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ON THE RAMANUJAN-NAGELL TYPE DIOPHANTINE EQUATION $Dx^2 + k^n = B$

Zhongfeng Zhang and Alain Togbé
Zhaoqing University, China and Purdue University Northwest, USA

Abstract. In this paper, we prove that the Ramanujan-Nagell type Diophantine equation $Dx^2+k^n = B$ has at most three nonnegative integer solutions (x, n) for k a prime and B, D positive integers.

1. Introduction

Studying some generalized Ramanujan-Nagell equations, Ulas ([3]) gave the following conjecture.

Conjecture 1.1 ([3, Conjecture 4.4]). *The Diophantine equation*

$$x^2 + k^n = B \tag{1.1}$$

has at most three nonnegative integers (x, n)*, for any given integers* $k \geq 2$ *and* $B \geq 1$.

Meng Bai and the first author ([1]) confirmed Conjecture 1.1 for $k = 2$ and the authors ([6]) of this paper for k an odd prime, i.e. they proved the following theorem.

Theorem 1.2. *For any prime* p *and any positive integer* B*, the Diophantine equation*

$$x^2 + p^n = B$$

has at most three solutions (x, n) *in nonnegative integers. Furthermore, if* $p \geq 3$ *and* $p^2 \nmid B$*, we can replace three by two.*

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Their result and our previous results (see [5]-[7]) give us the motivation to consider the following equation

$$Dx^2 + k^n = B \tag{1.2}$$

and to prove the following result.

THEOREM 1.3. *Let p be a prime, B and D be positive integers. Then, the Diophantine equation*

$$Dx^2 + p^n = B \tag{1.3}$$

has at most three nonnegative integer solutions (x, n). Furthermore, if $p^2 \nmid B$, then we can replace three by two when $p \geq 3$ or when $p = 2$ with $D \neq 1$ when B is odd and $D \neq 2$ when B is even.

REMARK 1.4. The result in Theorem 1.3 is the best possible.

(1) Choose D so that $4D \pm 1 = p^r$, where p is a prime and $r \geq 1$. Then, for $B = 64D^3 \pm 48D^2 + 13D \pm 1$, we have $p^2 \nmid B$ and the equation (1.3) has the solutions $(x, n) = (1, 3r)$, $(8D \pm 3, r)$, where the sign agrees with the sign in $4D \pm 1$.
(2) For $(p, D, B) = \left(2, 3, \frac{4}{3}(2^{4m} + 2^{2m} + 1)\right), m > 1$, the equation (1.3) has the solutions

$$(x, n) = \left(\frac{1}{3}(2^{2m+1} + 1), 0\right),$$
$$\left(\frac{1}{3}(2^{2m+1} - 2), 2m + 2\right), \left(\frac{2}{3}(2^{2m-1} + 1), 4m\right).$$

2. PRELIMINARIES

First, we recall a result on Pell equation, which was proved by Walker ([4]) and a slightly improved version with a short and straightforward proof by Luo and Yuan ([2]).

LEMMA 2.1. *Let (x, y) be a positive integer solution of the Diophantine equation*

$$ux^2 - vy^2 = 1, \tag{2.1}$$

where $u > 1$ and v are coprime positive integers with uv nonsquare.

If every prime divisor of x divides u, then either

$$x\sqrt{u} + y\sqrt{v} = \varepsilon$$

or

$$x\sqrt{u} + y\sqrt{v} = \varepsilon^3, x = 3^t x_1, 3 \nmid x_1, 3^t + 3 = 4ux_1^2,$$

where $\varepsilon = x_1\sqrt{u} + y_1\sqrt{v}$ is the minimal positive solution of (2.1) and t is a positive integer.

Now, we will prove a series of three results that will be useful for the proof of Theorem 1.3. The first result in this series is the following.

LEMMA 2.2. *Let D be a nonsquare positive integer and A a positive integer. Let p be a prime. Then, the Diophantine equation*

$$Ap^{2m} - Dy^2 = 1 \tag{2.2}$$

has at most one positive integer solution (m, y).

PROOF. Let $(m, y) = (r, a)$ be the least positive integer solution of (2.2). Consider (2.2) as an example of (2.1): letting u and v be as in Lemma 2.1, let

$$u = Ap^{2r}, \quad v = D.$$

Let $(m, y) = (s, b)$ be any positive integer solution to (2.2). Let $\varepsilon = \sqrt{Ap^{2r}} + a\sqrt{D}$ and let $\alpha = p^{s-r}\sqrt{Ap^{2r}} + b\sqrt{D}$. By Lemma 2.1 either $\alpha = \varepsilon$ or $\alpha = \varepsilon^3$. If $\alpha = \varepsilon^3$ then, by Lemma 2.1, $p^{s-r} = 3^t$, so that $p = 3$. But then the equation $3^t + 3 = 4Ap^{2r}$, which is required by Lemma 2.1, is impossible modulo 9. So by Lemma 2.1, we must have $\alpha = \varepsilon$ and then $s = r$, which completes the proof of Lemma 2.2. ☐

We will now prove the second preliminary result. Here, we deal with the case where p is an odd prime with $p^2 \nmid B$.

LEMMA 2.3. *Let B, D be positive integers with $D > 1$ and $B \geq 4D$. Let p be an odd prime with $p^2 \nmid B$. Then, the Diophantine equation (1.3) has at most two nonnegative integer solutions (x, n).*

PROOF. We will consider two cases according to the divisibility of B by p.

(1) $p \nmid B$. At this level, we will also study the problem according to the divisibility of D by p.

(i) If $p|D$, then n can only take the value 0 since $p \nmid B$. So, Diophantine equation (1.3) has at most one nonnegative integer solution (x, n).

(ii) If $p \nmid D$, then here we will study the following two claims.

CLAIM 1. *There is at most one nonnegative integer solution (x, n) satisfying $p^n < 2\sqrt{D(B-1)} - D + 1$.*

Assume that (x_1, n_1) and (x_2, n_2) are two distinct integer solutions of equation (1.3) satisfying $x_1 > x_2 \geq 0$, $p^{n_1} < p^{n_2} < 2\sqrt{D(B-1)} - D + 1$. Thus, we get

$$D(x_1^2 - x_2^2) = p^{n_2} - p^{n_1} \leq p^{n_2} - 1$$

and

$$D(x_1^2 - x_2^2) = D(x_1 + x_2)(x_1 - x_2) \geq D(x_1 + x_2) \geq D(2x_2 + 1) \geq 2Dx_2 + D.$$

This means that $p^{n_2} - (D + 1) \geq 2Dx_2$, which yields

$$p^{2n_2} - 2(D + 1)p^{n_2} + (D + 1)^2 \geq 4D^2x_2^2 = 4D(B - p^{n_2}).$$

Therefore, we obtain

$$p^{2n_2} + 2(D-1)p^{n_2} + (D-1)^2 + 4D \geq 4DB,$$

i.e.

$$(p^{n_2} + D - 1)^2 \geq 4D(B-1),$$

which yields $p^{n_2} \geq 2\sqrt{D(B-1)} - D + 1$. This leads to a contradiction and finishes the proof of the first claim.

Claim 2. *There is at most one nonnegative integer solution* (x, n) *satisfying* $p^n \geq 2\sqrt{D(B-1)} - D + 1$.

In this case, we have $n > 0$ since $2\sqrt{D(B-1)} - D + 1 > 1$, $B \geq 4D$, and $D > 1$. Assume that (x_1, n_1) and (x_2, n_2) are two distinct integer solutions of equation (1.3) satisfying $x_1 > x_2 \geq 0$, $p^{n_2} > p^{n_1} \geq 2\sqrt{D(B-1)} - D + 1$. We have $p \nmid x_1 x_2$ as $p \nmid B$. So, $p \geq 3$ leads to $p \nmid \gcd(x_1 + x_2, x_1 - x_2)$. Then, from

$$D(x_1 + x_2)(x_1 - x_2) = D(x_1^2 - x_2^2) = p^{n_2} - p^{n_1} = p^{n_1}(p^{n_2 - n_1} - 1)$$

and $p \nmid D$, we deduce that $p^{n_1} | x_1 + x_2$ or $p^{n_1} | x_1 - x_2$. Therefore, we get

$$2x_1 - 1 \geq x_1 + x_2 \geq p^{n_1}.$$

This implies that

$$B - p^{n_1} = Dx_1^2 \geq D\left(\frac{p^{n_1} + 1}{2}\right)^2.$$

Thus, we deduce that

$$4BD + 4D + 4 \geq (Dp^{n_1} + D + 2)^2,$$

which yields

$$p^{n_1} \leq \sqrt{\frac{4B}{D} + \frac{4}{D} + \frac{4}{D^2}} - 1 - \frac{2}{D}.$$

Recall that $D > 1$ and $B \geq 4D$. Thus, we have

$$\begin{aligned} 2\sqrt{D(B-1)} - D + 1 &= \sqrt{D(B-1)} + \sqrt{D(B-1)} - D + 1 \\ &\geq \sqrt{2(B-1)} + \sqrt{D(4D-1)} - D + 1 \\ &> \sqrt{2(B-1)} + 2D - 1 - D + 1 \\ &= \sqrt{2(B-1)} + D \geq \sqrt{2(B-1)} + 2 \end{aligned}$$

and

$$\sqrt{\frac{4B}{D} + \frac{4}{D} + \frac{4}{D^2}} - 1 - \frac{2}{D} < \sqrt{2B+3} - 1 < \sqrt{2(B-1)} + 2.$$

This leads to a contradiction and completes the proof of the second claim.

(2) $p || B$, that is $p | B$, but $p^2 \nmid B$. At this level also, we will also study the problem according to the divisibility of D by p.

(i) Suppose that $p|D$. Let $D = pD_1$. If $p|D_1$, then $n = 1$ since $p^2 \nmid B$. If $p \nmid D_1$, let $B = pB_1$, then $p \nmid B_1$. It is obvious that $n \geq 1$ and the Diophantine equation (1.3) turns into $D_1x^2 + p^{n_1} = B_1$, with $n_1 = n-1$. By the result of (1) for $D_1 > 1$ and Theorem 1.2 for $D_1 = 1$, this equation has at most two nonnegative integer solutions (x, n_1), then the Diophantine equation (1.3) has at most two nonnegative integer solutions (x, n).

(ii) Finally, suppose that $p \nmid D$. If $n \geq 2$, then $p|x$ and we get $p^2|B$, which is a contradiction. So we have $n \leq 1$ and then the Diophantine equation (1.3) has at most two nonnegative integer solutions (x, n).

□

The last preliminary result deals with the case $p = 2$. The proof will follow the line of that of Lemma 2.3. But for the sake of completeness, we will give some details.

Lemma 2.4. *Let B, D be positive integers with $4 \nmid B$, $B \geq 4D$, $D \neq 1$ when B is odd and $D \neq 2$ when B is even. Then, the Diophantine equation*

$$Dx^2 + 2^n = B \tag{2.3}$$

has at most two nonnegative integer solutions (x, n).

Proof. We will also consider two cases.

(1) $2 \nmid B$, then $D > 1$ since $D \neq 1$. Here will also distinguish two cases according to the parity of D.

(i) If $2|D$, then n can only take the value 0 since $2 \nmid B$. Therefore, Diophantine equation (1.3) has at most one nonnegative integer solution (x, n).

(ii) If $2 \nmid D$, then we will study the following two claims.

Claim 1. *There is at most one nonnegative integer solution (x, n) satisfying $2^n < 2\sqrt{D(B-1)} - D + 1$.*

The proof of this claim is similar to that of Lemma 2.3, Claim 1. Then, we leave it to the reader.

Claim 2. *There is at most one nonnegative integer solution (x, n) satisfying $2^n \geq 2\sqrt{D(B-1)} - D + 1$.*

In this case, we have $n > 0$ since $2\sqrt{D(B-1)} - D + 1 > 1$, $B \geq 4D$, and $D > 1$. Assume that (x_1, n_1) and (x_2, n_2) are two distinct integer solutions of equation (1.3) satisfying $x_1 > x_2 \geq 0$, $2^{n_2} > 2^{n_1} \geq 2\sqrt{D(B-1)} - D + 1$. One can see that $2 \nmid x_1x_2$ since $2 \nmid B$. So, we get $2 \| \gcd(x_1 + x_2, x_1 - x_2)$. Then, from

$$D(x_1 + x_2)(x_1 - x_2) = D(x_1^2 - x_2^2) = 2^{n_2} - 2^{n_1} = 2^{n_1}(2^{n_2 - n_1} - 1),$$

we deduce that $2^{n_1-1}|x_1+x_2$ or $2^{n_1-1}|x_1-x_2$. Hence, we obtain

$$2x_1-2\geq x_1+x_2\geq 2^{n_1-1}.$$

This implies that

$$B-2^{n_1}=Dx_1^2\geq D(2^{n_1-2}+1)^2.$$

Thus, we deduce that

$$BD+4D+4\geq (2^{n_1-2}D+D+2)^2,$$

which yields

$$2^{n_1}\leq 4\sqrt{\frac{B}{D}+\frac{4}{D}+\frac{4}{D^2}}-4-\frac{8}{D}.$$

Recall that $D>1$ and $2\nmid D$. We have $D=3$ or $D\geq 5$. As $B\geq 4D$, if $D=3$, then a straightforward calculation shows that

$$4\sqrt{\frac{B}{D}+\frac{4}{D}+\frac{4}{D^2}}-4-\frac{8}{D}<2\sqrt{D(B-1)}-D+1.$$

For $D\geq 5$, we can make a discussion similar that of Lemma 2.3, Claim 2. This leads to a contradiction.

(2) If $2||B$, that is $2|B$, but $4\nmid B$, then one can use a method similar to that of Lemma 2.3 (2) for $D>2$, but the case $D=2$ will leads to $D_1=1$ which is not handled by Theorem 1.2. If $D=1$ and $n\geq 2$, then $2|x$, which leads to $4|B$, so we have $n\leq 1$. We conclude that the Diophantine equation (1.3) has at most two nonnegative integer solutions (x,n) in this case for $D\neq 2$.

□

3. Proof of Theorem 1.3

Let us start the proof by studying some particular cases:

- If $B<4D$, then $x\leq 1$ and therefore equation (1.3) has at most two nonnegative integer solutions (x,n).
- If $D=d^2D_1$, we can rewrite Dx^2 as $D_1(dx)^2=D_1z^2$. If $D_1=1$, we can use Theorem 1.2, with the exceptional case $p=2, 2||B$ by Lemma 2.4.

Therefore, for the remainder of the proof, we assume that $B\geq 4D$ and $D>1$ squarefree. Moreover, we will consider two cases: $p^2\nmid B$ and $p^2\mid B$.

Case 1: $p^2\nmid B$. Combining Lemma 2.3 and Lemma 2.4, we see that equation (1.3) has at most two nonnegative integer solutions (x,n) in this case.

Case 2: $p^2|B$. Here also, we will consider two cases according to the divisibility of D by p.

(i) If $p \nmid D$, then we will use Lemma 2.2 to prove that equation (1.3) has at most three nonnegative integer solutions (x, n). Assume that $p^{2k}|B$ and $p^{2(k+1)} \nmid B$. Let $B = p^{2k}B_0$. We will prove that there is at most one nonnegative integer solution (x, n) satisfying $n < 2k$ and at most two nonnegative integer solutions (x, n) satisfying $n \geq 2k$.

If (x, n) is a nonnegative integer solution of (1.3) with $n < 2k$, then from $Dx^2 + p^n = B = p^{2k}B_0$, we deduce that $2|n$. Put $n = 2m$. Then, $p^m|x$. Put $x = p^m z$. Thus, we have

$$Dz^2 + 1 = B_0 p^{2(k-m)},$$

with $k - m = l \geq 1$, i.e.

$$B_0 p^{2l} - Dz^2 = 1.$$

By Lemma 2.2, the above equation has most one positive integer solution (z, l). This means that equation (1.3) has at most one nonnegative integer solution (x, n) satisfying $n < 2k$.

If $n \geq 2k$, then $p^k|x$. Put $x = p^k z$, $u = n - 2k$, $B = p^{2k}B_0$. Then, equation (1.3) becomes

$$Dz^2 + p^u = B_0,$$

with $p^2 \nmid B_0$. By Case 1, this equation has at most two nonnegative integer solution (z, u), i.e. equation (1.3) has at most two nonnegative integer solutions (x, n) satisfying $n \geq 2k$.

(ii) If $p|D$, then it is obvious that $n \geq 1$. Let $D = pD_1, n_1 = n-1, B = pB_1$, then $p \nmid D_1$ and equation (1.3) becomes

$$D_1 x^2 + p^{n_1} = B_1.$$

If $p||B_1$, then $n_1 \leq 1$, and equation (1.3) has at most two nonnegative integer solutions (x, n). If $p^2|B_1$, then equation (1.3) has at most three nonnegative integer solutions (x, n) for $D_1 = 1$ by Theorem 1.2 and for $D_1 > 1$ by Case 2 (i). This completes the proof of Theorem 1.3.

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Z. Zhang
School of Mathematics and Statistics
Zhaoqing University
Zhaoqing 526061
China
E-mail: bee2357@163.com

A. Togbé
Department of Mathematics and Statistics
Purdue University Northwest
1401 S. U.S. 421 Westville, IN 46391
USA
E-mail: atogbe@pnw.edu

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