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Metaplectic representations of Hecke algebras, Weyl group actions, and associated polynomials

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Abstract

We construct a family of representations of affine Hecke algebras, which depend on a number of auxiliary parameters g_i, and which we refer to as *metaplectic* representations. We realize these representations as quotients of certain parabolically induced modules, and we apply the method of Baxterization (localization) to obtain actions of corresponding Weyl groups on rational functions on the torus. Our construction both generalizes and provides a conceptual proof of earlier results of Chinta, Gunnells, and Puskas, which had depended on a crucial computer verification. A key motivation is that when the parameters g_i are specialized to certain Gauss sums, the resulting representation and its localization arise naturally in the consideration of p-parts of Weyl group multiple Dirichlet series. In this special case, similar results have been previously obtained in the literature by the study of Iwahori Whittaker functions for principal series of metaplectic covers of reductive p-adic groups. However this technique is not available for generic parameters g_i. It turns out that the metaplectic representations can be extended to the double affine Hecke algebra, where they share many important properties with Cherednik's basic polynomial representation, which they generalize. This allows us to introduce families of metaplectic polynomials, which depend on the g_i, and which generalize Macdonald polynomials. In this paper we discuss in some detail the situation for type A, which is of considerable interest in algebraic combinatorics. We postpone some of the proofs, as well as a discussion of other types, to the sequel.

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1 Introduction

This paper contains two main results concerning a somewhat mysterious action of the Weyl group of a reductive Lie group on the algebra of rational functions on its torus. This action was first introduced in type A by Kazhdan and Patterson [24], and in full generality by Chinta and Gunnells [15,16], who used it to obtain formulas for the local parts (p-parts) of Weyl group multiple Dirichlet series. The action involves an integer n and parameters $g_1, \ldots, g_{n-1}$, which in the application are specialized to certain Gauss sums; however it remains a group action even without this specialization. Chinta and Gunnells verified this fact through a computer check and they asked for a conceptual proof. Our first main result provides such a proof in complete generality. The key role in the proof is played by a certain representation of the affine Hecke algebra that we construct in Theorem 3.7 below, and which we refer to as the *metaplectic* representation.

There is a striking analogy between the Chinta–Gunnells setting and the theory of Macdonald polynomials [12,29,31]. The latter are a family of orthogonal polynomials on the torus that depend on two or three "root-length" parameters, and which generalize many important polynomials in representation theory and algebraic combinatorics, including spherical functions for real and p-adic groups. We show that there is much more to this analogy. Our second main result is the construction of a family of polynomials that we refer to as *metaplectic* polynomials. These depend on the root-length parameters as well as the $g_1, \ldots, g_{n-1}$, and are a common generalization of nonsymmetric Macdonald polynomials [11,28] and of the p-parts of Weyl group multiple Dirichlet series. A key point in our construction is extending the metaplectic representation from the affine Hecke algebra to the *double* affine Hecke algebra.

In the present paper we introduce, without proofs, the metaplectic polynomials in type A, where many of the essential ideas already appear. This is the setting of [24] and of Macdonald's book on symmetric functions [30], which is of considerable independent interest in algebraic combinatorics. The consideration of the metaplectic polynomials for arbitrary type requires some additional ideas. This will be presented in a forthcoming paper [36], which will also include the detailed proofs.

1.1 The Chinta–Gunnells action

We recall now briefly the Chinta–Gunnells Weyl group action, referring the reader to [4,5,9,14–16] and especially the survey [8] for the connection to Weyl group multiple Dirichlet series. Let W be the Weyl group of an irreducible root system Φ, with Coxeter generators $\{s_i\}_{i=1}^r$ corresponding to a choice of simple roots $\{\alpha_i\}_{i=1}^r$. Let P be the weight lattice of Φ. The Weyl group canonically acts on the fraction field

$\mathbb{C}(P)$ of the group algebra $\mathbb{C}[P]$ by field automorphisms. Chinta and Gunnells have constructed a deformation of this action, which depends on the choice of a W-invariant quadratic form $\mathbf{Q} : P \to \mathbb{Q}$ taking integer values on the root lattice Q of Φ, a natural number n, and on parameters $v, g_0, \ldots, g_{n-1}$ satisfying

$$g_0 = -1, \quad g_j g_{n-j} = v^{-1}, \qquad j = 1, \ldots, n-1.$$

Let $0 \leq \mathbf{r}_m(j) \leq m-1$ denote the remainder on dividing j by the natural number m, and define g_j for arbitrary $j \in \mathbb{Z}$ by setting $g_j = g_{r_n(j)}$, let $\mathbf{B}(\lambda, \mu) = \mathbf{Q}(\lambda + \mu) - \mathbf{Q}(\lambda) - \mathbf{Q}(\mu)$ be the bilinear form associated to $\mathbf{Q}$, and put $m(\alpha) = n / \gcd(n, \mathbf{Q}(\alpha))$. It defines a new root system $\Phi^m := \{m(\alpha)\alpha\}_{\alpha \in \Phi}$, which is either isomorphic to Φ or to $\Phi^\vee$. The weight lattice $P^m \subseteq P$ of Φ^m is

$$P^m = \{\lambda \in P \mid \mathbf{B}(\lambda, \alpha) \equiv 0 \mod n \quad \forall \alpha \in \Phi\}$$

(see Lemma 2.2). Then the Chinta–Gunnells action $\sigma_i = \sigma(s_i)$ of the simple reflection $s_i \in W$ on $\mathbb{C}(P)$ is given by the formula

$$\begin{aligned}\sigma_i\left(f x^\lambda\right) := \frac{(s_i f) x^{s_i \lambda}}{1 - v x^{m(\alpha_i)\alpha_i}} \Bigg[& x^{-\mathbf{r}_{m(\alpha_i)}\left(-\frac{\mathbf{B}(\lambda,\alpha_i)}{\mathbf{Q}(\alpha_i)}\right)\alpha_i} (1 - v) \\ & - v g_{\mathbf{Q}(\alpha_i) - \mathbf{B}(\lambda, \alpha_i)} x^{(1 - m(\alpha_i))\alpha_i} \left(1 - x^{m(\alpha_i)\alpha_i}\right) \Bigg] \end{aligned} \tag{1.1}$$

for $f \in \mathbb{C}(P^m)$ and $\lambda \in P$.

It is non-trivial to show that the formula (1.1) defines a representation of W. The main issue is to verify that the braid relations are satisfied. Although this reduces to a rank 2 computation, the calculations become rather formidable, and in [16] the details are only presented for A_2. Trying to find a natural interpretation of this representation was one of the main motivations for our work.

Chinta and Gunnells [16] employed the action (1.1) to give an explicit construction of the "local" parts of certain Weyl group multiple Dirichlet series, and to establish thus the analytic continuation and functional equations for these series. In this situation, the g_i are n-th order Gauss sums for the local field, and $v = p^{-1}$ with p the cardinality of the residue field. Subsequently, Chinta–Offen [18] for type A, and McNamara [32] in general, showed that these local parts are essentially Whittaker functions for principal series of certain n-fold "metaplectic" covers of quasi-split reductive groups. The resulting explicit expression for the Whittaker function in terms of the action (1.1) is the metaplectic generalization of the Casselman–Shalika formula. This result is in line with the fact that multiple Dirichlet series should themselves be Whittaker coefficients attached to metaplectic Eisenstein series [6,9].

Still more recently, Chinta–Gunnells–Puskas [17] have shown that the W-action (1.1) gives rise to a Cherednik [12] type Demazure–Lusztig action of the Hecke algebra of W. It leads to an expression of the metaplectic Whittaker functions in terms of metaplectic Demazure–Lusztig operators. Their work was partly motivated by Brubaker–Bump–Licata [7], who gave formulas for (nonmetaplectic) Iwahori–Whittaker functions in terms of Hecke operators and nonsymmetric Macdonald polynomials. The recent work of Patnaik–Puskas [33] uses the Chinta–Gunnells–Puskas Hecke algebra action to study metaplectic Iwahori–Whittaker functions. It leads to a conceptual proof [33, App. B] that (1.1) defines a representation of W when the g_i are n-th order Gauss sums for a local field, and $v = p^{-1}$ with p the cardinality of the corresponding residue field.

1.2 Our results

In Sects. 3 and 4, we give a uniform construction of a Weyl group representation (Theorem 3.21) and an associated Hecke algebra representation (Theorem 4.2) that generalize the Chinta–Gunnells [16] and Chinta–Gunnells–Puskas [17] representations, respectively. Our construction does not involve case-by-case considerations, and it yields a representation for the *generic* Hecke algebra $H(\mathbf{k})$, which has independent Hecke parameters for each root length in Φ. Our method also allows us to incorporate extra freedom in the definition of g_i by allowing them to depend on the root length (see Definition 3.5 of the *representation parameters*). The Chinta–Gunnells and Chinta–Gunnells–Puskas representations are recovered in the equal Hecke and representation parameter case of our constructions.

Our starting point was the observation that (1.1) has many features in common with formulas obtained by the process of "Baxterization" [12]. The key idea behind this process is that the group algebra of the *affine* Weyl group and the *affine* Hecke algebra become *isomorphic* after a suitable localization, which allows one to relate certain representations of the two algebras. This inspired our search for a natural representation of the affine Hecke algebra whose associated localized affine Weyl group representation produces (1.1) for its W-action. Its first form can be recovered from the Chinta–Gunnells–Puskas Hecke algebra action as follows.

Note that the Chinta–Gunnells W-action (1.1) has an obvious extension to a representation of the extended affine Weyl group $\widetilde{W}^m := W \ltimes P^m$ with $\mu \in P^m$ acting on $\mathbb{C}(P)$ by multiplication by x^μ. Let $\widetilde{H}^m(\mathbf{k})$ be the associated extended affine Hecke algebra with single Hecke parameter $\mathbf{k}$ satisfying $\mathbf{k}^2 = v$. If the affine extension of the Chinta–Gunnells W-action on $\mathbb{C}(P)$ arises from a $\widetilde{H}^m(\mathbf{k})$-action on $\mathbb{C}[P]$ by localization, then the generators $\{T_i\}_{i=1}^r$ of the finite Hecke algebra $H(\mathbf{k})$ act on $\mathbb{C}[P]$ by the Chinta–Gunnells–Puskas metaplectic Demazure–Lusztig operators associated to σ_i (cf. Proposition 4.1). It follows that the underlying $H(\mathbf{k})$-representation is equivalent to the $H(\mathbf{k})$-representation on $\mathbb{C}[P]$ defined by

$$\pi(T_i)x^\lambda := (\mathbf{k} - \mathbf{k}^{-1})\overline{\nabla}_i(x^\lambda) - \mathbf{k}g_{-\mathbf{B}(\lambda,\alpha_i)}x^{s_i\lambda}, \qquad \lambda \in P, \tag{1.2}$$

with $\overline{\nabla}_i$ the following metaplectic version of the divided-difference operator

$$\overline{\nabla}_i(x^\lambda) := \frac{x^\lambda - x^{s_i\lambda + \mathbf{r}_{m(\alpha_i)}((\lambda,\alpha_i^\vee))\alpha_i}}{1 - x^{m(\alpha_i)\alpha_i}}.$$

But now we want to have an *a priori* proof that (1.2) defines a $H(\mathbf{k})$-action on $\mathbb{C}[P]$ and *conclude* from it that (1.1) defines a W-action on $\mathbb{C}(P)$ via the localization technique.

Although the formulas (1.2) are much simpler than (1.1), a direct case-by-case check that it defines a $H(\mathbf{k})$-representation will be close to being as cumbersome as for the Chinta–Gunnells action. Our first result is to circumvent the case-by-case check by proving that π is isomorphic to a quotient of the induced module $\widetilde{H}^m(\mathbf{k}) \otimes_{H(\mathbf{k})} V_C$ for an appropriate $H(\mathbf{k})$-representation V_C. This isomorphism in addition allows us to generalize π and the Chinta–Gunnells Weyl group action to the context of generic affine Hecke algebras.

The $H(\mathbf{k})$-representation V_C is defined as follows. Let $V = \bigoplus_{\lambda \in P} \mathbb{C}v_\lambda$ be the complex vector space with basis the weight lattice P. It has a natural left $H(\mathbf{k})$-module structure reducing to the canonical $\mathbb{C}[W]$-module structure when $\mathbf{k} = 1$ (see Lemma 3.1). We call V the reflection representation of $H(\mathbf{k})$. For each W-invariant subset $D \subseteq P$, the subspace $V_D := \bigoplus_{\lambda \in D} \mathbb{C}v_\lambda$ is a $H(\mathbf{k})$-submodule of V. In particular, $V_{\{0\}}$ is the trivial representation of $H(\mathbf{k})$. The appropriate choice of W-invariant subset C of P in the above realization of π now turns out to be

$$C := \{\lambda \in P \mid (\lambda, \alpha^\vee) \leq m(\alpha) \quad \forall\, \alpha \in \Phi\}.$$

Note that C contains a complete set of coset representatives of P/P^m.

The following trivial example is instructive to get a feeling for what is going on. Suppose that $m(\alpha) = 1$ for all $\alpha \in \Phi$. Then $P^m = P$ and $\overline{\nabla}_i$ is the standard divided-difference operator on $\mathbb{C}[P]$. In this case it is well known that (1.2) is equivalent to the induced module $\widetilde{H}^m(\mathbf{k}) \otimes_{H(\mathbf{k})} V_{\{0\}}$ by the Bernstein–Zelevinsky [27] presentation of $\widetilde{H}^m(\mathbf{k})$. The W-subset C in this case is oversized, with $C \setminus \{0\}$ being the set of nonzero minuscule weights in P.

In Sect. 5 we construct the metaplectic polynomials in type A. The extension to arbitrary types will be treated in the forthcoming paper [36]. The GL_r double affine Hecke algebra $\mathbb{H}^{(m)}$ has generators $T_0, \ldots, T_{r-1}, \omega^{\pm 1}, x_1^{\pm m}, \ldots, x_r^{\pm m}$, with $T_0, \ldots, T_{r-1}, \omega^{\pm 1}$. Coxeter type generators of a copy of the GL_r affine Hecke algebra in $\mathbb{H}^{(m)}$ (ω is the generator of the abelian group of group elements of length zero), and $T_1, \ldots, T_{r-1}, x_1^{\pm m}, \ldots, x_r^{\pm m}$ Bernstein–Zelevinsky type generators of the second copy of the GL_r affine Hecke algebra in $\mathbb{H}^{(m)}$ (the $x_j^{\pm m}$ ($j = 1, \ldots, r$) are generating its commutative subalgebra). The metaplectic representation of the second copy of the GL_r affine Hecke algebra is acting on Laurent polynomials in $x_1^{\pm 1}, \ldots, x_r^{\pm 1}$, where T_i for $1 \leq i < r$ act by (1.2) and x^ν ($\nu \in m\mathbb{Z}^r$) act by multiplication. It extends to a representation $\widehat{\pi}$ of $\mathbb{H}^{(m)}$, with ω acting as a twisted-cyclic permutation of the variables and T_0 by an appropriate affine version of the metaplectic Demazure–Lusztig operator (see Theorem 5.4). The representation $\widehat{\pi}$ is a metaplectic generalization of Cherednik's basic representation [12,31], which we call the metaplectic basic representation.

The GL_r affine Hecke algebra generated by $T_0, \ldots, T_{r-1}, \omega^{\pm 1}$ in its Bernstein–Zelevinsky presentation contains an abelian subalgebra generated by elements $Y^{\pm m\epsilon_i}$ ($i = 1, \ldots, r$). We define the metaplectic polynomials $E_\mu^{(m)}$ ($\mu \in \mathbb{Z}^r$) in Theorem 5.7 as the simultaneous eigenfunctions of $\widehat{\pi}(Y^{m\epsilon_i})$ ($i = 1, \ldots, r$). It depends, besides the standard Macdonald parameters, on the additional representation parameters g_j. The subfamily indexed by $m\mathbb{Z}^r$ recovers the nonsymmetric Macdonald polynomials in the variables $x_1^m, \ldots, x_r^m$ (see Remark 5.10). At the end of Sect. 5 we provide examples of

GL_3-metaplectic polynomials, highlighting some important phenomena. In a followup paper [36] other important properties, such as triangularity and orthogonality will be established in the context of arbitrary root systems.

1.3 The structure of the paper

We now briefly discuss the content of the paper. We introduce in Sect. 2 the appropriate metaplectic structures on the root systems and affine Weyl and Hecke algebras. Section 3 is devoted to the metaplectic representation theory of the affine Weyl groups and generic affine Hecke algebras. We introduce the reflection representation in Sect. 3.1. Section 3.2 forms the heart of our approach: we introduce the analogue of (1.2) for generic Hecke and representation parameters and establish that it defines a representation of the generic affine Hecke algebra by identifying it with a quotient of the induced module $\widetilde{H}^m(\mathbf{k}) \otimes_{H(\mathbf{k})} V_C$ (see Theorem 3.7). In Sect. 3.3 we explain the localization technique and apply it to π (Theorem 3.7) to obtain the generalized Chinta–Gunnells W-action (Theorem 3.21).

In Sect. 4 we form the associated metaplectic Demazure–Lusztig operators and generalize some of the results from [17] to the setting of unequal Hecke and representation parameters. We also simplify some of the proofs from that paper by using the standard symmetrizer and antisymmetrizer elements in the Hecke algebra. This allows us to define a natural class of "Whittaker functions" for generic Hecke algebras. It is natural to ask whether these more general functions arise as actual matrix coefficients for some class of representations of p-adic groups. This question is of particular interest since generic Hecke algebras have begun to play an increasing role in the study of the Bernstein components within the categories of smooth representations of p-adic groups, see, e.g., [10,21] and references therein.

In Sect. 5, we construct the metaplectic polynomials in type A. We begin by setting up the notation and modifications specific to the GL_r case. The double affine Hecke algebra $\mathbb{H}^m$ is presented in Sect. 5.2, and the metaplectic basic representation in Sect. 5.3. The characterization of the metaplectic polynomials as eigenfunctions of the metaplectic operators $\widehat{\pi}(Y^\nu)$ ($\nu \in m\mathbb{Z}^r$) may be found in Sect. 5.4. We also discuss the dependence on parameters, showing that we do not lose any generality by taking the quadratic form $\mathbf{Q}$ to satisfy $\mathbf{Q}(\alpha) = 1$ for α a root. Finally, in the "Appendix", we provide a list of metaplectic polynomials for $r = 3$ and $1 \leq m \leq 5$.

Let us conclude with remarking that the localization procedure we use in this paper is instrumental in Cherednik's construction of quantum affine Knizhnik–Zamolodchikov equations attached to affine Hecke algebra modules. Closely related to it is the role of the localization procedure for type A in the context of integrable vertex models with $U_q(\widehat{\mathfrak{sl}}_n)$-symmetry, in the special cases that the associated braid group action descends to an affine Hecke algebra action, in which case the localization procedure is often referred to as Baxterization (see, e.g., [12,37] and references therein). This is exactly the context in which the metaplectic Whittaker function can be realized as a partition function, the corresponding integrable model being "metaplectic ice", see [1–3]. It is an intriguing open question whether there is a conceptual connection with the current interpretation of the Chinta–Gunnells action through localization.

2 The extended affine Hecke algebra

2.1 The root system

Let E be an Euclidean space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $\Phi \subset E$ be an irreducible reduced root system, and $W \subset O(E)$ its Weyl group. The reflection in $\alpha \in \Phi$ is denoted by $s_\alpha \in W$, and its co-root is $\alpha^\vee := 2\alpha/\|\alpha\|^2$.

Fix a base $\{\alpha_1,\ldots,\alpha_r\}$ of Φ. Let Φ^+ be the corresponding set of positive roots and write $s_i := s_{\alpha_i}$ for $i = 1,\ldots,r$. Let

$$P := \{\lambda \in V \mid (\lambda,\alpha^\vee) \in \mathbb{Z} \quad \forall \alpha \in \Phi\} = \bigoplus_{i=1}^{r} \mathbb{Z}\varpi_i$$

be the weight lattice of Φ with $\varpi_i \in E$ the fundamental weights, defined by $(\varpi_i,\alpha_j^\vee) = \delta_{i,j}$. Let

$$Q = \mathbb{Z}\Phi = \bigoplus_{i=1}^{r} \mathbb{Z}\alpha_i$$

be the root lattice of Φ.

2.2 The metaplectic structure

In the theory of metaplectic Whittaker functions, a new root system Φ^m is attached to the metaplectic covering data of the reductive group over the non-archimedean local field, cf. [16,17] and references therein. We recall in this subsection this additional metaplectic data on the root system.

Fix a W-invariant quadratic form $\mathbf{Q} : P \to \mathbb{Q}$ which takes integral values on Q and write $\mathbf{B} : P \times P \to \mathbb{Q}$ for the associated symmetric bilinear pairing

$$\mathbf{B}(\lambda,\mu) := \mathbf{Q}(\lambda+\mu) - \mathbf{Q}(\lambda) - \mathbf{Q}(\mu), \qquad \lambda,\mu \in P.$$

Then $\mathbf{Q}(\cdot) = \frac{\kappa}{2}\|\cdot\|^2$ for some $\kappa \in \mathbb{R}^\times$, and hence $\mathbf{B}(\lambda,\mu) = \kappa(\lambda,\mu)$ for all $\lambda,\mu \in P$. In particular, for all $\lambda \in P$ and $\alpha \in \Phi$,

$$\frac{\mathbf{B}(\lambda,\alpha)}{\mathbf{Q}(\alpha)} = (\lambda,\alpha^\vee). \tag{2.1}$$

Let $n \in \mathbb{Z}_{>0}$ and define

$$m(\alpha) := \frac{n}{\gcd(n,\mathbf{Q}(\alpha))} = \frac{\mathrm{lcm}(n,\mathbf{Q}(\alpha))}{\mathbf{Q}(\alpha)} \qquad \forall\, \alpha \in \Phi.$$

Note that $m : \Phi \to \mathbb{Z}_{>0}$ is W-invariant.

Set $\Phi^m := \{\alpha^m := m(\alpha)\alpha\}_{\alpha\in\Phi} \subset E$. Then Φ^m is a root system. In fact, if m is constant then Φ^m is isomorphic to Φ, while if m is nonconstant then Φ^m is isomorphic to the co-root system $\Phi^\vee = \{\alpha^\vee\}_{\alpha\in\Phi}$ (this follows from the definition of $m(\alpha)$ and the fact that $\mathbf{Q}(\cdot) = \frac{\kappa}{2}\|\cdot\|^2$). In particular, $\{\alpha_1^m, \ldots, \alpha_r^m\}$ is a base of Φ^m and W is the Weyl group of Φ^m.

Write Q^m for the root lattice of Φ^m and P^m for the weight lattice of Φ^m. Since $(\alpha^m)^\vee = m(\alpha)^{-1}\alpha^\vee$ for $\alpha \in \Phi$, we have

$$Q^m = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^m, \qquad P^m = \bigoplus_{i=1}^r \mathbb{Z}\varpi_i^m$$

with $\varpi_i^m := m(\alpha_i)\varpi_i$ the fundamental weights of P^m.

Lemma 2.1 (a) *For $\alpha \in \Phi$ and $\lambda \in P$ we have*

$$\mathbf{B}(\lambda, \alpha^m) = \mathrm{lcm}(n, \mathbf{Q}(\alpha))\big(\lambda, \alpha^\vee\big).$$

(b) *For $\alpha \in \Phi$ and $\lambda \in P$ we have*

$$\mathbf{B}(\lambda, \alpha) \equiv 0 \mod n \;\Leftrightarrow (\lambda, \alpha^\vee) \equiv 0 \mod m(\alpha).$$

Proof. (a) For $\lambda \in P$ and $\alpha \in \Phi$ we have

$$\begin{aligned}\mathbf{B}(\lambda, \alpha^m) &= m(\alpha)\mathbf{B}(\lambda, \alpha)\\ &= \frac{n\,\mathbf{B}(\lambda, \alpha)}{\gcd(n, \mathbf{Q}(\alpha))}\\ &= \frac{n\mathbf{Q}(\alpha)}{\gcd(n, \mathbf{Q}(\alpha))}\frac{\mathbf{B}(\lambda, \alpha)}{\mathbf{Q}(\alpha)} = \mathrm{lcm}(n, \mathbf{Q}(\alpha))(\lambda, \alpha^\vee).\end{aligned}$$

(b) For $\lambda \in P$ and $\alpha \in \Phi$ we have

$$\begin{aligned}\mathbf{B}(\lambda, \alpha) \equiv 0 \mod n \;&\Leftrightarrow\; \mathbf{Q}(\alpha)(\lambda, \alpha^\vee) \equiv 0 \mod n\\ &\Leftrightarrow\; (\lambda, \alpha^\vee) \equiv 0 \mod \frac{n}{\gcd(n, \mathbf{Q}(\alpha))}\\ &\Leftrightarrow\; (\lambda, \alpha^\vee) \equiv 0 \mod m(\alpha). \quad \square\end{aligned}$$

Lemma 2.2

$$\begin{aligned}P^m &= \{\lambda \in P \mid (\lambda, \alpha^\vee) \equiv 0 \;\; mod\;\, m(\alpha) \;\; \forall \alpha \in \Phi\}\\ &= \{\lambda \in P \mid \mathbf{B}(\lambda, \alpha) \equiv 0 \;\; mod \;\; n \;\; \forall \alpha \in \Phi\}.\end{aligned}$$

Proof The first equality follows from the fact that $(\alpha^m)^\vee = m(\alpha)^{-1}\alpha^\vee$ for $\alpha \in \Phi$. The second equality follows immediately from part (b) of Lemma 2.1. $\square$

2.3 The extended affine Hecke algebra

We start with the definition of the finite Hecke algebra. Let $\mathbf{k} : \Phi \to \mathbb{C}^\times$ be a W-invariant function and write $\mathbf{k}_\alpha$ for the value of $\mathbf{k}$ at $\alpha \in \Phi$. Set $\mathbf{k}_i := \mathbf{k}_{\alpha_i}$ for $i = 1, \ldots, r$.

Definition 2.3 The Hecke algebra $H(\mathbf{k})$ associated to the root system Φ is the unital associative algebra over $\mathbb{C}$ generated by $T_1, \ldots, T_r$ with defining relations

(a) $(T_i - \mathbf{k}_i)(T_i + \mathbf{k}_i^{-1}) = 0$ for $i = 1, \ldots, r$,
(b) For $1 \leq i \neq j \leq r$ the braid relation $T_i T_j T_i \cdots = T_j T_i T_j \cdots$ (m_{ij} factors on each side, with m_{ij} the order of $s_i s_j$ in W).

Define the length of $w \in W$ by

$$\ell(w) := \#\big(\Phi^+ \cap w^{-1}\Phi^-\big).$$

For $w = s_{i_1} \ldots s_{i_\ell}$ ($1 \leq i_j \leq r$) a reduced expression of $w \in W$ (i.e. $\ell = \ell(w)$), set

$$T_w := T_{i_1} \ldots T_{i_t} \in H(\mathbf{k}).$$

The T_w ($w \in W$) are well defined and form a linear basis of $H(\mathbf{k})$.

We now introduce the extended affine Hecke algebra $\widetilde{H}^m(\mathbf{k})$ associated to the finite root system Φ^m through its Bernstein–Zelevinsky presentation (see [27]). It contains as subalgebras the finite Hecke algebra $H(\mathbf{k})$ and the group algebra $\mathbb{C}[P^m]$ of the weight lattice P^m of Φ^m. We write the canonical basis elements of $\mathbb{C}[P^m]$ in exponential form x^μ ($\mu \in P^m$), so that $x^\mu x^\nu = x^{\mu+\nu}$ and $x^0 = 1$. The Weyl group W acts naturally on $\mathbb{C}[P^m]$ by algebra automorphisms.

For $1 \leq i \leq r$ there exists a well defined linear operator ∇_i^m on $\mathbb{C}[P^m]$ satisfying

$$\nabla_i^m(x^\nu) := \frac{x^\nu - x^{s_i\nu}}{1 - x^{\alpha_i^m}}$$

for $\nu \in P^m$ (note that $x^\nu - x^{s_i\nu}$ is divisible by $1 - x^{\alpha_i^m}$ in $\mathbb{C}[P^m]$). It is called the divided difference operator associated to the simple root α_i^m.

Definition 2.4 The extended affine Hecke algebra $\widetilde{H}^m(\mathbf{k})$ is the unital associative algebra over $\mathbb{C}$ generated by the algebras $H(\mathbf{k})$ and $\mathbb{C}[P^m]$, with additional defining relations

$$T_i x^\nu - x^{s_i\nu} T_i = \big(\mathbf{k}_i - \mathbf{k}_i^{-1}\big)\nabla_i^m(x^\nu) \tag{2.2}$$

for $i = 1, \ldots, r$ and $\nu \in P^m$.

It is well known that the multiplication map defines a linear isomorphism

$$\mathbb{C}[P^m] \otimes H(\mathbf{k}) \xrightarrow{\sim} \widetilde{H}^m(\mathbf{k}). \tag{2.3}$$

3 Metaplectic representations

3.1 The reflection representation of $H(\mathbf{k})$

Set

$$V := \bigoplus_{\lambda \in P} \mathbb{C} v_\lambda.$$

It inherits a left W-action by the linear extension of the canonical action of W on P. For a W-invariant subset $D \subset P$ we write

$$V_D := \bigoplus_{\lambda \in D} \mathbb{C} v_\lambda$$

for the corresponding W-submodule of V. Then $V = \bigoplus_{\lambda \in P^+} V_{\mathcal{O}_\lambda}$ with $\mathcal{O}_\lambda = W\lambda$ the W-orbit of λ in P and $P^+ \subset P$ the cone of dominant weights of Φ with respect to the base $\{\alpha_1, \ldots, \alpha_r\}$. In this subsection we deform the W-action on V_D and V to a $H(\mathbf{k})$-action.

Fix $\lambda \in P^+$. The stabilizer subgroup

$$W_\lambda := \{w \in W \mid w\lambda = \lambda\}$$

is a standard parabolic subgroup of W. It is generated by the simple reflections s_i ($i \in I_\lambda$), with I_λ the index subset

$$I_\lambda := \{i \in \{1, \ldots, r\} \mid s_i \lambda = \lambda\}.$$

Note that $V_{\mathcal{O}_\lambda} \simeq \mathbb{C}[W] \otimes_{\mathbb{C}[W_\lambda]} \mathbb{C}$ as W-modules, with $\mathbb{C}$ regarded as the trivial W_λ-module. This description leads to a natural Hecke deformation of the W-action on $V_{\mathcal{O}_\lambda}$ as follows.

Let W^λ be the minimal coset representatives of W/W_λ, which can be characterized by

$$W^\lambda = \{w \in W \mid \ell(ws_i) = \ell(w) + 1 \quad \forall i \in I_\lambda\}.$$

For $\lambda \in P^+$ let $H_\lambda(\mathbf{k}) \subset H(\mathbf{k})$ be the subalgebra generated by the T_i ($i \in I_\lambda$). It is straightforward to check that the specialization $T_i \mapsto \mathbf{k}_i$ ($i \in I_\lambda$) satisfies the braid relations, and hence defines a one-dimensional $H_\lambda(\mathbf{k})$-module, which we denote $\mathbb{C}_\lambda$. Consider now the linear isomorphism

$$\phi_\lambda : H(\mathbf{k}) \otimes_{H_\lambda(\mathbf{k})} \mathbb{C}_\lambda \xrightarrow{\sim} V_{\mathcal{O}_\lambda}$$

defined by $\phi_\lambda(T_w \otimes_{H_\lambda(\mathbf{k})} 1) = v_{w\lambda}$ for $w \in W^\lambda$. Transporting the canonical $H(\mathbf{k})$-module structure on $H(\mathbf{k}) \otimes_{H_\lambda(\mathbf{k})} \mathbb{C}_\lambda$ to $V_{\mathcal{O}_\lambda}$ through the linear isomorphism ϕ_λ turns $V_{\mathcal{O}_\lambda}$ into a $H(\mathbf{k})$-module. The resulting direct sum $H(\mathbf{k})$-module structure on $V = \bigoplus_{\lambda \in P^+} V_{\mathcal{O}_\lambda}$ can be explicitly described as follows.

Lemma 3.1 *For $\mu \in P$ we have*

$$T_i v_\mu = \begin{cases} v_{s_i\mu} & \text{if} \quad (\mu, \alpha_i^\vee) > 0, \\ \mathbf{k}_i v_\mu & \text{if} \quad (\mu, \alpha_i^\vee) = 0, \\ \big(\mathbf{k}_i - \mathbf{k}_i^{-1}\big) v_\mu + v_{s_i\mu} & \text{if} \quad (\mu, \alpha_i^\vee) < 0 \end{cases}$$

for $i = 1, \ldots, r$.

Proof Write $\mu = w\lambda$ with $\lambda \in P^+$ and $w \in W^\lambda$. We claim that

(1) $\big(\mu, \alpha_i^\vee\big) > 0 \Leftrightarrow \ell(s_i w) = \ell(w) + 1$ and $s_i w \in W^\lambda$;
(2) $\big(\mu, \alpha_i^\vee\big) = 0 \Leftrightarrow \ell(s_i w) = \ell(w) + 1$ and $s_i w \notin W^\lambda$;
(3) $\big(\mu, \alpha_i^\vee\big) < 0 \Leftrightarrow \ell(s_i w) = \ell(w) - 1$. In this case we have $s_i w \in W^\lambda$.

Since each $w \in W^\lambda$ satisfies exactly one of these three conditions, it suffices to prove the $\Leftarrow$'s.
Case (1) $\ell(s_i w) = \ell(w) + 1$ and $s_i w \in W^\lambda$. Then $\ell(s_i w) = \ell(w) + 1$ implies $w^{-1}\alpha_i \in \Phi^+$, hence $(\mu, \alpha_i^\vee) = (\lambda, w^{-1}\alpha_i^\vee) \geq 0$. The assumption $s_i w \in W^\lambda$ implies $s_i w\lambda \neq w\lambda$, in particular $(\mu, \alpha_i^\vee) \neq 0$. Hence $(\mu, \alpha_i^\vee) > 0$.
Case (2) $\ell(s_i w) = \ell(w) + 1$ and $s_i w \notin W^\lambda$. Then $s_i w = w s_j$ for some $j \in I_\lambda$ by [19, Lem. 3.2]. Hence $s_i w\lambda = w\lambda$ and consequently $(\mu, \alpha_i^\vee) = 0$.
Case (3) $\ell(s_i w) = \ell(w) - 1$. Then $(\mu, \alpha_i^\vee) = (\lambda, w^{-1}\alpha_i^\vee) \leq 0$ since $w^{-1}\alpha_i \in \Phi^-$. If $s_i w\lambda = w\lambda$ then $s_i w$ would be a representative of wW_λ of smaller length than w, which is absurd. Hence $s_i w\lambda \neq w\lambda$, and consequently $(\mu, \alpha_i^\vee) < 0$. If $s_i w \notin W^\lambda$ then the minimal length representative $w' \in W^\lambda$ of the coset $s_i w W_\lambda$ has length strictly smaller than $\ell(s_i w) = \ell(w) - 1$. But then wW_λ contains an element of length strictly smaller than $\ell(w)$, which is absurd. Hence $s_i w \in W^\lambda$.

It is now easy to conclude the proof of the lemma:
Case (1) $\ell(s_i w) = \ell(w) + 1$ and $s_i w \in W^\lambda$. Then

$$T_i v_\mu = \phi_\lambda\big(T_i T_w \otimes_{H_\lambda(\mathbf{k})} 1\big) = \phi_\lambda\big(T_{s_i w} \otimes_{H_\lambda(\mathbf{k})} 1\big) = v_{s_i\mu}.$$

Case (2) $\ell(s_i w) = \ell(w) + 1$ and $s_i w \notin W^\lambda$. Let $j \in I_\lambda$ such that $s_i w = w s_j$. Note that $\alpha_j \in W\alpha_i$, hence $\mathbf{k}_i = \mathbf{k}_j$, and that $\ell(w s_j) = \ell(w) + 1$, so that $T_i T_w = T_{s_i w} = T_{w s_j} = T_w T_j$. Then

$$T_i v_\mu = \phi_\lambda\big(T_i T_w \otimes_{H_\lambda(\mathbf{k})} 1\big) = \phi_\lambda\big(T_w T_j \otimes_{H_\lambda(\mathbf{k})} 1\big) = \mathbf{k}_i v_\mu.$$

Case (3) $l(s_i w) = l(w) - 1$. Using $T_i^2 = (\mathbf{k}_i - \mathbf{k}_i^{-1})T_i + 1$ we get

$$T_i T_w = T_i^2 T_{s_i w} = \big(\mathbf{k}_i - \mathbf{k}_i^{-1}\big) T_w + T_{s_i w}$$

in $H(\mathbf{k})$, and hence

$$T_i v_\mu = \big(\mathbf{k}_i - \mathbf{k}_i^{-1}\big) v_\mu + v_{s_i\mu}$$

since $s_i w \in W^\lambda$. □

3.2 The metaplectic affine Hecke algebra representation

For $s \in \mathbb{Z}_{>0}$ and $t \in \mathbb{Z}$ let $\mathbf{r}_s(t) \in \{0, \ldots, s-1\}$ be the remainder of t modulo s. Define $\mathbf{q}, \mathbf{r} : P \to P$ by

$$\begin{aligned}\mathbf{r}(\lambda) &:= \sum_{i=1}^{r} \mathbf{r}_{m(\alpha_i)}\big((\lambda, \alpha_i^\vee)\big)\varpi_i,\\ \mathbf{q}(\lambda) &:= \lambda - \mathbf{r}(\lambda).\end{aligned}$$

Lemma 3.2 $\mathbf{q}(P) \subseteq P^m$.

Proof. For $i = 1, \ldots, r$ and $\lambda \in P$ we have

$$\begin{aligned}\big(\mathbf{q}(\lambda), \alpha_i^{m\vee}\big) &= m(\alpha_i)^{-1}(\mathbf{q}(\lambda), \alpha_i^\vee)\\ &= m(\alpha_i)^{-1}\big((\lambda, \alpha_i^\vee) - \mathbf{r}_{m(\alpha_i)}((\lambda, \alpha_i^\vee))\big) \in \mathbb{Z}. \quad \square\end{aligned}$$

Let $\mathbb{C}[P] = \mathrm{span}\{x^\lambda\}_{\lambda \in P}$ be the group algebra of the weight lattice P. The Weyl group W acts naturally on $\mathbb{C}[P]$ by algebra automorphisms.

Note that the divided difference operator ∇_i^m featuring in the Bernstein–Zelevinsky cross relations (2.2) of the extended affine Hecke algebra $\widetilde{H}(\mathbf{k})$ satisfies

$$\nabla_i^m(x^\nu) = \frac{x^\nu - x^{s_i \nu}}{1 - x^{\alpha_i^m}} = \left(\frac{1 - x^{-(\nu, \alpha_i^{m\vee})\alpha_i^m}}{1 - x^{\alpha_i^m}}\right) x^\nu, \qquad \nu \in P^m$$

for $i = 1, \ldots, r$.

Lemma 3.3 *For $i = 1, \ldots, r$ there exists a unique linear map*

$$\overline{\nabla}_i : \mathbb{C}[P] \to \mathbb{C}[P]$$

satisfying

$$\overline{\nabla}_i(x^\lambda) := \left(\frac{1 - x^{-(\mathbf{q}(\lambda), \alpha_i^{m\vee})\alpha_i^m}}{1 - x^{\alpha_i^m}}\right) x^\lambda \tag{3.1}$$

for $\lambda \in P$. Furthermore,

$$\overline{\nabla}_i|_{\mathbb{C}[P^m]} = \nabla_i^m.$$

Proof. Note that $\overline{\nabla}_i : \mathbb{C}[P] \to \mathbb{C}[P]$ is a well defined linear operator by the previous lemma. In fact,

$$\overline{\nabla}_i(x^\lambda) := \begin{cases} -x^{\lambda - \alpha_i^m} - \cdots - x^{\lambda - (\mathbf{q}(\lambda), \alpha_i^{m\vee})\alpha_i^m}, & \text{if } (\mathbf{q}(\lambda), \alpha_i^{m\vee}) > 0,\\ 0 & \text{if } (\mathbf{q}(\lambda), \alpha_i^{m\vee}) = 0,\\ x^\lambda + x^{\lambda + \alpha_i^m} + \cdots + x^{\lambda - (1 + (\mathbf{q}(\lambda), \alpha_i^{m\vee}))\alpha_i^m}, & \text{if } (\mathbf{q}(\lambda), \alpha_i^{m\vee}) < 0. \end{cases} \tag{3.2}$$

The second statement follows from the observation that

$$P^m = \{\lambda \in P \mid \mathbf{q}(\lambda) = \lambda\}. \qquad \square$$

Remark 3.4 Note that the action of $\overline{\nabla}_i$ can alternatively be described by

$$\overline{\nabla}_i(x^\lambda) = \frac{x^\lambda - x^{s_i\lambda + (\mathbf{r}(\lambda), \alpha_i^{m\vee})\alpha_i^m}}{1 - x^{\alpha_i^m}}, \qquad \lambda \in P.$$

Write $\Phi^m = \Phi^m_{sh} \cup \Phi^m_{lg}$ for the division of Φ^m into short and long roots, with the convention $\Phi^m = \Phi^m_{lg}$ if all roots have the same length. Write

$$\mathrm{size} : \Phi^m \to \{sh, lg\}$$

for the function on Φ^m satisfying $\mathrm{size}(\alpha) = \mathrm{sh}$ iff $\alpha \in \Phi^m_{sh}$. Write $\mathbf{k}_{sh}$ and $\mathbf{k}_{lg}$ for the value of $\mathbf{k}$ on Φ^m_{sh} and Φ^m_{lg} respectively.

Definition 3.5 (*Representation parameters*) Let $g_j(y) \in \mathbb{C}^\times$ for $j \in \mathbb{Z}$ and $y \in \{sh, lg\}$ be parameters satisfying the following conditions:

(a) $g_j(y) = -1$ if $j \in n\mathbb{Z}$,
(b) $g_j(y) = g_{\mathbf{r}_n(j)}(y)$,
(c) $g_j(y) g_{n-j}(y) = \mathbf{k}_y^{-2}$ if $j \in \mathbb{Z} \setminus n\mathbb{Z}$.

Remark 3.6 The special case where $g_j(y) = g_j$, i.e., the parameters do not depend on root length, was considered in [15,16] and motivated the generalization above. In the applications considered in those papers, the g_i are taken to be certain Gauss sums.

Write $\overline{\lambda}$ for the class of $\lambda \in P$ in the finite abelian quotient group P/P^m. By Lemma 2.2,

$$\mathbf{p}_i(\overline{\lambda}) := -\mathbf{k}_i\, g_{-\mathbf{B}(\lambda,\alpha_i)}(\mathrm{size}(\alpha_i^m)) \tag{3.3}$$

is a well defined function $\mathbf{p}_i : P/P^m \to \mathbb{C}^\times$ for $i = 1, \ldots, r$. Note that $\mathbf{p}_i(\overline{\lambda}) = \mathbf{k}_i$ if $m(\alpha_i) \mid (\lambda, \alpha_i^\vee)$ by Lemma 2.1(b). The following theorem is the main result of this subsection.

Theorem 3.7 *The formulas*

$$\begin{aligned} \pi(T_i)x^\lambda &:= \left(\mathbf{k}_i - \mathbf{k}_i^{-1}\right)\overline{\nabla}_i(x^\lambda) + \mathbf{p}_i(\overline{\lambda})x^{s_i\lambda}, \\ \pi(x^\nu)x^\lambda &:= x^{\lambda+\nu} \end{aligned} \tag{3.4}$$

for $\lambda \in P$, $i = 1, \ldots, r$ and $\nu \in P^m$ turn $\mathbb{C}[P]$ into a left $\widetilde{H}^m(\mathbf{k})$-module.

Remark 3.8 (i) Note that $\mathbb{C}[P^m] \subseteq \mathbb{C}[P]$ is a $\widetilde{H}^m(\mathbf{k})$-submodule with respect to the action (3.4). The action on $\mathbb{C}[P^m]$ simplifies to

$$\begin{aligned} \pi(T_i)x^\nu &= \left(\mathbf{k}_i - \mathbf{k}_i^{-1}\right)\nabla_i^m(x^\nu) + \mathbf{k}_i x^{s_i\nu}, \\ \pi(x^\mu)x^\nu &= x^{\mu+\nu} \end{aligned}$$

for $i = 1, \ldots, r$ and $\mu, \nu \in P^m$. It follows that $\mathbb{C}[P^m] \simeq \widetilde{H}^m(\mathbf{k}) \otimes_{H(\mathbf{k})} \mathbb{C}_0$ as $\widetilde{H}^m(\mathbf{k})$-modules. In particular for $m \equiv 1$ (which happens for instance when $n = 1$), the representation π itself is isomorphic to $\widetilde{H}^m(\mathbf{k}) \otimes_{H(\mathbf{k})} \mathbb{C}_0$.
(ii) Let Λ be a lattice in E satisfying $Q \subseteq \Lambda \subseteq P$. Note that Λ is automatically W-invariant. The lattice $\Lambda_0 := \Lambda \cap P^m$ then satisfies $Q^m \subseteq \Lambda_0 \subseteq P^m$, and

$$\Lambda_0 = \{\lambda \in \Lambda \mid \mathbf{B}(\lambda, \alpha) \equiv 0 \mod n \;\; \forall \alpha \in \Phi\}$$

by Lemma 2.2. Furthermore, $\mathbb{C}[\Lambda] \subseteq \mathbb{C}[P]$ is a $\widetilde{H}^m(\mathbf{k}, \Lambda_0)$-submodule for the action (3.4), with $\widetilde{H}^m(\mathbf{k}, \Lambda_0)$ the subalgebra of $\widetilde{H}^m(\mathbf{k})$ generated by $H(\mathbf{k})$ and $\mathbb{C}[\Lambda_0] := \mathrm{span}\{x^\nu\}_{\nu \in \Lambda_0}$. We write $\pi_\Lambda : \widetilde{H}^m(\mathbf{k}, \Lambda_0) \to \mathrm{End}(\mathbb{C}[\Lambda])$ for the corresponding representation map.

The remainder of this subsection is devoted to the proof of Theorem 3.7. The strategy is to realize the $\widetilde{H}^m(\mathbf{k})$-module $(\pi, \mathbb{C}[P])$ as a quotient of the induced $\widetilde{H}^m(\mathbf{k})$-module

$$N_C := \widetilde{H}^m(\mathbf{k}) \otimes_{H(\mathbf{k})} V_C$$

for an appropriate choice of W-invariant subset $0 \in C \subseteq P$. Note that the subrepresentation $N_{\{0\}}$ is isomorphic to $\mathbb{C}[P^m]$ viewed as module over $\widetilde{H}^m(\mathbf{k})$ by Remark 3.8(i).

The elements

$$x^\nu \otimes_{H(\mathbf{k})} v_\lambda \qquad (\nu \in P^m, \lambda \in C)$$

form a linear basis of N_C and, by the Bernstein–Zelevinsky commutation relations (2.2), the $\widetilde{H}^m(\mathbf{k})$-action on N_C is explicitly given by

$$\begin{aligned} T_i\big(x^\nu \otimes_{H(\mathbf{k})} v_\lambda\big) &= \big(\mathbf{k}_i - \mathbf{k}_i^{-1}\big)\nabla_i^m(x^\nu) \otimes_{H(\mathbf{k})} v_\lambda + x^{s_i\nu} \otimes_{H(\mathbf{k})} T_i v_\lambda, \\ x^\mu\big(x^\nu \otimes_{H(\mathbf{k})} v_\lambda\big) &= x^{\mu+\nu} \otimes_{H(\mathbf{k})} v_\lambda \end{aligned} \tag{3.5}$$

for $\lambda \in C$, $i = 1, \ldots, r$ and $\mu, \nu \in P^m$.

Note that the group algebra $\mathbb{C}[P] := \mathrm{span}\{x^\lambda\}_{\lambda \in P}$ is a free left $\mathbb{C}[P^m]$-module via the action

$$x^\nu \cdot x^\lambda := x^{\lambda+\nu}, \qquad \nu \in P^m, \;\; \lambda \in P.$$

This $\mathbb{C}[P^m]$-module structure on $\mathbb{C}[P]$ coincides with the $\mathbb{C}[P^m]$-structure that will arise from the desired $\widetilde{H}^m(\mathbf{k})$-action (3.4) by restriction.

Let $C \subseteq P$ be a W-invariant subset and let

$$\mathbf{c} : C \to \mathbb{C}^\times$$

be a (for the moment, arbitrary) non-vanishing complex-valued function on C.

Definition 3.9 We write

$$\psi_C^{\mathbf{c}} : N_C \to \mathbb{C}[P]$$

for the morphism of $\mathbb{C}[P^m]$-modules satisfying

$$\psi_C^{\mathbf{c}}(x^\nu \otimes_{H(\mathbf{k})} v_\lambda) := \mathbf{c}(\lambda)^{-1} x^{\lambda+\nu}, \qquad \lambda \in C, \ \nu \in P^m.$$

We fix from now on the W-invariant subset $C \subseteq P$ to be

$$C := \{\lambda \in P \mid |(\lambda, \alpha^\vee)| \leq m(\alpha) \quad \forall \alpha \in \Phi\}. \tag{3.6}$$

Lemma 3.10 *$\psi_C^{\mathbf{c}} : N_C \to \mathbb{C}[P]$ is an epimorphism of $\mathbb{C}[P^m]$-modules.*

Proof. We need to show that $\psi_C^{\mathbf{c}}$ is surjective. Consider the action of $\widetilde{W}^m = W \ltimes P^m$ on P and E by reflections and translations. Since C is W-invariant it suffices to show that each $\widetilde{W}^m$-orbit in P intersects C. We prove the stronger statement that each $W \ltimes Q^m$-orbit in P intersects $C \cap P^+$ in exactly one point.

Write

$$E^+ := \{v \in E \mid (v, \alpha^\vee) \geq 0 \quad \forall \alpha \in \Phi^+\}$$

for the closure of the fundamental Weyl chamber of E with respect to Φ^+. Let $\theta^m \in \Phi^{m+}$ be the highest short root with respect to the base $\{\alpha_1^m, \ldots, \alpha_r^m\}$ of Φ^m. Then $\theta^{m\vee} \in \Phi^{m\vee+}$ is the highest root of $\Phi^{m\vee}$.

By [22, §4.3] each $W \ltimes Q^m$-orbit in E intersects the fundamental alcove

$$A_o := \{v \in E^+ \mid (v, \theta^{m\vee}) \leq 1\}$$

in exactly one point. Hence each $W \ltimes Q^m$-orbit in P intersects $A_o \cap P$ in exactly one point. Now note that

$$\begin{aligned} A_o \cap P &= \{\lambda \in P^+ \mid (\lambda, \theta^{m\vee}) \leq 1\} \\ &= \{\lambda \in P^+ \mid (\lambda, \alpha^{m\vee}) \leq 1 \quad \forall \alpha \in \Phi^+\} \\ &= \{\lambda \in P^+ \mid (\lambda, \alpha^\vee) \leq m(\alpha) \quad \forall \alpha \in \Phi^+\} \\ &= C \cap P^+. \end{aligned}$$

□

The map $\psi_C^{\mathbf{c}}$ gives rise to an isomorphism

$$\overline{\psi}_C^{\mathbf{c}} : N_C/\ker(\psi_C^{\mathbf{c}}) \xrightarrow{\sim} \mathbb{C}[P] \tag{3.7}$$

of $\mathbb{C}[P^m]$-modules by Lemma 3.10. We now show how to fine-tune the normalizing factor $\mathbf{c}$ so that the kernel $\ker(\psi_C^{\mathbf{c}}) \subseteq N_C$ is in fact a $\widetilde{H}^m(\mathbf{k})$-submodule of N_C. We start with deriving some elementary properties of the metaplectic divided difference operators $\overline{\nabla}_i$ $(i = 1, \ldots, r)$.

Lemma 3.11 *Let $i \in \{1, \ldots, r\}$.*
(i) *For $\lambda \in P$ and $\nu \in P^m$ we have*

$$x^\lambda \nabla_i^m(x^\nu) = \overline{\nabla}_i(x^{\lambda+\nu}) - \overline{\nabla}_i(x^\lambda)x^{s_i\nu}.$$

(ii) *For $\lambda \in P$ and $\nu \in P^m$ we have*

$$x^\lambda \nabla_i^m(x^\nu) = \begin{cases} \overline{\nabla}_i(x^{\lambda+\nu}) - x^{\lambda+s_i\nu} & \textit{if } -m(\alpha_i) \leq (\lambda, \alpha_i^\vee) < 0, \\ \overline{\nabla}_i(x^{\lambda+\nu}) & \textit{if } 0 \leq (\lambda, \alpha_i^\vee) < m(\alpha_i), \\ \overline{\nabla}_i(x^{\lambda+\nu}) + x^{s_i(\lambda+\nu)} & \textit{if } (\lambda, \alpha_i^\vee) = m(\alpha_i). \end{cases}$$

Proof (i) This follows by a direct computation.
(ii) Note that

$$\big(\mathbf{q}(\lambda), \alpha_i^{m\vee}\big) = \begin{cases} -1 & \text{if } -m(\alpha_i) \leq (\lambda, \alpha_i^\vee) < 0, \\ 0 & \text{if } 0 \leq (\lambda, \alpha_i^\vee) < m(\alpha_i), \\ 1 & \text{if } (\lambda, \alpha_i^\vee) = m(\alpha_i), \end{cases}$$

hence

$$\overline{\nabla}_i(x^\lambda) = \begin{cases} x^\lambda & \text{if } -m(\alpha_i) \leq (\lambda, \alpha_i^\vee) < 0, \\ 0 & \text{if } 0 \leq (\lambda, \alpha_i^\vee) < m(\alpha_i), \\ -x^{\lambda-\alpha_i^m} = -x^{s_i\lambda} & \text{if } (\lambda, \alpha_i^\vee) = m(\alpha_i). \end{cases}$$

Now use (i). □

The following lemma will play an important role in finding the proper choice of normalizing factor $\mathbf{c}$.

Lemma 3.12 *For $\nu \in P^m$, $\lambda \in C$ and $i = 1, \ldots, r$ we have*

$$\psi_C^{\mathbf{c}}\big(T_i x^\nu \otimes_{H(\mathbf{k})} v_\lambda\big) = \mathbf{c}(\lambda)^{-1}\big((\mathbf{k}_i - \mathbf{k}_i^{-1})\overline{\nabla}_i(x^{\lambda+\nu}) + \mathbf{d}_i(\lambda)x^{s_i(\lambda+\nu)}\big)$$

with $\mathbf{d}_i : C \to \mathbb{C}^\times$ given by

$$\mathbf{d}_i(\lambda) := \begin{cases} \mathbf{c}(\lambda)/\mathbf{c}(s_i\lambda) & \textit{if } -m(\alpha_i) \leq (\lambda, \alpha_i^\vee) < 0, \\ \mathbf{k}_i\mathbf{c}(\lambda)/\mathbf{c}(s_i\lambda) & \textit{if } (\lambda, \alpha_i^\vee) = 0, \\ \mathbf{c}(\lambda)/\mathbf{c}(s_i\lambda) & \textit{if } 0 < (\lambda, \alpha_i^\vee) < m(\alpha_i), \\ \mathbf{k}_i - \mathbf{k}_i^{-1} + \mathbf{c}(\lambda)/\mathbf{c}(s_i\lambda) & \textit{if } (\lambda, \alpha_i^\vee) = m(\alpha_i). \end{cases} \tag{3.8}$$

Proof By a direct computation using (3.5), we have

$$\psi_C^{\mathbf{c}}(T_i x^\nu \otimes_{H(\mathbf{k})} v_\lambda) = \mathbf{c}(\lambda)^{-1}\big(\mathbf{k}_i - \mathbf{k}_i^{-1}\big)x^\lambda\nabla_i^m(x^\nu) + x^{s_i\nu}\psi_C^{\mathbf{c}}(1 \otimes_{H_0(\mathbf{k})} T_i v_\lambda) \tag{3.9}$$

for $\lambda \in C$ and $\nu \in P^m$. We analyze the right hand side using Lemma 3.1. We now consider four cases.
Case 1: $-m(\alpha_i) \leq (\lambda, \alpha_i^\vee) < 0$.
Then

$$\psi_C^{\mathbf{c}}(1 \otimes_{H(\mathbf{k})} T_i v_\lambda) = \mathbf{c}(\lambda)^{-1}\big(\mathbf{k}_i - \mathbf{k}_i^{-1}\big)x^\lambda + \mathbf{c}(s_i\lambda)^{-1}x^{s_i\lambda}.$$

Substituting into (3.9) and using Lemma 3.11 we get the desired formula

$$\psi_C^{\mathbf{c}}(T_i x^\nu \otimes_{H(\mathbf{k})} v_\lambda) = \mathbf{c}(\lambda)^{-1}\Big(\big(\mathbf{k}_i - \mathbf{k}_i^{-1}\big)\overline{\nabla}_i(x^{\lambda+\nu}) + \frac{\mathbf{c}(\lambda)}{\mathbf{c}(s_i\lambda)}x^{s_i(\lambda+\nu)}\Big).$$

Case 2: $(\lambda, \alpha_i^\vee) = 0$.
Now we have

$$\psi_C^{\mathbf{c}}(1 \otimes_{H(\mathbf{k})} T_i v_\lambda) = \mathbf{c}(\lambda)^{-1}\mathbf{k}_i x^\lambda = \mathbf{c}(s_i\lambda)^{-1}\mathbf{k}_i x^{s_i\lambda}.$$

Substituting into (3.9) and using Lemma 3.11 we now get the desired formula

$$\psi_C^{\mathbf{c}}(T_i x^\nu \otimes_{H(\mathbf{k})} v_\lambda) = \mathbf{c}(\lambda)^{-1}\Big(\big(\mathbf{k}_i - \mathbf{k}_i^{-1}\big)\overline{\nabla}_i(x^{\lambda+\nu}) + \mathbf{k}_i\frac{\mathbf{c}(\lambda)}{\mathbf{c}(s_i\lambda)}x^{s_i(\lambda+\nu)}\Big).$$

Case 3: $0 < (\lambda, \alpha_i^\vee) < m(\alpha_i)$.
Then

$$\psi_C^{\mathbf{c}}(1 \otimes_{H(\mathbf{k})} T_i v_\lambda) = \mathbf{c}(s_i\lambda)^{-1}x^{s_i\lambda}.$$

Substitution into (3.9) and using Lemma 3.11 gives the desired formula

$$\psi_C^{\mathbf{c}}(T_i x^\nu \otimes_{H(\mathbf{k})} v_\lambda) = \mathbf{c}(\lambda)^{-1}\Big(\big(\mathbf{k}_i - \mathbf{k}_i^{-1}\big)\overline{\nabla}_i(x^{\lambda+\nu}) + \frac{\mathbf{c}(\lambda)}{\mathbf{c}(s_i\lambda)}x^{s_i(\lambda+\nu)}\Big).$$

Case 4: $(\lambda, \alpha_i^\vee) = m(\alpha_i)$.
In this case

$$\psi_C^{\mathbf{c}}(1 \otimes_{H(\mathbf{k})} T_i v_\lambda) = \mathbf{c}(s_i\lambda)^{-1}x^{s_i\lambda},$$

hence substitution into (3.9) and using Lemma 3.11 gives

$$\psi_C^{\mathbf{c}}(T_i x^\nu \otimes_{H_0(\mathbf{k})} v_\lambda) = \mathbf{c}(\lambda)^{-1}\Big(\big(\mathbf{k}_i - \mathbf{k}_i^{-1}\big)\overline{\nabla}_i(x^{\lambda+\nu}) + \Big(\mathbf{k}_i - \mathbf{k}_i^{-1} + \frac{\mathbf{c}(\lambda)}{\mathbf{c}(s_i\lambda)}\Big)x^{s_i(\lambda+\nu)}\Big),$$

as desired. □

We now continue with the proof of Theorem 3.7. Define parameters $h_j(y) \in \mathbb{C}^\times$ for $j \in \mathbb{Z}$ and $y \in \{sh, lg\}$ by

$$h_j(y) := \begin{cases} \mathbf{k}_y & \text{if } j \in n\mathbb{Z}_{<0}, \\ -\mathbf{k}_y^{-1} g_j(y)^{-1} & \text{if } j \in \mathbb{Z}_{<0} \setminus n\mathbb{Z}_{<0}, \\ 1 & \text{if } j \in \mathbb{Z}_{\geq 0}. \end{cases}$$

Then $h_j(y) = h_{-n+j}(y)$ if $j \in \mathbb{Z}_{<0}$, and $h_j(y)h_{-sn-j}(y) = 1$ for $j \in \mathbb{Z}_{<0} \setminus n\mathbb{Z}_{<0}$ and $s \in \mathbb{Z}_{>0}$ such that $-sn < j < 0$.

Choose $\mathbf{c} : C \to \mathbb{C}^\times$ by

$$\mathbf{c}(\lambda) := \prod_{\alpha \in \Phi^+} h_{\mathbf{Q}(\alpha)(\lambda,\alpha^\vee)}(\mathrm{size}(\alpha^m)), \qquad \lambda \in C. \tag{3.10}$$

Using

$$\frac{\mathbf{c}(\lambda)}{\mathbf{c}(s_i\lambda)} = \frac{h_{\mathbf{Q}(\alpha_i)(\lambda,\alpha_i^\vee)}\big(\mathrm{size}(\alpha_i^m)\big)}{h_{-\mathbf{Q}(\alpha_i)(\lambda,\alpha_i^\vee)}\big(\mathrm{size}(\alpha_i^m)\big)}, \tag{3.11}$$

Equations (2.1), (3.8) and Lemma 2.1b one verifies that for $i = 1, \ldots, r$ and $\lambda \in C$,

$$\mathbf{d}_i(\lambda) = \begin{cases} h_{-\mathbf{Q}(\alpha_i)\mathbf{r}_{m(\alpha_i)}((\lambda,\alpha_i^\vee))}(\mathrm{size}(\alpha_i^m))^{-1} & \text{if } m(\alpha_i) \nmid (\lambda, \alpha_i^\vee), \\ \mathbf{k}_i & \text{if } m(\alpha_i) \mid (\lambda, \alpha_i^\vee). \end{cases}$$

Rewriting in terms of the representation parameters $g_j(y)$ and using Lemma 2.1(b) we get

$$\mathbf{d}_i(\lambda) = \mathbf{p}_i(\overline{\lambda}) \tag{3.12}$$

for $i = 1, \ldots, r$ and $\lambda \in C$, with $\mathbf{p}_i(\overline{\lambda})$ given by (3.3).

Now let $S_i : \mathbb{C}[P] \to \mathbb{C}[P]$ be the linear map defined by

$$S_i(x^\lambda) := \big(\mathbf{k}_i - \mathbf{k}_i^{-1}\big)\overline{\nabla}_i(x^\lambda) + \mathbf{p}_i(\overline{\lambda})x^{s_i\lambda}, \qquad \lambda \in P,$$

then Lemma 3.12 and (3.12) show that for $i = 1, \ldots, r$ and $\lambda \in C$, $\nu \in P^m$,

$$S_i\big(\psi_C^{\mathbf{c}}(x^\nu \otimes_{H(\mathbf{k})} v_\lambda)\big) = \psi_C^{\mathbf{c}}\big(T_i x^\nu \otimes_{H(\mathbf{k})} v_\lambda\big). \tag{3.13}$$

Hence the kernel of the epimorphism $\psi_C^{\mathbf{c}} : N_C \twoheadrightarrow \mathbb{C}[P]$ is a $\widetilde{H}^m(\mathbf{k})$-submodule. By (3.13) it follows that the $\widetilde{H}^m(\mathbf{k})$-module structure on $\mathbb{C}[P]$, inherited from the quotient $\widetilde{H}^m(\mathbf{k})$-module $N_C/\ker(\psi_C^{\mathbf{c}})$ by the $\mathbb{C}[P^m]$-module isomorphism $\overline{\psi}_C^{\mathbf{c}}$ (see (3.7)), is explicitly given by (3.4). This completes the proof of Theorem 3.7.

In subsequent sections, we will work with some conjugations of π, so the following lemma will be useful.

Lemma 3.13 *Let Λ be a lattice in E satisfying $Q \subseteq \Lambda \subseteq P$. Let $h \in \widetilde{H}^m(\mathbf{k}, \Lambda_0)$ and $\mu \in P$. Then $x^{-\mu}\pi(h)x^\mu$ preserves $\mathbb{C}[\Lambda]$.*

Proof Since $\pi(x^\nu)$ for $\nu \in \Lambda_0$ commutes with multiplication by x^μ, we need only check that $x^{-\mu}\pi(T_i)x^\mu$ preserves $\mathbb{C}[\Lambda]$ for $1 \leq i \leq r$. Let $\lambda \in \Lambda$. By Theorem 3.7, we have

$$x^{-\mu}\pi(T_i)(x^{\lambda+\mu}) = x^{-\mu}(\mathbf{k}_i - \mathbf{k}_i^{-1})\overline{\nabla}_i(x^{\lambda+\mu}) + x^{-\mu}\mathbf{p}_i(\overline{\lambda+\mu})x^{s_i(\lambda+\mu)}.$$

We have

$$x^{-\mu+s_i(\lambda+\mu)} = x^{\lambda-(\lambda+\mu,\alpha_i^\vee)\alpha_i} \in \mathbb{C}[\Lambda],$$

since $(\lambda+\mu,\alpha_i^\vee)\alpha_i \in Q$. For the other term, by (3.1) and Lemma 3.2, we have

$$\overline{\nabla}_i(x^{\lambda+\mu}) = x^{\lambda+\mu}g,$$

where $g \in \mathbb{C}[Q^m]$. So now $x^{-\mu}\overline{\nabla}_i(x^{\lambda+\mu}) \in \mathbb{C}[\Lambda]$. □

3.3 The metaplectic Weyl group representation

Let $\widetilde{H}^m_{\mathrm{loc}}(\mathbf{k})$ be algebra obtained by localizing the extended affine Hecke algebra $\widetilde{H}^m(\mathbf{k})$ at the multiplicative subset $\mathbb{C}[P^m]\setminus\{0\}$ (which satisfies the right Ore condition). The canonical algebra embedding $\mathbb{C}[P^m] \hookrightarrow \widetilde{H}^m_{\mathrm{loc}}(\mathbf{k})$ uniquely extends to an algebra embedding $\mathbb{C}(P^m) \hookrightarrow \widetilde{H}^m_{\mathrm{loc}}(\mathbf{k})$, with $\mathbb{C}(P^m)$ the quotient field of $\mathbb{C}[P^m]$. Furthermore, the multiplication map (2.3) extends to a linear isomorphism

$$\mathbb{C}(P^m) \otimes H(\mathbf{k}) \xrightarrow{\sim} \widetilde{H}^m_{\mathrm{loc}}(\mathbf{k}). \tag{3.14}$$

The defining relations of $\widetilde{H}^m_{\mathrm{loc}}(\mathbf{k})$ with respect to the decomposition (3.14) are captured by the extended cross relations

$$T_i f = (s_i f)T_i + \left(\mathbf{k}_i - \mathbf{k}_i^{-1}\right)\left(\frac{f - s_i f}{1 - x^{\alpha_i^m}}\right)$$

for $i = 1,\ldots,r$ and $f \in \mathbb{C}(P^m)$, where we use the extension of the W-action on $\mathbb{C}[P^m]$ to $\mathbb{C}(P^m)$ by field automorphisms.

If the multiplicity function $\mathbf{k}$ is identically equal to one then $\widetilde{H}^m_{\mathrm{loc}}(\mathbf{k})$ is isomorphic to the semi-direct product algebra

$$W \ltimes \mathbb{C}(P^m) := \mathbb{C}[W] \otimes \mathbb{C}(P^m)$$

with algebra structure given by $(v \otimes f)(w \otimes g) := vw \otimes (w^{-1}f)g$ for $v, w \in W$ and $f, g \in \mathbb{C}(P^m)$. We write gw for the element $(1 \otimes g)(w \otimes 1) = w \otimes w^{-1}g$ in $W \ltimes \mathbb{C}(P^m)$ if no confusion can arise.

Define for $\alpha \in \Phi$ the c-functions $c_\alpha = c_\alpha^m \in \mathbb{C}(Q^m)$ by

$$c_\alpha := \frac{1 - \mathbf{k}_\alpha^2 x^{\alpha^m}}{1 - x^{\alpha^m}}. \tag{3.15}$$

We write $c_i := c_{\alpha_i}$ $(i = 1, \ldots, r)$ for the c-functions at the simple roots. Note that $w(c_\alpha) = c_{w\alpha}$ for $w \in W$ and $\alpha \in \Phi$.

By [23] we have the following result.

Theorem 3.14 *There exists a unique algebra isomorphism*

$$\varphi : W \ltimes \mathbb{C}(P^m) \xrightarrow{\sim} \widetilde{H}^m_{loc}(\mathbf{k}) \tag{3.16}$$

given by $\varphi(f) = f$ *for* $f \in \mathbb{C}(P^m)$ *and*

$$\varphi(s_i) := \frac{\mathbf{k}_i}{c_i} T_i + 1 - \frac{\mathbf{k}_i^2}{c_i} \tag{3.17}$$

for $i = 1, \ldots, r$.

The $\varphi(s_i)$ are the so-called *normalized intertwiners* of the extended affine Hecke algebra $\widetilde{H}^m(\mathbf{k})$ (see [23] and, e.g., [12, §3.3.3]). They play an instrumental role in the representation theory of $\widetilde{H}^m(\mathbf{k})$.

Note that for $i = 1, \ldots, r$ we have

$$\varphi^{-1}(T_i) = \mathbf{k}_i + \mathbf{k}_i^{-1} c_i (s_i - 1)$$

in $W \ltimes \mathbb{C}(P^m)$, which are the Demazure–Lusztig operators [27].

Remark 3.15 The localization isomorphism (3.16) extends to the double affine Hecke algebra, see [12, §3.3.3]. In its most natural form it involves normalized intertwiners dual to $\varphi(s_i)$, as well as an additional dual intertwiner naturally attached to the simple affine reflection of the affine Weyl group $W \ltimes Q^m$.

Definition 3.16 Let (ρ, M) be a left $\widetilde{H}^m(\mathbf{k})$-module. Write $(\rho_{\mathrm{loc}}, M_{\mathrm{loc}})$ for the associated localized $W \ltimes \mathbb{C}(P^m)$-module

$$M_{\mathrm{loc}} := \widetilde{H}^m_{\mathrm{loc}}(\mathbf{k}) \otimes_{\widetilde{H}^m(\mathbf{k})} M$$

with representation map $\rho_{\mathrm{loc}} : W \ltimes \mathbb{C}(P^m) \to \mathrm{End}(M_{\mathrm{loc}})$ defined by

$$\rho_{\mathrm{loc}}(X)\big(h \otimes_{\widetilde{H}^m(\mathbf{k})} m\big) := (\varphi(X)h) \otimes_{\widetilde{H}^m(\mathbf{k})} m$$

for $X \in W \ltimes \mathbb{C}(P^m)$, $h \in \widetilde{H}^m_{\mathrm{loc}}(\mathbf{k})$ and $m \in M$.

Note that $M_{\mathrm{loc}} \simeq \mathbb{C}(P^m) \otimes_{\mathbb{C}[P^m]} M$ as vector spaces with the isomorphism mapping $fh \otimes_{\widetilde{H}^m(\mathbf{k})} m$ to $f \otimes_{\mathbb{C}[P^m]} \rho(h)m$ for $f \in \mathbb{C}(P^m)$, $h \in H(\mathbf{k})$ and $m \in M$ (the map is well defined by the Bernstein–Zelevinsky presentation of $\widetilde{H}^m_{\mathrm{loc}}(\mathbf{k})$).

Remark 3.17 Identifying M as subspace of M_{loc} by the linear embedding $M \hookrightarrow M_{\mathrm{loc}}$, $m \mapsto 1 \otimes_{\widetilde{H}^m(\mathbf{k})} m$, we have

$$\rho_{\mathrm{loc}}(\varphi^{-1}(h))m = \rho(h)m, \qquad h \in \widetilde{H}^m(\mathbf{k}), \ m \in M.$$

Remark 3.18 A Bethe integrable system with extended affine Hecke algebra symmetry is a $\widetilde{H}^m(\mathbf{k})$-module V endowed with the integrable structure obtained from the action of the associated dual intertwiners on $\mathbb{C}(P^m) \otimes V$. The integrable structure is thus encoded by solutions of (braid versions of) generalized quantum Yang–Baxter equations with spectral parameter. In the literature on integrable systems one sometimes says that the integrable structure arises from Baxterizing the affine Hecke algebra module structure on the quantum state space. See e.g. [37] for an example involving the Heisenberg XXZ spin-$\frac{1}{2}$ chain.

The intertwiners are also instrumental in the construction of the quantum affine KZ equations, see, e.g., [12, §1.3.2].

Recall the metaplectic affine Hecke algebra representation $(\pi, \mathbb{C}[P])$ from Theorem 3.7. In the following proposition we explicitly describe $(\pi_{\mathrm{loc}}, \mathbb{C}[P]_{\mathrm{loc}})$.

Proposition 3.19 (i) $\mathbb{C}(P) = \bigoplus_{\overline{\lambda}\in P/P^m} \mathbb{C}(P^m)x^\lambda$.
(ii) $\mathbb{C}[P]_{loc} \simeq \mathbb{C}(P)$ *as vector spaces by*

$$fh \otimes_{\widetilde{H}^m(\mathbf{k})} g \mapsto f \cdot \pi(h)g, \qquad f \in \mathbb{C}(P^m), \;\; g \in \mathbb{C}[P], \;\; h \in H(\mathbf{k}).$$

(iii) *The* π_{loc}*-action of* $W \ltimes \mathbb{C}(P^m)$ *on* $\mathbb{C}(P)$ *(identifying* $\mathbb{C}[P]_{loc}$ *with* $\mathbb{C}(P)$ *using the linear isomorphism from (ii)) is explicitly given by*

$$\pi_{loc}(s_i)(fx^\lambda) = (s_i f)\left(\left(\frac{(1-\mathbf{k}_i^2)x^{-(\mathbf{q}(\lambda),\alpha_i^{m\vee})\alpha_i^m}}{1-\mathbf{k}_i^2 x^{\alpha_i^m}}\right)x^\lambda + \left(\frac{\mathbf{k}_i \mathbf{p}_i(\overline{\lambda})(1-x^{\alpha_i^m})}{1-\mathbf{k}_i^2 x^{\alpha_i^m}}\right)x^{s_i\lambda}\right),$$
$$\pi_{loc}(g)(fx^\lambda) = gfx^\lambda$$

for $f, g \in \mathbb{C}(P^m)$*,* $\lambda \in P$ *and* $i = 1, \ldots, r$ *(recall that* $\mathbf{p}_i(\overline{\lambda})$ *is given by* (3.3)*)*.

Proof (i) Let G be the group of characters of the finite abelian group P/P^m. It acts by field automorphisms on $\mathbb{C}(P)$ by

$$\chi \cdot x^\lambda := \chi(\overline{\lambda})x^\lambda, \qquad \lambda \in P, \;\; \chi \in G.$$

Decomposing $\mathbb{C}(P)$ in G-isotypical components yields

$$\mathbb{C}(P) = \bigoplus_{\overline{\lambda}\in P/P^m} \mathbb{C}(P)^G x^\lambda,$$

with $\mathbb{C}(P)^G$ the subfield of G-invariant elements in $\mathbb{C}(P)$. It remains to show that $\mathbb{C}(P)^G = \mathbb{C}(P^m)$, for which it suffices to show that $\mathbb{C}[P]^G = \mathbb{C}[P^m]$. The latter follows from the fact that

$$\mathrm{pr}(x^\lambda) = \delta_{\overline{\lambda},\overline{0}} x^\lambda, \qquad \lambda \in P$$

for the projection map $\mathrm{pr}: \mathbb{C}[P] \twoheadrightarrow \mathbb{C}[P]^G$ defined by

$$\mathrm{pr}(f) := \frac{1}{\#G} \sum_{\chi \in G} \chi \cdot f, \qquad f \in \mathbb{C}[P].$$

(ii) Using (3.14) we get

$$\begin{aligned}\mathbb{C}[P]_{\mathrm{loc}} &= \widetilde{H}^m_{\mathrm{loc}}(\mathbf{k}) \otimes_{\widetilde{H}^m(\mathbf{k})} \mathbb{C}[P] \\ &\simeq \mathbb{C}(P^m) \otimes_{\mathbb{C}[P^m]} \mathbb{C}[P] \simeq \mathbb{C}(P)\end{aligned}$$

with the last isomorphism mapping $f \otimes_{\mathbb{C}[P^m]} g$ to fg for $f \in \mathbb{C}(P^m)$ and $g \in \mathbb{C}[P]$. This is well defined and an isomorphism due to the second formula of (3.4) and due to part (i) of the proposition. The result now immediately follows.
(iii) For $f, g \in \mathbb{C}(P^m)$ and $\lambda \in P$ we have

$$\begin{aligned}\pi_{\mathrm{loc}}(g)(f x^\lambda) &= \pi_{\mathrm{loc}}(g)\big(f \otimes_{\widetilde{H}^m(\mathbf{k})} x^\lambda\big) \\ &= (gf) \otimes_{\widetilde{H}^m(\mathbf{k})} x^\lambda = gfx^\lambda = \pi_{\mathrm{loc}}(gf)x^\lambda,\end{aligned}$$

this establishes the second formula. For the first formula it then suffices to prove that

$$\pi_{\mathrm{loc}}(s_i)(x^\lambda) = \left(\frac{(1-\mathbf{k}_i^2)x^{-(\mathbf{q}(\lambda),\alpha_i^{m\vee})\alpha_i^m}}{1-\mathbf{k}_i^2 x^{\alpha_i^m}}\right) x^\lambda + \left(\frac{\mathbf{k}_i \mathbf{p}_i(\overline{\lambda})(1-x^{\alpha_i^m})}{1-\mathbf{k}_i^2 x^{\alpha_i^m}}\right) x^{s_i\lambda} \tag{3.18}$$

for $i = 1, \ldots, r$ and $\lambda \in P$. By the first formula of (3.4) we have

$$\begin{aligned}\pi_{\mathrm{loc}}(s_i)x^\lambda &= \frac{\mathbf{k}_i}{c_i}\pi(T_i)x^\lambda + \left(1 - \frac{\mathbf{k}_i^2}{c_i}\right)x^\lambda \\ &= \frac{\mathbf{k}_i}{c_i}\left(\left(\mathbf{k}_i - \mathbf{k}_i^{-1}\right)\left(\frac{x^\lambda - x^{\lambda-(\mathbf{q}(\lambda),\alpha_i^{m\vee})\alpha_i^m}}{1-x^{\alpha_i^m}}\right) + \mathbf{p}_i(\overline{\lambda})x^{s_i\lambda}\right) + \left(1 - \frac{\mathbf{k}_i^2}{c_i}\right)x^\lambda.\end{aligned}$$

Substituting the definition of the c-function c_i (see (3.15)) gives

$$\pi_{\mathrm{loc}}(s_i)x^\lambda = \frac{\left(\mathbf{k}_i^2 - 1\right)\left(x^\lambda - x^{\lambda-(\mathbf{q}(\lambda),\alpha_i^{m\vee})\alpha_i^m}\right) + \mathbf{k}_i\mathbf{p}_i(\overline{\lambda})\left(1-x^{\alpha_i^m}\right)x^{s_i\lambda} + \left(1-\mathbf{k}_i^2\right)x^\lambda}{1-\mathbf{k}_i^2 x^{\alpha_i^m}}.$$

Simplifying the expression gives (3.18). □

Remark 3.20 Since $\mathbf{q}(0) = 0$ and $\mathbf{p}_i(\overline{0}) = \mathbf{k}_i$ we have $\pi_{\mathrm{loc}}(s_i)1 = 1$. Hence $\mathbb{C}(P^m)$ is a π_{loc}-submodule of $\mathbb{C}(P)$ with the $W \ltimes \mathbb{C}(P^m)$-action reducing to the standard one,

$$\pi_{\mathrm{loc}}(s_i)f = s_i f, \qquad \pi_{\mathrm{loc}}(g)f := gf$$

for $i = 1, \ldots, r$ and $f, g \in \mathbb{C}(P^m)$.

Recall the definition of the representation parameters $g_j(y)$ ($j \in \mathbb{Z}$, $y \in \{sh, lg\}$), see Definition 3.5. We conjugate the π_{loc}-action by a certain factor, in order to line it up with the Weyl group action of Chinta–Gunnells [15,16].

Theorem 3.21 (Metaplectic Weyl group representation) *The following formulas turn $\mathbb{C}(P)$ into a left $W \ltimes \mathbb{C}(P^m)$-module,*

$$
\begin{aligned}
\sigma(s_i)(fx^\lambda) &:= \frac{\big(1-\mathbf{k}_i^2\big)x^{(\mathbf{q}(-\lambda),\alpha_i^{m\vee})\alpha_i^m}}{\big(1-\mathbf{k}_i^2x^{\alpha_i^m}\big)}(s_if)x^\lambda \\
&\qquad + \mathbf{k}_i^2 g_{\mathbf{Q}(\alpha_i)-\mathbf{B}(\lambda,\alpha_i)}\big(size(\alpha_i^m)\big)\frac{\big(1-x^{-\alpha_i^m}\big)}{\big(1-\mathbf{k}_i^2x^{\alpha_i^m}\big)}(s_if)x^{\alpha_i+s_i\lambda}, \qquad (3.19)\\
\sigma(g)(fx^\lambda) &:= gfx^\lambda
\end{aligned}
$$

for $f, g \in \mathbb{C}(P^m)$, $\lambda \in P$ and $i = 1, \ldots, r$.

Proof Write $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ and $\rho^m := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha^m$ for the half sum of positive roots of Φ and Φ^m respectively. Then $s_i(\rho) = \rho - \alpha_i$ and $s_i(\rho^m) = \rho^m - \alpha_i^m$, in particular $\rho = \sum_{i=1}^r \varpi_i \in P$ and

$$
\rho^m = \sum_{i=1}^r \varpi_i^m = \sum_{i=1}^r m(\alpha_i)\varpi_i \in P^m.
$$

Consider now the action of $W \ltimes \mathbb{C}(P^m)$ on $\mathbb{C}(P)$ defined by

$$
\sigma(X)f := x^{\rho-\rho^m}\pi_{\mathrm{loc}}(X)(x^{\rho^m-\rho}f), \qquad X \in W \ltimes \mathbb{C}(P^m),\ \ f \in \mathbb{C}(P). \qquad (3.20)
$$

Then $\sigma(g)f = gf$ for $g \in \mathbb{C}(P^m)$ and $f \in \mathbb{C}(P)$, and

$$
\begin{aligned}
\sigma(s_i)x^\lambda &= \frac{(1-\mathbf{k}_i^2)x^{-(\mathbf{q}(\lambda+\rho^m-\rho),\alpha_i^{m\vee})\alpha_i^m}}{1-\mathbf{k}_i^2x^{\alpha_i^m}}x^\lambda \\
&\quad - \left(\frac{\mathbf{k}_i\mathbf{p}_i(\overline{\lambda-\rho})(1-x^{-\alpha_i^m})}{1-\mathbf{k}_i^2x^{\alpha_i^m}}\right)x^{\alpha_i+s_i\lambda}. \qquad (3.21)
\end{aligned}
$$

Note that

$$
\mathbf{r}(\lambda+\rho^m-\rho) = \rho^m - \rho - \mathbf{r}(-\lambda)
$$

since $r_s(t+s-1) = s-1-r_s(-t)$ for $s \in \mathbb{Z}_{>0}$ and $t \in \mathbb{Z}$, hence

$$
\mathbf{q}(\lambda+\rho^m-\rho) = -\mathbf{q}(-\lambda).
$$

Furthermore,

$$
-\mathbf{k}_i\mathbf{p}_i(\overline{\lambda-\rho}) = \mathbf{k}_i^2 g_{\mathbf{Q}(\alpha_i)-\mathbf{B}(\lambda,\alpha_i)}\big(\mathrm{size}(\alpha_i^m)\big)
$$

for $\lambda \in P$ since $-\mathbf{B}(\lambda-\rho,\alpha_i)=\mathbf{Q}(\alpha_i)-\mathbf{B}(\lambda,\alpha_i)$. Substituting these two formulas in (3.21) gives the desired result. □

As in Remark 3.8(ii), fix a lattice $\Lambda \subseteq E$ satisfying $Q \subseteq \Lambda \subseteq P$ and set $\Lambda_0 := \Lambda \cap P^m$. Then $Q^m \subseteq \Lambda_0 \subseteq P^m$ and recall that Λ_0 can alternatively be described as

$$\Lambda_0 = \{\lambda \in \Lambda \mid \mathbf{B}(\lambda,\alpha) \equiv 0 \mod n \ \ \forall \alpha \in \Phi\},$$

which places us directly in the context of [16]. Note that Λ and Λ_0 are automatically W-stable. In particular the subalgebra of $W \ltimes \mathbb{C}(P^m)$ generated by W and $\mathbb{C}(\Lambda_0)$ is isomorphic to the semi-direct product algebra $W \ltimes \mathbb{C}(\Lambda_0)$.

Let $\mathbb{C}(\Lambda_0)$ and $\mathbb{C}(\Lambda)$ be the subfields of $\mathbb{C}(P)$ generated by x^ν ($\nu \in \Lambda_0$) and x^λ ($\lambda \in \Lambda$) respectively. Similarly to Proposition 3.19(i) we have the decomposition

$$\mathbb{C}(\Lambda) = \bigoplus_{\overline{\lambda} \in \Lambda/\Lambda_0} \mathbb{C}(\Lambda_0) x^\lambda.$$

Then $\mathbb{C}(\Lambda) \subseteq \mathbb{C}(P)$ is a $W \ltimes \mathbb{C}(\Lambda_0)$-submodule with respect to the action σ. Writing

$$\sigma_\Lambda : W \ltimes \mathbb{C}(\Lambda_0) \to \mathrm{End}(\mathbb{C}(\Lambda))$$

for the resulting representation map, we get

Corollary 3.22 *In the setup as above, the representation map σ_Λ is explicitly given by*

$$\begin{aligned} \sigma_\Lambda(s_i)(fx^\lambda) &:= \frac{\left(1-\mathbf{k}_i^2\right)x^{(\mathbf{q}(-\lambda),\alpha_i^{m\vee})\alpha_i^m}}{\left(1-\mathbf{k}_i^2 x^{\alpha_i^m}\right)}(s_i f)x^\lambda \\ &\quad + \mathbf{k}_i^2 g_{\mathbf{Q}(\alpha_i)-\mathbf{B}(\lambda,\alpha_i)}\big(size\big(\alpha_i^m\big)\big)\frac{\left(1-x^{-\alpha_i^m}\right)}{\left(1-\mathbf{k}_i^2 x^{\alpha_i^m}\right)}(s_i f)x^{\alpha_i+s_i\lambda}, \\ \sigma(g)(fx^\lambda) &:= gfx^\lambda \end{aligned} \tag{3.22}$$

for $f, g \in \mathbb{C}(\Lambda_0)$, $\lambda \in \Lambda$ and $i = 1, \ldots, r$.

Remark 3.23 Consider the special case that $\mathbf{k} : \Phi^m \to \mathbb{C}^\times$ is constant and the representation parameters $g_j(y)$ satisfy $g_j(sh) = g_j(lg)$ for all $j \in \mathbf{Z}$. We call this the equal Hecke and representation parameter case. Then σ_Λ is exactly the Chinta–Gunnells [15,16] Weyl group action. This is immediately apparent by comparing (3.22) with [17, (7)] (the parameter v in [17] corresponds to $\mathbf{k}^2$). Note that our technique gives an independent and uniform proof that the formulas of Chinta–Gunnells do indeed give an action of the Weyl group.

Remark 3.24 Note that σ_Λ reduces at $n = 1$ to the standard W-action. However, it is in fact not the standard action on $\mathbb{C}(P^m)$, due to the fact that we have conjugated π_{loc} by $x^{\rho-\rho^m}$ (compare with Remark 3.20).

Set $\Phi(w) := \Phi^+ \cap w^{-1}\Phi^-$ $(w \in W)$ and let $w_0 \in W$ be the longest Weyl group element.

Definition 3.25 For $\lambda \in P^+$ define $\widetilde{\mathcal{W}}_\lambda \in \mathbb{C}(P)$ by

$$\widetilde{\mathcal{W}}_\lambda := \left(\prod_{\alpha\in\Phi^+} c_\alpha\right) \sum_{w\in W} (-1)^{\ell(w)} \left(\prod_{\alpha\in\Phi(w^{-1})} x^{\alpha^m}\right) \sigma(w)\big(x^{w_0\lambda}\big).$$

In the equal Hecke and representation parameter case, McNamara's [32, Thm. 15.2] metaplectic Casselman–Shalika formula relates $\widetilde{\mathcal{W}}_\lambda$ to the spherical Whittaker function of metaplectic covers of unramified reductive groups over local fields, see also [17, Thm. 16]. It is a natural open problem what the corresponding representation theoretic interpretation is of $\widetilde{\mathcal{W}}_\lambda$ in the unequal Hecke and/or representation parameter case.

In the following section we will obtain in Theorem 4.9 an expression of $\widetilde{\mathcal{W}}_\lambda$ in terms of metaplectic analogues of Demazure–Lusztig operators, generalizing [17, Thm. 16].

4 Metaplectic Demazure–Lusztig operators

In the previous section we used the localization isomorphism $\varphi : W \ltimes \mathbb{C}(P^m) \xrightarrow{\sim} \widetilde{H}^m_{\mathrm{loc}}(\mathbf{k})$ to obtain the metaplectic Weyl group representation σ from the metaplectic affine Hecke algebra representation π. In this section we use the localization isomorphism to turn the metaplectic Weyl group representation σ into a localized affine Hecke algebra representation involving metaplectic Demazure–Lusztig type operators. This leads to a generalization of some of the results in [17, §3] to unequal Hecke and representation parameters, and simplifies some of the proofs in [17, §3].

Define the algebra map

$$\tau : \widetilde{H}^m_{\mathrm{loc}}(\mathbf{k}) \to \mathrm{End}(\mathbb{C}(P))$$

by $\tau := \sigma \circ \varphi^{-1}$.

Proposition 4.1 *For $h \in \widetilde{H}^m(\mathbf{k})$ and $g \in \mathbb{C}[P]$, we have*

$$\tau(h)(g) = x^{\rho-\rho^m}\pi(h)x^{\rho^m-\rho}g.$$

In particular, the restriction of τ to $\widetilde{H}^m(\mathbf{k})$ preserves $\mathbb{C}[P]$, and the restriction of τ to $\widetilde{H}^m(\mathbf{k}, \Lambda_0)$ preserves $\mathbb{C}[\Lambda]$.

Proof The formula follows from (3.20), Proposition 3.19(ii) and Remark 3.17, and then the statements about restrictions follow from Theorem 3.7 and Lemma 3.13. □

Proposition 4.2 *We have*

$$\tau(T_i)(fx^\lambda) = \mathbf{k}_i fx^\lambda + \mathbf{k}_i^{-1}c_i\big(\sigma(s_i)(fx^\lambda) - fx^\lambda\big),$$
$$\tau(g)(fx^\lambda) = gfx^\lambda$$

for $f, g \in \mathbb{C}(P^m)$, $\lambda \in P$ *and* $i = 1, \ldots, r$.

Proof This is immediate from the fact that $\varphi^{-1}(T_i) = \mathbf{k}_i + \mathbf{k}_i^{-1} c_i(s_i - 1)$ and $\varphi^{-1}(g) = g$ for $i = 1, \ldots, r$ and $g \in \mathbb{C}(P^m)$. □

Define the linear operator

$$\mathcal{T}_i := -\mathbf{k}_i \tau\big(x^{\rho^m} T_i^{-1} x^{-\rho^m}\big) \in \mathrm{End}(\mathbb{C}(P)). \tag{4.1}$$

Definition 4.3 We call $\mathcal{T}_i \in \mathrm{End}(\mathbb{C}(P))$ ($i = 1, \ldots, r$) the metaplectic Demazure–Lusztig operators.

By a direct computation,

$$\mathcal{T}_i(f) = \big(1 - \mathbf{k}_i^2 x^{\alpha_i^m}\big)\left(\frac{f - x^{\alpha_i^m}\sigma(s_i)f}{1 - x^{\alpha_i^m}}\right) - f, \qquad f \in \mathbb{C}(P). \tag{4.2}$$

They restrict to well-defined linear operators on $\mathbb{C}(\Lambda)$ for any lattice Λ in V satisfying $Q \subseteq \Lambda \subseteq P$. In case of equal Hecke and representation parameters they reduce to the Demazure–Lusztig operators [17, (11)].

Lemma 4.4 *The metaplectic Demazure–Lusztig operator* $\mathcal{T}_i$ *stabilizes* $\mathbb{C}[P]$ *and* $\mathbb{C}[\Lambda]$ *for* $i = 1, \ldots, r$.

Proof Follows from (4.1), Proposition 4.1, and Lemma 3.13. □

The realization (4.1) of the $\mathcal{T}_i$'s through the $\widetilde{H}^m_{\mathrm{loc}}(\mathbf{k})$-representation τ directly imply that the metaplectic Demazure–Lusztig operators $\mathcal{T}_i$ ($i = 1, \ldots, r$) satisfy the braid relations of W and the quadratic Hecke relations

$$\mathcal{T}_i^2 = \big(\mathbf{k}_i^2 - 1\big)\mathcal{T}_i + \mathbf{k}_i^2, \qquad i = 1, \ldots, r \tag{4.3}$$

(this in particular provides an alternative and uniform proof of the braid relations and quadratic Hecke relations of the metaplectic Demazure–Lusztig operators in [17], see [17, Prop. 5(ii)] and formula (13) in [17, Prop. 7]). For $w = s_{i_1} \cdots s_{i_r} \in W$ a reduced expression we write $\mathcal{T}_w := \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_r} \in \mathrm{End}(\mathbb{C}(P))$.

Remark 4.5 Using that $\sigma(s_i)f = x^{\rho - \rho^m}\pi_{\mathrm{loc}}(s_i)(x^{\rho^m - \rho} f)$ we have

$$x^{-\rho}\tau\big(x^{\rho^m} T_i x^{-\rho^m}\big)(x^\rho f) = \mathbf{k}_i f + \mathbf{k}_i^{-1} c_i(\pi_{\mathrm{loc}}(s_i) f - f) = \pi_{\mathrm{loc}}\big(\varphi^{-1}\big(T_i\big)\big) f \tag{4.4}$$

for $f \in \mathbb{C}(P)$. Hence

$$\mathcal{T}_i = -\mathbf{k}_i\, x^\rho \circ \pi_{\mathrm{loc}}\big(\varphi^{-1}\big(T_i^{-1}\big)\big) \circ x^{-\rho}. \tag{4.5}$$

Remark 4.6 Let Λ be a lattice in E satisfying $Q \subseteq \Lambda \subseteq P$. The localization isomorphism φ restricts to an isomorphism of algebras

$$\varphi_\Lambda : W \ltimes \mathbb{C}(\Lambda_0) \xrightarrow{\sim} \widetilde{H}^m_{\mathrm{loc}}(\mathbf{k}, \Lambda_0),$$

with $\widetilde{H}^m_{\mathrm{loc}}(\mathbf{k}, \Lambda_0)$ the subalgebra of $\widetilde{H}^m_{\mathrm{loc}}(\mathbf{k})$ generated by $H(\mathbf{k})$ and the quotient field $\mathbb{C}(\Lambda_0)$ of $\mathbb{C}[\Lambda_0]$. The algebra map

$$\tau_\Lambda : \widetilde{H}^m_{\mathrm{loc}}(\mathbf{k}, \Lambda_0) \to \mathrm{End}(\mathbb{C}(\Lambda))$$

defined by $\tau_\Lambda := \sigma_\Lambda \circ \varphi_\Lambda^{-1}$ then satisfies

$$\begin{aligned}
\tau_\Lambda(T_i)(fx^\lambda) &= \mathbf{k}_i f x^\lambda + \mathbf{k}_i^{-1} c_i \big(\sigma_\Lambda(s_i)(fx^\lambda) - fx^\lambda\big),\\
\tau_\Lambda(g)(fx^\lambda) &= g f x^\lambda
\end{aligned}$$

for $f, g \in \mathbb{C}(\Lambda_0)$, $\lambda \in \Lambda$ and $i = 1, \ldots, r$, where $\Lambda_0 := \Lambda \cap P^m$. Note that

$$\tau_\Lambda(X) = \tau(X)|_{\mathbb{C}(\Lambda)}, \qquad X \in \widetilde{H}^m_{\mathrm{loc}}(\mathbf{k}, \Lambda_0).$$

The metaplectic Demazure–Lusztig operators $\mathcal{T}_i$ then restrict to the following linear operators on $\mathbb{C}(\Lambda)$,

$$\mathcal{T}_i|_{\mathbb{C}(\Lambda)} = -\mathbf{k}_i \tau_\Lambda\big(\mathrm{Ad}_{x^{\rho^m}}\big(T_i^{-1}\big)\big),$$

where $\mathrm{Ad}_{x^{\rho^m}} \in \mathrm{Aut}(\widetilde{H}^m_{\mathrm{loc}}(\mathbf{k}, \Lambda_0))$ is the restriction of the inner automorphism $X \mapsto x^{\rho^m} X x^{-\rho^m}$ of $\widetilde{H}^m_{\mathrm{loc}}(\mathbf{k})$ to the subalgebra $\widetilde{H}^m_{\mathrm{loc}}(\mathbf{k}, \Lambda_0)$.

We now use these results to generalize results from [17, §3] to the case of unequal Hecke and representation parameters. We first analyze certain symmetrizer and antisymmetrizer elements in $\widetilde{H}^m_{\mathrm{loc}}(\mathbf{k})$. We then use the metaplectic Weyl group representation σ to obtain generalizations of the formula [17, Thm. 16] for the metaplectic Whittaker function.

Recall from Sect. 2.3 that $\mathbf{k} : \Phi \to \mathbb{C}^\times$ is a W-invariant function and $\mathbf{k}_j := \mathbf{k}_{a_j^m}$ for $j = 0, \ldots, r$. For $w = s_{i_1} \cdots s_{i_m} \in W$ a reduced word ($1 \leq i_j \leq r$), we define

$$\mathbf{k}_w := \prod_{j=1}^{m} \mathbf{k}_{i_j}. \tag{4.6}$$

Note that, in the special case that $\mathbf{k}$ is a constant function (the equal Hecke algebra parameters case), we have $\mathbf{k}_w = \mathbf{k}^{\ell(w)}$. Also let

$$W(\mathbf{k}^{\pm 2}) := \sum_{w \in W} \mathbf{k}_w^{\pm 2}. \tag{4.7}$$

Define the symmetrizer $\mathbf{1}_+ \in H(\mathbf{k})$ and antisymmetrizer $\mathbf{1}_- \in H(\mathbf{k})$ by

$$\mathbf{1}_+ := \sum_{w\in W} \mathbf{k}_w T_w, \qquad \mathbf{1}_- := \sum_{w\in W} (-1)^{l(w)} \mathbf{k}_w^{-1} T_w. \tag{4.8}$$

It is well known (see e.g., [23, 1.19.1] and [12]) that the symmetrizer $\mathbf{1}_+$ and antisymmetrizer $\mathbf{1}_-$ satisfy the following properties.

Proposition 4.7 *We have the following identities in* $H(\mathbf{k})$*:*

$$\begin{aligned} T_i \mathbf{1}_\pm &= \pm \mathbf{k}_i^{\pm 1} \mathbf{1}_\pm = \mathbf{1}_\pm T_i, \\ \mathbf{1}_\pm^2 &= W(\mathbf{k}^{\pm 2}) \mathbf{1}_\pm \end{aligned} \tag{4.9}$$

for $i = 1, \ldots, r$*.*

The equations $T_i \mathbf{1}_\pm = \pm \mathbf{k}_i^{\pm 1} \mathbf{1}_\pm$ for $i = 1, \ldots, r$ characterize $\mathbf{1}_\pm$ as an element in $H(\mathbf{k})$ up to a multiplicative constant. It follows from this observation that

$$\mathbf{1}_+ = \mathbf{k}_{w_0}^2 \sum_{w\in W} \mathbf{k}_w^{-1} T_{w^{-1}}^{-1}, \qquad \mathbf{1}_- = \mathbf{k}_{w_0}^{-2} \sum_{w\in W} (-1)^{\ell(w)} \mathbf{k}_w T_{w^{-1}}^{-1}. \tag{4.10}$$

The multiplicative constant is determined by comparing the coefficient of T_{w_0} in the linear expansion in terms of the basis $\{T_w\}_{w\in W}$ of $H(\mathbf{k})$.

Recall the definition (3.15) of the c-functions c_α ($\alpha \in \Phi$).

Proposition 4.8 *We have the following identities in* $W \ltimes \mathbb{C}(P^m)$*:*

$$\begin{aligned} \varphi^{-1}(\mathbf{1}_+) &= \left(\sum_{w\in W} w\right) \prod_{\alpha\in\Phi^+} c_{-\alpha}, \\ \varphi^{-1}(\mathbf{1}_-) &= \mathbf{k}_{w_0}^{-2} \left(\prod_{\alpha\in\Phi^+} c_\alpha\right) \sum_{w\in W} (-1)^{\ell(w)} w. \end{aligned}$$

Proof See [31, (5.5.14)]. □

We now obtain the following main result of this section.

Theorem 4.9 *We have the following identity of operators in* $\mathrm{End}(\mathbb{C}(P))$*:*

$$\begin{aligned} \sum_{w\in W} \mathcal{T}_w &= \left(\prod_{\alpha\in\Phi^+} c_\alpha\right) x^{\rho^m} \left(\sum_{w\in W} (-1)^{\ell(w)} \sigma(w)\right) x^{-\rho^m} \\ &= \left(\prod_{\alpha\in\Phi^+} c_\alpha\right) \sum_{w\in W} (-1)^{\ell(w)} \left(\prod_{\alpha\in\Phi(w^{-1})} x^{\alpha^m}\right) \sigma(w). \end{aligned}$$

In particular, for $\lambda \in P^+$ we have

$$\widetilde{\mathcal{W}}_\lambda = \sum_{w \in W} \mathcal{T}_w(x^{w_0\lambda}).$$

Proof By (4.1) and (4.10) we have

$$\begin{aligned}\sum_{w \in W} \mathcal{T}_w &= \sum_{w \in W} (-1)^{\ell(w)} \mathbf{k}_w \tau \left(x^{\rho^m} T_{w^{-1}}^{-1} x^{-\rho^m} \right) \\ &= \mathbf{k}_{w_0}^2 \tau \left(x^{\rho^m} \mathbf{1}_- x^{-\rho^m} \right).\end{aligned}$$

The first formula now follows directly using $\tau = \sigma \circ \varphi^{-1}$ and the previous proposition. The second formula follows from the observation that

$$\rho^m - w\rho^m = \sum_{\alpha \in \Phi(w^{-1})} \alpha^m$$

for $w \in W$. □

Corollary 4.10 *Let $\Lambda \subset E$ be a lattice satisfying $Q \subseteq \Lambda \subseteq P$. Then $\widetilde{\mathcal{W}}_\lambda \in \mathbb{C}[\Lambda]$ for $\lambda \in \Lambda^+ := P^+ \cap \Lambda$.*

Proof This follows from Lemma 4.4 and the previous theorem. □

Remark 4.11 Note that the symmetric variant $\tau(\mathbf{1}_+)(x^\lambda)$ of $\widetilde{\mathcal{W}}_\lambda$ for $\lambda \in \Lambda^+$ may also be of interest. These are polynomials (again by Lemma 4.4), symmetric with respect to the Chinta–Gunnells W-action σ (by Proposition 4.8(a)), which reduce for $m \equiv 1$ to Hall–Littlewood polynomials [29, §10] (by e.g., Remark 3.20).

5 Metaplectic polynomials

In this section we present metaplectic variants of GL_r Macdonald polynomials. Full proofs and additional results will be provided in the forthcoming paper [36], in which we will also introduce the metaplectic polynomials for arbitrary root systems.

5.1 The metaplectic data (n, Q)

Let $r \geq 2$. Fix the standard orthonormal basis $\{\epsilon_i\}_{i=1}^r$ of $\mathbb{R}^r$. The associated scalar product is denoted by $(\cdot, \cdot)$ and the corresponding norm by $\|\cdot\|$. Then

$$\Phi = \{\epsilon_i - \epsilon_j\}_{1 \leq i \neq j \leq r}$$

is the root system of type A_{r-1}, with basis Δ of simple roots and associated set Φ^+ of positive roots given by

$$\Delta := \{\alpha_1, \ldots, \alpha_{r-1}\} \subset \Phi^+ = \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq r}$$

with $\alpha_i := \epsilon_i - \epsilon_{i+1}$. The associated highest root is $\theta = \epsilon_1 - \epsilon_r$. The root lattice is $Q := \mathbb{Z}\Phi$, which is contained in the GL_r weight lattice $\bigoplus_{i=1}^r \mathbb{Z}\epsilon_i \simeq \mathbb{Z}^r$. The Weyl group is the symmetric group S_r in r letters.

Let $\mathbf{Q} : \mathbb{Z}^r \to \mathbb{Q}$ be a non-zero S_r-invariant quadratic form which is integral-valued on Q. Then

$$\mathbf{Q}(\gamma) = \frac{\kappa}{2}\|\gamma\|^2 \qquad \forall\, \gamma \in Q$$

for some nonzero integer $\kappa = \kappa_{\mathbf{Q}}$ (we suppress the dependence of κ on $\mathbf{Q}$ if it is clear from the context). In particular, $\mathbf{Q}(\alpha) = \kappa$ for all $\alpha \in \Phi$. Write

$$\mathbf{B}(\lambda, \mu) := \mathbf{Q}(\lambda + \mu) - \mathbf{Q}(\lambda) - \mathbf{Q}(\mu), \qquad \lambda, \mu \in \mathbb{Z}^r$$

for the associated symmetric S_r-invariant bilinear form $\mathbf{B} : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Q}$. By the S_r-invariance of $\mathbf{B}$ we then have

$$\mathbf{B}(\lambda, \alpha) = \kappa\big(\lambda, \alpha^\vee\big) \qquad \forall\, \lambda \in \mathbb{Z}^r, \;\; \forall\, \alpha \in \Phi \tag{5.1}$$

(in the present context $\alpha^\vee = \alpha$ for $\alpha \in \Phi$, but we distinguish them in anticipation of the results for arbitrary root systems in our followup paper [36]). In particular, $\mathbf{B}(\lambda, \alpha) \in \mathbb{Z}$ for $\lambda \in \mathbb{Z}^r$ and $\alpha \in \Phi$.

Fix $n \in \mathbb{Z}_{>0}$ once and for all. Given a quadratic form $\mathbf{Q}$ as in the previous paragraph with associated normalisation scalar $\kappa = \kappa_{\mathbf{Q}}$, we define positive integers $\kappa' = \kappa'_{\mathbf{Q}}$ and $m = m_{\mathbf{Q}}$ by

$$\kappa' := \gcd(n, \kappa), \qquad m := \frac{n}{\kappa'} = \frac{n}{\gcd(n, \kappa)}. \tag{5.2}$$

Note that $m = n/\gcd(n, \mathbf{Q}(\alpha))$ for all $\alpha \in \Phi$, in particular $\Phi^m = m\Phi$. Furthermore,

$$m\mathbb{Z}^r \subseteq \{\lambda \in \mathbb{Z}^r \mid \mathbf{B}(\lambda, \alpha) \equiv 0 \mod n \;\; \forall\, \alpha \in \Phi\}.$$

Set $\mathbb{F} := \mathbb{C}(q, k)$, and let $\mathbb{K}^{(n)}$ be the field extension of $\mathbb{F}$ obtained by adjoining $\lfloor \frac{n-1}{2} \rfloor$ additional indeterminates $g_1^{(n)}, \ldots, g_{\lfloor \frac{n-1}{2} \rfloor}^{(n)}$. Thus $\mathbb{K}^{(n)} = \mathbb{F}$ for $n = 1, 2$.

For $n \geq 1$ we now define representation parameters $g_j^{(n)} \in \mathbb{K}^{(n)}$ for all integers $j \in \mathbb{Z}$ as follows (it depends on a choice of a sign $\epsilon \in \{\pm 1\}$ when n is even, which we fix once and for all). We set $g_0^{(n)} := -1$. The representation parameters $g_j^{(n)}$ for indices $\frac{n}{2} < j < n$ are defined by $g_j^{(n)} := k^{-2}(g_{n-j}^{(n)})^{-1}$. For n even, we set $g_{\frac{n}{2}}^{(n)} := \epsilon^{-1}k^{-1}$. Finally, the representation parameters $g_j^{(n)} \in \mathbb{K}^{(n)}$ are extended to indices $j \in \mathbb{Z}$ by $g_j^{(n)} := g_{r_n(j)}^{(n)}$, with $r_n(j) \in \{0, \ldots, n-1\}$ the remainder modulo n. Note that $g_j^{(n)} g_{n-j}^{(n)} = k^{-2}$ in $\mathbb{K}^{(n)}$ for all $j \in \mathbb{Z} \setminus n\mathbb{Z}$, and $g_j^{(n)} = -1$ if $j \in n\mathbb{Z}$.

Lemma 5.1 *There exists a unique $\mathbb{F}$-homomorphism $\iota_\kappa : \mathbb{K}^{(m)} \hookrightarrow \mathbb{K}^{(n)}$ mapping $g_j^{(m)}$ to $g_{\kappa j}^{(n)}$ for all $j \in \mathbb{Z}$.*

Proof This is an easy check. □

We write $\mathbb{K}^{(n,\kappa)}$ for the image of $\mathbb{K}^{(m)} = \mathbb{K}^{(n/\kappa')}$ under $\iota_\kappa : \mathbb{K}^{(m)} \hookrightarrow \mathbb{K}^{(n)}$. It is the subfield of $\mathbb{K}^{(n)}$ obtained by adjoining the elements $g^{(n)}_{\kappa' j}$ to $\mathbb{F}$ for $1 \leq j < \frac{m}{2}$. Note that $\mathbb{K}^{(n,1)} = \mathbb{K}^{(n)}$.

We finish this subsection by introducing metaplectic analogues $(p_j)_{0\leq j<r}$ of multiplicity functions. For $\lambda \in \mathbb{Z}^r$ write $\overline{\lambda}^m = \lambda + m\mathbb{Z}^r$ for the class of λ in $\big(\mathbb{Z}/m\mathbb{Z}\big)^r$. Then we define

$$p_j : \big(\mathbb{Z}/m\mathbb{Z}\big)^r \to \mathbb{K}^{(n)} \qquad (0 \leq j < r)$$

by

$$p_i(\overline{\lambda}^m) := -k g^{(n)}_{-\mathbf{B}(\lambda,\alpha_i)}, \qquad p_0(\overline{\lambda}^m) := -k g^{(n)}_{\mathbf{B}(\lambda,\theta)}$$

for $1 \leq i < r$. By (5.1), the dependence of p_j on the metaplectic data is a dependence on (n, κ) (and ϵ if n is even). If we want to emphasize it, we will write $p_j = p_j^{(n,\kappa)}$ (we always suppress ϵ from the notations). Note that by (5.1), the functions p_j take values in the subfield $\mathbb{K}^{(n,\kappa)}$ of $\mathbb{K}^{(n)}$.

5.2 The double affine Hecke algebra $\mathbb{H}^{(m)}$

Consider the extended affine Weyl group $W^{(m)} := S_r \ltimes m\mathbb{Z}^r$. We denote its elements by $\sigma\tau(\nu)$ ($\sigma \in S_r$, $\nu \in m\mathbb{Z}^r$). We may also view $W^{(m)}$ as the subgroup of affine linear transformations of $\mathbb{R}^r$ of the form $\sigma\tau(\nu)$ ($\sigma \in S_r$, $\nu \in m\mathbb{Z}^r$), acting on $\mathbb{R}^r$ by

$$(\sigma\tau(\nu))(v) := \sigma(v + \nu), \qquad v \in \mathbb{R}^r, \;\; \sigma \in S_r, \;\; \nu \in m\mathbb{Z}^r.$$

View $\mathbb{R}^r \oplus \mathbb{R}$ as the space of real-valued affine linear functionals on $\mathbb{R}^r$ by associating to $(v, x) \in \mathbb{R}^r \oplus \mathbb{R}$ the affine linear functional $\mathbb{R}^r \ni u \mapsto (v, u) + x$. The extended affine Weyl group $W^{(m)}$ acts on $\mathbb{R}^r \oplus \mathbb{R}$ by

$$(\sigma\tau(\nu))(v, x) := (\sigma v, x - (\nu, v)).$$

The affine root system is

$$\widetilde{\Phi}^{(m)} = \{(m\alpha, tm^2) \mid \alpha \in \Phi, \;\; t \in \mathbb{Z}\} \subset \mathbb{R}^r \oplus \mathbb{R},$$

which is stabilized by $W^{(m)}$. We identify $m\Phi$ with the subset of affine linear roots $\{(m\alpha, 0)\}_{\alpha\in\Phi}$ in $\widetilde{\Phi}^{(m)}$. For $a = (m\alpha, tm^2) \in \widetilde{\Phi}^{(m)}$ let $s_a \in W^{(m)}$ be the orthogonal reflection in the affine hyperplane $a^{-1}(0)$. Then

$$s_a = \tau(-tm\alpha^\vee)s_\alpha \in W^{(m)}$$

with $s_\alpha \in S_r$ the orthogonal reflection in the hyperplane $\alpha^\perp \subset \mathbb{R}^r$.

Write $b_0^{(m)} := (-m\theta, m^2)$ and $b_i^{(m)} := m\alpha_i$ $(1 \le i < r)$. Then $\{b_0^{(m)}, b_1^{(m)}, \ldots, b_{r-1}^{(m)}\}$ is a set of simple roots for $\widetilde{\Phi}^{(m)}$. Write $s_j^{(m)} := s_{b_j} \in W^{(m)}$ $(j = 0, \ldots, r-1)$ for the associated simple reflections. Then

$$s_0^{(m)} = \tau(m\theta^\vee)s_\theta,$$

and $s_i^{(m)} = s_{\alpha_i} \in S_r$ $(1 \le i < r)$ are the simple neighbouring transpositions. Since the latter do not depend on m, we will write $s_i = s_i^{(m)}$ for $1 \le i < r$.

The subgroup $W_{\mathrm{Cox}}^{(m)} := \langle s_0^{(m)}, \ldots, s_{r-1}^{(m)} \rangle$ of $W^{(m)}$ is the affine Weyl group of type $\widehat{\mathrm{A}}_{r-1}$. Its defining relations in terms of the simple reflections are $(s_j^{(m)})^2 = 1$ and the type $\widehat{A}_{r-1}$ braid relations. Then $W^{(m)} \simeq \mathbb{Z} \ltimes W_{\mathrm{Cox}}^{(m)}$, with $1 \in \mathbb{Z}$ acting on $W_{\mathrm{Cox}}^{(m)}$ by $s_j^{(m)} \mapsto s_{j+1}^{(m)}$ (indices modulo r), which corresponds under the isomorphism $W^{(m)} \simeq \mathbb{Z} \ltimes W_{\mathrm{Cox}}^{(m)}$ with the extended affine Weyl group element

$$\omega^{(m)} := s_1 s_2 \cdots s_{r-1} \tau(m\epsilon_r).$$

Note that $\omega^{(m)}(b_j^{(m)}) = b_{j+1}^{(m)}$ for $0 \le j < r$ (with the indices taken modulo r).

We write x^v $(v \in \mathbb{R}^r)$ for the canonical basis of the group algebra $\mathbb{F}[\mathbb{R}^r]$ of $\mathbb{R}^r$ over $\mathbb{F}$, so that $x^u x^v = x^{u+v}$ and $x^0 = 1$. We write for $c \in \mathbb{Z}$ and $v \in \mathbb{R}^r$,

$$x^{(v,c)} = q^c x^v \in \mathbb{F}[\mathbb{R}^r].$$

Let $\mathbb{F}[x^{\pm 1}]$ be the $\mathbb{F}$-algebra of Laurent polynomials in $x_1, \ldots, x_r$, viewed as the $\mathbb{F}$-subalgebra of $\mathbb{F}[\mathbb{R}^r]$ generated by $\mathbb{Z}^r \subset \mathbb{R}^r$ via $x_i := x^{\epsilon_i}$ $(1 \le i \le r)$. The extended affine Weyl group $W^{(m)}$ acts by $\mathbb{F}$-algebra automorphisms on $\mathbb{F}[x^{\pm 1}]$ by

$$w\big(x^{(\lambda,c)}\big) := x^{w(\lambda,c)} \tag{5.3}$$

for $w \in W^{(m)}$ and $(\lambda, c) \in \mathbb{Z}^r \oplus \mathbb{Z}$. In particular, for $\lambda \in \mathbb{Z}^r$, $\sigma \in S_r$ and $\nu \in \mathbb{Z}^r$,

$$(\sigma\tau(\nu))x^\lambda = q^{-(\nu,\lambda)} x^{\sigma\lambda}. \tag{5.4}$$

For $\lambda \in \mathbb{Z}^r$ we thus have

$$x^{\omega^{(m)}\lambda} = q^{-m\lambda_r} x^{s_1 \ldots s_{r-1}\lambda}, \qquad x^{s_0^{(m)}\lambda} = q^{m(\lambda,\theta^\vee)} x^{s_\theta\lambda}$$

and $x^{b_0^{(m)}} = q^{m^2} x^{-m\theta}$.

Definition 5.2 The GL_r double affine Hecke algebra $\mathbb{H}^{(m)}$ is the unital associative $\mathbb{F}$-algebra generated by $T_0, \ldots, T_{r-1}$, $\omega^{\pm 1}$ and $\mathbb{F}[x^{\pm m}] := \mathbb{F}[x_1^{\pm m}, \ldots, x_r^{\pm m}]$ with defining relations:

(1) The type $\widehat{\mathrm{A}}_{r-1}$ braid relations for $T_0, \ldots, T_{r-1}$.
(2) The Hecke relations $(T_j - k)(T_j + k^{-1}) = 0$.

(3) $\omega\omega^{-1} = 1 = \omega^{-1}\omega$ and $\omega T_j = T_{j+1}\omega$ (indices modulo r).
(4) The cross relations

$$T_j x^\lambda - x^{s_j^{(m)}\lambda} T_j = (k - k^{-1}) \left(\frac{x^\lambda - x^{s_j^{(m)}\lambda}}{1 - x^{b_j^{(m)}}} \right), \qquad \omega x^\lambda = x^{\omega^{(m)}\lambda}\omega \tag{5.5}$$

for $\lambda \in m\mathbb{Z}^r$ and $0 \leq j < r$.

Consider the subalgebras $\widetilde{H}_Y^{(m)} := \mathbb{F}\langle T_0, \ldots, T_{r-1}, \omega^{\pm 1}\rangle$ and $H^{(m)} = \mathbb{F}\langle T_1, \ldots, T_{r-1}\rangle$ of $\mathbb{H}^{(m)}$. The subalgebra $H^{(m)}$ is the finite Hecke algebra (of type A_{r-1}). A key structure theoretic fact (see e.g. §3.2.1 in [12]) is that the elements

$$Y_i := T_{i-1}^{-1} \ldots T_1^{-1} \omega T_{r-1} \ldots T_i, \qquad i = 1, \ldots, r. \tag{5.6}$$

commute pairwise and are invertible in $\widetilde{H}_Y^{(m)}$. The assignment

$$x^\nu \mapsto Y^\nu := Y_1^{\kappa_1} \ldots Y_r^{\kappa_r}, \qquad \nu = m\kappa \in m\mathbb{Z}^r$$

defines an injective algebra map $\mathbb{F}[x^{\pm m}] \hookrightarrow \widetilde{H}_Y^{(m)}$, whose image we denote by $\mathbb{F}[Y^{\pm m}]$. The multiplication map

$$H^{(m)} \otimes_{\mathbb{F}} \mathbb{F}[Y^{\pm m}] \to \widetilde{H}_Y^{(m)}, \qquad h \otimes Y^\lambda \mapsto hY^\lambda$$

is a $\mathbb{F}$-linear isomorphism. The defining relations of $\widetilde{H}_Y^{(m)}$ in terms of the subalgebras $H^{(m)}$ and $\mathbb{F}[Y^{\pm m}]$ are the Bernstein–Zelevinsky cross relations

$$T_i Y^\mu - Y^{s_i\mu} T_i = (k - k^{-1}) \left(\frac{Y^\mu - Y^{s_i\mu}}{1 - Y^{-m\alpha_i}} \right)$$

for $1 \leq i < r$ and $\mu \in m\mathbb{Z}^r$.

Remark 5.3 Let $\delta : \mathbb{H}^{(m)} \to \mathbb{H}^{(m)}$ be the $\mathbb{F}$-linear antialgebra isomorphism satisfying $\delta(T_i) := T_i$ $(1 \leq i < r)$, $\delta(Y^\mu) := x^{-\mu}$ and $\delta(x^\mu) := Y^{-\mu}$ for $\mu \in m\mathbb{Z}^r$. It provides an anti-isomorphism between the subalgebras

$$\widetilde{H}_X^{(m)} := \mathbb{F}\langle T_1, \ldots, T_{r-1}, x_1^{\pm m}, \ldots, x_r^{\pm m}\rangle$$

and $\widetilde{H}_Y^{(m)}$. The corresponding Coxeter type presentation of $\widetilde{H}_X^{(m)}$ thus involves $\delta(T_0)$ and $\delta(\omega^{-1})$ as the generator for the simple affine root and the element corresponding to the generator of the affine Dynkin diagram automorphisms. Note that

$$\delta(\omega^{-1}) = x_1^m T_1 \ldots T_{r-1}$$

and that $\delta(\omega^{-1})Y^\lambda = q^{m\lambda_r} Y^{s_1\cdots s_{r-1}\lambda}\delta(\omega^{-1})$ for $\lambda \in m\mathbb{Z}^r$ in $\mathbb{H}^{(m)}$.

5.3 The metaplectic basic representation

Set $t_m(s) := s - r_m(s) \in m\mathbb{Z}$. The metaplectic divided difference operators $\overline{\nabla}_j^{(m)}$ $(0 \leq j < r)$ are the $\mathbb{F}$-linear operators on $\mathbb{F}[x^{\pm 1}]$ defined by

$$\overline{\nabla}_i^{(m)}(x^\lambda) := \left(\frac{1 - x^{-t_m((\lambda,\alpha_i^\vee))\alpha_i}}{1 - x^{m\alpha_i}} \right) x^\lambda$$

and

$$\overline{\nabla}_0^{(m)}(x^\lambda) := \left(\frac{1 - q^{-mt_m(-(\lambda,\theta^\vee))} x^{t_m(-(\lambda,\theta^\vee))\theta}}{1 - q^{m^2} x^{-m\theta}} \right) x^\lambda$$

for $\lambda \in \mathbb{Z}^r$. Note that the $\overline{\nabla}_j^{(m)}|_{\mathbb{F}[x^{\pm m}]}$ are the usual divided-difference operators,

$$\overline{\nabla}_j^{(m)}(x^\lambda) = \frac{x^\lambda - x^{s_j^{(m)}\lambda}}{1 - x^{b_j^{(m)}}}, \qquad \lambda \in m\mathbb{Z}^r$$

for $0 \leq j < r$. For a field extension $\mathbb{F} \subseteq \mathbb{K}$, the $\mathbb{K}$-linear extension of $\overline{\nabla}_j^{(m)}$ to a linear operator on $\mathbb{K}[x^{\pm 1}]$ will also be denoted by $\overline{\nabla}_j^{(m)}$.

Recall that the metaplectic data $(n, \mathbf{Q})$ provide us with the nonzero integer $\kappa := \mathbf{Q}(\alpha)$ $(\alpha \in \Phi)$, from which κ' and m are determined by (5.2). In addition, we have fixed a sign $\epsilon \in \{\pm 1\}$ in case n is even, through the definition of the representation parameter $g_{\frac{n}{2}}^{(n)} = \epsilon^{-1} k^{-1}$.

Theorem 5.4 *The formulas*

$$\begin{aligned}
\widehat{\pi}^{(n,\kappa)}(T_j)x^\lambda &:= \big(k - k^{-1}\big)\overline{\nabla}_j^m(x^\lambda) + p_j^{(n,\kappa)}(\overline{\lambda}^m)x^{s_j^{(m)}\lambda}, \\
\widehat{\pi}^{(n,\kappa)}(x^\mu)x^\lambda &:= x^{\lambda+\mu}, \\
\widehat{\pi}^{(n,\kappa)}(\omega)x^\lambda &:= x^{\omega^{(m)}\lambda}
\end{aligned} \tag{5.7}$$

for $j = 0, \ldots, r-1$, $\mu \in m\mathbb{Z}^r$ and $\lambda \in \mathbb{Z}^r$ turn $\mathbb{K}^{(n,\kappa)}[x^{\pm 1}]$ into a left $\mathbb{H}^{(m)}$-module. We write $\widehat{\pi}^{(m)} := \widehat{\pi}^{(m,1)}$ (we suppress here the dependence on ϵ).

Proof Set

$$\Lambda := \mathbb{Z}^r + \mathbb{Z}\mathfrak{n}$$

with $\mathfrak{n} := \frac{1}{r}(\epsilon_1 + \cdots + \epsilon_r)$. Note that Λ contains the weight lattice P of Φ. The quadratic form $\mathbf{Q}$ has a unique extension to a $\mathbb{Q}$-valued S_r-invariant quadratic form $\Lambda \to \mathbb{Q}$, which we also denote by $\mathbf{Q}$. We write $\mathbf{B} : \Lambda \times \Lambda \to \mathbb{Q}$ for the associated symmetric S_r-invariant bilinear form.

Adjoin a rth root $q^{\frac{1}{r}}$ of q to $\mathbb{K}^{(n,\kappa)}$ (by abuse of notation, we denote it again by $\mathbb{K}^{(n,\kappa)}$). Let $\mathbb{K}^{(n,\kappa)}[\Lambda]$ be the $\mathbb{K}^{(n,\kappa)}$-submodule of $\mathbb{K}^{(n,\kappa)}[\mathbb{R}^r] := \mathbb{K}^{(n,\kappa)} \otimes_{\mathbb{F}} \mathbb{F}[\mathbb{R}^r]$ generated by x^λ ($\lambda \in \Lambda$). The extended affine Weyl group $W^{(m)}$ acts on $\mathbb{K}^{(n,\kappa)}[\Lambda]$ by $\mathbb{K}^{(n,\kappa)}$-algebra automorphisms by the formula (5.3). The $\mathbb{K}^{(n,\kappa)}$-subalgebra $\mathbb{K}^{(n,\kappa)}[P]$ generated by x^λ ($\lambda \in P$) is a $W^{(m)}$-submodule.

By Theorem 3.7 (which holds true with formal parameters), the first two lines of (5.7),

$$\begin{aligned}\widehat{\pi}(T_i)x^\lambda &:= \big(k-k^{-1}\big)\overline{\nabla}_i^{(m)}(x^\lambda) + p_i^{(n,\kappa)}(\overline{\lambda}^m)x^{s_i\lambda},\\ \widehat{\pi}(x^\mu)x^\lambda &:= x^{\lambda+\mu}\end{aligned} \tag{5.8}$$

for $1 \leq i < r$, $\mu \in mP$ and $\lambda \in P$ define a representation

$$\widehat{\pi} : \widetilde{H}_X^{(m)} \to \mathrm{End}_{\mathbb{K}^{(n,\kappa)}}\big(\mathbb{K}^{(n,\kappa)}[P]\big).$$

Using the decomposition

$$\mathbb{K}^{(n,\kappa)}[\Lambda] = \bigoplus_{s\in\mathbb{Z}} x^{s\mathfrak{n}}\mathbb{K}^{(n,\kappa)}[P]$$

it extends to a representation $\widehat{\pi} : \widetilde{H}_X^{(m)} \to \mathrm{End}_{\mathbb{K}^{(n,\kappa)}}\big(\mathbb{K}^{(n,\kappa)}[\Lambda]\big)$ by $\widehat{\pi}(h)\big(x^{s\mathfrak{n}}x^\lambda\big) := x^{s\mathfrak{n}}\widehat{\pi}(h)x^\lambda$ for $h \in \widetilde{H}_X^{(m)}$, $s \in \mathbb{Z}$ and $\lambda \in P$. The formulas (5.8) are then valid for all $\lambda \in \Lambda$. A direct check shows that the operators $\widehat{\pi}(T_j)$ and $\widehat{\pi}(x^\mu)$ on $\mathbb{K}^{(n,\kappa)}[\Lambda]$ satisfy the cross relation (5.5) for $1 \leq j < r$ and $\mu \in m\mathbb{Z}^r$. Furthermore, direct computations show that

$$\begin{aligned}p_{j+1}^{(n,\kappa)}\big(\overline{(s_1\cdots s_{r-1}\lambda)}^m\big) &= p_j(\overline{\lambda}^m),\\ \widehat{\pi}(\omega)\overline{\nabla}_j^{(m)} &= \overline{\nabla}_{j+1}^{(m)}\widehat{\pi}(\omega)\end{aligned}$$

for $\lambda \in \mathbb{Z}^r$ and $0 \leq j < r$, hence $\widehat{\pi}(\omega)\widehat{\pi}(T_j) = \widehat{\pi}(T_{j+1})\widehat{\pi}(\omega)$ as operators on $\mathbb{K}^{(n,\kappa)}[x^{\pm 1}]$, with the indices modulo r. From this the defining double affine Hecke algebra relations involving T_0 are easily verified. The result now follows directly. □

We call $\widehat{\pi}^{(n,\kappa)}$ the *metaplectic basic representation* of the double affine Hecke algebra $\mathbb{H}^{(m)}$.

Remark 5.5 (i) Since the double affine Hecke algebra $\mathbb{H}^{(m)}$ is defined over $\mathbb{F}$, the representation parameters should be thought of as representation parameters of the representation $\widehat{\pi}^{(n,\kappa)}$. Since the representation is defined over the subfield $\mathbb{K}^{(n,\kappa)}$ of $\mathbb{K}^{(n)}$, the representation $\widehat{\pi}^{(n,\kappa)}$ only depends on the representation parameters $g_{\kappa' j}^{(n)}$ ($1 \leq j < \frac{m}{2}$) and, if m is even, on ϵ.
(ii) It follows from

$$\iota_\kappa\big(p_j^{(m,1)}(\overline{\lambda}^m)\big) = p_j^{(n,\kappa)}(\overline{\lambda}^m)$$

for $0 \leq j < r$ and $\lambda \in \mathbb{Z}^r$ that

$$\bar{\iota}_\kappa\left(\sum_\mu c_\mu x^\mu\right) := \sum_\mu \iota_\kappa(c_\mu) x^\mu \qquad \left(c_\mu \in \mathbb{K}^{(m)}\right)$$

defines an isomorphism

$$\bar{\iota}_\kappa : \left(\mathbb{K}^{(m)}[x^{\pm 1}], \widehat{\pi}^{(m)}\right) \xrightarrow{\sim} \left(\mathbb{K}^{(n,\kappa)}[x^{\pm 1}], \widehat{\pi}^{(n,\kappa)}\right)$$

of $\mathbb{H}^{(m)}$-modules. In particular, $(\mathbb{K}^{(n,\kappa)}[x^{\pm 1}], \widehat{\pi}^{(n,\kappa)})$ only depends on ϵ if m is even.
(iii) $\widehat{\pi}^{(1)} : \mathbb{H}^{(1)} \to \mathrm{End}_{\mathbb{F}}\big(\mathbb{F}[x^{\pm 1}]\big)$ is Cherednik's basic representation for GL_r, see, e.g., [12, §3.7] and [20].

By the second part of the remark, the dependence of the metaplectic basic representation on the metaplectic data is essentially only a dependence on m. The metaplectic basic representation $\widehat{\pi}^{(m)}$ can be recovered from $\widehat{\pi}^{(n)}$ as follows.

By a direct check one verifies that the assignments

$$\phi_{\kappa'}(q) := q^{\kappa'^2}, \qquad \phi_{\kappa'}(T_j) := T_j, \qquad \phi_{\kappa'}(\omega) := \omega, \qquad \phi_{\kappa'}(x^\lambda) := x^{\kappa'\lambda}$$

for $1 \leq i < r$ and $\lambda \in m\mathbb{Z}^r$ define a morphism $\phi_{\kappa'} : \mathbb{H}^{(m)} \to \mathbb{H}^{(n)}$ of $\mathbb{C}(k)$-algebras. Note that $\phi_{\kappa'}(Y^\lambda) = Y^{\kappa'\lambda}$ for $\lambda \in m\mathbb{Z}^r$.

Let $\mathbf{J}_{\kappa'} : \mathbb{K}^{(m)} \hookrightarrow \mathbb{K}^{(n)}$ be the $\mathbb{C}(q)$-homomorphism mapping q to $q^{\kappa'^2}$ and $g_j^{(m)}$ to $g_{\kappa' j}^{(n)}$ for all $j \in \mathbb{Z}$. Note the difference with $\iota_{\kappa'} : \mathbb{K}^{(m)} \to \mathbb{K}^{(n)}$ (Lemma 5.1), which fixes q. The image $\mathbb{K}_{\mathbf{J}}^{(n,\kappa')}$ of the homomorphism $\mathbf{J}_{\kappa'} : \mathbb{K}^{(m)} \to \mathbb{K}^{(n)}$ is the subfield of $\mathbb{K}^{(n)}$ obtained by adjoining $q^{\kappa'^2}$ and $g_{\kappa' j}^{(n)}$ $(1 \leq j < \frac{m}{2})$ to $\mathbb{C}(k)$. We now have the following proposition.

Proposition 5.6 *$\mathbb{K}_{\mathbf{J}}^{(n,\kappa')}[x^{\pm\kappa'}] \subseteq \mathbb{K}^{(n)}[x^{\pm 1}]$ is a $(\widehat{\pi}_n \circ \phi_{\kappa'}, \mathbb{H}^m)$-submodule. Then*

$$\bar{\mathbf{J}}_{\kappa'}\left(\sum_\mu c_\mu x^\mu\right) := \sum_\mu \mathbf{J}_{\kappa'}(c_\mu) x^{\kappa'\mu} \qquad \left(c_\mu \in \mathbb{K}^{(m)}\right)$$

defines an isomorphism

$$\bar{\mathbf{J}}_{\kappa'} : (\mathbb{K}^{(m)}[x^{\pm 1}], \widehat{\pi}^{(m)}) \xrightarrow{\sim} (\mathbb{K}_{\mathbf{J}}^{(n,\kappa')}[x^{\pm\kappa'}], \widehat{\pi}^{(n)} \circ \phi_{\kappa'})$$

of $\mathbb{H}^{(m)}$-modules. In particular, $(\mathbb{K}_{\mathbf{J}}^{(n,n)}[x^{\pm n}], \widehat{\pi}^{(n)} \circ \phi_n)$ realizes Cherednik's basic representation $\widehat{\pi}^{(1)}$ with the role of q replaced by q^{n^2}.

5.4 The metaplectic polynomials

We keep the notation from the previous subsections. In particular, $(n, \mathbf{Q})$ is the fixed metaplectic data and $\kappa := \mathbf{Q}(\alpha)$ $(\alpha \in \Phi)$, leading to the positive integers κ' and m by (5.2); moreover the sign $\epsilon \in \{\pm 1\}$ is fixed through the definition of the representation parameter $g^{(n)}_{\frac{n}{2}} := \epsilon^{-1}k^{-1}$ if n is even.

The commuting linear operators $\widehat{\pi}^{(n,\kappa)}(Y^\mu) \in \mathrm{End}_{\mathbb{K}^{(n,\kappa)}}\big(\mathbb{K}^{(n,\kappa)}[x^{\pm 1}]\big)$ $(\mu \in m\mathbb{Z}^r)$ are metaplectic analogues of Cherednik's Y-operators. The following theorem establishes the existence of a family of Laurent polynomials which are simultaneous eigenfunctions of the metaplectic Y-operators.

For $\mu \in \mathbb{Z}^r$ define $\gamma^{(n,\kappa)}_\mu \in \mathrm{Hom}_{\mathbb{Z}}\big(m\mathbb{Z}^r, \mathbb{K}^{(n)\times}\big)$ by

$$\gamma^{(n,\kappa)}_\mu := q^{-\mu} \prod_{\alpha \in \Phi^+} \big(\sigma^{(n,\kappa)}((\mu, \alpha^\vee))\big)^{\frac{\alpha^\vee}{m}},$$

with $\sigma^{(n,\kappa)} : \mathbb{Z} \to \mathbb{K}^{(n)}$ defined by

$$\sigma^{(n,\kappa)}(s) := \begin{cases} k^{-1} & \text{if } s \in m\mathbb{Z}_{>0}, \\ -kg_{-\kappa s} & \text{if } s \in \mathbb{Z} \setminus m\mathbb{Z}_{>0}. \end{cases}$$

In other words, the value $\big(\gamma^{(n,\kappa)}_\mu\big)^\lambda$ of $\gamma^{(n,\kappa)}_\mu$ at $\lambda \in m\mathbb{Z}^r$ is

$$\big(\gamma^{(n,\kappa)}_\mu\big)^\lambda = q^{-(\lambda,\mu)} \prod_{\alpha \in \Phi^+} \big(\sigma^{(n,\kappa)}((\mu, \alpha^\vee))\big)^{\frac{(\lambda,\alpha^\vee)}{m}}.$$

Note that $\gamma^{(n,\kappa)}_\mu$ takes values in $\mathbb{K}^{(n,\kappa)}$.

The proof of the following theorem, including its extension to arbitrary root systems, will be given in the forthcoming paper [36].

Theorem 5.7 *There exists a unique family of Laurent polynomials $\{E^{(n,\kappa)}_\mu(x)\}_{\mu \in \mathbb{Z}^r}$ in $\mathbb{K}^{(n,\kappa)}[x^{\pm 1}]$ such that for $\mu \in \mathbb{Z}^r$,*

(i) *For all $\lambda \in m\mathbb{Z}^r$ we have*

$$\widehat{\pi}^{(n,\kappa)}(Y^\lambda)E^{(n,\kappa)}_\mu(x) = \big(\gamma^{(n,\kappa)}_\mu\big)^\lambda E^{(n,\kappa)}_\mu(x).$$

(ii) *The coefficient of x^μ in the expansion of $E^{(n,\kappa)}_\mu(x)$ in the monomial basis $\{x^\nu\}_{\nu \in \mathbb{Z}^r}$, is one.*

We will write $E^{(n)}_\mu(x) := E^{(n,1)}_\mu(x)$ for $\mu \in \mathbb{Z}^r$.

Remark 5.8 Many key properties of Macdonald polynomials can be proved via the technique of intertwiners, introduced in [25,26,34] for type A, in [13] for arbitrary root systems and in [35] for the Koornwinder setting. In our forthcoming paper [36] we develop the metaplectic analog of the theory of intertwiners, which allows us to prove the above result and also establish a key triangularity property of the polynomials $E_\mu^{(n)}(x)$.

The following proposition is a consequence of Remark 5.5(ii) and Proposition 5.6.

Proposition 5.9 *For all $\mu \in \mathbb{Z}^r$,*

$$\begin{aligned}\bar{\iota}_\kappa\big(E_\mu^{(m)}(x)\big) &= E_\mu^{(n,\kappa)}(x),\\ \bar{\jmath}_{\kappa'}\big(E_\mu^{(m)}(x)\big) &= E_{\kappa'\mu}^{(n)}(x).\end{aligned} \tag{5.9}$$

By the first line of (5.9), the metaplectic polynomial $E^{(n,\kappa)}(x)$ essentially only depends on m, q, k, the representation parameters $g_{\kappa' j}^{(n)}$ $(1 \leq j < \frac{m}{2})$ and, if m is even, on ϵ.

Remark 5.10 By Remark 5.5(iii), $E_\mu^{(1)}(x)$ is the monic nonsymmetric Macdonald polynomial of degree μ (compared to the standard conventions on nonsymmetric Macdonald polynomials as in e.g. [20], k^2 corresponds to t). Furthermore, as a special case of the second line of (5.9), $E_{n\mu}^{(n)}(x)$ realizes the monic nonsymmetric Macdonald polynomial of degree $\mu \in \mathbb{Z}^r$ in the variables $x_1^n, \ldots, x_r^n$, with the role of q replaced by q^{n^2}.

5.5 Appendix: Table of GL_3 metaplectic polynomials

We give formulas for $E_\lambda^{(m)}(x)$, where $1 \leq m \leq 5$ and $\lambda \in \mathbb{Z}^3$ has weight at most 2. For convenience of notation, we write g_j instead of $g_j^{(m)}$. The technique used to compute these polynomials will be provided in the forthcoming paper [36].

$$E_{(0,0,0)}^{(m)}(x) = 1 \qquad (m \geq 1)$$

$$E_{(1,0,0)}^{(m)}(x) = x_1 \qquad (m \geq 1)$$

$$E_{(0,1,0)}^{(1)}(x) = \frac{(k-1)(k+1)}{k^4 q - 1} x_1 + x_2$$

$$E_{(0,1,0)}^{(2)}(x) = \frac{(k-1)(k+1)}{k\left(kq^2 + \epsilon\right)} x_1 + x_2$$

$$E_{(0,1,0)}^{(3)}(x) = \frac{(k-1)(k+1)\, g_1}{k^4 g_1^3 q^3 + 1} x_1 + x_2$$

$$E_{(0,1,0)}^{(4)}(x) = \frac{(k-1)(k+1)\, g_1}{k^4 g_1^3 q^4 + 1} x_1 + x_2$$

$$E_{(0,1,0)}^{(5)}(x) = \frac{(k-1)(k+1)\, g_1}{k^4 g_1^3 q^5 + 1} x_1 + x_2$$

$$E_{(0,0,1)}^{(1)}(x) = \frac{(k-1)(k+1)}{qk^2 - 1} x_1 + \frac{(k-1)(k+1)}{qk^2 - 1} x_2 + x_3$$

$$E^{(2)}_{(0,0,1)}(x) = -\frac{(k-1)(k+1)}{k\left(k+\epsilon q^2\right)}x_1 + \frac{(k-1)(k+1)}{q^2+\epsilon k}x_2 + x_3$$

$$E^{(3)}_{(0,0,1)}(x) = -\frac{(k-1)(k+1)\,g_1^2}{k^2g_1^3q^3+1}x_1 + \frac{(k-1)(k+1)\,g_1}{k^2g_1^3q^3+1}x_2 + x_3$$

$$E^{(4)}_{(0,0,1)}(x) = -\frac{(k-1)(k+1)\,g_1^2}{k^2g_1^3q^4+1}x_1 + \frac{(k-1)(k+1)\,g_1}{k^2g_1^3q^4+1}x_2 + x_3$$

$$E^{(5)}_{(0,0,1)}(x) = -\frac{(k-1)(k+1)\,g_1^2}{k^2g_1^3q^5+1}x_1 + \frac{(k-1)(k+1)\,g_1}{k^2g_1^3q^5+1}x_2 + x_3$$

$$E^{(1)}_{(0,1,1)}(x) = \frac{(k-1)(k+1)}{qk^2-1}x_1x_2 + \frac{(k-1)(k+1)}{qk^2-1}x_3x_1 + x_3x_2$$

$$E^{(2)}_{(0,1,1)}(x) = -\frac{(k-1)(k+1)}{k\left(k+\epsilon q^2\right)}x_1x_2 + \frac{(k-1)(k+1)}{q^2+\epsilon k}x_3x_1 + x_3x_2$$

$$E^{(3)}_{(0,1,1)}(x) = -\frac{(k-1)(k+1)\,g_1^2}{k^2g_1^3q^3+1}x_1x_2 + \frac{(k-1)(k+1)\,g_1}{k^2g_1^3q^3+1}x_3x_1 + x_3x_2$$

$$E^{(4)}_{(0,1,1)}(x) = -\frac{(k-1)(k+1)\,g_1^2}{k^2g_1^3q^4+1}x_1x_2 + \frac{(k-1)(k+1)\,g_1}{k^2g_1^3q^4+1}x_3x_1 + x_3x_2$$

$$E^{(5)}_{(0,1,1)}(x) = -\frac{(k-1)(k+1)\,g_1^2}{k^2g_1^3q^5+1}x_1x_2 + \frac{(k-1)(k+1)\,g_1}{k^2g_1^3q^5+1}x_3x_1 + x_3x_2$$

$$E^{(1)}_{(1,0,1)}(x) = \frac{(k-1)(k+1)}{k^4q-1}x_1x_2 + x_3x_1$$

$$E^{(2)}_{(1,0,1)}(x) = \frac{(k-1)(k+1)}{k\left(kq^2+\epsilon\right)}x_1x_2 + x_3x_1$$

$$E^{(3)}_{(1,0,1)}(x) = \frac{(k-1)(k+1)\,g_1}{k^4g_1^3q^3+1}x_1x_2 + x_3x_1$$

$$E^{(4)}_{(1,0,1)}(x) = \frac{(k-1)(k+1)\,g_1}{k^4g_1^3q^4+1}x_1x_2 + x_3x_1$$

$$E^{(5)}_{(1,0,1)}(x) = \frac{(k-1)(k+1)\,g_1}{k^4g_1^3q^5+1}x_1x_2 + x_3x_1$$

$$E^{(m)}_{(1,1,0)}(x) = x_1x_2 \qquad (m \geq 1)$$

$$E^{(1)}_{(2,0,0)}(x) = x_1^2 + \frac{q(k-1)(k+1)}{qk^2-1}x_1x_2 + \frac{q(k-1)(k+1)}{qk^2-1}x_3x_1$$

$$E^{(m)}_{(2,0,0)}(x) = x_1^2 \qquad (m \geq 2)$$

$$E^{(1)}_{(0,2,0)}(x) = \frac{(k-1)(k+1)}{\left(qk^2-1\right)\left(qk^2+1\right)}x_1^2 + \frac{(k-1)(k+1)\left(k^4q^2+qk^2-q-1\right)}{\left(qk^2+1\right)\left(qk^2-1\right)^2}x_1x_2$$

$$+\frac{(k-1)^2(k+1)^2 q}{\left(qk^2+1\right)\left(qk^2-1\right)^2}x_3x_1+x_2^2+\frac{q(k-1)(k+1)}{qk^2-1}x_3x_2$$

$$E^{(2)}_{(0,2,0)}(x)=\frac{(k-1)(k+1)}{\left(q^2k^2-1\right)\left(q^2k^2+1\right)}x_1^2+x_2^2$$

$$E^{(3)}_{(0,2,0)}(x)=\frac{(k-1)(k+1)g_1^2}{k^2g_1^3+q^6}x_1^2+x_2^2$$

$$E^{(4)}_{(0,2,0)}(x)=\frac{(k-1)(k+1)}{k\left(q^8k+\epsilon\right)}x_1^2+x_2^2$$

$$E^{(5)}_{(0,2,0)}(x)=\frac{(k-1)(k+1)g_2}{k^4g_2^3q^{10}+1}x_1^2+x_2^2$$

$$E^{(1)}_{(0,0,2)}(x)=\frac{(k-1)(k+1)}{(kq-1)(kq+1)}x_1^2+\frac{(q+1)(k-1)^2(k+1)^2}{(kq-1)(kq+1)\left(qk^2-1\right)}x_1x_2$$
$$+\frac{(q+1)(k-1)(k+1)}{(kq-1)(kq+1)}x_3x_1+\frac{(k-1)(k+1)}{(kq-1)(kq+1)}x_2^2$$
$$+\frac{(q+1)(k-1)(k+1)}{(kq-1)(kq+1)}x_3x_2+x_3^2$$

$$E^{(2)}_{(0,0,2)}(x)=\frac{(k-1)(k+1)}{\left(kq^2-1\right)\left(kq^2+1\right)}x_1^2+\frac{(k-1)(k+1)}{\left(kq^2-1\right)\left(kq^2+1\right)}x_2^2+x_3^2$$

$$E^{(3)}_{(0,0,2)}(x)=-\frac{(k-1)(k+1)g_1}{k^4g_1^3+q^6}x_1^2+\frac{(k-1)(k+1)k^2g_1^2}{k^4g_1^3+q^6}x_2^2+x_3^2$$

$$E^{(4)}_{(0,0,2)}(x)=-\frac{(k-1)(k+1)}{k\left(\epsilon q^8+k\right)}x_1^2+\frac{(k-1)(k+1)}{q^8+\epsilon k}x_2^2+x_3^2$$

$$E^{(5)}_{(0,0,2)}(x)=-\frac{(k-1)(k+1)g_2^2}{k^2g_2^3q^{10}+1}x_1^2+\frac{(k-1)(k+1)g_2}{k^2g_2^3q^{10}+1}x_2^2+x_3^2$$

Remark 5.11 We mention a few general properties of the metaplectic polynomials which can be observed in the table above.

(1) The following are monic nonsymmetric Macdonald polynomials: $E^{(1)}_\lambda(x)$ (for any λ), $E^{(2)}_{(2,0,0)}(x)$, $E^{(2)}_{(0,2,0)}(x)$, and $E^{(2)}_{(0,0,2)}(x)$ (see Remark 5.10). In particular, the formulas given above for these polynomials match the ones provided in the appendix of [20] (with k^2 replaced by t).

(2) More generally, for any $a\in\mathbb{Z}_{\geq 1}$, the metaplectic polynomial $E^{(am)}_{a\lambda}(x)$ may be obtained from $E^{(m)}_\lambda(x)$ via the substitutions $x_i\to x_i^a$, $q\to q^{a^2}$ and $g_j^{(m)}\to g_{aj}^{(am)}$. This follows directly from Proposition 5.9 with $\kappa'=a$. We list the pairs $\left(E^{(a)}_\lambda(x), E^{(am)}_{m\lambda}(x)\right)$ from the table with $a\neq 1$ for which this applies:

   (a) $\left(E^{(2)}_{(0,0,0)}(x), E^{(4)}_{(0,0,0)}(x)\right)$

   (b) $\left(E^{(2)}_{(1,0,0)}(x), E^{(4)}_{(2,0,0)}(x)\right)$

(c) $\big(E^{(2)}_{(0,1,0)}(x), E^{(4)}_{(0,2,0)}(x)\big)$

(d) $\big(E^{(2)}_{(0,0,1)}(x), E^{(4)}_{(0,0,2)}(x)\big)$.

(3) For GL_3, we have

$$C^m_+ = C^m \cap P_+ = \{\lambda \in \mathbb{Z}^3 : \lambda_1 \geq \lambda_2 \geq \lambda_3, \lambda_1 - \lambda_3 \leq m\}$$

(see (3.6) for the definition of C^m). If $\lambda \in C^m_+$, the metaplectic polynomial $E^{(m)}_\lambda(x)$ is equal to the monomial x^λ. This will be proved in the followup paper [36] in the context of arbitrary root systems. Note that this result applies to the following polynomials listed above: $E^{(m)}_{(0,0,0)}(x)$, $E^{(m)}_{(1,0,0)}(x)$, $E^{(m)}_{(1,1,0)}(x)$ (any $m \in \mathbb{Z}_{\geq 1}$) and $E^{(m)}_{(2,0,0)}(x)$ (any $m \in \mathbb{Z}_{\geq 2}$). Note that for $\lambda \in m\mathbb{Z}^3 \cap C^m_+$, this recovers the well-known fact that the nonsymmetric Macdonald polynomial corresponding to the minuscule weight λ is a monomial.

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