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Some Remarks on Non-Symmetric Interpolation Macdonald Polynomials

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We provide elementary identities relating the three known types of non-symmetric interpolation Macdonald polynomials. In addition we derive a duality for non-symmetric interpolation Macdonald polynomials. We consider some applications of these results, in particular to binomial formulas involving non-symmetric interpolation Macdonald polynomials.

1 Introduction

The symmetric interpolation Macdonald polynomials $R_\lambda(x;q,t) = R_\lambda(x_1,\ldots,x_n;q,t)$ form a distinguished inhomogeneous basis for the algebra of symmetric polynomials in n variables over the field $\mathbb{F} := \mathbb{Q}(q,t)$. They were first introduced in [4, 13], building on joint work by one of the authors with Knop [5] and earlier work with Kostant [6, 7, 12]. These polynomials are indexed by the set of partitions with at most n parts

$$\mathcal{P}_n := \left\{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \right\}.$$

For a partition $\mu \in \mathcal{P}_n$ we define $|\mu| = \mu_1 + \cdots + \mu_n$ and write

$$\overline{\mu} = \left(q^{\mu_1}\tau_1,\ldots,q^{\mu_n}\tau_n\right) \text{ where } \tau := (\tau_1,\ldots,\tau_n) \text{ with } \tau_i := t^{1-i}.$$

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Then $R_\lambda(x) = R_\lambda(x;q,t)$ is, up to normalization, characterized as the unique nonzero symmetric polynomial of degree at most $|\lambda|$ satisfying the vanishing conditions

$$R_\lambda(\overline{\mu}) = 0 \text{ for } \mu \in \mathcal{P}_n \text{ such that } |\mu| \leq |\lambda|, \mu \neq \lambda.$$

The normalization is fixed by requiring that the coefficient of $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in the monomial expansion of $R_\lambda(x)$ is 1. In spite of their deceptively simple definition, these polynomials possess some truly remarkable properties. For instance, as shown in [4, 13], the top homogeneous part of $R_\lambda(x)$ is the Macdonald polynomial $P_\lambda(x)$ [9] and $R_\lambda(x)$ satisfies the extra vanishing property $R_\lambda(\overline{\mu}) = 0$ unless $\lambda \subseteq \mu$ as Ferrer diagrams. Other key properties of $R_\lambda(x)$, which were proven by Okounkov [10], include the binomial theorem, which gives an explicit expansion of $R_\lambda(ax) = R_\lambda(ax_1, \ldots, ax_n; q, t)$ in terms of the $R_\mu(x; q^{-1}, t^{-1})$'s over the field $\mathbb{K} := \mathbb{Q}(q,t,a)$, and the *duality* or *evaluation symmetry*, which involves the evaluation points

$$\widetilde{\mu} = \left(q^{-\mu_n}\tau_1, \ldots, q^{-\mu_1}\tau_n\right), \qquad \mu \in \mathcal{P}_n$$

and takes the form

$$\frac{R_\lambda(a\widetilde{\mu})}{R_\lambda(a\tau)} = \frac{R_\mu(a\widetilde{\lambda})}{R_\mu(a\tau)}.$$

The interpolation polynomials have natural non-symmetric analogs $G_\alpha(x) = G_\alpha(x;q,t)$, which were also defined in [4, 13]. These are indexed by the set of compositions with at most n parts, $\mathcal{C}_n := \left(\mathbb{Z}_{\geq 0}\right)^n$. For a composition $\beta \in \mathcal{C}_n$ we define

$$\overline{\beta} := w_\beta(\overline{\beta_+}),$$

where w_β is the shortest permutation such that $\beta_+ = w_\beta^{-1}(\beta)$ is a partition. Then $G_\alpha(x)$ is, up to normalization, characterized as the unique polynomial of degree at most $|\alpha| := \alpha_1 + \cdots + \alpha_n$ satisfying the vanishing conditions

$$G_\alpha(\overline{\beta}) = 0 \text{ for } \beta \in \mathcal{C}_n \text{ such that } |\beta| \leq |\alpha|, \beta \neq \alpha.$$

The normalization is fixed by requiring that the coefficient of $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in the monomial expansion of $G_\alpha(x)$ is 1.

Many properties of the symmetric interpolation polynomials $R_\lambda(x)$ admit non-symmetric counterparts for the $G_\alpha(x)$. For instance, the top homogeneous part of $G_\alpha(x)$

is the non-symmetric Macdonald polynomial $E_\alpha(x)$ and $G_\alpha(x)$ satisfies an extra vanishing property [4]. An analog of the binomial theorem, proved by one of us in [14, Thm. 1.1], gives an explicit expansion of $G_\alpha(ax;q,t)$ in terms of a 2nd family of interpolation polynomials $G'_\alpha(x) = G'_\alpha(x;q,t)$. These latter polynomials are characterized by having the same top homogeneous part as $G_\alpha(x)$, namely the non-symmetric polynomial $E_\alpha(x)$, and the following vanishing conditions at the evaluation points $\widetilde{\beta} := \overline{(-w_0\beta)}$, with w_0 the longest element of the symmetric group S_n:

$$G'_\alpha(\widetilde{\beta}) = 0 \text{ for } |\beta| < |\alpha|\,.$$

The 1st result of the present paper is a Demazure-type formula for the primed interpolation polynomials $G'_\alpha(x)$ in terms of $G_\alpha(x)$, which involves the symmetric group action on the algebra of polynomials in n variables over $\mathbb{F}$ by permuting the variables, as well as the associated Hecke algebra action in terms of Demazure-Lusztig operators H_w ($w \in S_n$) as described in the next section.

Theorem A. Write $\mathrm{I}(\alpha) := \#\{i < j \mid \alpha_i \geq \alpha_j\}$. Then we have

$$G'_\alpha(t^{n-1}x;q^{-1},t^{-1}) = t^{(n-1)|\alpha|-\mathrm{I}(\alpha)}w_0H_{w_0}G_\alpha(x;q,t).$$

This is restated and proved in Theorem 1 below.

The 2nd result is the following duality theorem for $G_\alpha(x)$, which is the non-symmetric analog of Okounkov's duality result.

Theorem B. For all compositions $\alpha,\beta \in \mathcal{C}_n$ we have

$$\frac{G_\alpha(a\widetilde{\beta})}{G_\alpha(a\tau)} = \frac{G_\beta(a\widetilde{\alpha})}{G_\beta(a\tau)}.$$

This is a special case of Theorem 17 below.

We now recall the interpolation O-polynomials introduced in [14, Thm. 1.1]. Write x^{-1} for $(x_1^{-1},\ldots,x_n^{-1})$. Then it was shown in [14, Thm. 1.1] that there exists a unique polynomial $O_\alpha(x) = O_\alpha(x;q,t;a)$ of degree at most $|\alpha|$ with coefficients in the field $\mathbb{K}$ such that

$$O_\alpha(\overline{\beta}^{-1}) = \frac{G_\beta(a\widetilde{\alpha})}{G_\beta(a\tau)} \text{ for all } \beta.$$

Our 3rd result is a simple expression for the O-polynomials in terms of the interpolation polynomials $G_\alpha(x)$.

Theorem C. For all compositions $\alpha \in \mathcal{C}_n$ we have

$$O_\alpha(x) = \frac{G_\alpha(t^{1-n} a w_0 x)}{G_\alpha(a\tau)}.$$

This is deduced in Proposition 22 below as a direct consequence of non-symmetric duality. We also obtain new proofs of Okounkov's [10] duality theorem, as well as the dual binomial theorem of Lascoux et al. [8], which gives an expansion of the primed-interpolation polynomials $G'_\alpha(x)$ in terms of the $G_\beta(ax)$'s.

2 Demazure-Lusztig Operators and the Primed Interpolation Polynomials

We use the notations from [14]. The correspondence with the notations from the other important references [4], [13] and [10] is listed in [14, Section 2] (directly after Lemma 2.8).

Let S_n be the symmetric group in n letters and $s_i \in S_n$ the permutation that swaps i and $i+1$. The s_i $(1 \le i < n)$ are Coxeter generators for S_n. Let $\ell : S_n \to \mathbb{Z}_{\ge 0}$ be the associated length function. Let S_n act on $\mathbb{Z}^n$ and $\mathbb{K}^n$ by $s_i v := (\cdots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \ldots)$ for $v = (v_1, \ldots, v_n)$. Write $w_0 \in S_n$ for the longest element, given explicitly by $i \to n+1-i$ for $i = 1, \ldots, n$.

For $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ define $\overline{v} = (\overline{v}_1, \ldots, \overline{v}_n) \in \mathbb{F}^n$ by $\overline{v}_i := q^{v_i} t^{-k_i(v)}$ with

$$k_i(v) := \#\{k < i \mid v_k \ge v_i\} + \#\{k > i \mid v_k > v_i\}.$$

If $v \in \mathbb{Z}^n$ has non-increasing entries $v_1 \ge v_2 \ge \cdots \ge v_n$, then $\overline{v} = (q^{v_1}\tau_1, \ldots, q^{v_n}\tau_n)$. For arbitrary $v \in \mathbb{Z}^n$ we have $\overline{v} = w_v(\overline{v_+})$ with $w_v \in S_n$ the shortest permutation such that $v_+ := w_v^{-1}(v)$ has non-increasing entries, see [4, Section 2]. We write $\widetilde{v} := \overline{-w_0 v}$ for $v \in \mathbb{Z}^n$.

Note that $\overline{\alpha}_n = t^{1-n}$ if $\alpha \in \mathcal{C}_n$ with $\alpha_n = 0$.

For a field F we write $F[x] := F[x_1, \ldots, x_n]$, $F[x^{\pm 1}] := F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $F(x)$ for the quotient field of $F[x]$. The symmetric group acts by algebra automorphisms on $\mathbb{F}[x]$ and $\mathbb{F}(x)$, with the action of s_i by interchanging x_i and x_{i+1} for $1 \le i < n$. Consider the $\mathbb{F}$-linear operators

$$H_i = ts_i - \frac{(1-t)x_i}{x_i - x_{i+1}}(1 - s_i) = t + \frac{x_i - tx_{i+1}}{x_i - x_{i+1}}(s_i - 1)$$

on $\mathbb{F}(x)$ $(1 \le i < n)$ called Demazure-Lusztig operators, and the automorphism Δ of $\mathbb{F}(x)$ defined by

$$\Delta f(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).$$

Note that H_i $(1 \le i < n)$ and Δ preserve $\mathbb{F}[x^{\pm 1}]$ and $\mathbb{F}[x]$. Cherednik [1, 2] showed that the operators H_i $(1 \le i < n)$ and Δ satisfy the defining relations of the type A extended affine Hecke algebra,

$$\begin{aligned}(H_i - t)(H_i + 1) &= 0,\\ H_iH_j &= H_jH_i, \qquad |i-j| > 1,\\ H_iH_{i+1}H_i &= H_{i+1}H_iH_{i+1},\\ \Delta H_{i+1} &= H_i\Delta,\\ \Delta^2 H_1 &= H_{n-1}\Delta^2\end{aligned}$$

for all the indices such that both sides of the equation make sense (see also [4, Section 3]). For $w \in S_n$ we write $H_w := H_{i_1}H_{i_2}\cdots H_{i_\ell}$ with $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ a reduced expression for $w \in S_n$. It is well defined because of the braid relations for the H_i's. Write $\overline{H}_i := H_i + 1 - t = tH_i^{-1}$ and set

$$\xi_i := t^{1-n}\overline{H}_{i-1}\cdots\overline{H}_1\Delta^{-1}H_{n-1}\cdots H_i, \qquad 1 \le i \le n. \tag{1}$$

The operators ξ_i's are pairwise commuting invertible operators, with inverses

$$\xi_i^{-1} = \overline{H}_i\cdots\overline{H}_{n-1}\Delta H_1\cdots H_{i-1}.$$

The ξ_i^{-1} $(1 \le i \le n)$ are the Cherednik operators [2, 4].

The monic non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ of degree $\alpha \in \mathcal{C}_n$ is the unique polynomial satisfying

$$\xi_i^{-1}E_\alpha = \overline{\alpha}_i E_\alpha, \qquad i = 1, \ldots, n$$

and normalized such that the coefficient of x^α in E_α is 1.

Let ι be the field automorphism of $\mathbb{K}$ inverting q, t and a. It restricts to a field automorphism of $\mathbb{F}$, inverting q and t. We extend ι to a $\mathbb{Q}$-algebra automorphism of $\mathbb{K}[x]$

and $\mathbb{F}[x]$ by letting ι act on the coefficients of the polynomial. Write

$$G^\circ_\alpha := \iota\big(G_\alpha\big), \qquad E^\circ_\alpha := \iota\big(E_\alpha\big)$$

for $\alpha \in \mathcal{C}_n$. Note that $\overline{v}^{-1} = (\iota(\overline{v}_1), \ldots, \iota(\overline{v}_n))$.

Put H°_i, H°_w, $\overline{H}^\circ_i$, Δ° and ξ°_i for the operators H_i, H_w, $\overline{H}_i$, Δ and ξ_i with q, t replaced by their inverses. For instance,

$$H^\circ_i = t^{-1}s_i - \frac{(1-t^{-1})x_i}{x_i - x_{i+1}}(1-s_i),$$

$$\Delta^\circ f(x_1, \ldots, x_n) = f(qx_n, x_1, \ldots, x_{n-1}).$$

We then have $\xi^\circ_i E^\circ_\alpha = \overline{\alpha}_i E^\circ_\alpha$ for $i = 1, \ldots, n$, which characterizes E°_α up to a scalar factor.

Theorem 1. For $\alpha \in \mathcal{C}_n$ we have

$$G'_\alpha(x) = t^{(1-n)|\alpha| + \mathrm{I}(\alpha)} w_0 H^\circ_{w_0} G^\circ_\alpha(t^{n-1}x) \tag{2}$$

with $\mathrm{I}(\alpha) := \#\{i < j \,|\, \alpha_i \geq \alpha_j\}$.

Remark. Formally set $t = q^r$, replace x by $1 + (q-1)x$, divide both sides of (2) by $(q-1)^{|\alpha|}$ and take the limit $q \to 1$. Then

$$G'_\alpha(x; r) = (-1)^{|\alpha|}\sigma(w_0) w_0 G_\alpha(-x - (n-1)r; r) \tag{3}$$

for the non-symmetric interpolation Jack polynomial $G_\alpha(\cdot\,; r)$ and its primed version (see [14]). Here σ denotes the action of the symmetric group with $\sigma(s_i)$ the rational degeneration of the Demazure-Lusztig operators H_i, given explicitly by

$$\sigma(s_i) = s_i + \frac{r}{x_i - x_{i+1}}(1 - s_i),$$

see [14, Section 1]. To establish the formal limit (3) one uses that $\sigma(w_0)w_0 = w_0\sigma^\circ(w_0)$ with σ° the action of the symmetric group defined in terms of the rational degeneration

$$\sigma^\circ(s_i) = s_i - \frac{r}{x_i - x_{i+1}}(1 - s_i)$$

of H°_i. Formula (3) was obtained before in [14, Thm. 1.10].

Proof. We show that the right-hand side of (2) satisfies the defining properties of G'_α. For the vanishing property, note that

$$t^{n-1} w_0 \widetilde{\beta} = \overline{\beta}^{-1} \tag{4}$$

(this is the q-analog of [14, Lem. 6.1(2)]); hence,

$$\big(w_0 H^\circ_{w_0} G^\circ_\alpha(t^{n-1}x)\big)|_{x=\widetilde{\beta}} = \big(H^\circ_{w_0} G^\circ_\alpha(x)\big)|_{x=\overline{\beta}^{-1}}.$$

This expression is zero for $|\beta| < |\alpha|$ since it is a linear combination of the evaluated interpolation polynomials $G^\circ_\alpha(\overline{w\beta}^{-1})$ ($w \in S_n$) by [14, Lem. 2.1(2)].

It remains to show that the top homogeneous terms of both sides of (2) are the same, that is, that

$$E_\alpha = t^{\mathrm{I}(\alpha)} w_0 H^\circ_{w_0} E^\circ_\alpha. \tag{5}$$

Note that $\Psi := w_0 H^\circ_{w_0}$ satisfies the intertwining properties

$$\begin{aligned} H_i \Psi &= t\Psi \overline{H}^\circ_i, \\ \Delta\Psi &= t^{n-1}\Psi \overline{H}^\circ_{n-1}\cdots\overline{H}^\circ_1(\Delta^\circ)^{-1}H^\circ_{n-1}\cdots H^\circ_1 \end{aligned} \tag{6}$$

for $1 \le i < n$ (use e.g., [2, Prop. 3.2.2]). It follows that $\xi_i^{-1}\Psi = \Psi\xi_i^\circ$ for $i = 1,\ldots,n$. Therefore,

$$E_\alpha(x) = c_\alpha \Psi E^\circ_\alpha(x)$$

for some constant $c_\alpha \in \mathbb{F}$. But the coefficient of x^α in Ψx^α is $t^{-\mathrm{I}(\alpha)}$; hence, $c_\alpha = t^{\mathrm{I}(\alpha)}$. ■

Consider the Demazure operators H_i and the Cherednik operators ξ_j^{-1} as operators on the space $\mathbb{F}[x^{\pm 1}]$ of Laurent polynomials. For an integral vector $u \in \mathbb{Z}^n$, let $E_u \in \mathbb{F}[x^{\pm 1}]$ be the common eigenfunction of the Cherednik operators ξ_j^{-1} with eigenvalues $\overline{u}_j$ ($1 \le j \le n$), normalized such that the coefficient of $x^u := x_1^{u_1}\cdots x_n^{u_n}$ in E_u is 1. For $u = \alpha \in \mathcal{C}_n$ this definition reproduces the non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ as defined before. Note that

$$E_{u+(1^n)} = x_1\cdots x_n E_u(x).$$

It is now easy to check that formula (5) is valid with α replaced by an arbitrary integral vector u,

$$E_u = t^{\mathrm{I}(u)} w_0 H^{\circ}_{w_0} E^{\circ}_u \tag{7}$$

with $E^{\circ}_u := \iota(E_u)$. Furthermore, one can show in the same vein as the proof of (5) that

$$w_0 E_{-w_0 u}(x^{-1}) = E_u(x)$$

for an integral vector u, where $p(x^{-1})$ stands for inverting all the parameters $x_1, \ldots, x_n$ in the Laurent polynomial $p(x) \in \mathbb{F}[x^{\pm 1}]$. Combining this equality with (7) yields

$$E_{-w_0 u}(x^{-1}) = t^{\mathrm{I}(u)} H^{\circ}_{w_0} E^{\circ}_u(x),$$

which is a special case of a known identity for non-symmetric Macdonald polynomials (see [2, Prop. 3.3.3]).

3 Evaluation Formulas

In [14, Thm. 1.1] the following combinatorial evaluation formula

$$G_\alpha(a\tau) = \prod_{s\in\alpha}\Big(\frac{t^{1-n} - q^{a'(s)+1}t^{1-l'(s)}}{1 - q^{a(s)+1}t^{l(s)+1}}\Big)\prod_{s\in\alpha}(at^{l'(s)} - q^{a'(s)}) \tag{8}$$

was obtained, with $a(s)$, $l(s)$, $a'(s)$ and $l'(s)$ the arm, leg, coarm and coleg of $s = (i,j) \in \alpha$, defined by

$$\begin{aligned} a(s) &:= \alpha_i - j, & l(s) &:= \#\{k > i \mid j \le \alpha_k \le \alpha_i\} + \#\{k < i \mid j \le \alpha_k + 1 \le \alpha_i\},\\ a'(s) &:= j - 1, & l'(s) &:= \#\{k > i \mid \alpha_k > \alpha_i\} + \#\{k < i \mid \alpha_k \ge \alpha_i\}. \end{aligned}$$

By (8) we have

$$E_\alpha(\tau) = \lim_{a\to\infty} a^{-|\alpha|} G_\alpha(a\tau) = \prod_{s\in\alpha}\Big(\frac{t^{1-n+l'(s)} - q^{a'(s)+1}t}{1 - q^{a(s)+1}t^{l(s)+1}}\Big),$$

which is the well-known evaluation formula [1, 2] for the non-symmetric Macdonald polynomials. Note that for $\alpha \in \mathcal{C}_n$,

$$\ell(w_0) - I(\alpha) = \#\{i < j \mid \alpha_i < \alpha_j\}.$$

Lemma 2. For $\alpha \in \mathcal{C}_n$ we have

$$G'_\alpha(a\tau) = t^{(1-n)|\alpha|+\mathrm{I}(\alpha)-\ell(w_0)} G^\circ_\alpha(a\tau^{-1}).$$

Proof. Since $t^{n-1}w_0\tau = \tau^{-1} = \overline{0}^{-1}$ we have by Theorem 1,

$$\begin{aligned} G'_\alpha(a\tau) &= t^{(1-n)|\alpha|+\mathrm{I}(\alpha)}\big(H^\circ_{w_0}G^\circ_\alpha\big)(a\overline{0}^{-1}) \\ &= t^{(1-n)|\alpha|+\mathrm{I}(\alpha)-\ell(w_0)} G^\circ_\alpha(a\overline{0}^{-1}), \end{aligned}$$

where we have used [14, Lem. 2.1(2)] for the 2nd equality. ■

We now derive a relation between the evaluation formulas for $G_\alpha(x)$ and $G^\circ_\alpha(x)$. To formulate this we write, following [8],

$$n(\alpha) := \sum_{s\in\alpha} l(s), \qquad n'(\alpha) := \sum_{s\in\alpha} a(s).$$

Note that $n'(\alpha) = \sum_{i=1}^n \binom{\alpha_i}{2}$; hence, it only depends on the S_n-orbit of α, while

$$n(\alpha) = n(\alpha^+) + \ell(w_0) - I(\alpha). \tag{9}$$

The following lemma is a non-symmetric version of the 1st displayed formula on [10, page 537].

Lemma 3. For $\alpha \in \mathcal{C}_n$ we have

$$G_\alpha(a\tau) = (-a)^{|\alpha|} t^{(1-n)|\alpha|-n(\alpha)} q^{n'(\alpha)} G^\circ_\alpha(a^{-1}\tau^{-1}).$$

Proof. This follows from the explicit evaluation formula (8) for the non-symmetric interpolation Macdonald polynomial G_α. ■

Following [8, (3.9)] we define $\tau_\alpha \in \mathbb{F}$ ($\alpha \in \mathcal{C}_n$) by

$$\tau_\alpha := (-1)^{|\alpha|} q^{n'(\alpha)} t^{-n(\alpha^+)}. \tag{10}$$

It only depends on the S_n-orbit of α.

Corollary 4. For $\alpha \in \mathcal{C}_n$ we have

$$G'_\alpha(a^{-1}\tau) = \tau_\alpha^{-1} a^{-|\alpha|} G_\alpha(a\tau).$$

Proof. Use Lemmas 2 and 3 and (9). ■

4 Normalized Interpolation Macdonald Polynomials

We need the basic representation of the (double) affine Hecke algebra on the space of $\mathbb{K}$-valued functions on $\mathbb{Z}^n$, which is constructed as follows.

For $v \in \mathbb{Z}^n$ and $y \in \mathbb{K}^n$ write $v^\natural := (v_2, \ldots, v_n, v_1 + 1)$ and $y^\natural := (y_2, \ldots, y_n, qy_1)$. Denote the inverse of $^\natural$ by $^\sharp$, so $v^\sharp = (v_n - 1, v_1, \ldots, v_{n-1})$ and $y^\sharp = (y_n/q, y_1, \ldots, y_{n-1})$. We have the following lemma (cf. [4, 13, 14]).

Lemma 5. Let $v \in \mathbb{Z}^n$ and $1 \leq i < n$. Then we have

1. $s_i(\overline{v}) = \overline{s_i v}$ if $v_i \neq v_{i+1}$.
2. $\overline{v}_i = t\overline{v}_{i+1}$ if $v_i = v_{i+1}$.
3. $\overline{v}^\natural = \overline{v^\natural}$.

Let $\mathbb{H}$ be the double affine Hecke algebra over $\mathbb{K}$. It is isomorphic to the subalgebra of $\mathrm{End}(\mathbb{K}[x^{\pm 1}])$ generated by the operators H_i $(1 \leq i < n)$, $\Delta^{\pm 1}$, and the multiplication operators $x_j^{\pm 1}$ $(1 \leq j \leq n)$.

For a unital $\mathbb{K}$-algebra A we write $\mathcal{F}_A$ for the space of A-valued functions $f : \mathbb{Z}^n \to A$ on $\mathbb{Z}^n$.

Corollary 6. Let A be a unital $\mathbb{K}$-algebra. Consider the A-linear operators $\widehat{H}_i$ $(1 \leq i < n)$, $\widehat{\Delta}$ and $\widehat{x}_j$ $(1 \leq j \leq n)$ on $\mathcal{F}_A$ defined by

$$\begin{aligned}
(\widehat{H}_i f)(v) &:= tf(v) + \frac{\overline{v}_i - t\overline{v}_{i+1}}{\overline{v}_i - \overline{v}_{i+1}}(f(s_i v) - f(v)),\\
(\widehat{\Delta} f)(v) &:= f(v^\sharp), \qquad (\widehat{\Delta}^{-1} f)(v) := f(v^\natural),\\
(\widehat{x}_j f)(v) &:= a\overline{v}_j f(v)
\end{aligned} \tag{11}$$

for $f \in \mathcal{F}_A$ and $v \in \mathbb{Z}^n$. Then $H_i \mapsto \widehat{H}_i$ $(1 \leq i < n)$, $\Delta \mapsto \widehat{\Delta}$ and $x_j \mapsto \widehat{x}_j$ $(1 \leq j \leq n)$ defines a representation $\mathbb{H} \to \mathrm{End}_A(\mathcal{F}_A)$, $X \mapsto \widehat{X}$ $(X \in \mathbb{H})$ of the double affine Hecke algebra $\mathbb{H}$ on $\mathcal{F}_A$.

Proof. Let $\mathcal{O} \subset \mathbb{K}^n$ be the smallest S_n-invariant and $\natural$-invariant subset that contains $\{a\overline{v} \mid v \in \mathbb{Z}^n\}$. Note that $\mathcal{O}$ is contained in $\{y \in \mathbb{K}^n \mid y_i \neq y_j \text{ if } i \neq j\}$. The Demazure–Lusztig operators H_i $(1 \leq i < n)$, $\Delta^{\pm 1}$ and the coordinate multiplication operators x_j $(1 \leq j \leq n)$ act A-linearly on the space $F_A^{\mathcal{O}}$ of A-valued functions on $\mathcal{O}$, and hence turns $F_A^{\mathcal{O}}$ into an $\mathbb{H}$-module. Define the surjective A-linear map

$$\mathrm{pr} : F_A^{\mathcal{O}} \to \mathcal{F}_A$$

by $\mathrm{pr}(g)(v) := g(a\overline{v})$ $(v \in \mathbb{Z}^n)$.

We claim that $\mathrm{Ker}(\mathrm{pr})$ is an $\mathbb{H}$-submodule of $F_A^{\mathcal{O}}$. Clearly $\mathrm{Ker}(\mathrm{pr})$ is x_j-invariant for $j = 1, \ldots, n$. Let $g \in \mathrm{Ker}(\mathrm{pr})$. Part 3 of Lemma 5 implies that $\Delta g \in \mathrm{Ker}(\mathrm{pr})$. To show that $H_i g \in \mathrm{Ker}(\mathrm{pr})$ we consider two cases. If $v_i \neq v_{i+1}$ then $s_i\overline{v} = \overline{s_i v}$ by part 1 of Lemma 5. Hence,

$$(H_i g)(a\overline{v}) = tg(a\overline{v}) + \frac{\overline{v}_i - t\overline{v}_{i+1}}{\overline{v}_i - \overline{v}_{i+1}}(g(a\overline{s_i v}) - g(a\overline{v})) = 0.$$

If $v_i = v_{i+1}$ then $\overline{v}_i = t\overline{v}_{i+1}$ by part 2 of Lemma 5. Hence,

$$(H_i g)(\overline{v}) = tg(a\overline{v}) + \frac{\overline{v}_i - t\overline{v}_{i+1}}{\overline{v}_i - \overline{v}_{i+1}}(g(as_i\overline{v}) - g(a\overline{v})) = tg(a\overline{v}) = 0.$$

Hence, $\mathcal{F}_A$ inherits the $\mathbb{H}$-module structure of $F_A^{\mathcal{O}}/\mathrm{Ker}(\mathrm{pr})$. It is a straightforward computation, using Lemma 5 again, to show that the resulting action of H_i $(1 \leq i < n)$, Δ and x_j $(1 \leq j \leq n)$ on $\mathcal{F}_A$ is by the operators $\widehat{H}_i$ $(1 \leq i < n)$, $\widehat{\Delta}$ and $\widehat{x}_j$ $(1 \leq j \leq n)$. ■

Remark 7. With the notations from (the proof of) Corollary 6, let $\widetilde{g} \in F_A^{\mathcal{O}}$ and set $g := \mathrm{pr}(\widetilde{g}) \in \mathcal{F}_A$. In other words, $g(v) := \widetilde{g}(a\overline{v})$ for all $v \in \mathbb{Z}^n$. Then

$$\big(\widehat{X}g\big)(v) = (X\widetilde{g})(a\overline{v}), \qquad v \in \mathbb{Z}^n$$

for $X = H_i, \Delta^{\pm 1}, x_j$.

Remark 8. Let $\mathcal{F}_A^+$ be the space of A-valued functions on $\mathcal{C}_n$. We sometimes will consider $\widehat{H}_i$ $(1 \leq i < n)$, $\widehat{\Delta}^{-1}$ and $\widehat{x}_j$ $(1 \leq j \leq n)$, defined by the formulas (11), as linear operators on $\mathcal{F}_A^+$.

Definition 9. We call

$$K_\alpha(x; q, t; a) := \frac{G_\alpha(x; q, t)}{G_\alpha(a\tau; q, t)} \in \mathbb{K}[x] \tag{12}$$

the normalized non-symmetric interpolation Macdonald polynomial of degree α.

We frequently use the shorthand notation $K_\alpha(x) := K_\alpha(x;q,t;a)$. We will see in a moment that formulas for non-symmetric interpolation Macdonald polynomials take the nicest form in this particular normalization.

Note that a cannot be specialized to 1 in (12) since $G_\alpha(\tau) = G_\alpha(\overline{0}) = 0$ if $\alpha \in \mathcal{C}_n$ is nonzero. Note furthermore that

$$\lim_{a\to\infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)} \tag{13}$$

since $\lim_{a\to\infty} a^{-|\alpha|}G_\alpha(ax) = E_\alpha(x)$.

Recall from [4] the operator $\Phi = (x_n - t^{1-n})\Delta \in \mathbb{H}$ and the inhomogeneous Cherednik operators

$$\Xi_j = \frac{1}{x_j} + \frac{1}{x_j}H_j\cdots H_{n-1}\Phi H_1\cdots H_{j-1} \in \mathbb{H}, \qquad 1 \le j \le n.$$

The operators H_i, Ξ_j and Φ preserve $\mathbb{K}[x]$ (see [4]); hence, they give rise to $\mathbb{K}$-linear operators on $\mathcal{F}^+_{\mathbb{K}[x]}$ (e.g., $(H_if)(\alpha) := H_i(f(\alpha))$ for $\alpha \in \mathcal{C}_n$). Note that the operators H_i, Ξ_j and Φ on $\mathcal{F}^+_{\mathbb{K}[x]}$ commute with the hat-operators $\widehat{H}_i$, $\widehat{x}_j$ and $\widehat{\Delta}^{-1}$ on $\mathcal{F}^+_{\mathbb{K}[x]}$ (cf. Remark 8). The same remarks hold true for the space $\mathcal{F}_{\mathbb{K}(x)}$ of $\mathbb{K}(x)$-valued functions on $\mathbb{Z}^n$ (in fact, in this case the hat-operators define a $\mathbb{H}$-action on $\mathcal{F}_{\mathbb{K}(x)}$).

Let $K \in \mathcal{F}^+_{\mathbb{K}[x]}$ be the map $\alpha \mapsto K_\alpha(\cdot)$ ($\alpha \in \mathcal{C}_n$).

Lemma 10. For $1 \le i < n$ and $1 \le j \le n$ we have in $\mathcal{F}^+_{\mathbb{K}[x]}$,

1. $H_iK = \widehat{H}_iK$.
2. $\Xi_jK = a\widehat{x}_j^{-1}K$.
3. $\Phi K = t^{1-n}(a^2\widehat{x}_1^{-1} - 1)\widehat{\Delta}^{-1}K$.

Proof. **1.** To derive the formula we need to expand H_iK_α as a linear combination of the K_β's. As a 1st step we expand H_iG_α as linear combination of the G_β's.

If $\alpha \in \mathcal{C}_n$ satisfies $\alpha_i < \alpha_{i+1}$ then

$$H_iG_\alpha(x) = \frac{(t-1)\overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}}G_\alpha(x) + G_{s_i\alpha}(x)$$

by [14, Lem. 2.2]. Using part 1 of Lemma 5 and the fact that H_i satisfies the quadratic relation $(H_i - t)(H_i + 1) = 0$, it follows that

$$H_iG_\alpha(x) = \frac{(t-1)\overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}}G_\alpha(x) + \frac{t(\overline{\alpha}_{i+1} - t\overline{\alpha}_i)(\overline{\alpha}_{i+1} - t^{-1}\overline{\alpha}_i)}{(\overline{\alpha}_{i+1} - \overline{\alpha}_i)^2}G_{s_i\alpha}(x)$$

if $\alpha \in \mathcal{C}_n$ satisfies $\alpha_i > \alpha_{i+1}$. Finally, $H_i G_\alpha(x) = tG_\alpha(x)$ if $\alpha \in \mathcal{C}_n$ satisfies $\alpha_i = \alpha_{i+1}$ by [4, Cor. 3.4].

An explicit expansion of $H_i K_\alpha$ as linear combination of the K_β's can now be obtained using the formula

$$G_\alpha(a\tau) = \frac{\overline{\alpha}_{i+1} - t\overline{\alpha}_i}{\overline{\alpha}_{i+1} - \overline{\alpha}_i} G_{s_i\alpha}(a\tau)$$

for $\alpha \in \mathcal{C}_n$ satisfying $\alpha_i > \alpha_{i+1}$, cf. the proof of [14, Lem 3.1]. By a direct computation the resulting expansion formula can be written as $H_i K = \widehat{H}_i K$.

2. See [4, Thm. 2.6].

3. Let $\alpha \in \mathcal{C}_n$. By [14, Lem. 2.2 (1)],

$$\Phi G_\alpha(x) = q^{-\alpha_1} G_{\alpha^\natural}(x).$$

By the evaluation formula (8) we have

$$\frac{G_{\alpha^\natural}(a\tau)}{G_\alpha(a\tau)} = at^{1-n+k_1(\alpha)} - q^{\alpha_1} t^{1-n}.$$

Hence,

$$\Phi K_\alpha(x) = t^{1-n}(a\overline{\alpha}_1^{-1} - 1) K_{\alpha^\natural}(x).$$

■

Remark 11. Note that

$$\Phi K_\alpha(x) = (a\widetilde{\alpha}_n - t^{1-n}) K_{\alpha^\natural}(x)$$

for $\alpha \in \mathcal{C}_n$ since $\overline{\alpha}^{-1} = t^{n-1} w_0 \widetilde{\alpha}$.

5 Interpolation Macdonald Polynomials with Negative Degrees

In this section we give the natural extension of the interpolation Macdonald polynomials $G_\alpha(x)$ and $K_\alpha(x)$ to $\alpha \in \mathbb{Z}^n$. It will be the unique extension of $K \in \mathcal{F}^+_{\mathbb{K}[x]}$ to a map $K \in \mathcal{F}_{\mathbb{K}(x)}$ such that Lemma 10 remains valid.

Lemma 12. For $\alpha \in \mathcal{C}_n$ we have

$$G_\alpha(x) = q^{-|\alpha|} \frac{G_{\alpha+(1^n)}(qx)}{\prod_{i=1}^n (qx_i - t^{1-n})},$$

$$K_\alpha(x) = \Big(\prod_{i=1}^n \frac{(1 - a\overline{\alpha}_i^{-1})}{(1 - qt^{n-1}x_i)}\Big) K_{\alpha+(1^n)}(qx).$$

Proof. Note that for $f \in \mathbb{K}[x]$,

$$\Phi^n f(x) = \Big(\prod_{i=1}^{n}(x_i - t^{1-n})\Big) f(q^{-1}x).$$

The 1st formula then follows by iteration of [14, Lem. 2.2(1)] and the 2nd formula from part 3 of Lemma 10. ■

For $m \in \mathbb{Z}_{\geq 0}$ we define $A_m(x; v) \in \mathbb{K}(x)$ by

$$A_m(x; v) := \prod_{i=1}^{n} \frac{\left(q^{1-m} a \overline{v}_i^{-1}; q\right)_m}{\left(q t^{n-1} x_i; q\right)_m} \qquad \forall\, v \in \mathbb{Z}^n, \tag{14}$$

with $\left(y; q\right)_m := \prod_{j=0}^{m-1}(1 - q^j y)$ the q-shifted factorial.

Definition 13. Let $v \in \mathbb{Z}^n$ and write $|v| := v_1 + \cdots + v_n$. Define $G_v(x) = G_v(x; q, t) \in \mathbb{F}(x)$ and $K_v(x) = K_v(x; q, t; a) \in \mathbb{K}(x)$ by

$$G_v(x) := q^{-m|v| - m^2 n} \frac{G_{v+(m^n)}(q^m x)}{\prod_{i=1}^{n} x_i^m \left(q^{-m} t^{1-n} x_i^{-1}; q\right)_m},$$

$$K_v(x) := A_m(x; v) K_{v+(m^n)}(q^m x),$$

where m is a nonnegative integer such that $v + (m^n) \in \mathcal{C}_n$ (note that G_v and K_v are well defined by Lemma 12).

Example 14. If $n = 1$ then for $m \in \mathbb{Z}_{\geq 0}$,

$$K_{-m}(x) = \frac{\left(qa; q\right)_m}{\left(qx; q\right)_m}, \qquad K_m(x) = \left(\frac{x}{a}\right)^m \frac{\left(x^{-1}; q\right)_m}{\left(a^{-1}; q\right)_m}.$$

Lemma 15. For all $v \in \mathbb{Z}^n$,

$$K_v(x) = \frac{G_v(x)}{G_v(a\tau)}.$$

Proof. Let $v \in \mathbb{Z}^n$. Clearly $G_v(x)$ and $K_v(x)$ only differ by a multiplicative constant, so it suffices to show that $K_v(a\tau) = 1$. Fix $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in \mathcal{C}_n$. Then

$$K_v(a\tau) = A_m(a\tau; v) K_{v+(m^n)}(q^m a\tau) = A_m(a\tau; v) \frac{G_{v+(m^n)}(q^m a\tau)}{G_{v+(m^n)}(a\tau)} = 1,$$

where the last formula follows from a direct computation using the evaluation formula (8). ∎

We extend the map $K : \mathcal{C}_n \to \mathbb{K}[x]$ to a map

$$K : \mathbb{Z}^n \to \mathbb{K}(x)$$

by setting $v \mapsto K_v(x)$ for all $v \in \mathbb{Z}^n$. Lemma 10 now extends as follows.

Proposition 16. We have, as identities in $\mathcal{F}_{\mathbb{K}(x)}$,

1. $H_iK = \widehat{H}_iK$.
2. $\Xi_jK = a\widehat{x}_j^{-1}K$.
3. $\Phi K = t^{1-n}(a^2\widehat{x}_1^{-1} - 1)\widehat{\Delta}^{-1}K$.

Proof. Write $A_m \in \mathcal{F}_{\mathbb{K}(x)}$ for the map $v \mapsto A_m(x; v)$ for $v \in \mathbb{Z}^n$. Consider the linear operator on $\mathcal{F}_{\mathbb{K}(x)}$ defined by $(A_mf)(v) := A_m(x; v)f(v)$ for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_{\mathbb{K}(x)}$. For $1 \leq i < n$ we have $[H_i, A_m] = 0$ as linear operators on $\mathcal{F}_{\mathbb{K}(x)}$, since $A_m(x; v)$ is a symmetric rational function in $x_1, \ldots, x_n$. Furthermore, for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_{\mathbb{K}(x)}$,

$$\big((\widehat{H}_i \circ A_m)f\big)(v) = \big((A_m \circ \widehat{H}_i)f\big)(v) \quad \text{if} \quad v_i \neq v_{i+1} \tag{15}$$

by part 2 of Lemma 5 and the fact that $A_m(x; v)$ is symmetric in $\overline{v}_1, \ldots, \overline{v}_n$. Fix $v \in \mathbb{Z}^n$ and choose $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in \mathcal{C}_n$. Since

$$K_v(x) = A_m(x; v)K_{v+(m^n)}(q^mx)$$

we obtain from $[H_i, A_m] = 0$ and (15) that $(H_iK)(v) = (\widehat{H}_iK)(v)$ if $v_i \neq v_{i+1}$. This also holds true if $v_i = v_{i+1}$ since then $(\widehat{H}_iK)(v) = tK_v$ and $H_iK_{v+(m^n)}(q^mx) = tK_{v+(m^n)}(q^mx)$. This proves part 1 of the proposition.

Note that $\Phi K_v(x) = t^{1-n}(a\overline{v}_1^{-1} - 1)K_{v^\natural}(x)$ for arbitrary $v \in \mathbb{Z}^n$ by Lemma 10 and the commutation relation

$$\Phi \circ A_m = A_m \circ \Phi^{(q^m)}, \tag{16}$$

where $\Phi^{(q^m)} := (q^mx_n - t^{1-n})\Delta$. This proves part 3 of the proposition.

Finally we have $\Xi_jK_v(x) = \overline{v}_j^{-1}K_v(x)$ for all $v \in \mathbb{Z}^n$ by $[H_i, A_m] = 0$, (16) and Lemma 10. This proves part 2 of the proposition. ∎

6 Duality of the Non-Symmetric Interpolation Macdonald Polynomials

Recall the notation $\widetilde{v} = \overline{-w_0 v}$ for $v \in \mathbb{Z}^n$.

Theorem 17. (Duality). For all $u, v \in \mathbb{Z}^n$ we have

$$K_u(a\widetilde{v}) = K_v(a\widetilde{u}). \tag{17}$$

Example 18. If $n = 1$ and $m, r \in \mathbb{Z}_{\geq 0}$ then

$$K_m(aq^{-r}) = q^{-mr} \frac{(a^{-1}; q)_{m+r}}{(a^{-1}; q)_m (a^{-1}; q)_r} \tag{18}$$

by the explicit expression for $K_m(x)$ from Example 14. The right-hand side of (18) is manifestly invariant under the interchange of m and r.

Proof. We divide the proof of the theorem in several steps. ■

Step 1. If $K_u(a\widetilde{v}) = K_v(a\widetilde{u})$ for all $v \in \mathbb{Z}^n$ then $K_{s_i u}(a\widetilde{v}) = K_v(a\widetilde{s_i u})$ for $v \in \mathbb{Z}^n$ and $1 \leq i < n$.

Proof of Step 1. Writing out the formula from part 1 of Proposition 16 gives

$$\begin{aligned} &\frac{(t-1)\widetilde{v}_i}{(\widetilde{v}_i - \widetilde{v}_{i+1})} K_u(a\widetilde{v}) + \left(\frac{\widetilde{v}_i - t\widetilde{v}_{i+1}}{\widetilde{v}_i - \widetilde{v}_{i+1}}\right) K_u(a\widetilde{s_{n-i}v}) \\ &\qquad = \frac{(t-1)\overline{u}_i}{(\overline{u}_i - \overline{u}_{i+1})} K_u(a\widetilde{v}) + \left(\frac{\overline{u}_i - t\overline{u}_{i+1}}{\overline{u}_i - \overline{u}_{i+1}}\right) K_{s_i u}(a\widetilde{v}). \end{aligned} \tag{19}$$

Replacing in (19) the role of u and v and replacing i by $n - i$ we get

$$\begin{aligned} &\frac{(t-1)\widetilde{u}_{n-i}}{(\widetilde{u}_{n-i} - \widetilde{u}_{n+1-i})} K_v(a\widetilde{u}) + \left(\frac{\widetilde{u}_{n-i} - t\widetilde{u}_{n+1-i}}{\widetilde{u}_{n-i} - \widetilde{u}_{n+1-i}}\right) K_v(a\widetilde{s_i u}) \\ &\qquad = \frac{(t-1)\overline{v}_{n-i}}{(\overline{v}_{n-i} - \overline{v}_{n+1-i})} K_v(a\widetilde{u}) + \left(\frac{\overline{v}_{n-i} - t\overline{v}_{n+1-i}}{\overline{v}_{n-i} - \overline{v}_{n+1-i}}\right) K_{s_{n-i}v}(a\widetilde{u}). \end{aligned} \tag{20}$$

Suppose that $s_{n-i}v = v$. Then $\overline{v}_{n-i} = t\overline{v}_{n+1-i}$ by the 2nd part of Lemma 5. Since $\widetilde{v} = t^{1-n} w_0 \overline{v}^{-1}$, that is, $\widetilde{v}_i = t^{1-n} \overline{v}_{n+1-i}^{-1}$, we then also have $\widetilde{v}_i = t\widetilde{v}_{i+1}$. It then follows by a direct computation that (19) reduces to $K_{s_i u}(a\widetilde{v}) = K_u(a\widetilde{v})$ and (20) to $K_v(a\widetilde{s_i u}) = K_v(a\widetilde{u})$ if $s_{n-i}v = v$.

We now use these observations to prove Step 1. Assume that $K_u(a\widetilde{v}) = K_v(a\widetilde{u})$ for all v. We have to show that $K_{s_iu}(a\widetilde{v}) = K_v(a\widetilde{s_iu})$ for all v. It is trivially true if $s_iu = u$, so we may assume that $s_iu \neq u$. Suppose that v satisfies $s_{n-i}v = v$. Then it follows from the previous paragraph that

$$K_{s_iu}(a\widetilde{v}) = K_u(a\widetilde{v}) = K_v(a\widetilde{u}) = K_v(a\widetilde{s_iu}).$$

If $s_{n-i}v \neq v$ then (19) and the induction hypothesis can be used to write $K_{s_iu}(a\widetilde{v})$ as an explicit linear combination of $K_v(a\widetilde{u})$ and $K_{s_{n-i}v}(a\widetilde{u})$. Then (20) can be used to rewrite the term involving $K_{s_{n-i}v}(a\widetilde{u})$ as an explicit linear combination of $K_v(a\widetilde{u})$ and $K_v(a\widetilde{s_iu})$. Hence, we obtain an explicit expression of $K_{s_iu}(a\widetilde{v})$ as linear combination of $K_v(a\widetilde{u})$ and $K_v(a\widetilde{s_iu})$, which turns out to reduce to $K_{s_iu}(a\widetilde{v}) = K_v(a\widetilde{s_iu})$ after a direct computation. ■

Step 2. $K_0(a\widetilde{v}) = 1 = K_v(a\widetilde{0})$ for all $v \in \mathbb{Z}^n$.

Proof of Step 2. Clearly $K_0(x) = 1$ and $K_v(a\widetilde{0}) = K_v(a\tau) = 1$ for $v \in \mathbb{Z}^n$ by Lemma 15. ■

Step 3. $K_\alpha(a\widetilde{v}) = K_v(a\widetilde{\alpha})$ for $v \in \mathbb{Z}^n$ and $\alpha \in \mathcal{C}_n$.

Proof of Step 3. We prove it by induction. It is true for $\alpha = 0$ by Step 2. Let $m \in \mathbb{Z}_{>0}$ and suppose that $K_\gamma(a\widetilde{v}) = K_v(a\widetilde{\gamma})$ for $v \in \mathbb{Z}^n$ and $\gamma \in \mathcal{C}_n$ with $|\gamma| < m$. Let $\alpha \in \mathcal{C}_n$ with $|\alpha| = m$.

We need to show that $K_\alpha(a\widetilde{v}) = K_v(a\widetilde{\alpha})$ for all $v \in \mathbb{Z}^n$. By Step 1 we may assume without loss of generality that $\alpha_n > 0$. Then $\gamma := \alpha^\sharp \in \mathcal{C}_n$ satisfies $|\gamma| = m - 1$, and $\alpha = \gamma^\natural$. Furthermore, note that we have the formula

$$(a\overline{v}_1^{-1} - 1)K_u(a\widetilde{v^\natural}) = (a\overline{u}_1^{-1} - 1)K_{u^\natural}(a\widetilde{v}) \tag{21}$$

for all $u, v \in \mathbb{Z}^n$, which follows by writing out the formula from part 3 of Lemma 16. Hence, we obtain

$$\begin{aligned} K_\alpha(a\widetilde{v}) = K_{\gamma^\natural}(a\widetilde{v}) &= \frac{(a\overline{v}_1^{-1} - 1)}{(a\overline{\gamma}_1^{-1} - 1)} K_\gamma(a\widetilde{v^\natural}) \\ &= \frac{(a\overline{v}_1^{-1} - 1)}{(a\overline{\gamma}_1^{-1} - 1)} K_{v^\natural}(a\widetilde{\gamma}) = K_v(a\widetilde{\gamma^\natural}) = K_v(a\widetilde{\alpha}), \end{aligned}$$

where we used the induction hypothesis for the 3rd equality and (21) for the 2nd and 4th equality. This proves the induction step. ■

Step 4. $K_u(a\widetilde{v}) = K_v(a\widetilde{u})$ for all $u, v \in \mathbb{Z}^n$.

Proof of Step 4. Fix $u, v \in \mathbb{Z}^n$. Let $m \in \mathbb{Z}_{\geq 0}$ such that $u + (m^n) \in \mathcal{C}_n$. Note that $q^m\widetilde{v} = \widetilde{v - (m^n)}$ and $q^{-m}\widetilde{u} = \widetilde{u + (m^n)}$. Then

$$\begin{aligned}
K_u(a\widetilde{v}) &= A_m(a\widetilde{v}; u)K_{u+(m^n)}(q^m a\widetilde{v}) \\
&= A_m(a\widetilde{v}; u)K_{u+(m^n)}\big(a(\widetilde{v - (m^n)})\big) \\
&= A_m(a\widetilde{v}; u)K_{v-(m^n)}\big(a(\widetilde{u + (m^n)})\big) \\
&= A_m(a\widetilde{v}; u)K_{v-(m^n)}(q^{-m}a\widetilde{u}) = A_m(a\widetilde{v}; u)A_m(q^{-m}a\widetilde{u}; v - (m^n))K_v(a\widetilde{u}),
\end{aligned}$$

where we used Step 3 in the 3rd equality. The result now follows from the fact that

$$A_m(a\widetilde{v}; u)A_m(q^{-m}a\widetilde{u}; v - (m^n)) = 1,$$

which follows by a straightforward computation using (4). ∎

7 Some Applications of Duality

7.1 Non-symmetric Macdonald polynomials

Recall that the (monic) non-symmetric Macdonald polynomial $E_\alpha(x)$ of degree α is the top homogeneous component of $G_\alpha(x)$, i.e.,

$$E_\alpha(x) = \lim_{a\to\infty} a^{-|\alpha|}G_\alpha(ax), \qquad \alpha \in \mathcal{C}_n.$$

The normalized non-symmetric Macdonald polynomials are

$$\overline{K}_\alpha(x) := \lim_{a\to\infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}, \qquad \alpha \in \mathcal{C}_n.$$

We write $\overline{K} \in \mathcal{F}^+_{\mathbb{F}[x]}$ for the resulting map $\alpha \mapsto \overline{K}_\alpha$. Taking limits in Lemma 10 we get the following.

Lemma 19. We have for $1 \leq i < n$ and $1 \leq j \leq n$,

1. $H_i\overline{K} = \widehat{H}_i\overline{K}$.
2. $\xi_j\overline{K} = \widehat{x}_j^{-1}\overline{K}$.
3. $x_n\Delta\overline{K} = t^{1-n}\widehat{x}_1^{-1}\widehat{\Delta}^{-1}\overline{K}$.

Note that

$$(x_n\Delta)^n f(x) = \Big(\prod_{i=1}^n x_i\Big) f(q^{-1}x).$$

Then repeated application of part 3 of Lemma 19 shows that for $\alpha \in \mathcal{C}_n$,

$$E_\alpha(x) = \frac{E_{\alpha+(1^n)}(x)}{x_1\cdots x_n},$$
$$\overline{K}_\alpha(x) = q^{|\alpha|}t^{(1-n)n}\Big(\prod_{i=1}^n (\overline{\alpha}_i x_i)^{-1}\Big)\overline{K}_{\alpha+(1^n)}(x). \qquad (22)$$

As is well known and already noted in Section 2, the 1st equality allows to relate the non-symmetric Macdonald polynomials $E_v(x) := E_v(x; q, t) \in \mathbb{F}[x^{\pm 1}]$ for arbitrary $v \in \mathbb{Z}^n$ to those labeled by compositions through the formula

$$E_v(x) = \frac{E_{v+(m^n)}(x)}{(x_1\cdots x_n)^m}.$$

The 2nd formula of (22) can now be used to explicitly define the normalized non-symmetric Macdonald polynomials for degrees $v \in \mathbb{Z}^n$.

Definition 20. Let $v \in \mathbb{Z}^n$ and $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in \mathcal{C}_n$. Then $\overline{K}_v(x) := \overline{K}_v(x; q, t) \in \mathbb{F}[x^{\pm 1}]$ is defined by

$$\overline{K}_v(x) := q^{m|v|}t^{(1-n)nm}\Big(\prod_{i=1}^n (\overline{v}_i x_i)^{-m}\Big)\overline{K}_{v+(m^n)}(x).$$

Using

$$\lim_{a\to\infty} A_m(ax; v) = q^{-m^2 n}t^{(1-n)nm}\prod_{i=1}^n (\overline{v}_i x_i)^{-m}$$

and the definitions of $G_v(x)$ and $K_v(x)$ it follows that

$$\lim_{a\to\infty} a^{-|v|}G_v(ax) = E_v(x),$$
$$\lim_{a\to\infty} K_v(ax) = \overline{K}_v(x)$$

for all $v \in \mathbb{Z}^n$, so in particular

$$\overline{K}_v(x) = \frac{E_v(x)}{E_v(\tau)} \qquad \forall\, v \in \mathbb{Z}^n.$$

Lemma 19 holds true for the extension of $\overline{K}$ to the map $\overline{K} \in \mathcal{F}_{\mathbb{F}[x^{\pm 1}]}$ defined by $v \mapsto \overline{K}_v$ ($v \in \mathbb{Z}^n$). Taking the limit in Theorem 17 we obtain the well-known duality [1] of the Laurent polynomial versions of the normalized non-symmetric Macdonald polynomials.

Corollary 21. For all $u, v \in \mathbb{Z}^n$,

$$\overline{K}_u(\widetilde{v}) = \overline{K}_v(\widetilde{u}).$$

7.2 O-polynomials

We now show that the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17) directly implies the existence of the O-polynomials O_α (which is the nontrivial part of the proof of [14, Thm. 1.2]), and that it provides an explicit expression for O_α in terms of the non-symmetric interpolation Macdonald polynomial K_α.

Proposition 22. For all $\alpha \in \mathcal{C}_n$ we have

$$O_\alpha(x) = K_\alpha(t^{1-n} a w_0 x).$$

Proof. The polynomial $\widetilde{O}_\alpha(x) := K_\alpha(t^{1-n} a w_0 x)$ is of degree at most $|\alpha|$ and

$$\widetilde{O}_\alpha(\overline{\beta}^{-1}) = K_\alpha(t^{1-n} a w_0 \overline{\beta}^{-1}) = K_\alpha(a\widetilde{\beta}) = K_\beta(a\widetilde{\alpha})$$

for all $\beta \in \mathcal{C}_n$ by (4) and Theorem 17. Hence, $\widetilde{O}_\alpha = O_\alpha$. ■

7.3 Okounkov's duality

Write $F[x]^{S_n}$ for the symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in a field F. Write $C_+ := \sum_{w \in S_n} H_w$. The symmetric interpolation Macdonald polynomial $R_\lambda(x) \in \mathbb{F}[x]^{S_n}$ is the multiple of $C_+ G_\lambda$ such that the coefficient of x^λ is one (see, e.g., [13]). We write

$$K_\lambda^+(x) := \frac{R_\lambda(x)}{R_\lambda(a\tau)} \in \mathbb{K}[x]^{S_n}$$

for the normalized symmetric interpolation Macdonald polynomial. Then

$$C_+ K_\alpha(x) = \Big(\sum_{w \in S_n} t^{\ell(w)}\Big) K_{\alpha_+}^+(x) \tag{23}$$

for $\alpha \in \mathcal{C}_n$. Okounkov's [10, Section 2] duality result now reads as follows.

Theorem 23. For partitions $\lambda, \mu \in \mathcal{P}_n$ we have

$$K_\lambda^+(a\overline{\mu}^{-1}) = K_\mu^+(a\overline{\lambda}^{-1}).$$

Let us derive Theorem 23 as consequence of Theorem 17. Write $\widehat{C}_+ = \sum_{w\in S_n} \widehat{H}_w$, with $\widehat{H}_w := \widehat{H}_{i_1}\cdots\widehat{H}_{i_r}$ for a reduced expression $w = s_{i_1}\cdots s_{i_r}$. Write $f_\mu \in \mathcal{F}_{\mathbb{K}}$ for the function $f_\mu(u) := K_u(a\widetilde{\mu})$ $(u \in \mathbb{Z}^n)$. Then

$$\Big(\sum_{w\in S_n} t^{\ell(w)}\Big)K_\lambda^+(a\widetilde{\mu}) = (C_+K_\lambda)(a\widetilde{\mu}) = (\widehat{C}_+f_\mu)(\lambda) \tag{24}$$

by part 1 of Proposition 16. The duality (17) of K_u and (4) imply that

$$f_\mu(u) = K_\mu(a\widetilde{u}) = \big(Jw_0K_\mu(t^{1-n}x)\big)|_{x=a^{-1}\overline{u}} \tag{25}$$

with $(Jf)(x) := f(x_1^{-1},\ldots,x_n^{-1})$ for $f \in \mathbb{K}(x)$. A direct computation shows that

$$JH_iJ = (H_i^\circ)^{-1}, \qquad w_0H_iw_0 = (H_{n-i}^\circ)^{-1} \tag{26}$$

for $1 \le i < n$. In particular, $Jw_0C_+ = C_+Jw_0$. Combined with Remark 7 we conclude that

$$(\widehat{C}_+f_\mu)(\lambda) = \big(Jw_0C_+K_\mu(t^{1-n}x)\big)|_{x=a^{-1}\overline{\lambda}}.$$

By (23) and (4) this simplifies to

$$(\widehat{C}_+f_\mu)(\lambda) = \Big(\sum_{w\in S_n} t^{\ell(w)}\Big)K_\mu^+(a\widetilde{\lambda}).$$

Returning to (24) we conclude that $K_\lambda^+(a\widetilde{\mu}) = K_\mu^+(a\widetilde{\lambda})$. Since K_λ^+ is symmetric we obtain from (4) that

$$K_\lambda^+(a\overline{\mu}^{-1}) = K_\mu^+(a\overline{\lambda}^{-1}),$$

which is Okounkov's duality result.

7.4 A primed version of duality

We first derive the following twisted version of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17).

Lemma 24. For $u, v \in \mathbb{Z}^n$ we have

$$\big(H_{w_0}K_u\big)(a\widetilde{v}) = \big(H_{w_0}K_v\big)(a\widetilde{u}). \tag{27}$$

Proof. We proceed as in the previous subsection. Set $f_v(u) := K_u(a\widetilde{v})$ for $u, v \in \mathbb{Z}^n$. By part 1 of Proposition 16,

$$\big(H_{w_0}K_u\big)(a\widetilde{v}) = \big(\widehat{H}_{w_0}f_v\big)(u).$$

Since $f_v(u) = \big(Iw_0K_v\big)(a^{-1}t^{n-1}\overline{u})$ by (4), Remark 7 implies that

$$\big(\widehat{H}_{w_0}f_v\big)(u) = \big(H_{w_0}Jw_0K_v\big)(a^{-1}t^{n-1}\overline{u}).$$

Now $H_{w_0}Jw_0 = Jw_0H_{w_0}$ by (26); hence,

$$\big(\widehat{H}_{w_0}f_v\big)(u) = \big(Jw_0H_{w_0}K_v\big)(a^{-1}t^{n-1}\overline{u}) = (H_{w_0}K_v)(a\widetilde{u}),$$

which completes the proof. ■

Recall from Theorem 1 that

$$G'_\beta(x) = t^{(1-n)|\beta|+I(\beta)}\Psi G^\circ_\beta(t^{n-1}x)$$

with $\Psi := w_0H^\circ_{w_0}$. We define normalized versions by

$$K'_\beta(x) := \frac{G'_\beta(x)}{G'_\beta(a^{-1}\tau)} = t^{\ell(w_0)}\Psi K^\circ_\beta(t^{n-1}x), \qquad \beta \in \mathcal{C}_n,$$

with $K^\circ_v := \iota(K_v)$ for $v \in \mathbb{Z}^n$ (the 2nd formula follows from Lemma 2). More generally, we define for $v \in \mathbb{Z}^n$,

$$K'_v(x) := t^{\ell(w_0)}\Psi K^\circ_v(t^{n-1}x). \tag{28}$$

We write $K' : \mathbb{Z}^n \to \mathbb{K}(x)$ for the map $v \mapsto K'_v$ ($v \in \mathbb{Z}^n$). Since $H_i\Psi = \Psi H^\circ_i$, part 1 of Proposition 16 gives $H_iK' = \widehat{H}^\circ_iK'$. Considering the action of $((x_n-1)\Delta^\circ)^n$ on $K'_\beta(x)$ we get, using the fact that $((x_n-1)\Delta^\circ)^n$ commutes with Ψ and part 3 of Proposition 16,

$$K'_v(x) = \Big(\prod_{i=1}^n \frac{(1-a^{-1}\overline{v}_i)}{(1-q^{-1}x_i)}\Big)K'_{v+(1^n)}(q^{-1}x),$$

in particular

$$K'_v(x) = \Big(\prod_{i=1}^{n} \frac{(a^{-1}\overline{v}_i;q)_m}{(q^{-m}x_i;q)_m}\Big) K'_{v+(m^n)}(q^{-m}x).$$

Example 25. For $n=1$ we have $K'_v(x) = K^\circ_v(x)$ for $v \in \mathbb{Z}$; hence,

$$K'_{-m}(x) = \frac{(q^{-1}a^{-1};q^{-1})_m}{(q^{-1}x;q^{-1})_m} = (ax)^{-m}\frac{(qa;q)_m}{(qx^{-1};q)_m},$$
$$K'_m(x) = (ax)^m \frac{(x^{-1};q^{-1})_m}{(a;q^{-1})_m} = \frac{(x;q)_m}{(a^{-1};q)_m}$$

for $m \in \mathbb{Z}_{\geq 0}$ by Example 14.

Proposition 26. For all $u, v \in \mathbb{Z}^n$ we have

$$K'_v(a^{-1}\overline{u}) = K'_u(a^{-1}\overline{v}).$$

Proof. Note that

$$K'_v(a^{-1}\overline{u}) = t^{\ell(w_0)}\Psi K^\circ_v(t^{n-1}x)|_{x=a^{-1}\overline{u}} = t^{\ell(w_0)}\big(H^\circ_{w_0}K^\circ_v\big)(a^{-1}\widetilde{u}^{-1})$$

by (4). By (27) the right-hand side is invariant under the interchange of u and v. ■

7.5 Binomial formula and dual binomial formula

In [14] the existence and uniqueness of O_α was used to prove the following binomial theorem [14, Thm. 1.3]. Define for $\alpha, \beta \in \mathcal{C}_n$ the generalized binomial coefficient by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} := \frac{G_\beta(\overline{\alpha})}{G_\beta(\overline{\beta})}. \tag{29}$$

Applying the automorphism ι of $\mathbb{F}$ to (29) we get

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1},t^{-1}} = \frac{G^\circ_\beta(\overline{\alpha}^{-1})}{G^\circ_\beta(\overline{\beta}^{-1})}.$$

Theorem 27. For $\alpha, \beta \in \mathcal{C}_n$ we have the binomial formula

$$K_\alpha(ax) = \sum_{\beta \in \mathcal{C}_n} a^{|\beta|} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1},t^{-1}} \frac{G'_\beta(x)}{G_\beta(a\tau)}. \tag{30}$$

Remark 28. **1.** Note that the sum in (30) is finite, since the generalized binomial coefficient (29) is zero unless $\beta \subseteq \alpha$, with $\beta \subseteq \alpha$ meaning $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.
2. By Corollary 4 and (28) the binomial formula (30) can be alternatively written as

$$\begin{aligned} K_\alpha(ax) &= \sum_{\beta \in \mathcal{C}_n} \tau_\beta^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1},t^{-1}} K'_\beta(x) \\ &= \sum_{\beta \in \mathcal{C}_n} \frac{K^\circ_\beta(\overline{\alpha}^{-1}) K'_\beta(x)}{\tau_\beta K^\circ_\beta(\overline{\beta}^{-1})} \\ &= t^{\ell(w_0)} \sum_{\beta \in \mathcal{C}_n} \frac{K^\circ_\beta(\overline{\alpha}^{-1}) \Psi K^\circ_\beta(t^{n-1}x)}{\tau_\beta K^0_\beta(\overline{\beta}^{-1})} \end{aligned} \tag{31}$$

with $\Psi = w_0 H^\circ_{w_0}$ (note that the dependence on a in the right-hand side of (31) is through the normalization factors of the interpolation polynomials $K^\circ_\beta(x)$ and $K'_\beta(x)$).
3. The binomial formula (30) and Theorem 1 imply the twisted duality (27) of K_α as follows. By the identity $H_{w_0}\Psi = w_0$ the binomial formula (31) implies the finite expansion

$$\big(H_{w_0}K_\alpha\big)(ax) = t^{\ell(w_0)} \sum_\beta \frac{K^\circ_\beta(\overline{\alpha}^{-1}) K^\circ_\beta(t^{n-1}w_0x)}{\tau_\beta K^\circ_\beta(\overline{\beta}^{-1})}.$$

Substituting $x = \widetilde{\gamma}$ and using (4) we obtain

$$\big(H_{w_0}K_\alpha\big)(a\widetilde{\gamma}) = \sum_{\beta \in \mathcal{C}_n} \frac{K^\circ_\beta(\overline{\alpha}^{-1}) K^\circ_\beta(\overline{\gamma}^{-1})}{\tau_\beta K^\circ_\beta(\overline{\beta}^{-1})}.$$

The right-hand side is manifestly invariant under interchanging α and γ, which is equivalent to twisted duality (27).

In [8, Section 4] it is remarked that an explicit identity relating G'_α and G_α is needed to provide a proof of the dual binomial formula [8, Thm. 4.4] as a direct consequence of the binomial formula (30). We show here that Theorem 1 is providing the required identity. Instead of Theorem 1 we use its normalized version, encoded by (28).

The dual binomial formula [8, Thm. 4.4] in our notations reads as follows.

Theorem 29. For all $\alpha \in \mathcal{C}_n$ we have

$$K'_\alpha(x) = \sum_{\beta \in \mathcal{C}_n} \tau_\beta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} K_\beta(ax). \tag{32}$$

The starting point of the alternative proof of (32) is the binomial formula in the form

$$K_\alpha(ax) = t^{\ell(w_0)} \sum_{\beta \in \mathcal{C}_n} \frac{G^\circ_\beta(\overline{\alpha}^{-1})\Psi K^\circ_\beta(t^{n-1}x)}{\tau_\beta G^\circ_\beta(\overline{\beta}^{-1})},$$

see (31). Replace (a, x, q, t) by $(a^{-1}, at^{n-1}x, q^{-1}, t^{-1})$ and act by $w_0 H_{w_0}$ on both sides. Since $w_0 H_{w_0} \Psi = \mathrm{Id}$ we obtain

$$\Psi K^\circ_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_\beta \tau_\beta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} K_\beta(ax).$$

Now use (28) to complete the proof of (32).

Remark 30. It follows from this proof of (32) that the dual binomial formula (32) can be rewritten as

$$\Psi K^\circ_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_\beta \frac{\tau_\beta K_\beta(\overline{\alpha}) K_\beta(ax)}{K_\beta(\overline{\beta})}. \tag{33}$$

As observed in [8, (4.11)], the binomial and dual binomial formula directly imply the orthogonality relations

$$\sum_{\beta \in \mathcal{C}_n} \frac{\tau_\beta}{\tau_\alpha} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} \begin{bmatrix} \beta \\ \gamma \end{bmatrix}_{q^{-1},t^{-1}} = \delta_{\alpha,\gamma}.$$

Since $\begin{bmatrix} \delta \\ \epsilon \end{bmatrix}_{q,t} = 0$ unless $\delta \supseteq \epsilon$, the terms in the sum are zero unless $\gamma \subseteq \beta \subseteq \alpha$.

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