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Inverse Problems with Microlocal Observations

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Academic dissertation

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[1]For more details, see the included articles I-III.

List of Included Articles

I Caro, P., Helin, T., Kujanpää, A., & Lassas, M. (2019). Correlation imaging in inverse scattering is tomography on probability distributions. Inverse Problems, 35(1), [015010]. https://doi.org/10.1088/1361-6420/aaece1

II Balehowsky, T., Kujanpää, A., Lassas, M., Liimatainen, T. An Inverse Problem for the Relativistic Boltzmann Equation. arXiv:2011.09312 (ArXiv preprint)

III Kujanpää, A. (2021). Recovering a Riemannian Metric from Cherenkov Radiation in Inhomogeneous Anisotropic Medium. arXiv:2106.05447 (ArXiv preprint)

Contents

Errata for *Inverse Problems with Microlocal Observations*

Antti Kujanpää

January 24, 2022

(Jan 15, 2022) §3.1, page 7: The assumption $n = 1$ should be included in "Further, one checks that the wave front set..." (cf. Figure 1).

(Jan 24, 2022) Article II, page 49: The equation after "Thus, for any $(x, p) \in \overline{P}^+ M$, we have that..." should read

$$\lambda^{-1} F_{x,p}(\lambda) \longrightarrow \frac{d}{d\lambda}\Big|_{\lambda=0} F_{x,p}(\lambda) = 0.$$

(Jan 24, 2022) Article II, page 49: Absolute values are missing in (B.120).

1. Introduction

Inverse problems aim to recover a physical quantity (e.g. the density distribution of tissues inside a human body) from inside a target object using a set of data that is collected by making measurements in the exterior of the object. In order to reconstruct an image of the interior, which cannot be accessed for direct observations, one studies responses of the medium by probing it with various physical fields, such as X-rays, electromagnetic fields or acoustic waves. Inverse problems arise in many fields of science and technology, and a huge part of the information we have about the world today was obtained by solving them. For instance, we know about the interior structure of the Earth thanks to solving seismic inverse problems, and the structure of DNA was determined by solving X-ray diffraction problems. In today's hospitals patients are often examined and diagnosed using non-invasive techniques such as CT scans, MRIs and ultrasound imaging, all of which rely heavily on the theory of inverse problems.

To introduce the standard framework, let us consider a physical model which consists of a phase space and a set of equations that describe the physical reality. Let u_f stand for a solution (e.g. a wave, particle, moving object) to the equations with some additional property f. An inverse problem related to the system studies a map that links the controllable input f to an observation $m(u_f)$ that captures information about the solution u_f. The map $f \mapsto m(u_f)$ is interpreted as a mathematical model for measurement data. Typically $m(u_f)$ is a restriction of u_f, or some derivatives of it, to a subset or a topological boundary. The main objective in inverse problems is to recover the underlying physical system, or some properties of it, from this map. Determination of a potential, geometric structure or the conformal class of the configuration space are perhaps the most common topics. In this dissertation the main focus is on the uniqueness of such reconstructions.

As an example, let us consider an inverse problem related to a wave interacting with a smooth compactly supported potential V on a bounded domain $\Omega \subset \mathbb{R}^n$. Mathematically the wave is a function (or a distribution) u of space and time that satisfies for $(x,t) \in \overline{\Omega} \times [0,T]$ the equation

$$(\partial_t^2 - \Delta + V(x))u(x,t) = 0, \tag{1}$$

where $\Delta := \sum_{j=1}^n \partial_j^2$. For some fixed initial values $u(x,0) = \phi(x)$, $u(x,0) = \psi(x)$ the solution $u = u_f$ is uniquely defined by the Neumann boundary condition $f = N \cdot \nabla u|_{\partial X \times [0,T]}$, where N is the normal of $\partial\Omega$, and we can consider the Neumann-to-Dirichlet map $\Lambda_V : f \mapsto u|_{\partial\Omega\times[0,T]}$ as the measurement data (i.e. $m(u_f) := u|_{\partial\Omega\times[0,T]}$). The associated inverse problem of unique determinability is whether the data determine the potential V:

$$\text{Does } \Lambda_V = \Lambda_W \text{ for two potentials } V \text{ and } W \text{ imply } V = W?$$

A positive answer (for sufficiently large $T > 0$) to this particular question was given by Rakesh and Symes in [RS88]. The result indicates that at least in theory it is possible to recover the potential from active measurements on the surface of Ω by sending waves into the region and observing scattering on the surface $\partial\Omega$.

The object of study in the example above is the potential, whereas the kinematics of waves are in fact trivial in the sense that waves propagate along straight lines at a fixed constant speed. This is not always the case. As an example, let us consider the same model except (1) replaced with the equation

$$(\partial_t^2 - c^2(x)\Delta)u(x,t) = 0, \tag{2}$$

where $c > 0$ is a strictly positive function on $\overline{\Omega}$. The equation describes motion of waves in isotropic inhomogeneous medium. In fact, the model can be taken as a premise in ultrasound imaging with the quantity $c(x)$ corresponding to the speed of sound at x. Waves move along curved trajectories in the non-trivial geometry described by the function c. Naturally, the objective is to reconstruct the function c from the data. Let Λ_c stand for the data associated with (2). One asks the following:

Does $\Lambda_c = \Lambda_{\tilde{c}}$ for two strictly positive functions c and $\tilde{c}$ imply $c = \tilde{c}$?

A coarser analysis is often more convenient in applications, mainly because of stability and shorter computation times. If one is interested in recovering the main discontinuities in c it is enough to ask whether $c - \tilde{c}$ is sufficiently smooth. In fact, the irregularities in c scatter quite well and in medical ultrasound imaging one takes advantage of the echoes generated in this way. As an outcome, a reconstruction is computed from the travel times and amplitudes of the reflections. More sophisticated methods for the geometric problem exist. Unique determination from travel times of waves is theoretically possible even for smoothly inhomogeneous anisotropic materials. The techniques rely on Gaussian beams. [KKL01]

There are, of course, many systems that are not uniquely determined by the associated measurements. Non-trivial settings typically arise from reduction of data or discretisation but may also be caused by geometric effects such as conjugate points. For some models the uniqueness question remains unanswered. Perhaps the most famous open problems are the anisotropic Calderón problem in dimension $n \geq 3$, the boundary rigidity problem on simple manifolds of dimension $n \geq 3$, and the backscattering problem for smooth potentials in $\mathbb{R}^n$.

2. Lagrangian Distributions and Fourier Integral Operators

Some of the basic concepts of the theory of Fourier integral operators are briefly discussed below. For more detailed introduction, see e.g. the textbooks [GS94], [Dui96] and the original works [Ḧ71], [DH72], [MU79], [GU81] and [GU93].

2.1. **Symbols.** Let M be a smooth n dimensional manifold. The standard symbol class $S^m(M \times \mathbb{R}^k)$ of order $m \in \mathbb{R}$ and type $(1,0)$ is defined as the space of smooth functions $a \in C^\infty(M \times \mathbb{R}^k)$ that satisfy the following condition: For every $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^k$ and compact $K \subset M$ there is $C = C_{K,\alpha,\beta,a} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C\langle \xi \rangle^{m-|\beta|}, \quad \forall (x,\xi) \in K \times \mathbb{R}^k.$$

Moreover, we define $S^{-\infty}(M \times \mathbb{R}^k) \subset C^\infty(M \times \mathbb{R}^k)$ by

$$S^{-\infty}(M \times \mathbb{R}^k) := \bigcap_{m \in \mathbb{R}} S^m(M \times \mathbb{R}^k).$$

The space $S^{m_1,m_2}(M \times (\mathbb{R}^{k_1} \setminus \{0\}) \times \mathbb{R}^{k_2})$ is defined as functions $a \in C^\infty(M \times (\mathbb{R}^{k_1} \setminus \{0\}) \times \mathbb{R}^{k_2})$ that satisfy the following: For every $(\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$ and compact $K \subset M$ there is $C = C_{K,\alpha,\beta,\gamma,a} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma a(x,\xi,\eta)| \leq C\langle \xi, \eta \rangle^{m_1-|\beta|}\langle \eta \rangle^{m_2-|\gamma|}, \quad \forall (x,\xi,\eta) \in K \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}.$$

2.2. **The Wave Front Set.** Let M be a smooth manifold of dimension n. The singular support singsupp(u) of $u \in \mathcal{D}'(M)$ is defined as the set of points in M where u fails to be smooth. That is; a point $x_0 \in M$ lies in singsupp(u) if and only if for every $\phi \in C_c^\infty(M)$ with $\phi(x_0) \neq 0$ we have that $\phi u \notin C_c^\infty(M)$. If $x_0 \in$ singsupp(u) we say that u has a singularity at x_0. A more detailed characterisation of singularities is given by the Paley-Wiener theorem which implies that for a test function ϕ supported in a coordinate neighbourhood $U \subset M$ around x_0 the product ϕu is smooth if and only if

$$|\widehat{\phi u}(\tau\theta)| \leq O(\tau^{-N}), \tag{3}$$

for every N and $\theta \in \mathbb{S}^{n-1}$ as $\tau \longrightarrow \infty$. Here the hat refers to the Fourier transform in local coordinates. The wave front set $WF(u) \subset T^*M \setminus \{0\}$ of u is defined via local coordinatisations as the complement of all those cone axes $(x_0, \mathbb{R}_+\theta)$ that satisfy (3) for every N and some smooth local test function $\phi \in C_c^\infty(M)$ with $\phi(x_0) \neq 0$. In particular,

$$\text{singsupp}(u) = \pi WF(u),$$

where $\pi : T^*M \to M$ is the canonical projection $\pi(x, \xi) = x$.

2.3. **Distribution Classes.** Let M be a smooth manifold of dimension n and let $\sigma \in \wedge^2 T^*M$ be the canonical 2-form, given in local canonical coordinates $x^1, \ldots, x^n, \xi_1, \ldots, \xi_n$ by

$$\sigma = d\xi_j \wedge dx^j.$$

The cotangent bundle T^*M equipped with σ is a symplectic manifold. A submanifold Λ of $T^*M \setminus \{0\}$ is said to be Lagrangian if

$$T_\lambda \Lambda^\sigma = T_\lambda \Lambda, \quad \forall \lambda \in \Lambda,$$

where $T_\lambda \Lambda^\sigma$ stands for the symplectic orthogonal complement of $T_\lambda \Lambda$ in $T_\lambda T^*M$ with respect to σ. For example, the conormal bundle N^*V of a submanifold $V \subset M$ is Lagrangian.

A real-valued $\varphi \in C^\infty(V)$ on a conic neighbourhood $V \subset M \times \mathbb{R}^k$ is called a non-degenerate phase function if it is positively homogeneous of degree 1 in ξ, does not have critical points, and the condition $d_\xi \phi(x_0, \xi_0) = 0$ implies that the differentials $d\frac{\partial \varphi}{\partial \xi_1}, \ldots, d\frac{\partial \varphi}{\partial \xi_k}$ are linearly independent at (x_0, ξ_0). A Lagrangian distribution of order r associated with a conic Lagrangian manifold $\Lambda \subset T^*M \setminus \{0\}$ is an element $u \in \mathcal{D}'(M)$ that can be expressed as a locally finite sum of oscillatory integrals

$$I(a, \varphi) := \int_{\mathbb{R}^k} e^{i\varphi(x,\xi)} a(x, \xi) d\xi, \quad a \in S^{r-k/2+n/4}(M \times \mathbb{R}^k), \tag{4}$$

where φ is a non-degenerate phase function such that the manifold Λ locally coincides with

$$\Lambda_\varphi := \{(x, d_x\varphi(x, \xi)) : d_\xi \varphi(x, \xi) = 0\}.$$

The class of such distributions is denoted by $I^r(M; \Lambda)$. An important subclass of $I^r(M; \Lambda)$ is $I^r_{cl}(M; \Lambda)$ which is defined by requiring symbols in the oscillatory integrals to be classical. In this dissertation it suffices to focus on such distributions only.

A distribution u conormal to a submanifold $W \subset M$ is by definition an element of $I^r(M; N^*W)$ for some $r \in \mathbb{R} \cup \{-\infty\}$. In local coordinates $(x', x'') \in \mathbb{R}^{n-\dim W} \times$

$\mathbb{R}^{\dim W}$ such that W coincides with the level set $\{x : x' = 0\}$ one derives the following local expression:

$$(5) \qquad u(x) = \int_{\mathbb{R}^{n-\dim W}} e^{ix'\cdot\xi'} a(x,\xi')d\xi', \quad a \in S^{r+\dim W/2-n/4}(M \times \mathbb{R}^{n-\dim W}).$$

For example, a delta-distribution

$$\delta(x') \simeq \int_{\mathbb{R}} e^{ix'\xi'} d\xi', \quad x = (x', x'') \in \mathbb{R} \times \mathbb{R}^{n-1}$$

is conormal to $W = \{0\} \times \mathbb{R}^{n-1}$ and hence a Lagrangian distribution of degree $r = 1/2 - n/4$ with $\Lambda = N^*W = \{(0, x'', \xi', 0) : x'' \in \mathbb{R}^{n-1},\ \xi' \in \mathbb{R} \setminus \{0\}\}$.

In some local coordinates $x^1, \dots, x^n$ of M a Lagrangian manifold Λ has a local representation of the form $\{(-d_\xi H(\tilde{\xi}), \tilde{\xi}) : \tilde{\xi} \in \mathbb{R}^n\}$ for some positively homogeneous real-valued H and hence locally $\Lambda = \Lambda_\varphi$ for the phase function $\varphi(\tilde{x}, \tilde{\xi}) = \tilde{x}^j\tilde{\xi}_j + H(\tilde{\xi})$. Applying the method of stationary phase yields

$$\widehat{I(a,\varphi)}(\tilde{\xi}) = e^{-iH(\tilde{\xi})} A(\tilde{\xi}),$$

where

$$A(\tilde{\xi}) \equiv (2\pi)^{-\frac{n+k}{2}} |\det \varphi''|^{-1/2} e^{i\frac{\pi}{4}\mathrm{sgn}\varphi''} a_\mu(x(\tilde{\xi}), \xi(\tilde{\xi})) \quad \mathrm{mod}\ S^{m-n/4-1}$$

and a_μ, $\mu = m - k/2 + n/4$ is the leading term in the asymptotic expansion $a \sim \sum_{j=0}^\infty a_{\mu-j}^\infty$. (See [GS94, §11]) For $d_\varphi = |\det \varphi''||d\tilde{\xi}_1 \wedge \cdots \wedge d\tilde{\xi}_n|$ one derives the following transition law:

$$(6) \qquad \begin{cases} I(a,\varphi) = I(\tilde{a}, \tilde{\varphi}) \\ \Lambda_\varphi = \Lambda_{\tilde{\varphi}} \end{cases} \quad \text{microlocally near} \quad (x_0, \xi_0)$$

$$(7) \qquad \Rightarrow \quad a_\mu \sqrt{d_\varphi} = e^{i\frac{\pi}{4}(\mathrm{sgn}\tilde{\varphi}'' - \mathrm{sgn}\varphi'')} \tilde{a}_\mu \sqrt{d_{\tilde{\varphi}}} \quad \text{at} \quad (x_0, \xi_0)$$

By the principal symbol of $u \in I^r(M;\Lambda)$ one usually refers to the highest order term in the asymptotic expansion $a \sim \sum_{j=0}^\infty a_{\mu-j}^\infty$ of a in the oscillatory integral representation (4). However, such a definition depends on the associated local representation in T^*M. To define the principal symbol of $u \in I^m_{cl}(M;\Lambda)$ invariantly one instead considers $\frac{1}{2}$-densities $a_\mu\sqrt{d_\varphi}$ on Λ that obey the transition law (7). This leads to the global definition of $\sigma(u)$ as a section in the Keller-Maslov line bundle $\Omega_{\frac{1}{2}}(\Lambda) \otimes \mathcal{L}$ over Λ. The associated map from $I^m_{cl}(M;\Lambda)/I^{m-1}_{cl}(M;\Lambda)$ to the smooth homogeneous of degree $m + n/4$ sections of the bundle $\Omega_{\frac{1}{2}}(\Lambda) \otimes \mathcal{L}$ is bijective.

In close relation to the standard Lagrangian distributions there is also the space

$$I^{r_1,r_2}(M; \Lambda_0, \Lambda_1) \subset \mathcal{D}'(M)$$

of distributions associated with a cleanly intersecting pair (Λ_0, Λ_1) of conic Lagrangian manifolds $\Lambda_0, \Lambda_1 \subset T^*M \setminus \{0\}$. The elements of $I^{r_1,r_2}(M;\Lambda_0,\Lambda_1)$ are defined as locally finite sums of distributions in $I^{r_1+r_2}(M;\Lambda_0) + I^{r_2}(M;\Lambda_1)$ and oscillatory integrals

$$\int_{\mathbb{R}^{k_1+k_2}} e^{\varphi(x,\xi,\eta)} a(x,\xi,\eta) d\xi d\eta,$$
$$a \in S^{r_1+r_2-\frac{k_1+k_2}{2}+\frac{n}{4},\ -r_2+\frac{k_2}{2}}(M \times (\mathbb{R}^{k_1} \setminus \{0\}) \times \mathbb{R}^{k_2}),$$

where $\varphi(x,\xi,\eta)$ is a multiphase near a point in $\Lambda_0 \cap \Lambda_1$ such that the functions $\varphi(x,\xi,\eta)$ and $\varphi(x,\xi,0)$ parametrise the manifolds Λ_0 and Λ_1, respectively. Microlocally away from $\Lambda_0 \cap \Lambda_1$, each element of $I^{r_1,r_2}(M;\Lambda_0,\Lambda_1)$ belongs to $I^{r_1+r_2}(M;\Lambda_0) + I^{r_2}(M;\Lambda_1)$.

2.4. **Fourier Integral Operators.** Let M and N be smooth manifolds. A submanifold $\tilde{\Lambda} \subset T^*M \times T^*N$ is called a canonical relation if the manifold

$$\tilde{\Lambda}' := \{(x,\xi;y,\eta) : (x,\xi;y,-\eta) \in \tilde{\Lambda}\}$$

is Lagrangian with respect to the canonical form $\sigma_M + \sigma_N$, or equivalently, if $\tilde{\Lambda}$ is Lagrangian with respect to the twisted symplectic form $\sigma_M - \sigma_N$. A Fourier integral operator (FIO) of order m associated with the conic canonical relation $\tilde{\Lambda}$ is defined as an operator $F : C^\infty(N) \to \mathcal{D}'(M)$ with a Schwartz kernel k_F that is an element of $I^m(M \times N; \tilde{\Lambda}')$. For a closed cone $\Gamma \subset T^*Y \setminus \{0\}$ that does not meet the projection $WF'_N(F) := \pi_N^* WF(k_F)$, $\pi_N(x,y) := y$, the domain extends to $\mathcal{E}'_\Gamma(N)$ and further to $\mathcal{D}'_\Gamma(N)$ if the operator is properly supported. An operator with $M = N$ and $\tilde{\Lambda} = \operatorname{diag}(T^*M \setminus \{0\})$ is called a pseudo-differential operator. Another important class of Fourier integral operators is the one associated with homogeneous canonical transformations between conic neighbourhoods.

Analogously to the standard FIOs there is also the class of operators corresponding to a cleanly intersecting pair (Λ_1, Λ_2) of canonical relations, defined by assuming that the Schwartz kernels are Lagrangian distributions associated with the pair (Λ_1', Λ_2'). Such operators were first studied in [MU79], [GU81] by Melrose, Uhlmann and Guillemin in order to establish calculus for parametrices of pseudo-differential operators. In fact, it was shown by the authors that a parametrix of an operator of real principal type is associated with the pair (Δ, Λ_P) where $\Delta = \operatorname{diag}(T^*M \setminus \{0\})$ and Λ_P is a characteristic flow-out canonical relation, that is, the collection of pairs of characteristic covectors in a same bicharacteristic.

For example, the forwards propagating parametrix for $P := \frac{\partial}{\partial x^1}$ is

$$E(x;y) = H(x^1 - y^1)\delta(x^2 - y^2)\ldots\delta(x^n - y^n).$$

One checks that the bicharacteristic through a characteristic covector $(x^1, x''; 0, \xi'')$ is the line $\mathbb{R} \times \{(x'', 0, \xi'')\}$. Hence,

$$\Lambda_P = \{(x^1, x''; 0, \xi'' \mid \; y^1, x''; 0, \xi'') : x'' \in \mathbb{R}^{n-1},\; \xi'' \in \mathbb{R}^{n-1} \setminus \{0\},\; y^1, x^1 \in \mathbb{R}\}.$$

2.5. **Operator Calculus.** Composition calculus for the standard Fourier integral operators was developed in [H$\ddot{7}$1] and [DH72] by Hörmander and Duistermaat. The main requirements for the composition of two operators with canonical relations $\Lambda_1 \subset (T^*M \times T^*N) \setminus \{0,0\}$, $\Lambda_2 \subset (T^*N \times T^*Z) \setminus \{0,0\}$ to be valid in the framework are that the projection

$$(T^*M \times \operatorname{diag}(T^*N) \times T^*Z) \cap (\Lambda_1 \times \Lambda_2) \to T^*M \times T^*Z$$

is injective proper mapping and the Cartesian product $\Lambda_1 \times \Lambda_2$ meets the manifold $T^*M \times \operatorname{diag}(T^*N) \times T^*Z$ transversally. Hence, the theory is often referred to as transversal intersection calculus.

Provided that the conditions are satisfied, composing two Fourier integral operators F_1 and F_2 with canonical relations Λ_1 and Λ_2 yields a new Fourier integral operator $F_1 \circ F_2$ which admits as a canonical relation the manifold

$$\Lambda_1 \circ \Lambda_2 := \{(x,\xi;z,\zeta) : \exists (y,\eta);\; (x,\xi;y,\eta) \in \Lambda_1,\; (y,\eta;z,\zeta) \in \Lambda_2\}.$$

Moreover, the principal symbols of the operators are connected via the formula

$$\sigma(F_1 \circ F_2)(x,\xi;z,\zeta) = \sum \sigma(F_1)(x,\xi;y,\eta)\sigma(F_2)(y,\eta;z,\zeta),$$

where the sum is taken over those elements (y,η) in T^*N that satisfy both $(x,\xi,y,\eta) \in \Lambda_1$ and $(y,\eta,z,\zeta) \in \Lambda_2$. By viewing a Lagrangian distribution as a Fourier integral operator with trivial domain one obtains calculus also for Fourier integral operators acting on Lagrangian distributions. As pointed out earlier, the identities above were generalised in [MU79], [GU81] for operators with two cleanly intersecting canonical relations.

2.6. **Propagation of Singularities.** The composition identities above make it possible to compute singularities of solutions for partial differential equations. Provided that P is a pseudo-differential operator of real principal type and of order r, along with a Lagrangian manifold $\Lambda \subset T^*M$ that meets the characteristic set of P transversally and hits each bicharacteristic of P at most a finite number of times, a solution u of the equation

$$Pu = f, \quad f \in I^m_{comp}(M;\Lambda)$$

can be written (see [GU93, Proposition 2.1]) in the form

$$u = Ef + u_0 \in I^{m-r+1/2}(M;\Lambda,\Lambda_P \circ \Lambda), \tag{8}$$

where u_0 stands for a smooth residual and E is a parametrix of P which is associated with the pair Δ, Λ_P ($\Delta := \mathrm{diag} T^*M \setminus \{0\}$) of canonical relations. Moreover, we have

$$\sigma(u)(x,\xi) = \sum_{(x,\xi;y,\eta)\in\Lambda_P} \sigma(E)(x,\xi;y,\eta)\sigma(f)(y,\eta)$$

for $(x,\xi) \in (\Lambda_P \circ \Lambda) \setminus \Lambda$. Consequently, the principal symbol of u on $\Lambda_P \circ \Lambda$, and hence on $WF(u) \setminus \Lambda$, is completely described by the symbols of E and f. In fact, the principal symbol of u along $\Lambda_P \circ \Lambda$ is obtained (see [MU79, Proposition 5.4] and [DH72, Theorem 6.1.1]) by solving the first transport equation

$$(-i\mathscr{L}_{H_p} + c)\sigma(u)|_{\Lambda_P\circ\Lambda} = \sigma(f)|_{\Lambda_P\circ\Lambda}, \tag{9}$$

thus implying invariance of the wave front set $WF(u)$ along bicharacteristics (i.e. characteristic integral curves of H_p) in between intersections with the manifold Λ. Above, p stands for the principal symbol of P. If $(x,\xi) \in WF(u) \setminus \Lambda$ and $r \mapsto \Xi(r)$ is a bicharacteristic of P such that $\Xi(r_0) = (x,\xi)$, then

$$\Xi(r) \in WF(u)$$

holds for every r within any interval I that contains r_0 and satisfies $\Xi I \cap \Lambda = \emptyset$. In conclusion, the source f generates or annihilates singularities which elsewhere propagate according to the characteristic flow. This phenomenon is very useful as it can be used for transporting signals within the solution u.

For an elliptic operator P no propagation from $WF(f)$ takes place in the wave front set of a distribution u that solves $Pu = f$. That is; the singularities are static. In fact,

$$WF(u) = WF(f).$$

However, some elliptic systems are closely related to non-elliptic ones. For instance, a CGO solution $u(x,\tau) = e^{i\tau\eta}v(x,\tau)$ for the conductivity equation admits propagation within the Fourier transform $\widehat{v}(x,t)$. This was used by Greenleaf, Lassas, Santacesaria, Siltanen and Uhlmann in [GLS+18] for developing a new method for stroke imaging.

3. Applications in Inverse Problems

3.1. **Potential Scattering.** Potential scattering in Euclidean geometry is described by

$$(\partial_t^2 - \Delta - V(x))u(x,t) = 0, \quad (x,t) \in \mathbb{R}^{n+1} \tag{10}$$

$$u(x,t) = \delta(t - x\cdot\theta), \quad \text{for} \quad t \ll 0, \tag{11}$$

or, alternatively, by the frequency domain equivalent

$$(\omega^2 + \Delta + V(x))\hat{u}(x,\omega) = 0, \tag{12}$$

$$\hat{u}_{in}(x,\omega) = e^{-ix\cdot\theta\omega}, \tag{13}$$

equipped with the Sommerfeld radiation condition. Here Δ refers to the Euclidean Laplacian. A standard approach in the setting (10-11) is to write the scattered wave $u_{sc}(x,t) := u(x,t) - \delta(t - x\cdot\theta)$ in the form

$$u_{sc} = \sum_{j=0}^{N} u_j + r_N,$$

where the terms get more regular as N grows. For smooth V the method works for each N and one checks by substitution that the equations above are satisfied for

$$u_0(x,t) = EV(x)\delta(t - x\cdot\theta), \quad u_k(x,t) = EVu_{k-1}(x,t), \quad k = 1,2,3,\dots, \tag{14}$$

where $E \in I^{-3/2}(\mathbb{R}^n; \mathrm{diag}(T^*\mathbb{R}^n \setminus \{0\}), \Lambda_\Box)$ with

$$\begin{aligned}\Lambda_\Box := \{&(x+\sigma s, t+s; -\omega\sigma, \omega \mid x, t; -\omega\sigma, \omega) : \\ &\sigma \in \mathbb{S}^{n-1},\ \omega \in \mathbb{R}\setminus\{0\},\ s \in \mathbb{R},\ (x,t) \in \mathbb{R}^{n+1}\}\end{aligned}$$

is a future-propagating parametrix of the wave operator. Perhaps the most common approach is to analyse the first approximation

$$u_{sc}(x,t) \approx EV(x)\delta(t - x\cdot\theta),$$

which is valid also for less regular potentials. In view of (8),

$$WF(EV\delta(t - x\cdot\theta)) \subset WF(V\delta(t - x\cdot\theta)) \cup \Lambda_\Box \circ WF(V\delta(t - x\cdot\theta)).$$

Further, one checks that the wave front set $WF(V\delta(t-x\cdot\theta))$ lies in $(\mathrm{supp}(V)\times\mathbb{R}\times\mathbb{R}^{n+1})$ and meets the characteristic manifold $\{(x,t,\xi,\omega) \in T^*\mathbb{R}^{n+1}\setminus\{0\} : \omega^2 = |\xi|^2\}$ only in $Q := \{(x, x\cdot\theta; \pm\omega\theta, \omega) : x \in \mathrm{supp}(V),\ \omega \in \mathbb{R}\setminus\{0\}\}$ (see [Dui96, Theorem 1.3.6]). Thus,

$$\begin{aligned}WF(EV\delta(t - x\cdot\theta)) \setminus (\mathrm{supp}(V)\times\mathbb{R}\times\mathbb{R}^{n+1}) &\subset \Lambda_\Box \circ Q \\ &= \{(x \pm s\theta, x\cdot\theta + s; \mp\theta\omega, \omega) : x \in \mathrm{supp}(V),\ s \in \mathbb{R}\}.\end{aligned}$$

The two signs correspond to peak scattering and backscattering. The singularities in peak scattering move forwards along with the initial wave in direction θ, whereas backscattering corresponds to the reflected singularities (echoes) that propagate in the opposite direction. It follows from (9) that the amplitude in peak scattering is essentially the X-ray transform of V along the trajectories. In the frequency domain the propagating singularities manifest as slower decay, and hence non-trivial far-field asymptotes in these directions.

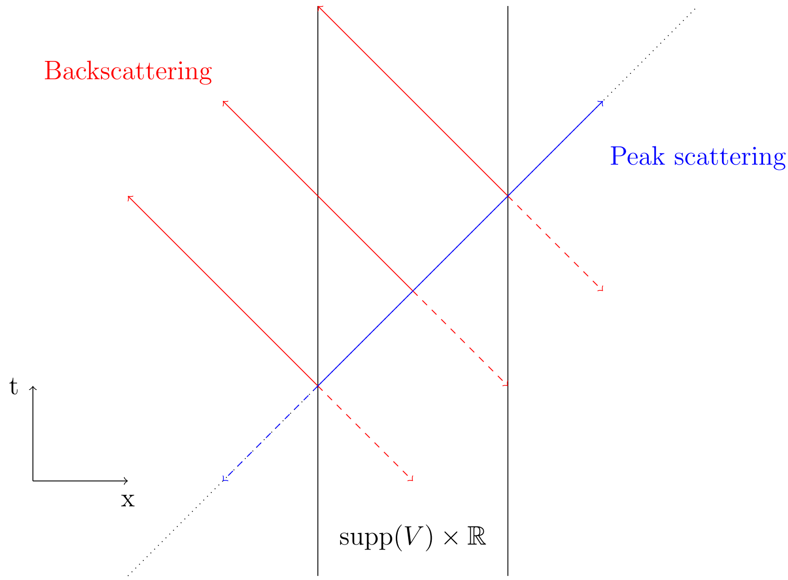

FIGURE 1. An schematic illustration of potential scattering in $\mathbb{R}^{1+1}$. The peak scattering and backscattering are illustrated in blue and red, respectively. The dashed arrows correspond to the backwards propagating bicharacteristic curves which lie in $\Lambda_\Box \circ Q$ but carry no singularities due to the initial condition $u = \delta(t - x \cdot \theta)$, $t \ll 0$. The dotted line in black stands for $\{(x, x \cdot \theta) : x \in \mathbb{R}^n\}$, that is, the (singular) support of the initial wave.

Inverse scattering, or more precisely, inverse potential scattering refers to the theory of recovering the potential or some features of it from scattering effects at a distance. Data are described using the scattering kernel, given in $n = 3$ by

$$\alpha_V(s, \theta, \omega) := \frac{1}{2\pi} \int_{x \cdot \theta = \tau} \partial_t u(x, \tau - s, \omega) dS_x, \quad (s, \theta, \omega) \in \mathbb{R} \times \mathbb{S}^2 \times \mathbb{S}^2 \tag{15}$$

(see e.g. [RU14], [Uhl01]) where $\tau > 0$ is large. The backscattering data is defined as $\alpha_V(s, \theta, -\theta)$. Alternative definitions exist. One may, for example, use the scattering amplitude

$$a_V(\lambda, \theta, \omega) = \int_{\mathbb{R}} e^{-is\lambda} \alpha_V(s, \theta, \omega) ds.$$

The recovery of singularities (e.g. discontinuities, peaks) in particular is very effective and a well studied topic in the field. See the works by Reyes and Ruiz [RR12], Barceló, Faraco, Ruiz and Vargas [BFRV10], Ruiz and Vargas [RV05], Ola, Päivärinta and Serov [OPS01], Päivärinta, Serov and Somersalo [PSS94], Greenleaf and Uhlmann [GU93], Päivärinta and Somersalo [PS91], Joshi [Jos98], Päivärinta and Serov [PS98]. Quite surprisingly, it remains unknown whether a smooth potential is uniquely determined by the backscattering data. For angularly controlled potentials the question was answered by Rakesh and Uhlmann [RU14]. The inverse backscattering problem has also been studied for other models. See, for example, the works on the acoustic equation by Stefanov and Uhlmann [SU97] and Wang [Wan98b], [Wan98c] and on isotropic Maxwell's equations by Wang [Wan98a].

In Article I peak scattering is considered for random potentials. For related studies we refer to the works by Lassas, Päivärinta and Saksmann [LPS08], Helin, Lassas and Päivärinta [HLP17], and Li, Liu and Ma [LLM21], [LLM19]

3.2. **Radon Transforms.** The theory of Fourier integral operators plays an important role in various reconstruction techniques. One approach is to apply the transport phenomenon (9), or even the lower order asymptotics, to obtain an integral transform of the source along the characteristic flow. This connects the theory to generalised Radon transforms of the form

$$\mathcal{R}(u)(x) = \int_{L_x} u(y)f(x,y)dy, \quad x \in X, \quad y \in Y,$$

where X and Y are topological spaces, the integral is considered in the sense of distributions and L_x is a submanifold of Y. A special yet very important case is when each L_x is a curve or even a geodesic. Such an operator is called a generalised X-ray transform as it extends the concept of the standard euclidean X-ray,

$$\mathcal{X}(u)(x,\theta) = \int_{l_{x,\theta}} u(y)dy. \tag{16}$$

Here $l_{x,\theta}$ stands for the line $x + \mathbb{R}\theta$ through $x \in \theta^{\perp}$.

Properties of Radon transforms, especially under the Bolker assumption (See [Qui94, Definition 2.1]) are well studied. The main topics are injectivity, inversion formulas and determination of range, kernel and support in specific function spaces. For an introduction, see [Qui06a]. Range, support and kernel theorems have been developed by various authors including Helgason [Hel81], [Hel11], Quinto [Qui82], [Qui08], [Qui06b], Richter [Ric86], [Ric90], Gonzalez [Gon90], Kurusa [Kur91], Kakehi [Kak92], Boman [Bom92], [Bom21], Boman and Quinto [BQ93], Zhou and Quinto [ZQ00], and Estrada and Rubin [ER16]. On inversion formulas and injectivity we emphasize works by Helgason [Hel65], [Hel07] Boman [Bom90] , [Bom12], Rouvière [Rou01], [Rou06], Grindberg and Rubin [GR04], along with studies on X-ray transforms by Greenleaf and Uhlmann [GU89], Gonzalez [Gon87], Paternain, Salo, Uhlmann and Zhou [PSUZ19], Au Yeung, Chung and Uhlmann [AYCU19], Stefanov, Uhlmann and Vasy [SUV18], Palacios, Uhlmann and Wang [PUW18], Pestov and Uhlmann [PU04]. For a study on light ray transform, see the study by Lassas, Oksanen, Stefanov and Uhlmann in [LOSU20]. See also the stability estimates by Caro, Dos Santos Ferreira and Ruiz [CDSFR14].

A generalised Radon transform $\mathcal{R}$ is usually analysed by studying microlocally the operator $\mathcal{R}^t\mathcal{R}$, where $\mathcal{R}^t$ is the transpose of $\mathcal{R}$. A standard question is whether $\mathcal{R}^t\mathcal{R}$ is an elliptic operator and hence stably invertible modulo a smoothing operator. In some cases the inverse operator is known. For the standard Radon transform

$$\mathcal{R}(\phi)(s,\theta) = \int_{\Gamma_{s,\theta}} \phi(x)dx, \quad (s,\theta) \in \mathbb{R} \times \mathbb{S}^{n-1}, \quad \Gamma_{s,\theta} := \{x \in \mathbb{R}^n : x \cdot \theta = s\},$$

that is,

$$\mathcal{R}(s,\theta;x) = \delta(s - x\cdot\theta), \quad (s,\theta,x) \in \mathbb{R} \times \mathbb{S}^{n-1} \times \mathbb{R}^n,$$

we have the Radon inversion formula

$$c_n \Delta^{\frac{n-1}{2}} \mathcal{R}^t \mathcal{R} = I.$$

This is essentially what CT devices are based on. In fact, the standard Radon transform can be constructed by integrating (16). After taking several X-ray images in various angles the formula (or a discretisation of it) can be applied to compute a cross-section of the target. In Article I the Radon inversion formula is applied to recover autocorrelations of a random potential. The Radon transform is used also in scattering theory. See [Uhl01] and references therein in the framework of Lax-Phillips.

3.3. **Travel Time Reconstructions and Non-linear Inverse Problems.** Let us consider the setup of Section 2.6 for a real principal type operator P with a characteristic flowout Λ_P and a Lagrangian source f associated with a manifold Λ. In addition to the transport phenomenon for symbols, studying the manifold $\Lambda_P \circ \Lambda$ itself provides information about the underlying system, especially in the form of travel times of characteristic curves. This is often more natural approach to take if instead of the source one is interested in determining the geometry that governs the characteristic flowout. A common approach in geometric inverse problems is to vary Λ in the composition $\Lambda_P \circ \Lambda$ by controlling the singularities of the source f and study on a submanifold (or a boundary) N the microlocal relation

$$f \mapsto WF(u|_N) \;\subset\; (\Lambda_P \circ \Lambda)|_{TN},$$

or even the local counterpart

$$f \mapsto \operatorname{singsupp}(u|_N) \;\subset\; \pi_M\big[(\Lambda_P \circ \Lambda)|_{TN}\big], \tag{17}$$

where $\pi_M : T^*M \to M$ stands for the canonical projection $\pi_M(x,\xi) := x$ and

$$(\Lambda_P \circ \Lambda)|_{TN} := \{(x, \xi|_{TN}) : x \in N,\ (x,\xi) \in \Lambda_P \circ \Lambda\} \subset T^*N.$$

For many physically relevant P such as the Lorentzian Laplace-Beltrami operator the associated bicharacteristic curves follow geodesics in the base manifold, thus providing information on the Riemannian or Lorentzian distances in form of travel times. In static geometries knowing the distances between the boundary and interior points is shown (See [KKL01] and references therein) to determine the metric. In fact, the method is applied in Article III. Analogously, in a globally hyperbolic Lorentzian setting one reconstructs the metric from relative travel times of null-geodesics which can be derived from earliest light observations. This was shown by Kurylev, Lassas, and Uhlmann:

Theorem 3.1. [KLU18, Theorem 1.2] *Let (M_j, g_j), $j = 1,2$ be two open, C^∞-smooth, globally hyperbolic Lorentzian manifolds of dimension $n \geq 0$, $\hat\mu : [-1,1] \to M_j$ be smooth time-like paths, and $p_j^\pm = \hat\mu_j(s_\pm)$. Let the observation sets $V_j \subset M_j$ be neighbourhoods of $\hat\mu_j[-1,1]$ and $W_j \subset M_j$ be relatively compact sets such that $W_j \subset J^-(p_j^+) \setminus I^-(p_j^-)$. Let $\mathcal{E}_{V_j}(W_j)$ be the families of the earliest light observations sets with source points at W_j. Assume that there is a conformal diffeomorphism $\Phi : V_1 \to V_2$ such that $\Phi(\hat\mu_1(s)) = \hat\mu_2(s)$, $\forall s \in [-1,1]$ and*

$$\widetilde{\Phi}(\mathcal{E}_{V_1}(W_1)) = \widetilde{\Phi}(\mathcal{E}_{V_2}(W_2)),$$

where $\widetilde{\Phi}$ is the power set extension of Φ. Then there is a diffeomorphism $\Psi : W_1 \to W_2$ such that the metric $\Psi^* g_2$ is conformal to g_1 and $\Psi|_{W_1 \cap V_1} = \Phi|_{W_1 \cap V_1}$.

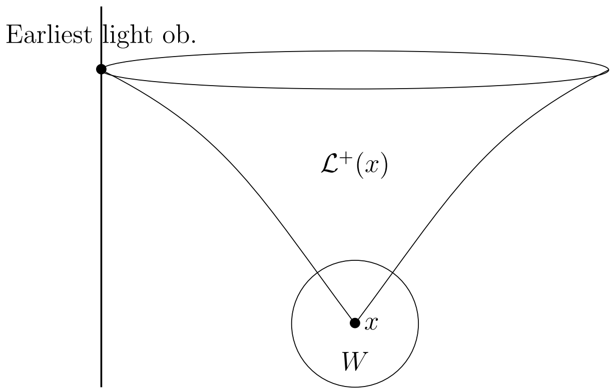

FIGURE 2. Schematic illustration of the earliest observation of light from $x \in W$ on a time-like curve.

In the reference this is referred to as the inverse problem for passive measurements. The result plays an important role in Article II. The data $\mathcal{E}_V(W)$ (j omitted) above consist of detecting the first (w.r.t. the intrinsic causality) intersections between the neighbourhood V and the flowouts $\mathcal{L}_x^+ = \exp_x(L_x^+ M)$ of cones

$$L_x^+ M := \{v \in T_x M \setminus \{0\} : g(v,v) = 0, \quad v \text{ future-pointing } \},$$

with a variable apex x in the region $W \subset M$ that is endowed with "unknown" topology and geometry. This is essentially the same as (17) for the Lorentzian Laplace-Beltrami operator

$$P = \Box_g, \quad \Box_g \phi(x) := \frac{1}{\sqrt{|\det(g(x))|}} \partial_j(\sqrt{|\det(g(x))|} g^{jk}(x) \partial_k \phi(x)),$$

$N = V$, point sources $f \in I^m(M; \Lambda)$, $\Lambda = T_x^* M$, $x \in W$ with $WF(f) = T_x^* M$, and associated family of solutions u with smooth or zero initial data outside the causal future of x. The main result for passive measurements says that the topology and conformal type of the metric in the target is identical up to a diffeomorphism for a pair of manifolds with identical data. A subsequent result for broken ray geometry was derived by Hintz and Uhlmann in [HU19, Theorem 1.2].

In addition to the passive measurement setup, Kurylev, Lassas, and Uhlmann studied in [KLU18] the so called active measurements

$$f \mapsto \operatorname{singsupp}(u|_V)$$

where u solves the nonlinear equation

$$\Box_g u(x) + a(x) u(x)^2 = f(x) \tag{18}$$

together with zero initial data outside the causal future of $\operatorname{supp}(f)$. They developed a method that applies nonlinearity of the equation to generate microlocally away from certain manifolds the wave equations $\Box_g u_\alpha \equiv \delta_x^{(r)}$ with controllable point sources $\delta_x^{(r)}$ for some of the coefficients u_α in the multivariate perturbation series $u(x) \approx u_\alpha(x)\varepsilon^\alpha$ associated with sources of the form $f = \varepsilon^j f_j$. Thus, it follows from the result for passive measurements that the space W and the conformal class of $g|_W$ on it are determined up to a diffeomorphism by the data of active measurements. The statement of the result is as follows:

Theorem 3.2. [KLU18, Theorem 1.5] *Let $(M^{(j)}, g^{(j)})$, $j = 1, 2$ be two smooth, globally hyperbolic Lorentzian manifolds of dimension $1+3$ that are represented in the form $M^{(j)} = \mathbb{R} \times N^{(j)}$ with a metric of the form $g^{(j)}(t, y) = -\beta^{(j)}(t, y)dt^2 + \kappa^{(j)}(t, y)$, where $\kappa^{(j)}$ is a Riemannian metric on $N^{(j)}$.*[2] *Let $\hat{\mu} : [-1, 1] \to (-\infty, T_0) \times N^{(j)}$ be smooth time-like paths, $p_j^+ = \hat{\mu}(s_+)$, $p_j^- = \hat{\mu}(s_-)$, where $-1 < s_- < s_+ < 1$, and $V_j \subset M^{(j)}$ are neighbourhoods of $\hat{\mu}[-1, 1]$.*

Let L_{V_j} be source-to-solution maps for the wave equation (18) *on manifolds $(M^{(j)}, g^{(j)})$ with nowhere vanishing smooth functions $a_j : M^{(j)} \to \mathbb{R}$, $j = 1, 2$. Assume that there is a diffeomorphism $\Phi : V_1 \to V_2$ such that $\Phi(p_1^-) = p_2^-$ and $\Phi(p_1^+) = p_2^+$ and the source-to-solution maps satisfy*

$$((\Phi^{-1})^* \circ L_{V_1} \circ \Phi^*)f = L_{V_2} f$$

for $f \in \mathcal{W}$, where $\mathcal{W}$ is a neighbourhood of the zero function in $C_c^6(V_2)$.

Then there is a diffeomorphism $\Psi : I(p_1^-, p_1^+) \to I(p_2^-, p_2^+)$ and the metric $\Psi^* g^{(2)}$ is conformal to $g^{(1)}$ in $I(p_1^-, p_1^+)$. Moreover, $(\Psi^* g^{(2)})|_{V_1} = g^{(1)}\ |_{V_1}$.

The techniques used for active measurements are based on the microlocal Fourier integral operator calculus. An important observation is that the product $v_1 v_2$ of distributions $v_j \in I^m(N^* H_j)$, $j = 1, 2$ conormal to transversally intersecting manifolds H_j has a wave front set not only in $N^* H_j$, $j = 1, 2$ but also in $N^*(H_1 \cap H_2)$, given that the singular supports meet. By adding more elements to the product one constructs a distribution that microlocally away from some Lagrangian submanifolds equals a point-source. The non-linear term $a(x)u(x)^2$ in the equation above generates such products into the approximative equations (higher order linearisations) by substituting the perturbation series as an ansatz in (18). In fact, one derives for any equation of the form $Pu + Q(u, u) = \varepsilon^j f_j$ with quadratic $Q(u, u)$ the following recursive identities

$$u = \sum_{m=1}^{N} \sum_{\alpha \in \{1, \ldots, k\}^m} u_\alpha\ \varepsilon^\alpha \tag{19}$$

$$Pu_j = f_j, \tag{20}$$

$$Pu_{j,k} = Q(u_j, u_k) \tag{21}$$

$$\ldots \tag{22}$$

$$Pu_\mu = \sum_{(\nu, \rho) = \mu} Q(u_\nu, u_\rho). \tag{23}$$

Contrary to the linear case, the quadratic interaction terms on the right provide a way to generate additional controllable sources. The construction above is algebraic, whereas the actual analysis requires additional assumptions on regularity.

In addition to the works mentioned above, the method has recently been applied to various non-linear models. This includes Einstein scalar field equations studied by Kurylev, Lassas, Oksanen and Uhlmann [KLOU18], Semilinear Lorentzian wave equations by Lassas, Uhlmann and Wang [LUW18], and quadratic derivative nonlinear wave equations by Wang and Zhou [WZ19]. See [UZ21] for more details on recent progress on the topic. In Article II a similar scheme is adopted to study the relativistic Boltzmann equation with a source. In that setting the non-linearity

[2] The existence of such representations follow from the global hyperbolicity.

arises from a quadratic integral operator, whereas the linear part P corresponds to a non-vanishing vector field. The model has points of resemblance to transport equations such as the ones studied by McDowall [LM08], [McD04], Langmore and McDowall [McD05], Mcdowall, Stefanov and Tamasan [MST10], and Choulli & Stefanov [CS99].

4. Overview of the Articles

4.1. **Article I.** Let $\Box$ stand for the Euclidean wave operator $\Box := \partial_t^2 - \Delta$ on $\mathbb{R}^n$, where $\Delta = \sum_{j=1}^n \partial_j^2$, the Laplacian in $\mathbb{R}^n$. The term "generalised function" (on X) is adopted for elements of $\mathcal{D}'(X)$ to avoid confusion with probability distributions. We study generalised functions u of the form

$$(\Box - V(x))u(x,t,\Theta) = 0 \tag{24}$$

$$u(x,t,\Theta) = \sum_{j=1}^{k} \delta(t - x\cdot\theta_j) + u_{sc}(x,t,\Theta), \tag{25}$$

$$u_{sc}(x,t,\Theta) = 0, \quad \text{for } t \ll 0, \tag{26}$$

where $\Theta = (\theta_1,\dots,\theta_k) \in (\mathbb{S}^{n-1})^k$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $V(x) = V(x,\omega)$ is a random potential in $H^2(\mathbb{R}^n)$, that is, a measurable function $\omega \mapsto V(\cdot,\omega)$ between a probability space Ω and the Sobolev space $H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ endowed with the Borel σ-algebra. Such solutions describe interaction between k plane waves and a single realisation of the random potential $V(x)$. As a simple example of $V(x)$, consider

$$V(x) = \int_{\mathbb{R}^n} k(x,y)W(y)dy,$$

where W is white noise and $k(x,y)$ is a convenient Schwartz kernel. The model is compatible with a randomly changing potential that varies relatively slowly compared to the speed of waves (e.g. the speed of light). The scattered wave u_{sc} on the opposite side of the target is known as forward scattering or peak scattering and the amplitude of the main singularity in it is essentially the X-ray transform of the potential. This correspondence, typically deduced for smooth $V(x)$ and a single incident wave, can be extended to the setting above as shown by us in the article. Thus, if one places a detector on the opposite side of the target for each incident wave then a collection of k X-ray images of the potential in that particular state is captured as a single measurement. We are interested in studying correlations in a large dataset of such measurements in several angles and random states of the potential. That is; we let the potential evolve randomly while taking samples in various angles. Under some a priori assumptions on ergodicity the data collected in this way give us the correlations

$$(x_1,\dots,x_k,\theta_1,\dots,\theta_k) \mapsto \mathbb{E}\left(\int_{x_1+\mathbb{R}\theta_1} V(x)dx \cdots \int_{x_k+\mathbb{R}\theta_k} V(x)dx\right)$$

between X-ray transforms. In Article I we first show that these correlations can be used to reconstruct the autocorrelations

$$M^k(x_1,\dots,x_k) = \mathbb{E}(V(x_1)\cdots V(x_k)) \in \mathcal{D}'(\mathbb{R}^{kn}),$$

given that the potential satisfies

$$V \in H^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{almost surely}$$

and there is a compact $K \subset \mathbb{R}^n$ such that $\operatorname{supp}(V) \subset K$ almost surely. The proof is based on the Radon inversion formula. The approach is constructive in the sense that it gives an actual algorithm for computing the moments. The correlations are then applied to show that the data uniquely determine the law $V_*\mathbb{P}$ if an additional condition on exponential boundedness is satisfied.

The main point in the article is to develop a method of applying peak scattering (X-ray) to analyse statistical behaviour of random potentials. The method is novel and extends the concept of tomography to probability distributions. As explained above, the technique can be used for recovering the distribution completely. The study is a starting point for probabilistic analysis of similar systems with more general FIO transforms such as generalised Radon transforms.

4.2. **Article II.** The relativistic Boltzmann equation with a source,

$$\mathcal{X}u(x,p) - \mathcal{Q}[u,u](x,p) = S(x,p), \tag{27}$$

is studied on the bundle $\overline{\mathcal{P}}M$ of causal tangent vectors (x,p) of a globally hyperbolic Lorentzian manifold (M,g). The operator $\mathcal{X}$ above stands for the geodesic vector field, given by

$$\mathcal{X}u(x,p) := \partial_t u(\gamma_{x,p}(t), \dot{\gamma}_{x,p}(t))\big|_{t=0}$$

where $\gamma_{x,p}$ refers to the geodesic that satisfies $(\dot{\gamma}_{x,p}(0), \dot{\gamma}_{x,p}(0)) = (x,p)$ for $(x,p) \in TM$. In the Minkowsky space-time one derives $\mathcal{X}u(x,p) = \partial_t u(x+tp,p)\big|_{t=0}$. The operator $\mathcal{Q}$ is a quadratic integral operator which takes the form

$$\mathcal{Q}[v,w](x,p) := \int_{\Sigma_{x,p}} v(x,p)w(x,q) - v(x,q')w(x,p')A(x,p,q,p',q')dV, \tag{28}$$

$$\Sigma_{x,p} := \{q,p',q' \in \overline{\mathcal{P}}_x M^4 \setminus \{0\} : p+q = p'+q'\} \tag{29}$$

where A is a convenient scattering kernel. Notice that the condition defining $\Sigma_{x,p}$ corresponds to the conservation of 4-momentum (i.e. energy-momentum). The solution u represents the density of a great number of particles moving along geodesics and interacting via elastic collisions. The particle interactions are described by $\mathcal{Q}$ which splits into two components (gain and loss) according to the two terms within the integral. The term S can be interpreted as an external source of energy that emits particles into the system. Although the model above arises from astronomy, there are also inhomogeneous, possibly time-dependent mediums that obey the Lorentzian geometry in smaller scales. In such a framework gravitation is substituted by properties of matter (cf. bending of light in inhomogeneous medium) and it is reasonable to predict that such geometric models are essential for the kinetic theory of inhomogeneous solids.

In the article we show that the source-to-solution map

$$S \mapsto u|_{L^+V}$$

which takes a regular source S of time-like particles into the corresponding solution of (27) on the future-pointing light-like bundle L^+V of a known open set $V \subset M$ is well defined and uniquely determines the Lorentzian metric g in the region $I^+(z_1) \cap I^-(z_2)$ between two chronologically related points $z_1, z_2 \in V$. The result shows that it is possible to reconstruct the geometry (including the conformal factor) of a Lorentzian space-time from a distance by sending particles with mass into the region and observing light (photons) that scatters in the collisions. In solids this could also refer to using quasiparticles as an input.

4.3. **Article III.** The third article studies a specific radiation phenomenon arising from electromagnetism in medium. We mention the works by Ola, Päivärinta and Somersalo [OPS93], Kurylev, Lassas and Somersalo [KLS06], McDowall [McD00], Joshi and McDowall [JM00], Liu, Rondi and Xiao [LRX19] and references therein as some works related to the topic. See the article for more references.

It is well known that a particle in material can move faster than electromagnetic waves. For a charge carrier, such as an electron, this leads to emission of light known as Cherenkov radiation or Vavilov-Cherenkov radiation. In experimental physics the phenomenon is applied in detection of charged particles by observing scattering in homogeneous dielectric material such as water. The mathematical model in that setting is simple from a geometric point of view and mainly reduces to solving the euclidean wave equation

$$(\partial_t^2 - k^2\Delta)u = f,$$

(or the associated Helmholtz equation) where $k > 0$ is a constant and the source f is conormal to the world line $\{(z + tv, t) : t \in \mathbb{R}\}$, $z, v \in \mathbb{R}^3$ of the charge carrier. However, in presence of anisotropic inhomogeneities the system becomes rather complicated even in the context of Maxwell's equations. This is due to the fact that the fields in general do not obey the standard wave equation of any simple form if the transformation

$$(E(x,t), H(x,t)) \mapsto (D(x,t), B(x,t)) = (\varepsilon(x)E(x,t), \mu(x)B(x,t))$$

between the standard and auxiliary fields are described by (1,1)-tensors

$$\varepsilon(x) = \varepsilon_j^k(x)\partial_k \otimes dx^j$$

and

$$\mu(x) = \mu_j^k(x)\partial_k \otimes dx^j$$

instead of scalar functions. However, assuming the tensors are conformally equivalent simplifies the system, as shown in [KLS06] by Kurylev Lassas and Somersalo. For a moving charge carrier one derives the wave equation

$$\begin{pmatrix} \Box_g & 0 & 0 \\ 0 & \Box_g & 0 \\ 0 & 0 & \Box_g \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} + \begin{pmatrix} F_1^1 & F_2^1 & F_3^1 \\ F_1^2 & F_2^2 & F_3^2 \\ F_1^3 & F_2^3 & F_3^3 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}, \tag{30}$$

where F_j^k, $j, k = 1, 2, 3$, are first order operators, each S_j, $j = 1, 2, 3$, is conormal to the world line of the particle and

$$\Box_g := \partial_t^2 - g^{jk}\partial_j\partial_k,$$

(i.e. $\Box_g = \partial_t^2 - \Delta_g$ with a different choice of F_j^k) for a Riemannian metric g which is conformally equivalent to the tensors ε and μ.

It is relevant to ask if Cherenkov radiation is generated also in a system of the form (30). A positive answer to this question is given in Article III. We show that in regions where a moving singularity exceeds the phase velocity of the material the source $S = S_j dx^j$ generates a cone of singularities that propagate in the field $E = E_j dx^j$ along the characteristic flowout of the scalar operator $\partial_t^2 - g^{jk}(x)\partial_j\partial_k$. This implies that whenever the moving source breaks the phase velocity barrier (cf. sound barrier) a phenomenon similar to Cherenkov radiation takes place in the level of singularities.

Since the bicharacteristics of a wave operator transport covectors along geodesics, a natural question is whether knowing the radiation on a boundary of a bounded region determines the metric g inside it. More precisely, we ask the following: For a bounded open set $W \subset \mathbb{R}^3$ with a smooth boundary ∂W is it possible to recover the metric g inside $U \subset W$ from the set of data that is collected by sending linearly moving point-like singularities (particles or quasiparticles) into U at various velocities while observing scattered singularities on a part of the boundary? As the main result of Article III we show that under some natural, physically relevant a priori conditions the metric in U is in fact uniquely determined by the data. The result covers even more general class of models, including some with complicated polarisations. In comparison to a stationary point source in space-time, Cherenkov radiation is mathematically more challenging, mainly because a single particle creates a radiation pattern that is simultaneously linked to many spatial points in the trajectory of it. However, by applying geometric arguments and data related to several incident angles we show that the distance to the boundary from any point in U can be derived.

The result is perhaps the first proposal for using Cherenkov radiation to reconstruct Riemannian structure of inhomogeneous anisotropic medium. Other existing methods (RICH, CLI) not only differ by purpose of use but rely strongly on flatness of the geometry.

4.4. **Author's Contribution.**

Article I: Antti T. P. Kujanpää has substantial contribution to the research and the main contribution to writing of the article.

Article II: Antti T. P. Kujanpää has substantial contribution to the research and writing of the article.

Article III: As the only author Antti T. P. Kujanpää has the main contribution to the research and writing of the article.

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