\partial^3}{\partial \omega_c^2 \partial \omega_b^1 \partial \omega_a^1}, & \Lambda_{11} &= \frac{1}{2} \epsilon^{\alpha\beta} \omega_b^2 \theta_\alpha^1 \theta_\beta^1 \epsilon_{ac} \frac{\partial^3}{\partial \omega_c^2 \partial \omega_b^1 \partial \omega_a^1}, \\ \Lambda_3 &= \frac{1}{2} \theta_\beta^1 (\omega_a^1 \omega_c^2 + \omega_c^1 \omega_a^2) \frac{\partial^3}{\partial \omega_c^2 \partial \omega_a^1 \partial \theta_\beta^1}, & \Lambda_{12} &= \epsilon^{\alpha\beta} \omega_c^2 \theta_\alpha^1 \theta_\beta^1 \epsilon_{\gamma\delta} \frac{\partial^3}{\partial \omega_a^1 \partial \theta_\delta^2 \partial \theta_\gamma^1}, \\ \Lambda_4 &= \frac{1}{2} \theta_\beta^1 (\omega_a^1 \omega_c^2 - \omega_c^1 \omega_a^2) \frac{\partial^3}{\partial \omega_c^2 \partial \omega_a^1 \partial \theta_\beta^1}, & \Lambda_{13} &= \epsilon^{\alpha\beta} \omega_b^1 \theta_\alpha^1 \theta_\beta^2 \epsilon_{ac} \frac{\partial^3}{\partial \omega_c^2 \partial \omega_b^1 \partial \omega_a^1}, \\ \Lambda_5 &= \frac{1}{2} \omega_a^1 \omega_b^1 \theta_\gamma^2 \frac{\partial^3}{\partial \omega_b^1 \partial \omega_a^1 \partial \theta_\gamma^2}, & \Lambda_{14} &= \frac{1}{2} \epsilon^{\alpha\beta} \omega_a^2 \theta_\alpha^1 \theta_\beta^1 \epsilon_{\gamma\delta} \frac{\partial^3}{\partial \omega_a^1 \partial \theta_\beta^2 \partial \theta_\gamma^1}, \\ \Lambda_6 &= \frac{1}{2} \omega_c^2 \theta_\alpha^1 \theta_\beta^1 \frac{\partial^3}{\partial \omega_c^2 \partial \theta_\beta^1 \partial \theta_\alpha^1}, & \Lambda_{15} &= \frac{1}{2} \epsilon^{\alpha\beta} \omega_c^1 \theta_\alpha^1 \theta_\beta^2 \epsilon_{\gamma\delta} \frac{\partial^3}{\partial \omega_c^2 \partial \theta_\delta^1 \partial \theta_\gamma^1}, \\ \Lambda_7 &= \frac{1}{2} \omega_a^1 (\theta_\beta^1 \theta_\gamma^2 + \theta_\gamma^1 \theta_\beta^2) \frac{\partial^3}{\partial \omega_a^1 \partial \theta_\gamma^2 \partial \theta_\beta^1}, & \Lambda_{16} &= \epsilon^{\alpha\gamma} \theta_\alpha^1 \theta_\beta^1 \theta_\gamma^2 \epsilon_{ac} \frac{\partial^3}{\partial \omega_c^2 \partial \omega_a^1 \partial \theta_\beta^1}, \\ \Lambda_8 &= \frac{1}{2} \omega_a^1 (\theta_\beta^1 \theta_\gamma^2 - \theta_\gamma^1 \theta_\beta^2) \frac{\partial^3}{\partial \omega_a^1 \partial \theta_\gamma^2 \partial \theta_\beta^1}, & \Lambda_{17} &= \epsilon^{kl} \omega_k^1 \omega_l^2 \theta_\beta^1 \epsilon_{\alpha\gamma} \frac{\partial^3}{\partial \theta_\gamma^2 \partial \theta_\beta^1 \partial \theta_\alpha^1}, \\ \Lambda_9 &= \theta_\alpha^1 \theta_\beta^1 \theta_\gamma^2 \frac{\partial^3}{\partial \theta_\gamma^2 \partial \theta_\beta^1 \partial \theta_\alpha^1}, & \Lambda_{18} &= \omega_b^1 \omega_a^2 \theta_\gamma^1 \frac{\partial^3}{\partial \omega_b^1 \partial \omega_a^1 \partial \theta_\gamma^1}, \\ & & \Lambda_{19} &= \omega_a^1 \omega_c^1 \theta_\beta^2 \frac{\partial^3}{\partial \omega_c^2 \partial \omega_a^1 \partial \theta_\beta^1}. \end{aligned} \quad (68)$$

The coefficients of the symmetric reflection matrix \mathcal{K}^{Ba} are:

$$\begin{aligned}
k_1^{(S)} &= 1 \\
k_2^{(S)} &= \frac{3x_B(x^-)^2 - x_B(x^+)^2(2 + 3(x^+)^2) + x^-x^+(x_B - 4x^+ + x_B(x^+)^2)}{2(x_B + (-1 + x_B^2)x^- - x_B(x^-)^2)(x^+)^2} \\
k_3^{(S)} &= -\frac{((x^-)^2 + x_Bx^+) \tilde{\eta}}{(x_B - x^-)x^- \eta}, \\
k_4^{(S)} &= -\frac{(x_B + x^+)(x^- + x_B(x^+)^2) \tilde{\eta}}{(x_B + (-1 + x_B^2)x^- - x_B(x^-)^2)x^+ \eta}, \\
k_5^{(S)} &= \frac{(x_Bx^- - (x^+)^2) \tilde{\eta}_B}{(x_B - x^-)x^- \eta_B}, \\
k_6^{(S)} &= \frac{(-x_B(x^-)^4 + x_B(x^+)^2 + x^-x^+(x_B + x_B(x^-)^2 + x^-(4 - 2x_Bx^+))) \tilde{\eta}^2}{2(x^-)^2(x_B + (-1 + x_B^2)x^- - x_B(x^-)^2) \eta^2}, \\
k_7^{(S)} &= -\frac{x^+(x_B + x^+) \tilde{\eta} \tilde{\eta}_B}{(x_B - x^-)x^- \eta \eta_B}, \\
k_8^{(S)} &= \frac{(2x_B^2(x^-)^3 + x_B(x^-)^2(x_B - x^+)x^+ + 2(x^+)^3 + x^-x^+(-x_B + x^+)) \tilde{\eta} \tilde{\eta}_B}{(x^-)^2(x_B + (-1 + x_B^2)x^- - x_B(x^-)^2) \eta \eta_B}, \\
k_9^{(S)} &= \frac{x^+(x_B + x^+)(-x_B(x^-)^2 + x^+) \tilde{\eta}^2 \tilde{\eta}_B}{(x^-)^2(x_B + (-1 + x_B^2)x^- - x_B(x^-)^2) \eta^2 \eta_B}, \\
k_{10}^{(S)} &= \frac{i\zeta((x^-)^2 - (x^+)^2)^2(x_Bx^+ + x_B(x^-)^2x^+ + x^-(x_B + 2x^+ - x_B(x^+)^2))}{4(x^-)^2(x_B + (-1 + x_B^2)x^- - x_B(x^-)^2)x^+(-1 + x_Bx^+)} \frac{1}{\eta^2}, \\
k_{11}^{(S)} &= -\frac{ix_B(x^- + x^+)^2}{2\zeta(x_B - x^-)x^-(1 + x_Bx^-)x^+} \tilde{\eta}^2, \\
k_{12}^{(S)} &= -\frac{i\zeta x_B(x_Bx^- - (x^+)^2)((x^-)^2 - (x^+)^2)}{\sqrt{2}x^-(x_B + (-1 + x_B^2)x^- - x_B(x^-)^2)x^+ \eta \eta_B} \frac{1}{\eta^2}, \\
k_{13}^{(S)} &= \frac{i(x^- + x^+)(x_Bx^- - (x^+)^2)}{\sqrt{2}\zeta(x_B - x^-)x^-(1 + x_Bx^-)x^+} \tilde{\eta} \tilde{\eta}_B, \\
k_{14}^{(S)} &= \frac{x_B(x_B(x^-)^2 - x^+)(x^- + x^+) \tilde{\eta}^2}{\sqrt{2}(x^-)^2(-x_B + x^+)(1 + x_Bx^-) \eta \eta_B}, \\
k_{15}^{(S)} &= \frac{(x_B(x^-)^2 - x^+)((x^-)^2 - (x^+)^2) \tilde{\eta} \tilde{\eta}_B}{\sqrt{2}(x^-)^2(x_B + (-1 + x_B^2)x^- - x_B(x^-)^2) \eta^2}, \\
k_{16}^{(S)} &= -\frac{i(x_B + x^+)(x^- + x^+) \tilde{\eta}^2 \tilde{\eta}_B}{\sqrt{2}\zeta x^-(x_B + x^+)(1 + x_Bx^-) \eta}, \\
k_{17}^{(S)} &= \frac{i\zeta x_B(x_B + x^+)(-(x^-)^2 + (x^+)^2) \tilde{\eta}}{\sqrt{2}x^-(x_B + (-1 + x_B^2)x^- - x_B(x^-)^2) \eta^2 \eta_B}, \\
k_{18}^{(S)} &= \frac{x_B(x^- + x^+) \tilde{\eta}}{\sqrt{2}(x_B - x^-)x^- \eta_B}, \\
k_{19}^{(S)} &= \frac{((x^-)^2 - (x^+)^2) \tilde{\eta}_B}{\sqrt{2}x^-(x_B + x^+) \eta}.
\end{aligned} \tag{69}$$

The anti-supersymmetric reflection K -matrix $\overline{\mathcal{K}}^{Ba}$ describing the reflection of the two-magnon bound states in the mirror bulk theory from the fundamental states on the boundary may be defined as a differential operator

$$\overline{\mathcal{K}}^{Ba}(p_1, x_B) = k_i^{(A)}(p_1, x_B) \overline{\Lambda}_i \quad (70)$$

acting on the mirror superspace, where $\overline{\Lambda}_i$ are the differential operators acting on the mirror superspace. They may be acquired from (68) by interchange of bosonic and fermionic indices, $(a, b) \leftrightarrow (\alpha, \beta)$. The reflection coefficients $k_i^{(A)}$ may be obtained from 69 using the relation $k_i^{(A)}(p, x_B) = k_i^{(S)}(-p, x_B)$.

D 1-loop computations in the dCFT

The 1-loop mixing matrix of anomalous dimensions for the scalar sector of the defect conformal field theory (dCFT) was obtained in [8]. Let us recall that the complex scalar defect fields ϕ^a transform in a $\mathbf{2}$ of $\mathfrak{so}(3)_H$, while the six scalar fields of the bulk theory are split into X_H^I and X_V^A , transforming in a $\mathbf{3}$ of $\mathfrak{so}(3)_H$ and in a $\mathbf{3}$ of $\mathfrak{so}(3)_V$ respectively.

In this appendix we are interested in the case when the reference state breaks the $\mathfrak{so}(3)_V$ symmetry. For definiteness let us take $\bar{\phi}_a Z \cdots Z \phi^b$, with $Z = X_V^2 + iX_V^3$, as the vacuum state. As already discussed, the 16 bulk impurities will be accommodated into a \boxtimes and a \boxminus of the diagonal $\mathfrak{su}(2|2)_D$. In particular, of the 4 scalar field impurities left after fixing the vacuum, the X_H^I give rise to the $\phi^{\{a,b\}}$ components of the symmetric representation and X_V^1 to the $\phi^{[a,b]}$ component of the antisymmetric representation.

In particular, the component $\phi^{\{1,1\}}$ is the combination $Y = X_H^1 + iX_H^2$, and its boundary reflection shall be diagonal if ϕ^1 defect fields (and their conjugate) are placed at the ends of the chain. Let us consider the superposition of left-moving single magnon Y with a right-moving one.

$$|\Psi^Y\rangle = \sum_{n=1}^L (e^{ipn} + K_L^Y(p) e^{-ipn}) |n\rangle, \quad |n\rangle \equiv |\bar{\phi}_2 Z^{n-1} Y Z^{L-n} \phi^1\rangle. \quad (71)$$

This superposition is an eigenstate with eigenvalue $8g^2 \sin^2(\frac{p}{2}) + 8g^2$ of the Hamiltonian given in [8] if

$$K_L^Y(p) = -\frac{1 - 3e^{ip}}{1 - 3e^{-ip}}, \quad e^{2ip(L+1)} = (K_L^Y(p))^2. \quad (72)$$

Another impurity that it is reflected diagonally is X_V^1 . In that case, the superposition

$$|\Psi^{X_V^1}\rangle = \sum_{n=1}^L (e^{ipn} + K_L^{X_V^1}(p) e^{-ipn}) |n\rangle, \quad |n\rangle \equiv |\bar{\phi}_2 Z^{n-1} X_V^1 Z^{L-n} \phi^1\rangle. \quad (73)$$

is an eigenstate with eigenvalue $8g^2 \sin^2(\frac{p}{2}) + 8g^2$ if

$$K_L^{X\check{V}}(p) = e^{ip}, \quad e^{2ip(L+1)} = (K_L^{X\check{V}}(p))^2. \quad (74)$$

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