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DETERMINANTS OF SOME PENTADIAGONAL MATRICES

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Abstract. In this paper we consider pentadiagonal $(n+1)\times(n+1)$ matrices with two subdiagonals and two superdiagonals at distances k and $2k$ from the main diagonal where $1 \le k < 2k \le n$. We give an explicit formula for their determinants and also consider the Toeplitz and "imperfect" Toeplitz versions of such matrices. Imperfectness means that the first and last k elements of the main diagonal differ from the elements in the middle. Using the rearrangement due to Egerváry and Szász we also show how these determinants can be factorized.

1. Introduction

Matrices have a special space in mathematics. Their theory is still actively researched and used by almost every mathematician and by several scientists working in various areas. The research on multidiagonal, in particular tridiagonal and pentadiagonal matrices, intensified in the past years. These matrices have important applications in optimization problems ([3]), autoregression modelling ([26]), approximation theory ([23]), Gauss-Markov random processes ([2]), orthogonal polynomials, solving elliptic and parabolic PDE's with finite difference methods ([14]), inequalities (quadratic, Wirtinger, Opial's type, [4, 18]).

Let n, k be given positive integers with $1 \le k < 2k \le n$ and denote by $\mathcal{M}_n$ the set of $n \times n$ complex matrices. Consider the pentadiagonal matrix

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$A_0 = (a_{ij}) \in \mathcal{M}_{n+1}$ where for $i, j = 0, 1, \ldots, n$,

$$a_{ij} = \begin{cases} L_j & \text{if } j - i = -2k, \\ l_j & \text{if } j - i = -k, \\ d_i & \text{if } j - i = 0, \\ r_i & \text{if } j - i = k, \\ R_i & \text{if } j - i = 2k, \\ 0 & \text{otherwise.} \end{cases}$$

Matrix A_0 has two subdiagonals and two superdiagonals at distances k and $2k$ from the main diagonal. Notice that the numbering of entries starts with zero. We also use $A_{n+1,k,2k}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R})$ and $D_0 = D_{n+1,k,2k}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R})$ to denote the matrix A_0 and its determinant, where the diagonal vectors $\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R}$ are defined by

$$\begin{aligned} \mathbf{L} &= (L_0, \ldots, L_{n-2k}), \quad & \mathbf{l} &= (l_0, \ldots, l_{n-k}), \\ \mathbf{d} &= (d_0, \ldots, d_n), & & \\ \mathbf{R} &= (R_0, \ldots, R_{n-2k}), \quad & \mathbf{r} &= (r_0, \ldots, r_{n-k}). \end{aligned}$$

If $\mathbf{L}, \mathbf{R}$ (or $\mathbf{l}, \mathbf{r}$) are zero vectors then our matrix becomes a tridiagonal one denoted by $A_{n+1,k}(\mathbf{l}, \mathbf{d}, \mathbf{r})$ and its determinant by $D_{n+1,k}(\mathbf{l}, \mathbf{d}, \mathbf{r})$.

We shall call $A_{n+1,k,2k}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R})$ a (general) $k, 2k$-pentadiagonal matrix while $A_{n+1,k}(\mathbf{l}, \mathbf{d}, \mathbf{r})$ will be termed as k-tridiagonal.

In case of Toeplitz pentadiagonal matrices the diagonal vectors are constant vectors, i.e. $L_j = L, R_j = R\,(j = 0, \ldots, n-2k)$, $l_j = l, r_j = r\,(j = 0, \ldots, n-k)$, $d_j = d\,(j = 0, \ldots, n)$ and for the matrix and its determinant the notations $A_{n+1,k,2k}(L, l, d, r, R)$ and $D_{n+1,k,2k}(L, l, d, r, R)$ will be used.

Marr and Vineyard ([20]) have shown that the product of two 1-tridiagonal Toeplitz matrix is an imperfect Toeplitz matrix $A^{(\alpha,\beta)}_{n+1,1,2}$ which is related to the corresponding Toeplitz matrix by a two-step recursion. Imperfectness means that the main diagonal is changed from $(d, \ldots, d) \in \mathbb{C}^{n+1}$ to

$$(\underbrace{d-\alpha, \ldots, d-\alpha}_{k}, \underbrace{d, \ldots, d}_{n+1-2k}, \underbrace{d-\beta, \ldots, d-\beta}_{k}) \in \mathbb{C}^{n+1}$$

where α, β are given reals. They found the determinants of Toeplitz and imperfect Toeplitz $1, 2$-pentadiagonal matrices in terms of Chebyshev polynomials of the second kind. A similar approach was used in [27] to find a formula for the inverse of a $1, 2$-pentadiagonal Toeplitz matrix.

Imperfect $k, 2k$-pentadiagonal Toeplitz matrices and their determinants will be denoted by $A^{(\alpha,\beta)}_{n+1,k,2k}(L, l, d, r, R)$ and by $D^{(\alpha,\beta)}_{n+1,k,2k}(L, l, d, r, R)$.

A number of papers studied general $1, 2$-pentadiagonal matrices and their Toeplitz versions. Algorithms and recursive formulas were found for their determinants in [24, 15, 5, 17] and inverses [16]. Explicit formulas were found for the determinants of symmetric ([8]), skew-symmetric ([7]) and general

([17]) Toeplitz $1,2$-pentadiagonal matrices. Perhaps the first appearance of k,ℓ-pentadiagonal (Toeplitz Hermitian) matrices (where the distances of the sub- and superdiagonals from the main one are k and ℓ) was in Egerváry and Szász paper [4] with $k+\ell=n+1$, while k-tridiagonal matrices appeared first in [6]. In [19] new sub/superdiagonals were added. A graph theoretical approach can be found in [9, 11] and some extensions in [21, 25]. An exhaustive list of recent references is given in the survey [10]. In spite of the large numbers of papers on pentadiagonal matrices there are only few of them in which determinants are given in terms of the entries.

In [12] a method was developed to reduce the determinant of k,ℓ-pentadiagonal matrices to tridiagonal determinants provided that $k+\ell \geq n+1$. If $k \geq (n+1)/3$ then by this method in [13] the determinants of general, Toeplitz and imperfect Toeplitz $k,2k$-pentadiagonal matrices were determined. There was proved (among others) the following result.

Theorem 1.1. *Assuming $(n+1)/3 \leq k \leq n/2$ the determinant of the general $k,2k$-pentadiagonal matrix $A_{n+1,k,2k}(\mathbf{L},\mathbf{l},\mathbf{d},\mathbf{r},\mathbf{R})$ is*

$$\prod_{j=0}^{n-2k}(d_j d_{j+k} d_{j+2k} - d_j l_{j+k} r_{j+k} - L_j R_j d_{j+k} - l_j r_j d_{j+2k} + l_j R_j l_{j+k} + r_j L_j r_{j+k})$$

$$\cdot \prod_{j=n+1-2k}^{k-1}(d_j d_{j+k} - l_j r_j).$$

If the condition $k+\ell \geq n+1$ is not satisfied then the method given in [12] is not applicable in general. However, if $\ell = 2k$ then with some modification it is applicable. In Section 2 we develop this modification and extend Theorem 1.1 to the case when the restriction $(n+1)/3 \leq k$ is dropped. In Section 3 we give two examples. Finally Section 4 shows how the determinants $D_{n+1,k,2k}(L,l,d,r,R)$ can be factorized and also discuss the case of Toeplitz determinants.

2. Reduction of general $k,2k$-pentadiagonal determinants to tridiagonal ones

Let $A_0 = A_{n+1,k,2k}(\mathbf{L},\mathbf{l},\mathbf{d},\mathbf{r},\mathbf{R})$ where $1 \leq k < 2k \leq n$. Further let $n+1 = kq+p$ where $0 \leq p < k$ and suppose that $k < (n+1)/3$ (or $q \geq 3$).

Now we explain the first process which reduces A_0 to a matrix whose first k rows and columns contain nonzero entries only in the diagonal and its lower $(n+1-k)\times(n+1-k)$ block is a pentadiagonal matrix whose structure is similar to that of A_0. We do this by multiplying A_0 by four suitable matrices. We describe each of these four matrices bellow.

(i) Let $B_1^{(l)} \in \mathcal{M}_{n+1}$ with entries

$$b_{ij}^{(1,l)} = \begin{cases} 1 & \text{if } i = j, \\ -l_j/d_j & \text{if } i = k, \ldots, 2k-1, j = i-k, \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

and multiply A_0 from the left by $B_1^{(l)}$.

The effect of this multiplication is the same as multiplying the rows $0, \ldots, k-1$ of A_0 by the numbers $-l_0/d_0, \ldots, -l_{k-1}/d_{k-1}$ and adding them to the rows $k, \ldots, 2k-1$, respectively. The elements $l_0, \ldots, l_{k-1}$ of the kth subdiagonal disappear and the diagonal elements $d_k, \ldots, d_{2k-1}$ change to

$$d_j^{(1)} := d_j - l_{j-k} r_{j-k}/d_{j-k} \quad \text{for } j = k, \ldots, 2k-1. \tag{2.2}$$

The $2k$th superdiagonal remains unchanged, but all its elements $R_0, \ldots, R_{k-1}$ multiplied by $l_0/d_0, \ldots, -l_{k-1}/d_{k-1}$ respectively, move down by k units and added to $r_k, \ldots, r_{2k-1}$ thus these elements change to

$$r_j^{(1)} = r_j - R_{j-k} l_{j-k}/d_{j-k}\,, \quad \text{if } j = k, \ldots, 2k-1. \tag{2.3}$$

Please note that the position of entries $-l_j/d_j\,(j = 0, \ldots, k-1)$ in the matrix $B_1^{(l)}$ is the same as the position of $l_j\,(j = 0, \ldots, k-1)$ in the matrix $A_{n+1,k,\ell}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R})$, the position of the entries we want to eliminate. The matrices $B_1^{(r)}, B_1^{(L)}, B_1^{(R)}$ in the following multiplications also have similar structures.

(ii) Multiply $B_1^{(l)} A_0$ from the right by $B_1^{(r)} \in \mathcal{M}_{n+1}$ with entries

$$b_{ij}^{(1,r)} := \begin{cases} 1 & \text{if } i = j, \\ -r_i/d_i & \text{if } i = 0, \ldots, k-1; j = i+k, \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

The effect of this is the following: the columns $0, \ldots, k-1$ of $B_1^{(l)} A_0$ are multiplied by $-r_0/d_0, \ldots, -r_{k-1}/d_{k-1}$ and these products are added to the columns $k, \ldots, 2k-1$, respectively. The elements $r_0, \ldots, r_{k-1}$ in these columns disappear, the diagonal elements remain unchanged. The $2k$th subdiagonal remains unchanged, however its elements $L_0, \ldots, L_{k-1}$ multiplied by $-r_0/d_0, \ldots, -r_{k-1}/d_{k-1}$ respectively, move to the right by k units and added to the elements $l_{2k}, \ldots, l_{3k-1}$, hence these elements change to

$$l_j^{(1)} = l_j - L_{j-k} r_{j-k}/d_{j-k}\,, \quad \text{if } j = 2k, \ldots, 3k-1. \tag{2.5}$$

(iii) Next multiply $B_1^{(l)} A_0 B_1^{(r)}$ from the left by $B_1^{(L)} \in \mathcal{M}_{n+1}$ with entries

$$b_{ij}^{(1,L)} := \begin{cases} 1 & \text{if } i = j, \\ -L_j/d_j & \text{if } i = 2k, \ldots, 3k-1; j = i-2k, \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

The effect of this on the matrix $B_1^{(l)} A_0 B_1^{(r)}$ is multiplication of its rows $0, \ldots, n-2k$ by $-L_0/d_0, \ldots, -L_{n-2k}/d_{n-2k}$ and adding these products to the rows $2k, \ldots, 3k-1$, respectively. The elements $L_0, \ldots, L_{k-1}$ of the $2k$th subdiagonal disappear, the elements $d_{2k}, \ldots, d_{3k-1}$ of the main diagonal change to

$$d_j^{(1)} = d_j - L_{j-2k} R_{j-2k}/d_{j-2k}\,, \quad \text{for } j = 2k, \ldots, 3k-1. \tag{2.7}$$

(iv) Finally multiply $B_1^{(L)} B_1^{(l)} A_0 B_1^{(r)}$ from the right by $B_1^{(R)} \in \mathcal{M}_{n+1}$ with entries

$$b_{ij}^{(1,R)} := \begin{cases} 1 & \text{if } i = j, \\ -R_i/d_i & \text{if } i = 0, \ldots, k-1; j = i + 2k, \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

The effect of this on the matrix $B_1^{(L)} B_1^{(l)} A_0 B_1^{(r)}$ is multiplication of its columns $0, \ldots, k-1$ by $-R_0/d_0, \ldots, -R_{k-1}/d_{k-1}$ and adding these products to the columns $2k, \ldots, 3k-1$ respectively. The elements $R_0, \ldots, R_{k-1}$ vanish and the main diagonal does not change.

With this the first process ended. The matrices $B_1^{(L)}, B_1^{(l)}, B_1^{(r)}, B_1^{(R)}$ are the same as in [12, 13] however the transformation rules for the entries are different. In those papers during the first (and subsequent) processes the diagonal vectors $\mathbf{l}, \mathbf{r}$ just shortened while the diagonal vectors $\mathbf{L}, \mathbf{R}$ transformed. Here just the opposite happened: the diagonals $\mathbf{L}, \mathbf{R}$ shortened and $\mathbf{l}, \mathbf{r}$ transformed.

The results of the first process are summarized in the following theorem.

THEOREM 2.1. *Suppose that* $1 \le k \le 2k \le n$, $k < (n+1)/3$. *Then*

$$A_1 := B_1^{(L)} B_1^{(l)} A_0 B_1^{(r)} B_1^{(R)} = E_1 \bigoplus A_1^* \tag{2.9}$$

where $E_1 \in \mathcal{M}_k$ *is a diagonal matrix with diagonal elements* $d_0, \ldots, d_{k-1}$ *and*

$$A_1^* = A_{n+1-k,k,2k}(\mathbf{L^{(1)}}, \mathbf{l^{(1)}}, \mathbf{d^{(1)}}, \mathbf{r^{(1)}}, \mathbf{R^{(1)}})$$

is a $k, 2k$*-pentadiagonal matrix with main and other diagonal vectors (called first iterated diagonals)*

$$\begin{aligned} \mathbf{L}^{(1)} &= (L_k, \ldots, L_{n-2k}),\ \mathbf{R}^{(1)} = (R_k, \ldots, R_{n-2k}) \in \mathbb{C}^{n+1-3k}, \\ \mathbf{l}^{(1)} &= (l_k^{(1)}, \ldots, l_{n-k}^{(1)}), \quad \mathbf{r}^{(1)} = (r_k^{(1)}, \ldots, r_{n-k}^{(1)}) \in \mathbb{C}^{n+1-2k}, \\ \mathbf{d}^{(1)} &= (d_k^{(1)}, \ldots, d_n^{(1)}) \in \mathbb{C}^{n+1-k}, \end{aligned}$$

where

$$\begin{aligned} l_j^{(1)} &= \begin{cases} l_j - L_{j-k} r_{j-k}/d_{j-k} & (j = k, \ldots, 2k-1), \\ l_j & (j = 2k, \ldots, n-k), \end{cases} \\ r_j^{(1)} &= \begin{cases} r_j - R_{j-k} l_{j-k}/d_{j-k} & (j = k, \ldots, 2k-1), \\ r_j & (j = 2k, \ldots, n-k), \end{cases} \end{aligned} \tag{2.10}$$

and

$$d_j^{(1)} = \begin{cases} d_j - l_{j-k}r_{j-k}/d_{j-k} & (j = k, \dots, 2k-1), \\ d_j - L_{j-2k}R_{j-2k}/d_{j-2k} & (j = 2k, \dots, 3k-1), \\ d_j & (j = 3k, \dots, n). \end{cases} \tag{2.11}$$

Proof. Relation (2.9) follows from the description of the matrices $B_1^{(L)}$, $B_1^{(l)}, B_1^{(r)}, B_1^{(R)}$. Relations (2.3), (2.5) justify (2.10) and (2.2) while (2.7) shows that (2.11) is valid. $\square$

For the convenience of later calculations all letters l_i, d_i, r_i referring to the original matrix will be labeled with superscripts $^{(0)}$ but occasionally we omit this label. We omit from this labeling L_i, R_i since these numbers did not change during our process.

Define for $s = 1, 2, \dots, q-1$ the matrices from $\mathcal{M}_{n+1}$ (for $s = 1$ these coincide with the matrices (2.1), (2.4), (2.6), (2.8)) by

$$B_s^{(l)} = \left(b_{ij}^{(s,l)}\right) = \begin{cases} 1 & \text{if } i = j, \\ -l_j^{(s-1)}/d_j^{(s-1)} & \text{if } i = sk, \dots, (s+1)k-1; j = i-k, \\ 0 & \text{otherwise}, \end{cases}$$

$$B_s^{(r)} = \left(b_{ij}^{(s,r)}\right) = \begin{cases} 1 & \text{if } i = j, \\ -r_i^{(s-1)}/d_i^{(s-1)} & \text{if } i = (s-1)k, \dots, sk-1; j = i+k, \\ 0 & \text{otherwise}, \end{cases}$$

$$B_s^{(L)} = \left(b_{ij}^{(s,L)}\right) = \begin{cases} 1 & \text{if } i = j, \\ -L_j^{(s-1)}/d_j^{(s-1)} & \text{if } i = (s+1)k, \dots, (s+2)k-1; \\ & \quad j = i-2k, \\ 0 & \text{otherwise}, \end{cases}$$

$$B_s^{(R)} = \left(b_{ij}^{(s,R)}\right) = \begin{cases} 1 & \text{if } i = j, \\ -R_i^{(s-1)}/d_i^{(s-1)} & \text{if } i = (s-1)k, \dots, sk-1; j = i+2k, \\ 0 & \text{otherwise}. \end{cases}$$

We define the first and second matrix for $s = q$ too, but in this case the index sets are restricted to $i = qk, \dots, n; j = i-k$ and $i = (q-1)k, \dots, n-k; j = i+k$ respectively, if $p = 0$ then both sets are empty and the first and second matrices degenerate to unit matrices. Similarly for the third and fourth matrix with $s = q-1$ the index domains are $i = qk, \dots, n; j = i-2k$ and $i = (q-2)k, \dots, n-2k; j = i+2k$ respectively and if $p = 0$ then both sets are empty and the third and fourth matrices degenerate to unit matrices.

The quantities $l_j^{(s-1)}, r_j^{(s-1)}, d_j^{(s-1)}$ for $s = 1, 2, \ldots, (q-2, q-1, q)$ can be obtained inductively by continuing the iterations of (2.10), (2.11) as follows

$$l_j^{(s)} = \begin{cases} l_j^{(s-1)} - L_{j-k} r_{j-k}^{(s-1)} / d_{j-k}^{(s-1)} & (j = sk, \ldots, (s+1)k-1), \\ l_j & (j = (s+1)k, \ldots, n-k), \end{cases}$$
(2.12)
$$r_j^{(s)} = \begin{cases} r_j^{(s-1)} - R_{j-k} l_{j-k}^{(s-1)} / d_{j-k}^{(s-1)} & (j = sk, \ldots, (s+1)k-1), \\ r_j & (j = (s+1)k, \ldots, n-k), \end{cases}$$

(2.13)
$$d_j^{(s)} = \begin{cases} d_j^{(s-1)} - l_{j-k}^{(s-1)} r_{j-k}^{(s-1)} / d_{j-k}^{(s-1)} & (j = sk, \ldots, (s+1)k-1), \\ d_j^{(s-1)} - L_{j-2k} R_{j-2k} / d_{j-2k}^{(s-1)} & (j = (s+1)k, \ldots, (s+2)k-1), \\ d_j & (j = (s+2)k, \ldots, n). \end{cases}$$

These definitions are valid if $s < q-2$. We put the last three values $q-2, q-1, q$ of s in parenthesis since for them the definitions should be modified. For these values of s the index groups in (2.12), (2.13) may run out of their ranges $j = sk, \ldots, n-k$ and $j = sk, \ldots, n$ therefore cannot be defined, or the index groups may be restricted.

For example (2.13) is valid for $s \le q-2$, for $s = q-1$ the second index group should be restricted to $j = qk, \ldots, n$ if $p > 0$ and empty if $p = 0$ with empty third index group. For $s = q$ the diagonal elements $d_j^{(q)}$ can be defined only if $p > 0$, in this case for $s = q$ in (2.13) the first index group should be restricted to $j = qk, \ldots, n$ with empty second and third index groups.

In the second process we calculate the product $A_2 := B_2^{(L)} B_2^{(l)} A_1 B_2^{(r)} B_2^{(R)}$ then we continue similarly $s-2$ $(s < q-1)$ times to get after s processes

$$A_s := \left(\prod_{j=1}^{j=s} B_{s+1-j}^{(L)} B_{s+1-j}^{(l)} \right) A_0 \left(\prod_{j=1}^{j=s} B_j^{(r)} B_j^{(R)} \right).$$

For A_s we have the decomposition

$$A_s = E_{sk} \bigoplus A_s^*$$

where $E_{sk} \in \mathcal{M}_{sk}$ is a diagonal matrix with diagonal elements

$$d_0, \ldots, d_{k-1}, d_k^{(1)}, \ldots, d_{2k-1}^{(1)}, \ldots, d_{(s-1)k}^{(s-1)}, \ldots, d_{sk-1}^{(s-1)}$$

and

$$A_s^* = A_{n+1-sk,k,2k}(\mathbf{L}^{(s)}, \mathbf{l}^{(s)}, \mathbf{d}^{(s)}, \mathbf{r}^{(s)}, \mathbf{R}^{(s)})$$

with

$$\begin{aligned} \mathbf{L}^{(s)} &= (L_{sk}, \ldots, L_{n-2k}), \quad \mathbf{R}^{(s)} = (R_{sk}, \ldots, R_{n-2k}) \in \mathbb{C}^{n+1-(s+2)k}, \\ \mathbf{l}^{(s)} &= (l_{sk}^{(s-1)}, \ldots, l_{n-k}^{(s-1)}), \quad \mathbf{r}^{(s)} = (r_{sk}^{(s-1)}, \ldots, r_{n-k}^{(s-1)}) \in \mathbb{C}^{n+1-(s+1)k}, \\ \mathbf{d}^{(s)} &= (d_{sk}^{(s-1)}, \ldots, d_n^{(s-1)}) \in \mathbb{C}^{n+1-sk}, \end{aligned}$$

and the elements of these iterated diagonals are given by (2.12), (2.13).

For $s = q-2$ the dimension of the vectors $\mathbf{L}^{(s)}, \mathbf{R}^{(s)}$ is p. This means that if $p = 0$ then A^*_{q-2} is tridiagonal, and if $p > 0$ then A^*_{q-1} is tridiagonal. Therefore the cases $p = 0$ and $p > 0$ should be treated differently.

THEOREM 2.2. *Let $1 \le k < 2k \le n$, $n+1 = kq$ (i.e. $p = 0$) $q > 3$. Then*

$$A_{q-1} := B^{(l)}_{q-2} A_{q-2} B^{(r)}_{q-2} = E_{qk} \in \mathcal{M}_{n+1}$$

is a diagonal matrix with diagonal elements

$$d_0, \dots, d_{k-1}, d^{(1)}_k, \dots, d^{(1)}_{2k-1}, \dots, d^{(q-1)}_{(q-1)k}, \dots, d^{(q-1)}_{qk-1}$$

defined by (2.13).

If $p > 0$ then multiplying A_{q-1} from the left by $B^{(l)}_q$ and the from the right by $B^{(r)}_q$ we get a diagonal matrix.

THEOREM 2.3. *Let $1 \le k < 2k \le n$, $n+1 = kq+p$, $0 \le p < k$, $q \ge 3$. Then*

$$A_q := B^{(l)}_q A_{q-1} B^{(r)}_q = E_{qk+p} \in \mathcal{M}_{n+1}$$

is a diagonal matrix with diagonal elements

$$d_0, \dots, d_{k-1}, d^{(1)}_k, \dots, d^{(1)}_{2k-1}, \dots, d^{(q-1)}_{(q-1)k}, \dots, d^{(q-1)}_{qk-1}, d^{(q)}_{qk}, \dots, d^{(q)}_{qk+p-1}$$

defined by (2.13).

For the determinant we have the following result.

THEOREM 2.4. *Let $1 \le k < 2k \le n$, $n+1 = kq+p$, $0 \le p < k$, and suppose that either $q = 3, p > 0$, or $q > 3$. Then*

$$\begin{aligned} D_{n+1,k,2k}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R}) &= \prod_{j=0}^{k-1} d^{(0)}_j d^{(1)}_{j+k} \cdots d^{(q-1)}_{j+(q-1)k} \prod_{j=0}^{p-1} d^{(q)}_{j+qk} \\ &= \prod_{j=0}^{p-1} d^{(0)}_j d^{(1)}_{j+k} \cdots d^{(q)}_{j+qk} \prod_{j=p}^{k-1} d^{(0)}_j d^{(1)}_{j+k} \cdots d^{(q-1)}_{j+(q-1)k}, \end{aligned} \tag{2.14}$$

where $d^{(s)}_j$ $(s=1,\dots,q; j=sk,\dots,n)$ are defined by (2.11), (2.13) and $\prod_{j=0}^{-1} := 1$.

3. EXAMPLES

Our first example is a numerical one showing how the calculations of Theorem 2.2 work. The smallest example for this is when $n+1 = 4, k =$

$1, q = 4, p = 0$. We use the same notations as in the previous section without referring to the size of the matrices. Let

$$A_0 = \begin{pmatrix} 1 & 2 & 3 & 0 \\ -1 & 4 & 1 & 2 \\ 5 & 2 & -2 & 4 \\ 0 & 1 & 3 & -1 \end{pmatrix}$$

It is easy to see that its determinant is -32. Since $k = 1$ by our method each multiplication makes just one diagonal element (not in the main diagonal) zero, hence we need 10 matrix multiplications to transform our matrix to a diagonal one. The first four multiplier matrices $B_1^{(L)}, B_1^{(l)}, B_1^{(r)}, B_1^{(R)}$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

After the multiplications we obtain

$$A_1 := B_1^{(L)} \cdot B_1^{(l)} \cdot A_0 \cdot B_1^{(r)} \cdot B_1^{(R)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 4 & 2 \\ 0 & -8 & -13 & 4 \\ 0 & 1 & 3 & -1 \end{pmatrix}.$$

The next four multiplier matrices $B_2^{(L)}, B_2^{(l)}, B_2^{(r)}, B_2^{(R)}$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{6} & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{8}{6} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{4}{6} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{2}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Performing the second process we get

$$A_2 := B_2^{(L)} \cdot B_2^{(l)} \cdot A_1 \cdot B_2^{(r)} \cdot B_2^{(R)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & -\frac{23}{3} & \frac{20}{3} \\ 0 & 0 & \frac{7}{3} & -\frac{4}{3} \end{pmatrix}.$$

The last two multiplier matrices $B_3^{(l)}, B_3^{(r)}$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{7}{23} & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{20}{23} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally we obtain

$$A_3 := B_3^{(l)} \cdot A_2 \cdot B_3^{(r)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & -\frac{23}{3} & 0 \\ 0 & 0 & 0 & \frac{16}{23} \end{pmatrix}.$$

One can see that the product $1 \cdot 6 \cdot -23/3 \cdot 16/23 = -32$ is the same as the determinant of A_0.

In the next example we express the determinant of $A_{n+1,k,2k}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R})$ in terms of its entries if $n+1 = 3k+p$ i.e. if $q = 3$. For $j = 0, \dots, k-1$ we have

$$d_j^{(0)} d_{j+k}^{(1)} = d_j \left(d_{j+k} - \frac{l_j r_j}{d_j} \right) = d_j d_{j+k} - l_j r_j$$

and

$$\begin{aligned}
d_{j+2k}^{(2)} &= d_{j+2k}^{(1)} - \frac{l_{j+k}^{(1)} r_{j+k}^{(1)}}{d_{j+k}^{(1)}} = d_{j+2k} - \frac{L_j R_j}{d_j} - \frac{\left(l_{j+k} - \frac{L_j r_j}{d_j}\right)\left(r_{j+k} - \frac{R_j l_j}{d_j}\right)}{d_{j+k} - \frac{l_j r_j}{d_j}} \\
&= \frac{d_j d_{j+2k} - L_j R_j}{d_j} - \frac{(l_{j+k} d_j - L_j r_j)(r_{j+k} d_j - R_j l_j)}{d_j (d_{j+k} d_j - l_j r_j)} \\
&= \frac{d_j d_{j+k} d_{j+2k} - d_{j+2k} l_j r_j - d_{j+k} L_j R_j - d_j l_{j+k} r_{j+k} + R_j l_j l_{j+k}}{d_j d_{j+k} - l_j r_j} \\
&\quad + \frac{L_j r_j r_{j+k}}{d_j d_{j+k} - l_j r_j},
\end{aligned}$$

$$\begin{aligned}
d_j^{(0)} d_{j+k}^{(1)} d_{j+2k}^{(2)} &= d_j d_{j+k} d_{j+2k} - d_{j+2k} l_j r_j - d_{j+k} L_j R_j - d_j l_{j+k} r_{j+k} \\
&\quad + R_j l_j l_{j+k} + L_j r_j r_{j+k}.
\end{aligned}$$

If $p > 0$ then for $j = 0, \dots, p$ we have to calculate $d_{j+3k}^{(3)}$ too. We get

$$d_{j+3k}^{(3)} = d_{j+3k}^{(2)} - \frac{l_{j+2k}^{(2)} r_{j+2k}^{(2)}}{d_{j+2k}^{(2)}} = \frac{d_{j+3k}^{(2)} d_{j+2k}^{(2)} - l_{j+2k}^{(2)} r_{j+2k}^{(2)}}{d_{j+2k}^{(2)}},$$

where

$$\begin{aligned}
d_{j+3k}^{(2)} &= d_{j+3k}^{(1)} - \frac{L_{j+k} R_{j+k}}{d_{j+k}^{(1)}} = d_{j+3k} - \frac{L_{j+k} R_{j+k}}{d_{j+k} - \frac{l_j r_j}{d_j}} \\
&= \frac{d_j d_{j+k} d_{j+3k} - d_{j+3k} l_j r_j - d_j L_{j+k} R_{j+k}}{d_j d_{j+k} - l_j r_j}, \\
l_{j+2k}^{(2)} &= l_{j+2k}^{(1)} - \frac{L_{j+k} r_{j+k}^{(1)}}{d_{j+k}^{(1)}} = l_{j+2k} - \frac{L_{j+k}\left(r_{j+k} - \frac{R_j l_j}{d_j}\right)}{d_{j+k} - \frac{l_j r_j}{d_j}} \\
&= \frac{d_j d_{j+k} l_{j+2k} - l_j l_{j+2k} r_j - d_j L_{j+k} r_{j+k} + L_{j+k} l_j R_j}{d_j d_{j+k} - l_j r_j}, \\
r_{j+2k}^{(2)} &= r_{j+2k}^{(1)} - \frac{R_{j+k} l_{j+k}^{(1)}}{d_{j+k}^{(1)}} = r_{j+2k} - \frac{R_{j+k}\left(l_{j+k} - \frac{L_j r_j}{d_j}\right)}{d_{j+k} - \frac{l_j r_j}{d_j}} \\
&= \frac{d_j d_{j+k} r_{j+2k} - r_j r_{j+2k} l_j - d_j R_{j+k} l_{j+k} + R_{j+k} r_j L_j}{d_j d_{j+k} - l_j r_j}.
\end{aligned}$$

Denote by n_1, n_2, n_3, n_4 the numerators by m_1, m_2, m_3, m_4 the denominators of the above fraction forms of $d^{(2)}_{j+3k}, d^{(2)}_{j+2k}, l^{(2)}_{j+2k}, r^{(2)}_{j+2k}$ respectively, then we have that

$$d^{(3)}_{j+3k} = \frac{n_1n_2 - n_3n_4}{(d_{j+k}d_j - l_jr_j)^2}\,\frac{d_{j+k}d_j - l_jr_j}{n_2} = \frac{n_1n_2 - n_3n_4}{(d_{j+k}d_j - l_jr_j)n_2}.$$

Therefore

$$d^{(0)}_j d^{(1)}_{j+k} d^{(2)}_{j+2k} d^{(3)}_{j+3k} = n_2 d^{(3)}_{j+3k} = \frac{n_1n_2 - n_3n_4}{d_{j+k}d_j - l_jr_j}.$$

Calculating and factorizing the numerator by Maple software we obtain

$$n_1n_2 - n_3n_4 = (d_{j+k}d_j - l_jr_j)M,$$

where

$$\begin{aligned}
M &= M_{n,k,j}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R}) \\
&:= d_jd_{j+k}d_{j+2k}d_{j+3k} - d_jd_{j+k}l_{j+2k}r_{j+2k} - d_jd_{j+2k}L_{j+k}R_{j+k} - d_jd_{j+3k}l_{j+k}r_{j+k} \\
&\quad - d_{j+k}d_{j+3k}L_jR_j - d_{j+2k}d_{j+3k}l_jr_j + d_jL_{j+k}r_{j+k}r_{j+2k} + d_jR_{j+k}l_{j+k}l_{j+2k} \\
&\quad + d_{j+3k}L_jr_jr_{j+k} + d_{j+3k}R_jl_jl_{j+k} + L_jL_{j+k}R_jR_{j+k} - L_jR_{j+k}l_{j+2k}r_j \\
&\quad - L_{j+k}R_jl_jr_{j+2k} + l_jl_{j+2k}r_jr_{j+2k},
\end{aligned}$$

therefore $M_{n,k,j}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R}) = d^{(0)}_j d^{(1)}_{j+k} d^{(2)}_{j+2k} d^{(3)}_{j+3k}$. Further let

$$\begin{aligned}
N_{n,k,j}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R}) &:= n_2 \\
&= d_jd_{j+k}d_{j+2k} - d_{j+2k}l_jr_j - d_{j+k}L_jR_j - d_jl_{j+k}r_{j+k} \\
&\quad + R_jl_jl_{j+k} + L_jr_jr_{j+k}
\end{aligned}$$

then we have

Theorem 3.1. *Let $1 \le k < 2k \le n$, $n+1 = 3k+p$, $0 \le p < k$ then*

(3.1)

$$\begin{aligned}
D_{n+1,k,2k}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R}) &= \prod_{j=0}^{p-1} d^{(0)}_j d^{(1)}_{j+k} d^{(2)}_{j+2k} d^{(3)}_{j+3k} \prod_{j=p}^{k-1} d^{(0)}_j d^{(1)}_{j+k} d^{(2)}_{j+2k} \\
&= \prod_{j=0}^{p-1} M_{n,k,j}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R}) \prod_{j=p}^{k-1} N_{n,k,j}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R})
\end{aligned}$$

where $\prod_{j=0}^{-1} := 1$.

4. $k, 2k$-pentadiagonal determinants, the Toeplitz case

To obtain the imperfect Toeplitz determinant corresponding to (3.1) substitute in it $L_j = L, l_j = l, r_j = r, R_j = R$ $(j = 0, \dots, k-1$ further in $M_{n,k,j}$ substitute $d_j = d - \alpha, d_{j+k} = d, d_{j+2k} = d, d_{j+3k} = d - \beta$ $(j = 0, \dots, p-1)$ and in $N_{n,k,j}$ substitute $d_j = d - \alpha, d_{j+k} = d, d_{j+2k} = d - \beta$ $(j = p, \dots, k-1)$.

Theorem 4.1. *Let* $1 \le k < 2k \le n$, $n+1 = 3k+p, 0 \le p < k$ *then the determinant of the imperfect pentadiagonal Toeplitz matrix* $A^{(\alpha,\beta)}_{n+1,k,2k}(L,l,d,r,R)$ *is*

$$\begin{aligned}D^{(\alpha,\beta)}_{n+1,k,2k} = \Big(&d^4 - (\alpha+\beta)d^3 - (3lr + 2LR - \alpha\beta)d^2 \\ &+ \big(2Lr^2 + 2Rl^2 + (\alpha+\beta)(2lr + LR)\big)d \\ &+ L^2R^2 - 2LRlr + l^2r^2 + (\alpha+\beta)(Lr^2 + Rl^2) - \alpha\beta lr\Big)^p \\ &\cdot \big(d^3 - (\alpha+\beta)d^2 - (2lr + LR - \alpha\beta)d + Lr^2 + Rl^2 + \alpha\beta lr\big)^{k-p}.\end{aligned}$$

In the Toeplitz case we immediately get by $\alpha = \beta = 0$ from the previous theorem.

Theorem 4.2. *Let* $1 \le k < 2k \le n$, $n+1 = 3k+p$, $0 \le p < k$ *then the determinant of* $A_{n+1,k,2k}(L,l,d,r,R)$ *is*

$$\begin{aligned}D_{n+1,k,2k} = &\big(d^4 - (3lr + 2LR)d^2 + (2Lr^2 + 2Rl^2)d + L^2R^2 - 2LRlr + l^2r^2\big)^p \\ &\cdot \big(d^3 - (2lr + LR)d + Lr^2 + Rl^2\big)^{k-p}.\end{aligned}$$

With a suitable rearrangement of $k, 2k$-pentadiagonal (general or Toeplitz) matrices they can be reduced to the direct sum of $1, 2$-pentadiagonal (general or Toeplitz) matrices. In this way we can say more about $k, 2k$-pentadiagonal determinants.

Let $n+1 = kq + p$ where $0 \le p < k$ and consider the permutation σ of the integers $0, 1, \ldots, n$ given by

$$\begin{aligned}\sigma(s + j(q+1)) &= sk + j \qquad \text{if } s = 0,1,\ldots,q; j = 0,\ldots,p-1, \\ \sigma(s + p + jq) &= sk + j \qquad \text{if } s = 0,1,\ldots,q-1; j = p,\ldots,k-1.\end{aligned}$$

Define the permutation matrix $P_\sigma = (p_{ij})$ by

$$p_{ij} = \begin{cases} 1 \text{ if } j = \sigma(i), \\ 0 \text{ otherwise.}\end{cases}$$

$P_\sigma A_0 P_\sigma^T$ rearranges the rows and columns of $A_0 = A_{n+1,k,2k}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R})$ in the order of the permutation σ. Thus we obtain

$$P_\sigma A_0 P_\sigma^T = \left(\bigoplus_{j=0}^{p-1} A^{(j)}_{q+1,1,2}\right)\left(\bigoplus_{j=p}^{k-1} A^{(j)}_{q,1,2}\right), \tag{4.1}$$

where $A^{(s)}_{t+1,1,2} = A^{(s)}_{t+1,1,2}(\mathbf{L}_{(s)}, \mathbf{l}_{(s)}, \mathbf{d}_{(s)}, \mathbf{r}_{(s)}, \mathbf{R}_{(s)})$ for $t = q, s = 0, 1, \ldots, p-1$ and $t = q-1, s = p, p+1, \ldots, k-1$ are $1, 2$-pentadiagonal matrices with diagonal vectors

$$\begin{aligned}\mathbf{L}_{(s)} &= (L_s, L_{s+k}, \ldots, L_{s+(t-2)k}), \quad \mathbf{R}_{(s)} = (R_s, R_{s+k}, \ldots, R_{s+(t-2)k}), \\ \mathbf{l}_{(s)} &= (l_s, l_{s+k}, \ldots, l_{s+(t-1)k}), \quad \mathbf{r}_{(s)} = (r_s, r_{s+k}, \ldots, r_{s+(t-1)k}),\end{aligned}$$

$\mathbf{d}_{(s)} = (d_s, d_{s+k}, \ldots, d_{s+tk})$.

We remark that σ is the same permutation as the one used by Egerváry and Szász in [4], see also N. Bebiano and S. Furtado [1] where a similar decomposition was given.

From (4.1) it follows immediately the following result.

THEOREM 4.3. *Let $1 \le k < 2k \le n$, then we have*

$$D_{n+1,k,2k}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R}) = \prod_{j=0}^{p-1} D^{(j)}_{q+1,1,2} \prod_{j=p}^{k-1} D^{(j)}_{q,1,2}$$

where $D^{(s)}_{t+1,1,2}$ denotes the determinant of the matrix $A^{(s)}_{t+1,1,2}$.

For Toeplitz matrices we get (with the notation $B^{\oplus j} := \underbrace{B \oplus \cdots \oplus B}_{j}$)

$$P_\sigma A_{n+1,k,2k}(L, l, d, r, R) P_\sigma^T = A_{q+1,1,2}(L, l, d, r, R)^{\oplus p} A_{q,1,2}(L, l, d, r, R)^{\oplus (k-p)}$$

as in this case $A^{(s)}_{t+1,1,2}(\mathbf{L}_{(s)}, \mathbf{l}_{(s)}, \mathbf{d}_{(s)}, \mathbf{r}_{(s)}, \mathbf{R}_{(s)}) = A_{t+1,1,2}(L, l, d, r, R)$ for $t = q, s = 0, 1, \ldots, p-1$ and $t = q-1, s = p, p+1, \ldots, k-1$.

For the determinants we get

$$D_{n+1,k,2k}(L, l, d, r, R) = D_{q+1,1,2}(L, l, d, r, R)^p D_{q,1,2}(L, l, d, r, R)^{k-p};$$

thus it is enough to calculate the determinants of $1, 2$-pentadiagonal matrices. To do this we could use the iteration formulae (2.11), (2.13) (which are now considerably simpler) to find the diagonal elements then by (2.14) to find the determinants. However it seems easier to apply existing recursion formulae for the determinants. The six term recursion of R.A. Sweet ([24]) is applicable for the determinants of Toeplitz matrices but its coefficients contain fractions and are more complicated than those of the seven term recursion found by J. Jia, B. Yang, S. Li in [17], thus we apply the latter.

Let $D(n+1) := D_{n+1,1,2}(L, l, d, r, R)$ $(n \ge 2)$ and let $D(-2) = 0, D(-1) = 0, D(0) = 1$, $D(1) = d, D(2) = d^2 - lr$. For the determinant $D(3)$ we easily obtain

$$D(3) = d^3 - d(LR + 2lr) + (Lr^2 + Rl^2). \tag{4.2}$$

The recursion of [17] with our notations is

$$\begin{aligned} D(n) = {} & dD(n-1) + (LR - lr)D(n-2) + (Lr^2 + Rl^2 \\ & - 2dLR)D(n-3) + LR(LR - lr)D(n-4) \\ & + dL^2R^2 D(n-5) - L^3R^3 D(n-6) \quad (n = 4, \ldots). \end{aligned} \tag{4.3}$$

Calculating $D(n)$ for $n = 4, \dots, 8$ using (4.2), (4.3) with Maple software we obtain

$$\begin{aligned}
D(4) &= d^4 - d^2(2LR + 3lr) + d(2Lr^2 + 2Rl^2) + (L^2R^2 - 2LRlr + l^2r^2),\\
D(5) &= d^5 - d^3(3LR + 4lr) + d^2(3Lr^2 + 3Rl^2) + d(2L^2R^2 - 2LRlr + 3l^2r^2)\\
&\quad + (L^2Rr^2 + R^2Ll^2 - 2Llr^3 - 2Rrl^3),\\
D(6) &= d^6 - d^4(4LR + 5lr) + d^3(4Lr^2 + 4Rl^2) + d^2(4L^2R^2 + 6l^2r^2)\\
&\quad + d(-6Llr^3 - 6Rl^3r) + (-4L^2R^2lr + L^2r^4 + 6LRl^2r^2 + R^2l^4 - l^3r^3),\\
D(7) &= d^7 - d^5(5LR + 6lr) + d^4(5Lr^2 + 5Rl^2) + d^3(7L^2R^2 + 4LRlr + 10l^2r^2)\\
&\quad - d^2(3L^2Rr^2 + 3R^2Ll^2 + 12Llr^3 + 12Rrl^3)\\
&\quad - d(2L^3R^3 + 6L^2R^2lr - 3L^2r^4 - 3R^2l^4 - 15LRl^2r^2 + 4l^3r^3)\\
&\quad + (3L^3R^2r^2 + 3R^3L^2l^2 - 6L^2Rlr^3 - 6R^2Lrl^3 + 3Ll^2r^4 + 3Rr^2l^4),\\
D(8) &= d^8 - d^6(6LR + 7lr) + d^5(6Lr^2 + 6Rl^2)\\
&\quad + d^4(11L^2R^2 + 10LRlr + 15l^2r^2)\\
&\quad + d^3(-8L^2Rr^2 - 8LR^2l^2 - 20Llr^3 - 20Rl^3r)\\
&\quad + d^2(-6L^3R^3 - 9L^2R^2lr + 6L^2r^4 + 24LRl^2r^2 + 6R^2l^4 - 10l^3r^3)\\
&\quad + d(6L^3R^2r^2 + 6L^2R^3l^2 - 12L^2Rlr^3 - 12LR^2l^3r + 12Ll^2r^4\\
&\quad + 12Rl^4r^2) + (R^4L^4 - 6L^3R^3lr + 2L^3Rr^4 + 15L^2R^2l^2r^2\\
&\quad - 3L^2lr^5 + 2LR^3l^4 - 12LRl^3r^3 - 3R^2l^5r + l^4r^4).
\end{aligned}$$

Observing these determinants we see that they are monic polynomials

$$p_n(L, l, d, r, R) = d^n + \sum_{j=0}^{n-2} A_{n,j}(L, l, r, R)d^j$$

of degree n in d where the coefficients $A_{n,j}(L, l, r, R)\,(j = 0, \dots, n-2)$ are polynomials of L, l, r, R of degree $n - k$ which are symmetric in L, R and l, r. Unfortunately for larger n the formulae for these polynomials are too long. With this we have proved the following theorem.

THEOREM 4.4. *Let $1 \le k < 2k \le n$, $n + 1 = kq + p$, $0 \le p < k$ then for $q = 3, 4, 5, 6, 7$ the determinant of $A_{n+1,k,2k}(L, l, d, r, R)$ is*

$$D_{n+1,k,2k}(L, l, d, r, R) = p_{q+1}(L, l, d, r, R)^p\ p_q(L, l, d, r, R)^{k-p}$$

where the polynomials p_n are given above and by (4.2).

We conjecture that this theorem is true for all possible values of q. The imperfect Toeplitz determinants $D^{(\alpha,\beta)}(n+1) := D^{(\alpha,\beta)}_{n+1,k,2k}(L, l, d, r, R)$ can be calculated by the recursion

$$D^{(\alpha,\beta)}(n) = D(n) - (\alpha + \beta)D(n-1) + \alpha\beta D(n-2) \tag{4.4}$$

found by Marr and Vineyard in [20] and using the previous theorem. Another possibility is using a nine term recursion based on (4.4) and (4.3).

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