# Modalities in the realm of questions: axiomatizing inquisitive epistemic logic

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# Modalities in the Realm of Questions: Axiomatizing Inquisitive Epistemic Logic

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**Abstract**

Building on ideas from inquisitive semantics, the recently proposed framework of *inquisitive epistemic logic* (IEL) provides the tools to model and reason about scenarios in which agents do not only have information, but also entertain issues. This framework has been shown to allow for a generalization to issues of important notions, such as common knowledge and public announcements, and it has been argued to form a suitable basis for the analysis of information exchange as an interactive process of raising and resolving issues. From an abstract point of view, the system is interesting, in that it implies extending the logical operations, including the modalities, beyond the truth-conditional realm, in such a way that they can embed not only standard declarative formulas, but also interrogatives. The present paper investigates the logic of IEL, building up to a completeness result. It is shown that the standard logical features of the logical constants extend smoothly beyond the truth-conditional realm, except for double negation, which is the hallmark of truth-conditionality. In particular, while the modalities of IEL operate in a crucially richer semantic space than Kripke modalities do, they retain entirely standard logical features.

*Keywords:* Epistemic logic, inquisitive semantics, logic of questions.

## 1 Introduction

Standard epistemic logic provides a framework to reason about scenarios comprising facts and information. In a large body of work (see [7], [1] for recent surveys), this framework has been taken to provide a basis for dynamic logics that aim at describing information exchange. However, an exchange of information is not a mere sequence of informative utterances; rather, it is best regarded as an orderly process in which participants try to achieve certain epistemic goals by raising and addressing issues. In order to formalize this idea, we need a framework that describes not only the information that agents have, but also the issues that they entertain, and that allows us to reason about them.

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Providing such a framework is the aim of the *inquisitive epistemic logic* proposed in [6], which extends epistemic logic with issues and interrogative formulas, building on ideas and techniques recently developed in *inquisitive semantics* ([5], [3], [4], among others). From an abstract perspective, this framework is interesting in that it shows that the standard account of the logical constants, including the modalities, can be extended smoothly and conservatively beyond the truth-conditional realm, to a richer setting where both declaratives and interrogatives receive a natural interpretation. Moreover, such a generalization has been shown in [6] to extend to other key notions of epistemic and dynamic logics, such as common knowledge and public announcements.

In the present paper we investigate the logic to which this framework gives rise, illustrating the significance of entailment in this richer semantic context, and building up to a completeness result. The paper is organized as follows: section 2 introduces inquisitive epistemic logic; section 3 discusses the combined notion of entailment that arises from this framework; finally, in section 4 a proof system for this logic is provided, and a completeness result is established.

## 2 Inquisitive epistemic logic

This section provides a concise overview of inquisitive epistemic logic. For a more detailed introduction to the system, and for proofs of the results stated in this section, the reader is referred to [6]. Our presentation will make use of two fundamental ingredients, the notions of *information states* and *issues*. The former notion is standard, while the latter comes from work on inquisitive semantics (for motivation, see e.g. [3], [6]). An information state represents a piece or body of information, identified with the set of worlds compatible with it. Similarly, an issue represents a certain desire, or request, for information, identified with those states in which it is *resolved*.

**Definition 2.1** [States and issues] If $\mathcal{W}$ is a set of possible worlds, then:

- an *information state* is a subset $s \subseteq \mathcal{W}$;
- an *issue* is a non-empty set $I$ of information states which is *downward closed*: if $s \in I$ and $t \subseteq s$, then $t \in I$. We denote by $\mathcal{I}$ the set of all issues.

Intuitively, downward closure corresponds to the following persistence property of resolution conditions: if $I$ is resolved in $s$, and $t$ is more informed than $s$, then $I$ is resolved in $t$ as well. Now, an issue $I$ can only be truthfully resolved if the actual world $w$ is located in some resolving state $s \in I$, that is, if $w \in \bigcup I$. Hence, an issue $I$ assumes the information corresponding to $\bigcup I$: we will say that it is an issue *over* the state $s = \bigcup I$.

With these notions in place, we are ready to define the models for our logic. Standard epistemic logic allows us to model and reason about certain facts, together with what certain agents know about these facts. Accordingly, a possible world $w$ is fully specified by two aspects: (i) a propositional valuation $V(w)$, which specifies which atomic sentences are true at $w$; and (ii) for every agent $a$, an information state $\sigma_a(w)$, representing the information available to $a$ in $w$. Thus, a model for epistemic logic is a triple $\langle \mathcal{W}, V, \{\sigma_a \,|\, a \in \mathcal{A}\}\rangle$, where the functions $\sigma_a : \mathcal{W} \to \wp(\mathcal{W})$, called *epistemic maps*, are constrained by certain requirements, the most standard being factivity and introspection.

In *inquisitive* epistemic logic, what matters is not only the information that agents have, but also the issues that they entertain. Thus, the description of a possible world will comprise a third aspect: for every agent $a$, our models will have to specify an issue $\Sigma_a(w)$ over $\sigma_a(w)$, which represents the inquisitive state of $a$, the agent's desire to locate the actual world more precisely inside her information state. Intuitively, if the inquisitive state of $a$ is $\Sigma_a(w)$, this means that $a$'s epistemic goals are to reach some information state $t \in \Sigma_a(w)$.

Now, since $\Sigma_a(w)$ is required to be an issue over $\sigma_a(w)$, we must have that $\sigma_a(w) = \bigcup \Sigma_a(w)$. Hence, the map $\Sigma_a$ by itself describes both information and issues of $a$, and we do not need $\sigma_a$ as a separate component of our models. Like in standard epistemic logic, the maps $\Sigma_a$ may be constrained by specific requirements. Since the choice of these requirements is rather orthogonal to the main novelties introduced, IEL builds on the most standard version of epistemic logic, requiring the maps to satisfy factivity and introspection, where the latter now concerns both information and issues.

**Definition 2.2** [Inquisitive epistemic models] An inquisitive epistemic model for a set $\mathcal{P}$ of atoms and a set $\mathcal{A}$ of agents is a triple $M = \langle \mathcal{W}, V, \Sigma_{\mathcal{A}} \rangle$ where:

- $\mathcal{W}$ is a set, whose elements we refer to as *possible worlds*.
- $V : \mathcal{W} \to \wp(\mathcal{P})$ is a *valuation map* that specifies for every world $w$ which atomic sentences are true at $w$.
- $\Sigma_{\mathcal{A}} = \{\Sigma_a \mid a \in \mathcal{A}\}$ is a set of *state maps* $\Sigma_a : \mathcal{W} \to \mathcal{I}$, each of which assigns to any world $w$ an issue $\Sigma_a(w)$, in accordance with:

  **Factivity** : for any $w \in \mathcal{W}$, $w \in \sigma_a(w)$
  **Introspection** : for any $w, v \in \mathcal{W}$, if $v \in \sigma_a(w)$, then $\Sigma_a(v) = \Sigma_a(w)$

  where $\sigma_a(w) := \bigcup \Sigma_a(w)$ represents the *information state* of agent $a$ in $w$.

Now that issues have entered the stage, it seems natural to equip the logical language with the means to talk about them. Following *dichotomous inquisitive semantics* ([4]), this is done in IEL by augmenting a standard logical language of *declaratives* with a new syntactic category, the category of *interrogatives*. The set $\mathcal{L}_!$ of declarative formulas and the set $\mathcal{L}_?$ of interrogative formulas of IEL are defined by simultaneous recursion as follows.

**Definition 2.3** [Syntax]
Let $\mathcal{P}$ be a set of atomic sentences and let $\mathcal{A}$ be a set of agents.

(i) For any $p \in \mathcal{P}$, $p \in \mathcal{L}_!$

(ii) $\bot \in \mathcal{L}_!$

(iii) If $\alpha_1, \ldots, \alpha_n \in \mathcal{L}_!$, then $?\{\alpha_1, \ldots, \alpha_n\} \in \mathcal{L}_?$

(iv) If $\varphi \in \mathcal{L}_\circ$ and $\psi \in \mathcal{L}_\circ$, then $\varphi \wedge \psi \in \mathcal{L}_\circ$, where $\circ \in \{!, ?\}$

(v) If $\varphi \in \mathcal{L}_! \cup \mathcal{L}_?$ and $\psi \in \mathcal{L}_\circ$, then $\varphi \to \psi \in \mathcal{L}_\circ$, where $\circ \in \{!, ?\}$

(vi) If $\varphi \in \mathcal{L}_! \cup \mathcal{L}_?$ and $a \in \mathcal{A}$, then $K_a\varphi \in \mathcal{L}_!$

(vii) If $\varphi \in \mathcal{L}_! \cup \mathcal{L}_?$ and $a \in \mathcal{A}$, then $E_a\varphi \in \mathcal{L}_!$

Importantly, conjunction, implication, and the modalities are allowed to apply to interrogatives as well as declaratives. Also, notice how the two syntactic categories are intertwined: from a sequence of declaratives, clause (iii) allows us to form a *basic interrogative*, from which more complex interrogatives may be formed by means of clauses (iv) and (v). On the other hand, clauses (vi) and (vii) allow us to embed an interrogative under a modality, resulting in a new declarative. In this way, we can form sentences such as $E_a?K_b?p$ (which will express the fact that $a$ wants to get to know whether $b$ knows whether $p$).

Besides our primitive connectives, we also make use of some defined ones. We write $\varphi \leftrightarrow \psi$ for $(\varphi \to \psi) \wedge (\psi \to \varphi)$ and $\neg\varphi$ for $\varphi \to \bot$. Moreover, for $\alpha$ and $\beta$ declaratives, we write $\alpha \vee \beta$ for $\neg(\neg\alpha \wedge \neg\beta)$ , and $?\alpha$ for $?\{\alpha, \neg\alpha\}$.

Throughout the paper, we adopt the following notational convention: $\alpha, \beta, \gamma$ range over declaratives, $\mu, \nu, \lambda$ over interrogatives, and $\varphi, \psi, \chi$ over the whole language. Moreover, $\Gamma$ ranges over sets of declaratives, $\Lambda$ over sets of interrogatives, and $\Phi$ over sets of formulas in the whole language.

We now have to specify a semantics for this language. Standardly, this means giving truth-conditions with respect to worlds in a model. However, our language now contains interrogatives as well as declaratives. We do not lay out the meaning of an interrogative by specifying at which worlds it is true, but rather by specifying what information is needed to resolve it. Thus, the natural evaluation points for interrogatives are not worlds, but rather information states. One option would then be to define by simultaneous recursion truth for declaratives and resolution for interrogatives. However, IEL adopts a solution which is both more practical and conceptually more insightful: it lifts the interpretation of *all* formulas from worlds to information states. This brings out the interesting fact that the logical operations—conjunction, implication, and the modalities—make a uniform semantic contribution, and display uniform logical properties, whether they apply to declaratives or to interrogatives. The semantics of IEL is thus defined by a relation of *support* between information states and formulas. Intuitively, for a declarative being supported amounts to being *established*, while for an interrogative it amounts to being *resolved*.

**Definition 2.4** [Support] Let $M$ be a model and $s$ an information state in $M$.

(i) $M, s \models p \iff p \in V(w)$ for all worlds $w \in s$

(ii) $M, s \models \bot \iff s = \emptyset$

(iii) $M, s \models ?\{\alpha_1, \dots, \alpha_n\} \iff M, s \models \alpha_1$ or $\dots$ or $M, s \models \alpha_n$

(iv) $M, s \models \varphi \wedge \psi \iff M, s \models \varphi$ and $M, s \models \psi$

(v) $M, s \models \varphi \to \psi \iff$ for every $t \subseteq s$, if $M, t \models \varphi$ then $M, t \models \psi$

(vi) $M, s \models K_a\varphi \iff$ for every $w \in s$, $M, \sigma_a(w) \models \varphi$

(vii) $M, s \models E_a\varphi \iff$ for every $w \in s$ and every $t \in \Sigma_a(w)$, $M, t \models \varphi$

The set of states in $M$ that support $\varphi$ is called the *proposition expressed* by $\varphi$

and denoted $[\varphi]_M$. Reference to the model $M$ will be dropped when possible.

Before turning to an explanation of the clauses, let us review some fundamental facts and notions. A first, crucial feature of IEL is that support is *persistent*.

**Fact 2.5 (Persistence)** *If* $M, s \models \varphi$ *and* $t \subseteq s$*, then* $M, t \models \varphi$.

Thus, more formulas are supported as information grows. In the limit, the empty state supports all formulas. Thus, we refer to $\emptyset$ as the *absurd* state.

**Fact 2.6** $M, \emptyset \models \varphi$ *for any* $M$ *and* $\varphi$.

Together, these two properties guarantee that $[\varphi]_M$ is always an issue, in the sense of Definition 2.1 (or, an *inquisitive proposition*, as non-empty downward closed sets of states are also called in inquisitive semantics).

Although our semantics is defined in terms of support, *truth* at a world can be recovered by defining it as support at the corresponding singleton state.

**Definition 2.7** [Truth]
We say that $\varphi$ is *true* at a world $w$, notation $M, w \models \varphi$, in case $M, \{w\} \models \varphi$. The set of worlds at which $\varphi$ is true is called the *truth-set* of $\varphi$, notation $|\varphi|_M$.

Writing out the support clauses for singletons, it is easy to see that the connectives get their standard truth-conditional clauses, even when their constituents are interrogative. Moreover, persistence implies that a world makes a formula true iff it is contained in some supporting state.

**Fact 2.8** $M, w \models \varphi \iff w \in s$ *for some state* $s$ *s.t.* $M, s \models \varphi$.

This fact also tells us how truth should be viewed for interrogatives: an interrogative $\mu$ is true in $w$ iff $w \in s$ for some $s$ which resolves $\mu$, that is, iff there is some body of information that resolves $\mu$ and is true at $w$. In other words, an interrogative is true at those worlds where it can be truthfully resolved.

In general, truth conditions do not determine support conditions. For instance, the polar interrogatives $?p$ and $?q$ are both true everywhere, but clearly, in general they have different support conditions. However, the semantics of declaratives *is*, as usual, fully determined by truth-conditions: for, a declarative is supported in a state iff it is true at all the worlds in the state.

**Fact 2.9** *For any* $M, s,$ *and* $\alpha$*:* $M, s \models \alpha \iff (M, w \models \alpha$ *for all* $w \in s)$.

Importantly, this does *not* mean that truth for declaratives can be defined *independently* of support. For, the truth of a modal formula depends crucially on the *support* conditions of its argument, as the following truth-clauses show.

- $M, w \models K_a\varphi \iff M, \sigma_a(w) \models \varphi$
- $M, w \models E_a\varphi \iff$ for all $t \in \Sigma_a(w),\ M, t \models \varphi$

Another notion that will play a crucial role below is that of *resolutions* of a formula; intuitively, the resolutions of a formula are declaratives that correspond to the different *ways* in which the formula may be settled.

**Definition 2.10** [Resolutions]
The set $\mathcal{R}(\varphi)$ of resolutions of a formula $\varphi$ is defined recursively as follows:

- $\mathcal{R}(\alpha) = \{\alpha\}$ if $\alpha$ is a declarative
- $\mathcal{R}(?\{\alpha_1, \ldots, \alpha_n\}) = \{\alpha_1, \ldots, \alpha_n\}$
- $\mathcal{R}(\mu \wedge \nu) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\mu) \text{ and } \beta \in \mathcal{R}(\nu)\}$
- $\mathcal{R}(\varphi \to \mu) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)} \alpha \to f(\alpha) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\mu)\}$

We may think of the resolutions of an interrogative as syntactic answers to it. [2] The next fact, provable by induction, says that to resolve an interrogative is to establish some resolution of it; so, each resolution provides sufficient information to resolve the interrogative, and, taken together, the resolutions exhaust the ways in which an interrogative may be resolved.

**Fact 2.11** *For any $M$, $s$ and $\varphi$, $M, s \models \varphi \iff M, s \models \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$*

As a corollary, we get the following normal form result: every formula $\varphi$ is equivalent to a basic interrogative having the resolutions of $\varphi$ as constituents. [3]

**Corollary 2.12 (Normal form)** *For any $\varphi$, $\varphi \equiv ?\mathcal{R}(\varphi)$.*

In terms of resolutions we define the notion of *presupposition* of an interrogative.

**Definition 2.13** [Presupposition of an interrogative]
The presupposition of an interrogative $\mu$ is the declarative $\pi_\mu = \bigvee \mathcal{R}(\mu)$.

Since the interrogative operator has the same truth conditions as a disjunction, it follows from Corollary 2.12 that $\mu$ and $\pi_\mu$ have the same truth conditions. The notion of resolution can be generalized to sets of formulas as follows.

**Definition 2.14** [Resolutions of a set]
The set $\mathcal{R}(\Phi)$ of resolutions of a set $\Phi$ contains those sets $\Gamma$ of declaratives s.t.:

- for all $\varphi \in \Phi$ there is an $\alpha \in \Gamma$ such that $\alpha \in \mathcal{R}(\varphi)$
- for all $\alpha \in \Gamma$ there is a $\varphi \in \Phi$ such that $\alpha \in \mathcal{R}(\varphi)$

That is, a resolution of a set $\Phi$ is a set of declaratives obtained by replacing every formula in $\Phi$ by one or more resolutions. Notice that, since a declarative has itself as unique resolution, the declarative component of $\Phi$ is inherited by any resolution. In particular, if $\Gamma$ is a set of declaratives, $\mathcal{R}(\Gamma) = \Gamma$. Fact 2.11 generalizes to sets: writing $M, s \models \Phi$ for '$M, s \models \varphi$ for all $\varphi \in \Phi$', we have:

**Fact 2.15** *For any $M$, $s$ and $\Phi$, $M, s \models \Phi \iff M, s \models \Gamma$ for some $\Gamma \in \mathcal{R}(\Phi)$*

Equipped with these basic facts and notions, we are ready to briefly explain the support clauses. Clause (i) simply says that an atom is established in a state iff it is true everywhere in the state, a fact that we have seen to hold for

[2] Indeed, our notion of *resolutions* is a general version of the notion of *basic answers* in the interrogative frameworks of Hintikka [8,9] and Wiśniewski [11]. We use the term *resolutions* as a reminder that this is only a specific technical notion, sufficient for the present purposes. More can be said, also in a logical framework, on the complex phenomenon of *answerhood*. Our notion of *presupposition* of a question is also shared by the mentioned theories.

[3] It may seem strange that a declarative $\alpha$ can be equivalent to an interrogative. However, the corresponding interrogative is the trivial interrogative $?\{\alpha\}$, which has a unique resolution $\alpha$. Also, while $\alpha$ and $?\{\alpha\}$ are semantically equivalent, they differ in pragmatics: see [6].

declaratives in general. Similarly for (ii), which says that the falsum is only established in the absurd state $\emptyset$. Clause (iii) lays out the resolution conditions for basic interrogatives: $?\{\alpha_1, \ldots, \alpha_n\}$ is resolved in a state $s$ iff some $\alpha_i$ is established in $s$. Clause (iv) says that a conjunction is established (resolved) in a state $s$ iff both conjuncts are established (resolved) in $s$.

The clause for implication requires some more explanation. First, if the antecedent is a declarative $\alpha$, clause (v) amounts to the simpler clause (v′):

(v′) $M, s \models \alpha \to \varphi \iff M, s \cap |\alpha| \models \varphi$

The conditional $\alpha \to \varphi$ is established (resolved) in $s$ iff $\varphi$ is established (resolved) in the state $s \cap |\alpha|$ which results from augmenting $s$ with the assumption that $\alpha$ is true. For a conditional *declarative*, this delivers a standard material implication. At the same time, this also yields conditional interrogatives like $p \to ?q$, which is resolved in a state iff either $p \to q$ or $p \to \neg q$ is established.

Now consider the case in which the antecedent is an interrogative $\mu$. If the consequent is a declarative $\alpha$, then $\mu \to \alpha$ is a declarative, and may be seen to be equivalent with $\pi_\mu \to \alpha$: so, interrogative antecedents may be substituted by their presuppositions when the consequent is declarative, and add no expressive power to the language. If the consequent is itself an interrogative $\nu$, on the other hand, the clause says that $\mu \to \nu$ is resolved in $s$ in case, if we extend $s$ so as to resolve $\mu$, the resulting state will resolve $\nu$. So, we can resolve $\mu \to \nu$ if we can resolve $\nu$ *conditionally* on having a resolution of $\mu$, i.e., if we have enough information to turn any resolution of $\mu$ into a resolution of $\nu$. E.g., the conditional interrogative $?p \to ?q$ is resolved precisely in case at least one of the following declaratives is established:

1. $(p \to q) \land (\neg p \to q) \equiv q$
2. $(p \to q) \land (\neg p \to \neg q) \equiv q \leftrightarrow p$
3. $(p \to \neg q) \land (\neg p \to q) \equiv q \leftrightarrow \neg p$
4. $(p \to \neg q) \land (\neg p \to \neg q) \equiv \neg q$

which are precisely the four ways to link the answer to $?q$ to the answer to $?p$.

To conclude the tour, let us consider the modalities, starting with $K_a$. Since $K_a\varphi$ is a declarative, Fact 2.9 guarantees that we need only look at truth-conditions. Now, $K_a\varphi$ is true at $w$ in case the information state $\sigma_a(w)$ of $a$ at $w$ supports $\varphi$. If $\varphi$ is a declarative $\alpha$, this simply means that $\alpha$ is true everywhere in $\sigma_a(w)$, and we recover the standard truth conditions familiar from epistemic logic: $K_a\alpha$ is true iff $\alpha$ is true everywhere in $a$'s information state. At the same time, $K_a$ is more general in IEL, since it also embeds interrogatives. For an interrogative $\mu$, we read $\sigma_a(w) \models \mu$ as "$\mu$ is resolved in $\sigma_a(w)$". Thus, $K_a\mu$ is true at $w$ in case the information available to $a$ at $w$ resolves $\mu$. For instance, we have: $w \models K_a?\alpha \iff \sigma_a(w) \subseteq |\alpha|$ or $\sigma_a(w) \subseteq |\neg\alpha| \iff w \models K_a\alpha \lor K_a\neg\alpha$.

Now consider the *entertain* modality $E_a$. The clause says that $E_a\varphi$ is true at $w$ iff any $t \in \Sigma_a(w)$ supports $\varphi$, that is, if $\varphi$ is supported in all the states where $a$'s epistemic goals are achieved. If $\varphi$ is a declarative $\alpha$, it is not difficult to see using Fact 2.9 that this holds iff $\sigma_a(w)$ supports $\alpha$, which means that $E_a\alpha$ boils down to $K_a\alpha$. On the other hand, if $\varphi$ is an interrogative $\mu$, then the clause says that $E_a\mu$ holds in case every state that $a$ wants to reach is a state where $\mu$ is resolved. Thus, $E_a\mu$ expresses that $a$ wants resolve $\mu$.

Notice that, if $K_a\mu$ holds, i.e., if $\sigma_a(w)$ *already* resolves $\mu$, then all enhancements of $\sigma_a(w)$ will resolve $\mu$ as well, so $E_a\mu$ will hold too. However, combining $K_a$ and $E_a$, we can define a modality $W_a$ which rules out this case, expressing *not knowing and wanting to know.* We read $W_a\varphi$ as "*a wonders about* $\varphi$".

- $W_a\varphi := \neg K_a\varphi \wedge E_a\varphi$

Finally, a remark about the mathematical workings of the modalities. In standard EL, the modality $K_a$ expresses a relation (inclusion) between two semantic objects of the same kind, namely, two sets of worlds: a state $\sigma_a(w)$ associated with the evaluation world, and a proposition $|\varphi|$ expressed by its argument. In general, Kripke modalities may be seen as expressing such a relation. The modalities of our system are not Kripke modalities—they are not quantifiers over accessible worlds—yet in a sense they behave in just the same way. Now both the state $\Sigma_a(w)$ associated with the evaluation world and the proposition $[\varphi]$ expressed by a sentence are more structured objects than simple sets of worlds, embodying both information and issues. Accordingly, more types of relations between them are possible. Our modalities express two such relations, as shown by the next reformulation of their truth-conditions:

- $M, w \models K_a\varphi \iff \bigcup \Sigma_a(w) \in [\varphi]$
- $M, w \models E_a\varphi \iff \Sigma_a(w) \subseteq [\varphi]$

To sum up, IEL is a conservative extension of standard EL, in that all EL-formulas in the language receive their standard truth conditions. At the same time, IEL allows us to talk not only about the facts that agents know, but also about the issues that they can resolve and that they entertain, including higher-order ones. Moreover, containing both declarative and interrogative sentences, IEL provides a suitable ground for a dynamics in which announcements may both provide information and raise issues, as spelled out in detail in [6].

## 3 Entailment

Entailment in IEL is defined in the natural way, as preservation of support.

**Definition 3.1** [Entailment]
$\Phi \models \psi \iff$ for any model $M$ and state $s$, if $M, s \models \Phi$ then $M, s \models \psi$.

To see what this notion captures, consider first entailment towards a declarative. Fact 2.9 implies that, in this case, only truth-conditional content matters.

**Fact 3.2** $\Phi \models \alpha \iff$ *for any* $M$ *and world* $w$ : *if* $M, w \models \Phi$ *then* $M, w \models \alpha$.

Thus, entailment among declaratives amounts as usual to preservation of truth. In particular, since formulas in the language of epistemic logic get their usual truth conditions, for them entailment amounts to entailment in epistemic logic.

**Fact 3.3 (Conservativity over epistemic logic)**

*Let* $\Gamma, \alpha$ *consist of formulas in* $\mathcal{L}_{\mathsf{EL}}$*. Then* $\Gamma \models \alpha \iff \Gamma \models_{\mathsf{EL}} \alpha$

However, the declarative fragment of IEL is strictly richer than epistemic logic, encompassing in particular a logic of entertaining issues. As an example, consider:

$$E_a?\{p, q, r\} \models K_a \neg p \to E_a?\{q, r\}$$

This reads: suppose $a$ wants to establish at least one of $p, q$, and $r$; it follows that if $a$ knows that $\neg p$, then $a$ wants to establish one of $q$ and $r$.

What about the case in which we also have some interrogative assumptions? Well, since only truth-conditions matter when the conclusion is a declarative, each interrogative assumption $\mu$ may be replaced by its presupposition $\pi_\mu$, which shares the same truth conditions. Thus, entailment towards declaratives is essentially a declarative business.

Let us now consider the case in which the conclusion is an interrogative. We first establish an important characterization of entailment in IEL. Recall that to support a formula, or a set, is to support some resolution of it (facts 2.11 and 2.15). From this, we get the following characterization, which shows how cross-categorial entailment is grounded in declarative entailment: $\Phi$ entails $\psi$ iff every resolution of $\Phi$ entails some resolution of $\psi$.

**Fact 3.4** $\Phi \models \psi \iff$ *for all* $\Gamma \in \mathcal{R}(\varphi)$ *there is an* $\alpha \in \mathcal{R}(\psi)$ *s.t.* $\Gamma \models \alpha$.

Now, decomposing $\Phi$ into a set $\Gamma$ of declaratives and a set $\Lambda$ of interrogatives, and assuming $\psi$ is an interrogative $\mu$, this tells us that $\Gamma, \Lambda \models \mu$ holds iff any resolution of all interrogatives in $\Lambda$, together with $\Gamma$, entails some resolution of $\mu$; that is, if *given* $\Gamma$*, any resolution of the interrogatives in* $\Lambda$ *determines some resolution of* $\mu$. For instance, the following entailment is valid

$$p \leftrightarrow q \wedge r,\ ?q \wedge ?r \models ?p$$

since, given $p \leftrightarrow q \wedge r$, any resolution of the conjunctive question $?q \wedge ?r$ determines a resolution of $?p$: the resolution $q \wedge r$ determines the resolution $p$, the resolution $q \wedge \neg r$ determines the resolution $\neg p$, and so on. Thus, entailment involving an interrogative conclusion and interrogative assumptions captures the notion of *interrogative dependency.* [4]

Finally, how about the case in which we have an interrogative conclusion and *no* interrogative assumption? Well, since a set of declaratives $\Gamma$ is the only resolution of itself, it follows form Fact 3.4 that $\Gamma$ entails an $\mu$ iff it establishes some resolution of $\mu$, i.e., in case it settles $\mu$ in a particular way.

**Fact 3.5** *If* $\Gamma$ *is a set of declaratives,* $\Gamma \models \psi \iff \Gamma \models \alpha$ *for some* $\alpha \in \mathcal{R}(\psi)$.

Summing up, the combined notion of entailment of IEL unifies three crucial and seemingly independent notions of a logic of information and issues: *standard declarative entailment*, *answerhood*, and *interrogative dependency.*

[4] For a discussion of this interesting aspect of inquisitive logic, and the way in which it relates to the framework of dependence logic ([10,12]), see [2].

Having clarified the significance of entailment, let us turn to the formal properties of the modalities. First, the modalities of IEL are distributive. This is easy to check, but not obvious, since our modalities are not Kripke modalities.

**Fact 3.6** $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ *is valid for* $\Box \in \{K_a, E_a \mid a \in \mathcal{A}\}$.

Moreover, it has been already mentioned above that Fact 2.9 implies that the two modalities are equivalent when applied to declaratives.

**Fact 3.7** *For any declarative* $\alpha$, $K_a\alpha \equiv E_a\alpha$

The next fact says that the knowledge modality distributes over the resolutions of a formula: that is, to know a formula is to know some resolution of it.

**Fact 3.8** *For any formula* $\varphi$, $K_a\varphi \equiv \bigvee_{\alpha \in \mathcal{R}(\varphi)} K_a\alpha$

**Proof.** Since we are dealing with declaratives, we just need to check identity of truth conditions. Now, $K_a\varphi$ is true at a world $w$ just in case $\varphi$ is supported by $\sigma_a(w)$. By Fact 2.11, this is the case iff $\sigma_a(w)$ supports some $\alpha \in \mathcal{R}(\varphi)$, which in turn is precisely what is needed for $\bigvee_{\alpha \in \mathcal{R}(\varphi)} K_a\alpha$ to be true at $w$. □

Taken together, the last two facts imply that the knowledge modality can be completely paraphrased away from our language: for, Fact 3.8 tell us that a $K_a$-formula is always equivalent to one in which $K_a$ applies only to declaratives; then, Fact 3.7 states that, on declaratives, $K_a$ may be replaced by $E_a$. Thus, any formula is equivalent to a $K_a$-free one. Notice, however, that $K_a$ is not *uniformly* definable in terms of $E_a$ and the connectives: for, the paraphrase of $K_a\varphi$ depends on the specific formula $\varphi$; moreover, it is possible to show that the size of the paraphrase may grow exponentially relative to the size of $\varphi$.

As for the modality $E_a$, it is worth remarking that $E_a\varphi$ is *not* in general equivalent to a formula where the modalities are applied only to declaratives. For, on declaratives, $E_a$ coincides with $K_a$: thus, the truth-conditions of any formula in which the modalities occur only on declaratives depend exclusively on the epistemic states of the agents, as well as the propositional valuation; but of course, in general, the truth of a formula $E_a\varphi$ depends crucially on $a$'s issues, not just on $a$'s information; hence, $E_a\varphi$ cannot in general be equivalent to any formula in which the modalities occur only applied to declaratives.

Intuitively, this witnesses that *entertaining* is a relation between an agent and an issue, which is not reducible to a more basic relation between the agent and a proposition, as in the case of *knowing*. Formally, it shows that the enrichment that comes about by letting modalities embed interrogatives is substantial, as we can express things that we could not express in epistemic logic.

Finally, it is easy to verify that the factivity and introspection conditions required from state maps render valid the usual schemes for both modalities.

**Fact 3.9** *The following are valid for* $\Box \in \{K_a, E_a \mid a \in \mathcal{A}\}$*, and* $\alpha$ *declarative:*

- $\Box\alpha \to \alpha$
- $\Box\varphi \to \Box\Box\varphi$
- $\neg\Box\varphi \to \Box\neg\Box\varphi$

**Conjunction**

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

**Implication**

$$\frac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}$$

**Interrogative**

$$\frac{\alpha_i}{?\{\alpha_1, \dots, \alpha_n\}} \qquad \frac{\begin{matrix}[\alpha_1] \\ \vdots \\ \varphi\end{matrix} \quad \dots \quad \begin{matrix}[\alpha_n] \\ \vdots \\ \varphi\end{matrix} \quad ?\{\alpha_1, \dots, \alpha_n\}}{\varphi}$$

**Falsum**

$$\frac{\bot}{\varphi}$$

**Kreisel-Putnam axiom**

$$(\alpha \to ?\{\beta_1, \dots, \beta_n\}) \to ?\{\alpha \to \beta_1, \dots, \alpha \to \beta_n\}$$

**Double negation**

$$\neg\neg\alpha \to \alpha$$

This concludes our short discussion of the significance and of the features of IEL-entailment. In the next section, the insights gained in the present section will be put to use to provide a sound and complete proof system.

## 4 Axiomatization

Since the propositional fragment of IEL coincides with dichotomous inquisitive semantics, let us start out with a proof system for this fragment. The table above describes a natural deduction system, proved in [4] to be sound and complete. The standard connectives—conjunction, implication, and falsum—are all assigned their standard inference rules. These inference rules are generalized to apply not only when the constituents are declarative, but also when they are interrogative. Thus, the core proof-theoretic features of the connectives are preserved when these operations are generalized to interrogatives.

There *is*, however, one element of the system which is restricted to declaratives, namely, the double negation axiom. Indeed, the following fact says that the double negation axiom is valid for $\varphi$ iff $\varphi$ enjoys the fundamental property of declaratives (Fact 2.9), for which support amounts to truth at each world.

**Fact 4.1 (Double negation characterizes truth-conditionality)**
$\neg\neg\varphi \to \varphi$ *is valid iff for all* $M, s$*:* $M, s \models \varphi \iff (M, w \models \varphi$ *for all* $w \in s)$

The rules for the interrogative operator are simply the usual ones for a disjunction. This is hardly surprising, since the semantics of ? is disjunctive. Intuitively, the introduction rule says that if we have established $\alpha_i$ for some $i$, then we have resolved $?\{\alpha_1, \dots, \alpha_n\}$. The elimination rule says that if we can

| | |
|---|---|
| $E$-distributivity<br>$E_a(\varphi \to \psi) \to (E_a\varphi \to E_a\psi)$ | $K$-distributivity<br>$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$ |
| $E$-factivity<br>$E_a\alpha \to \alpha$ | $K$-$E$ equivalence on declaratives<br>$E_a\alpha \leftrightarrow K_a\alpha$ |
| $E$-positive introspection<br>$E_a\varphi \to E_aE_a\varphi$ | $K$ distributes over interrogatives<br>$K_a?\{\alpha_1,\dots,\alpha_n\} \to K_a\alpha_1 \vee \dots \vee K_a\alpha_n$ |
| $E$-negative introspection<br>$\neg E_a\varphi \to E_a\neg E_a\varphi$ | Necessitation, for $\Box \in \{K_a, E_a \mid a \in \mathcal{A}\}$<br>$\dfrac{\begin{array}{c}\emptyset\\ \vdots\\ \varphi\end{array}}{\Box\varphi}$ |

infer $\varphi$ from the assumption that $\alpha_i$ is established for each $i$, then we can infer $\varphi$ from the assumption that $?\{\alpha_1,\dots,\alpha_n\}$ is resolved.[5] The last component of the system is the Kreisel-Putnam axiom, which distributes an implication over an interrogative consequent, provided the antecedent is a declarative.

These ingredients provide a complete axiomatization of the propositional fragment of IEL. We then need to extend this system with axioms and rules for the modalities, which are described in the table above. Each of these corresponds to some property discussed in the previous section: for the entertain modalities we have the distributivity axiom (valid by Fact 3.6) and the axioms rendered valid by the constraints of factivity and introspection (Fact 3.9); for the knowledge modalities we have again distributivity (Fact 3.6), coincidence of $K_a$ and $E_a$ on declaratives (Fact 3.7) and distributivity over the interrogative operator (a special case of Fact 3.8). Finally, we have a standard necessitation rule for both modalities: if $\varphi$ has been derived without undischarged assumptions, infer $\Box\varphi$. This completes the description of our deduction system.

**Definition 4.2** We write $P : \Phi \vdash \psi$ if $P$ is a proof in our deduction system, whose conclusion is $\psi$ and whose set of assumptions is included in $\Phi$. As usual, we then write $\Phi \vdash \psi$ if some proof $P : \Phi \vdash \psi$ exists. We say that two formulas $\varphi$ and $\psi$ are *provably equivalent*, notation $\varphi \dashv\vdash \psi$, in case $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

As customary, it is a tedious but straightforward matter to check that the proof system is *sound* for entailment in IEL.

[5] The standard rules for negation and disjunction, which are derived connectives in IEL, are admissible, with one caveat: a disjunction may only be eliminated towards a declarative. This restriction marks the difference between $\vee$ and $?$ and prevents unsound derivations such as $p \vee \neg p \vdash ?p$ (remember that $?p$ is defined as $?\{p, \neg p\}$).

**Theorem 4.3 (Soundness)** *If* $\Phi \vdash \psi$ *then* $\Phi \models \psi$*.*

We will now prove some facts about the proof system which, besides providing important insights into the logic, play a crucial role in the completeness proof. First, the normal form result of Corollary 2.12 is provable in the system.

**Lemma 4.4** *For any* $\varphi$, $\varphi \dashv\vdash ?\mathcal{R}(\varphi)$*.*

**Proof.** The lengthy but straightforward proof is essentially the same as that given for the propositional fragment in [4]. We omit it in the interest of space. □

As a corollary, a formula is always derivable from any of its resolutions.

**Corollary 4.5** *If* $\alpha \in \mathcal{R}(\varphi)$*, then* $\alpha \vdash \varphi$*.*

**Proof.** If $\alpha \in \mathcal{R}(\varphi)$ then by a simple application of ?-introduction we have $\alpha \vdash ?\mathcal{R}(\varphi)$, whence by the previous lemma, $\alpha \vdash \varphi$. □

Lemma 4.4 also implies that interrogatives always derive their presupposition.

**Corollary 4.6** *For any interrogative* $\mu$, $\mu \vdash \pi_\mu$*.*

Let us now mention a few basic facts about the modalities. First, we can prove as usual that the distributivity axioms and the necessitation rules together ensure that the modalities are monotonic.

**Lemma 4.7**
*If* $\varphi_1, \dots, \varphi_n \vdash \psi$ *then* $\Box\varphi_1, \dots, \Box\varphi_n \vdash \Box\psi$ *for* $\Box \in \{E_a, K_a \mid a \in \mathcal{A}\}$*.*

Moreover, the equivalences we have seen in facts 3.7 and 3.8 are provable in our system: that is, $K_a$ distributes over resolutions and coincides with $E_a$ on declaratives, which means that it can be paraphrased away from the language.

**Lemma 4.8** *For any* $\varphi$, $K_a\varphi \;\dashv\vdash\; \bigvee_{\alpha\in\mathcal{R}(\varphi)} K_a\alpha \;\dashv\vdash\; \bigvee_{\alpha\in\mathcal{R}(\varphi)} E_a\alpha$

**Proof.** Immediate from Lemma 4.4, using the axiom of $K$-distributivity over the interrogative operator and the axiom of $K$-$E$ equivalence on declaratives. □

The next thing to show is that derivability shares the fundamental property of entailment expressed by Fact 3.4: from $\Phi$ we can derive $\psi$ iff from any specific resolution $\Gamma$ of $\Phi$ we can derive some resolution $\alpha$ of $\psi$.

**Theorem 4.9 (Resolution theorem)**
$\Phi \vdash \psi \iff$ *for all* $\Gamma \in \mathcal{R}(\Phi)$ *there exists some* $\alpha \in \mathcal{R}(\psi)$ *s.t.* $\Gamma \vdash \alpha$*.*

The substantial proof is given in the appendix. In particular, the proof of the left-to-right direction has an interesting computational interpretation. For, suppose we have a proof $P : \Phi \vdash \psi$. By soundness, $\Phi \models \psi$, which by Fact 3.4 means that any resolution $\Gamma$ of $\Phi$ entails some resolution $\alpha$ of $\psi$: the proof of the theorem tells us to use $P$ to find such an $\alpha$ and to produce a proof $Q : \Gamma \vdash \alpha$. Thus, a proof $P : \Phi \vdash \psi$ essentially encodes how a resolution of $\psi$ may be obtained from a resolution of $\Phi$.

Notice that, since a set of declaratives has itself as unique resolution, the resolution theorem has the following corollary.

**Corollary 4.10 (Split)**
*Let $\Gamma$ be a set of declaratives. If $\Gamma \vdash \psi$ then $\Gamma \vdash \alpha$ for some $\alpha \in \mathcal{R}(\psi)$.*

On our way to the proof of the resolution theorem, in the appendix we will also establish the following fact.

**Lemma 4.11** *If $\Phi \not\vdash \psi$ then there exists some $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \not\vdash \psi$.*

Since consistency with a declarative $\alpha$ amounts to not deriving $\neg\alpha$, an immediate consequence of this lemma is that if $\Phi$ is consistent with $\alpha$, then some resolution $\Gamma$ of $\Phi$ is consistent with $\alpha$.

We will prove completeness by constructing a canonical model. The construction is similar to the familiar one from standard modal logic, but slightly more sophisticated, since the model we need to build has a richer structure than a standard Kripke model.

The possible worlds in our canonical model will be *complete theories of declaratives* (CTD), defined as sets $\Gamma$ of declaratives which are (i) closed under deduction of declaratives; (ii) consistent; and (iii) complete, in the sense that for any declarative $\alpha$, either $\alpha$ or $\neg\alpha$ is in $\Gamma$. The following features of CTDs are familiar from classical logic and modal logic.

**Fact 4.12 (Disjunction property)**
*If $\Gamma$ is a CTD and $\alpha_1 \vee \cdots \vee \alpha_n \in \Gamma$, then $\alpha_i \in \Gamma$ for some $i$.*

**Fact 4.13 (Lindenbaum's lemma)**
*If $\Theta$ is a consistent set of declaratives, then $\Theta \subseteq \Gamma$ for some CTD $\Gamma$.*

We are now ready to define our canonical model for inquisitive epistemic logic.

**Definition 4.14** [Canonical model for IEL]
The canonical model for IEL is the model $M^c = \langle \mathcal{W}^c, V^c, \Sigma^c_{\mathcal{A}} \rangle$, where:

- the elements of $\mathcal{W}^c$ are the complete theories of declaratives
- $V^c(\Gamma) = \{p \in \mathcal{P} \mid p \in \Gamma\}$
- $\Sigma^c_a(\Gamma)$ is the set of states $S \subseteq \mathcal{W}^c$ defined as follows:
  $S \in \Sigma^c_a(\Gamma) \iff \bigcap S \vdash \varphi$ whenever $E_a\varphi \in \Gamma$ [6]

Recall that the information state $\sigma^c_a(\Gamma)$ of an agent $a$ at a world $\Gamma$ is defined as the union of the inquisitive state $\Sigma^c_a(\Gamma)$ of the agent. The following lemma gives a direct characterization of $\sigma^c_a(\Gamma)$ in terms of the theories that it contains. The proof is given in the appendix.

**Lemma 4.15** $\sigma^c_a(\Gamma) = \{\Delta \mid \alpha \in \Delta$ *whenever* $E_a\alpha \in \Gamma\}$

We then have to show that the structure we defined is a proper inquisitive epistemic model, in the sense that the state maps $\Sigma^c_a$ satisfy the factivity and introspection requirements. This is precisely what the axioms of factivity and positive and negative introspection for $E$ are intended to enforce.

**Lemma 4.16** *$M^c$ is an inquisitive epistemic model.*

[6] The intersection of the empty state is defined as the set of all formulas, $\bigcap \emptyset = \mathcal{L}_{\mathcal{P}}$.

**Proof.** It is an easy exercise to check that the axiom $E_a\alpha \to \alpha$ ensures that the state maps are satisfy factivity, while the axioms $E_a\varphi \to E_aE_a\varphi$ and $\neg E_a\varphi \to E_a\neg E_a\varphi$ ensure that the state maps satisfy introspection. □

The bridge between derivability and semantics in the canonical model is usually provided by a *truth lemma* equating truth at a world in the canonical model with derivability from that world. In IEL, the fundamental semantic relation is not truth at a world, but support at a state. Accordingly, the bridge between derivability and semantics is given by the following *support lemma*, stating that support at a state $S$ in the canonical model amounts to derivability from the intersection $\bigcap S$ of all the theories in $S$. The proof is given in the appendix.

**Lemma 4.17 (Support lemma)**
*For any* $S \subseteq \mathcal{W}^c$ *and any* $\varphi$, $M^c, S \models \varphi \iff \bigcap S \vdash \varphi$.

We can then rely on the support lemma to prove the completeness theorem.

**Theorem 4.18 (Completeness theorem)** *If* $\Phi \models \psi$, *then* $\Phi \vdash \psi$.

**Proof.** Suppose $\Phi \nvdash \psi$. By Theorem 4.9, there is a resolution $\Theta$ of $\Phi$ which does not derive any resolution of $\psi$. Let $\mathcal{R}(\psi) = \{\alpha_1, \dots, \alpha_n\}$: for each $i$, since $\Theta \nvdash \alpha_i$, the set $\Theta \cup \{\neg\alpha_i\}$ is consistent, and thus extendible to a CTD $\Gamma_i \in W^c$. Now let $S = \{\Gamma_1, \dots, \Gamma_n\}$: we claim that $S \models \Phi$ but $S \not\models \psi$.

To see that $S \models \Phi$, notice that by construction, $\Theta \in \bigcap S$, whence by the support lemma $M^c, S \models \Theta$. But since $\Theta \in \mathcal{R}(\Phi)$, by Fact 2.15 we also have $M^c, S \models \Phi$. However, suppose $S$ supported $\psi$: then by Fact 2.11 it should also support $\alpha_i$ for some $i$. By the support lemma, that would mean that $\bigcap S \vdash \alpha_i$, and so also $\alpha_i \in \Gamma_i$, since $\bigcap S \subseteq \Gamma_i$ and $\Gamma_i$ is closed under declarative deduction. But that is impossible, since $\Gamma_i$ is consistent and contains $\neg\alpha_i$ by construction. Hence, $M^c, S \models \Phi$ but $M^c, S \not\models \psi$, which witnesses that $\Phi \not\models \psi$. □

**Conclusion** In this paper we have investigated and axiomatized the notion of entailment arising from *inquisitive epistemic logic*. Concretely, this gives us a conservative extension of standard epistemic logic in which we can reason not only about the agents' information, but also about the agents' issues, that is, their epistemic goals, thus providing the ground for a logical account of information exchange as a directed process of raising and resolving issues.

From an abstract standpoint, the main finding is that propositional and modal logics generalize smoothly beyond the truth-conditional realm, to a setting where both declaratives and interrogatives receive a uniform semantics. In the proof system, the connectives are handled by their standard rules. What does *not* generalize is the double negation axiom, which characterizes truth-conditionality, and must be restricted to declaratives. Finally, while the modalities of our system are not Kripke modalities, in that they operate on crucially richer semantic objects, they enjoy completely standard logical properties: distributivity holds, and the frame conditions of factivity and introspection are characterized by the familiar schemes.

## Appendix

**Proof of theorem 4.9** Let us first show the left-to-right direction of the theorem: if $\Phi$ derives $\psi$, any resolution $\Gamma$ of $\Phi$ derives some resolution $\alpha$ of $\psi$. The proof goes by induction on the complexity of the proof $P : \Phi \vdash \psi$. We distinguish a number of cases depending on the last rule applied in $P$. In the interest of space, the most straightforward inductive cases are omitted.

- $\psi$ is an undischarged assumption, $\psi \in \Phi$. In this case, any resolution $\Gamma$ of $\Phi$ contains a resolution $\alpha$ of $\psi$ by definition, so $\Gamma \vdash \alpha$.
- $\psi$ is an axiom. If $\psi$ is declarative, the claim is trivially true. If $\psi$ is interrogative, it must be an instance of the Kreisel-Putnam axiom, $(\beta \to ?\{\gamma_1, \dots, \gamma_n\}) \to ?\{\beta \to \gamma_1 \dots, \beta \to \gamma_n\}$, since all other axioms are declaratives. In this case, take $\alpha = \bigwedge_{1 \le i \le n}((\beta \to \gamma_i) \to (\beta \to \gamma_i))$: $\alpha$ is a resolution of $\psi$ and, being a classical tautology, we have $\Gamma \vdash \alpha$ for any set $\Gamma$ whatsoever.
- $\psi = \chi \to \mu$ was obtained by an implication introduction rule. Then the immediate subproof of $P$ is a proof of $\mu$ from the set of assumptions $\Phi \cup \{\chi\}$. Take any resolution $\Gamma$ of $\Phi$. Suppose $\alpha_1, \dots, \alpha_n$ are the resolutions of $\chi$. For any $1 \le i \le n$, then, $\Gamma \cup \{\alpha_i\}$ is a resolution of $\Phi \cup \{\chi\}$, whence by induction hypothesis we have a proof $Q_i : \Gamma \cup \{\alpha_i\} \vdash \beta_i$ for some resolution $\beta_i$ of $\mu$. But then, extending $Q_i$ by an application of implication introduction, we derive $\alpha_i \to \beta_i$ from $\Gamma$. And since this is the case for $1 \le i \le n$, from $\Gamma$ we can derive $(\alpha_1 \to \beta_1) \wedge \dots \wedge (\alpha_n \to \beta_n)$, which is a resolution of $\chi \to \mu = \psi$.
- $\psi$ was obtained by an implication elimination rule from $\chi$ and $\chi \to \psi$. Then the immediate subproofs of $P$ are a proof of $\chi$ from $\Phi$, and a proof of $\chi \to \psi$ from $\Phi$. Consider any $\Gamma \in \mathcal{R}(\Phi)$. By induction hypothesis we have a proof $Q_1 : \Gamma \vdash \beta$ where $\beta \in \mathcal{R}(\chi)$, and a proof $Q_2 : \Gamma \vdash \gamma$, where $\gamma \in \mathcal{R}(\chi \to \psi)$. Now, if $\mathcal{R}(\chi) = \{\beta_1, \dots, \beta_n\}$, then $\beta = \beta_i$ for some $i$, and, by definition of the resolutions of an implication, $\gamma = (\beta_1 \to \gamma_1) \wedge \dots \wedge (\beta_n \to \gamma_n)$ for some $\{\gamma_1, \dots, \gamma_n\} \subseteq \mathcal{R}(\psi)$. Extending $Q_2$ with a conjunction elimination rule we get a proof of $\beta_i \to \gamma_i$ from $\Gamma$. So, from $\Gamma$ we can derive both $\beta_i$ and $\beta_i \to \gamma_i$ whence, eliminating the implication, we can derive $\gamma_i$, a resolution of $\psi$.
- $\psi$ was obtained by a ?-elimination rule from $?\{\beta_1, \dots, \beta_m\}$. Then the immediate subproofs of $P$ are a proof $P_0 : \Phi \vdash ?\{\beta_1, \dots, \beta_m\}$ and, for $1 \le i \le n$ a proof $P_i : \Phi \cup \{\beta_i\} \vdash \psi$. Now consider a resolution $\Gamma$ of $\Phi$. By induction hypothesis we have a proof $Q_0 : \Gamma \vdash \beta$ for some $\beta \in \mathcal{R}(?\{\beta_1, \dots, \beta_m\})$. Moreover, for any $1 \le i \le n$, since $\Gamma \cup \{\beta_i\}$ is a resolution of $\Phi \cup \{\beta_i\}$, by induction hypothesis we have a proof $Q_i : \Gamma \cup \{\beta_i\} \vdash \alpha_i$ where $\alpha_i \in \mathcal{R}(\psi)$. Now since $\beta$ is a resolution of $?\{\beta_1, \dots, \beta_m\}$, by definition $\beta = \beta_i$ for some $i$. But then, combining the proof $Q_0 : \Gamma \vdash \beta_i$ with the proof $Q_i : \Gamma \cup \{\beta_i\} \vdash \alpha_i$ (more precisely, substituting any undischarged assumption of $\beta_i$ in $Q_i$ with an occurrence of the proof $Q_0$ with conclusion $\beta_i$) we obtain a proof of $\alpha_i$ from $\Gamma$, which is what we needed, since $\alpha_i$ is a resolution of $\psi$.

This case-by-case examination proves the left-to-right direction of the theorem. In order to establish the converse, let us make a detour to prove Fact 4.11.

**Proof of Lemma 4.11** First let us prove this for the case in which $\Phi$ is finite. We will prove by induction on the number of formulas in $\Phi$ the claim that *for any* $\psi$, if $\Phi \not\vdash \psi$ there is some $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \not\vdash \psi$.

If $\Phi = \emptyset$, the claim is trivially true. Now make the inductive hypothesis that the claim is true for sets of $n$ formulas, and let us consider a set $\Phi$ of $n+1$ formulas. Then $\Phi$ is of the form $\Psi \cup \{\chi\}$ for some set $\Psi$ of $n$ formulas and some formula $\chi$. Now consider a formula $\psi$ such that $\Psi, \chi \not\vdash \psi$. By Lemma 4.4, we must also have $\Psi, ?\mathcal{R}(\chi) \not\vdash \psi$ whence, by the ?-introduction rule, we must have $\Psi, \alpha \not\vdash \psi$ for some $\alpha \in \mathcal{R}(\chi)$. By the rules for implication, we must then have $\Psi \not\vdash \alpha_i \to \psi$, and so by induction hypothesis there is a $\Gamma \in \mathcal{R}(\Psi)$ such that $\Gamma \not\vdash \alpha_i \to \psi$. Finally, again by the rules for implication we have $\Gamma, \alpha_i \not\vdash \psi$, which proves the claim since $\Gamma \cup \{\alpha\}$ is a resolution of $\Psi \cup \{\chi\}$.

Our inductive proof is thus complete, and the claim is proved for the case in which $\Phi$ is finite. Now let us suppose that $\Phi$ is infinite and choose an enumeration of $\Phi$, so that $\Phi = \{\varphi_n \,|\, n \geq 1\}$. Now, for $n \in \mathbb{N}$, put:

$$T_n = \{ \langle \alpha_1, \ldots, \alpha_n \rangle \mid \alpha_i \in \mathcal{R}(\varphi_i) \text{ for } 1 \leq i \leq n\}$$

Now let $T = \bigcup_{n \in \mathbb{N}} T_n$ and, for $a, b \in T$, let $a \leq b$ in case $a$ is an initial segment of $b$. Clearly, $\langle T, \leq \rangle$ is a tree. Moreover, $T$ is finitely branching: this is because the immediate successors of $a = \langle \alpha_1, \ldots, \alpha_n \rangle$ are $a' = \langle \alpha_1, \ldots, \alpha_n, \alpha_{n+1} \rangle$ where $\alpha_{n+1} \in \mathcal{R}(\varphi_{n+1})$, and the set of resolutions of a sentence is always finite.

Now consider a formula $\psi$ such that $\Phi \not\vdash \psi$. To find a resolution $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \not\vdash \psi$, we first divide $T$ into two parts:

- $T_{\vdash \psi} = \{ \langle \alpha_1, \ldots, \alpha_n \rangle \in T \mid \{\alpha_1, \ldots, \alpha_n\} \vdash \psi\}$
- $T_{\not\vdash \psi} = \{ \langle \alpha_1, \ldots, \alpha_n \rangle \in T \mid \{\alpha_1, \ldots, \alpha_n\} \not\vdash \psi\}$

Cearly, $T_{\not\vdash \psi}$ and $T_{\vdash \psi}$ form a partition of $T$. Notice that $T_{\vdash \psi}$ is upward closed, that is, if $a \leq b$ and $a \in T_{\vdash \psi}$, then $b \in T_{\vdash \psi}$ as well: for, if $\psi$ is provable from a certain set, it is also provable from any superset. Conversely, $T_{\not\vdash \psi}$ is downward closed, that is, if $a \leq b$ and $b \in T_{\not\vdash \psi}$ then $a \in T_{\not\vdash \psi}$.

We claim that $T_{\not\vdash \psi}$ is infinite. For, if it were finite, it would only intersect finitely many of the $T_n$'s. For an index $k$ such that $T_{\not\vdash \psi} \cap T_k = \emptyset$, this would mean that $T_k \subseteq T_{\vdash \psi}$. But, recalling the definition of $T_k$, this means that every resolution $\Gamma$ of the set $\{\varphi_1, \ldots, \varphi_k\}$ derives $\psi$. Since we have already proved our claim for finite sets, we can conclude that $\{\varphi_1, \ldots, \varphi_k\}$ must derive $\psi$ as well. But this is a contradiction, since $\{\varphi_1, \ldots, \varphi_k\} \subseteq \Phi$ and $\Phi \not\vdash \psi$ by assumption.

So, $T_{\not\vdash \psi}$ must be infinite, and $\langle T_{\not\vdash \psi}, \leq \rangle$ is an infinite tree. Since $\langle T_{\not\vdash \psi}, \leq \rangle$ is finitely branching, by König's lemma it must have an infinite branch. But this means precisely that there exists an infinite sequence $\alpha_n, n \geq 1$ of declaratives (the limit of the finite sequences on the infinite branch of $T_{\not\vdash \psi}$) such that (i) $\alpha_n \in \mathcal{R}(\varphi_n)$ for any $n \geq 1$ and (ii) for any $n \geq 1$, $\{\alpha_1, \ldots, \alpha_n\} \not\vdash \psi$.

Consider $\Gamma = \{\alpha_n \,|\, n \geq 1\}$. Since $\alpha_n \in \mathcal{R}(\varphi_n)$ for every $n \geq 1$, $\Gamma \in \mathcal{R}(\Phi)$. Moreover, since for every $n \geq 1$, $\{\alpha_1, \ldots, \alpha_n\} \not\vdash \psi$, $\Gamma$ cannot derive $\psi$. So we have found a resolution $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \not\vdash \psi$. □

**Proof of theorem 4.9, right-to-left direction.** Suppose $\Phi \not\vdash \psi$. By Lemma

4.11 we have a $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \not\vdash \psi$. Since for any $\alpha \in \mathcal{R}(\psi)$ we have $\alpha \vdash \psi$ (Corollary 4.5), $\Gamma$ cannot derive any $\alpha \in \mathcal{R}(\psi)$, otherwise it would derive $\psi$. So, $\Gamma$ is a resolution of $\Phi$ which does not derive any resolution of $\psi$. □

**Proof of Lemma 4.15.** First assume $\Delta \in \sigma^c_a(\Gamma)$. Since $\sigma^c_a(\Gamma) = \bigcup \Sigma^c_a(\Gamma)$, this means that $\Delta \in S$ for some state $S$ such that $\bigcap S \vdash \varphi$ whenever $E_a\varphi \in \Gamma$. In particular, then, if $E_a\alpha \in \Gamma$ we have $\bigcap S \vdash \alpha$, whence also $\Delta \vdash \alpha$. Since $\Delta$ is closed under deduction of declaratives, this implies $\alpha \in \Delta$.

Conversely, suppose $\alpha \in \Delta$ whenever $E_a\alpha \in \Gamma$. We claim that the singleton state $\{\Delta\}$ belongs to $\Sigma^c_a(\Gamma)$, so that $\Delta \in \bigcup \Sigma^c_a(\Gamma) = \sigma^c_a(\Gamma)$. Since $\bigcap\{\Delta\} = \Delta$, to show that $\{\Delta\} \in \Sigma^c_a(\Gamma)$ we must show that $\Delta \vdash \varphi$ whenever $E_a\varphi \in \Gamma$.

So, suppose $E_a\varphi \in \Gamma$ and let us show $\Delta \vdash \varphi$. If $\varphi$ is a declarative, this is true by assumption. Now consider an interrogative $\mu$ such that $E_a\mu \in \Gamma$. Corollary 4.6 gives $\mu \vdash \pi_\mu$, whence by Lemma 4.7 we have $E_a\mu \vdash E_a\pi_\mu$. Since $E_a\mu \in \Gamma$ and $\Gamma$ is closed under deduction of declaratives, $E_a\pi_\mu \in \Gamma$. But since, unlike $\mu$, $\pi_\mu$ is a declarative, by assumption we have $\pi_\mu \in \Delta$. By definition, $\pi_\mu = \bigvee \mathcal{R}(\mu)$. Now, since $\Delta$ is a CTD, it has the disjunction property (Fact 4.12), which means that we must have $\alpha \in \Delta$ for some $\alpha \in \mathcal{R}(\mu)$. But then, since $\alpha \vdash \mu$ by Corollary 4.5, it follows $\Delta \vdash \mu$, as required. □

In preparation to the support lemma, we will prove three intermediate lemmata.

**Lemma A.19** *For any state $S \subseteq \mathcal{W}^c$ and any $\alpha \in \mathcal{L}_!$,* $\bigcap S \vdash \alpha \iff \alpha \in \bigcap S$

**Proof.** If $\alpha \in \bigcap S$ then obviously $\bigcap S \vdash \alpha$. For the converse, suppose $\bigcap S \vdash \alpha$. For any $\Gamma \in S$ we have $\bigcap S \subseteq \Gamma$, so also $\Gamma \vdash \alpha$. But then, because $\Gamma$ is closed under deduction of declaratives, we must have $\alpha \in \Gamma$. So, $\alpha \in \bigcap S$. □

**Lemma A.20**
*Let $\Gamma \in \mathcal{W}^c$. If $E_a\varphi \notin \Gamma$ there exists a state $T \in \Sigma^c_a(\Gamma)$ such that $\bigcap T \not\vdash \varphi$.*

**Proof.** Put $\Gamma^{E_a} = \{\psi \mid E_a\psi \in \Gamma\}$ (notice that $\Gamma^{E_a}$ does not only contain declaratives but also interrogatives). We claim that $\Gamma^{E_a}$ does not entail $\varphi$. Towards a contradiction, suppose $\Gamma^{E_a} \vdash \varphi$. Let $\psi_1, \dots, \psi_n \in \Gamma^{E_a}$ be assumptions such that $\psi_1, \dots, \psi_n \vdash \varphi$. By Lemma 4.7 we have $E_a\psi_1, \dots, E_a\psi_n \vdash E_a\varphi$. But the fact that $\psi_1, \dots, \psi_n$ are in $\Gamma^{E_a}$ means that $E_a\psi_1, \dots, E_a\psi_n$ are in $\Gamma$. Hence, we would also have $\Gamma \vdash E_a\varphi$, and so also $E_a\varphi \in \Gamma$, contrary to assumption.

We have thus proved $\Gamma^{E_a} \not\vdash \varphi$. But then, by Lemma 4.9 we know that there must be a resolution $\Theta$ pf $\Gamma^{E_a}$ which entails no resolution $\alpha$ of $\varphi$. But then, for any $\alpha \in \mathcal{R}(\varphi)$, the set $\Theta \cup \{\neg\alpha\}$ is a consistent set of declaratives, and so by Lindenbaum's lemma it can be extended to some CTD $\Delta_\alpha \in \mathcal{W}^c$.

Now consider the state $T = \{\Delta_\alpha \mid \alpha \in \mathcal{R}(\varphi)\}$. We claim that $T$ has the properties we need. First, since $\Theta \subseteq \Delta_\alpha$ for each $\alpha$, we have $\Theta \subseteq \bigcap T$. Now suppose $E_a\psi \in \Gamma$: then $\psi \in \Gamma^{E_a}$, and since $\Theta$ is a resolution of $\Gamma^{E_a}$, it contains some resolution $\beta$ of $\psi$. But then, since $\beta \in \Theta \subseteq \bigcap T$ and $\beta \vdash \psi$, we must also have $\bigcap T \vdash \psi$. So, $\bigcap T \vdash \psi$ whenever $E_a\psi \in \Gamma$, which means that $T \in \Sigma^c_a(\Gamma)$.

On the other hand, $\bigcap T \not\vdash \varphi$. For, if we had $\bigcap T \vdash \varphi$, by Corollary 4.10 we should have $\bigcap T \vdash \alpha$ for some resolution $\alpha$ of $\varphi$, which would entail $\Delta_\alpha \vdash \alpha$, since $\bigcap T \subseteq \Delta_\alpha$. But this is impossible, since by construction $\Delta_\alpha$ contains $\neg\alpha$ and is a consistent theory. Hence $\bigcap T \not\vdash \varphi$ and our lemma is proven. □

**Lemma A.21** *Let* $\Gamma \in \mathcal{W}^c$. *If* $K_a\varphi \notin \Gamma$, *then* $\bigcap \sigma_a^c(\Gamma) \not\vdash \varphi$.

**Proof.** Suppose $K_a\varphi \notin \Gamma$. Since $K_a\varphi \dashv\vdash \bigvee_{\alpha\in\mathcal{R}(\varphi)} E_a\alpha$, the latter formula is not in $\Gamma$ either. Since $\Gamma$ is closed under declarative deduction, this implies $E_a\alpha \notin \Gamma$ for every $\alpha \in \mathcal{R}(\varphi)$. Now consider any $\alpha \in \mathcal{R}(\varphi)$: since $E_a\alpha \notin \Gamma$, by Lemma A.20 there is a state $T_\alpha \in \Sigma_a^c(\Gamma)$ such that $\bigcap T_\alpha \not\vdash \alpha$. Now since $T_\alpha \in \Sigma_a^c(\Gamma)$ we have $T_\alpha \subseteq \bigcup \Sigma_a^c(\Gamma) = \sigma_a^c(\Gamma)$, whence $\bigcap \sigma_a^c(\Gamma) \subseteq \bigcap T_\alpha$. And since $\bigcap T_\alpha \not\vdash \alpha$, a fortiori $\bigcap \sigma_a^c(\Gamma) \not\vdash \alpha$. But as $\bigcap \sigma_a^c(\Gamma)$ does not derive any resolution of $\varphi$, by Corollary 4.10 it cannot derive $\varphi$ either: $\bigcap \sigma_a^c(\Gamma) \not\vdash \varphi$. □

**Proof of Lemma 4.17.** The proof goes by induction on the complexity of $\varphi$. The straightforward cases for atoms, falsum, and conjunction are omitted.

**Implication** Suppose $\bigcap S \vdash \varphi \to \psi$. Take any $T \subseteq S$: if $T \models \varphi$ then by induction hypothesis $\bigcap T \vdash \varphi$. Since $T \subseteq S$, we have $\bigcap T \supseteq \bigcap S$, and since $\bigcap S \vdash \varphi \to \psi$, also $\bigcap T \vdash \varphi \to \psi$. But from $\bigcap T \vdash \varphi \to \psi$ and $\bigcap T \vdash \varphi$ it follows $\bigcap T \vdash \psi$, which by induction hypothesis implies $T \models \psi$. So, every substate of $S$ that supports $\varphi$ also supports $\psi$, which proves that $S \models \varphi \to \psi$.

Viceversa, suppose $\bigcap S \not\vdash \varphi \to \psi$. By the introduction rule for implication, this means that $\bigcap S, \varphi \not\vdash \psi$. Now by Lemma 4.11 there is a a resolution of $(\bigcap S) \cup \{\varphi\}$ which does not derive $\psi$. Since $\bigcap S$ is a set of declaratives, this resolution must include a set of the form $(\bigcap S) \cup \{\alpha\}$ where $\alpha$ is a resolution of $\varphi$. Hence, there must exist a resolution $\alpha$ of $\varphi$ such that $\bigcap S, \alpha \not\vdash \psi$.

Now let $T = \{\Gamma \in S \mid \alpha \in \Gamma\}$. First, by definition we have $\alpha \in \bigcap T$, whence $\bigcap T \vdash \varphi$ by Corollary 4.5. By induction hypothesis we then have $T \models \varphi$. Now, if we can show that $\bigcap T \not\vdash \psi$ we are done. For then, the induction hypothesis gives $T \not\models \psi$, which means that $T$ is a substate of $S$ that supports $\varphi$ but not $\psi$, which shows that $S \not\models \varphi \to \psi$.

So, we are left to show that $\bigcap T \not\vdash \psi$. Towards a contradiction, suppose that $\bigcap T \vdash \psi$. Since $\bigcap T$ is a set of declaratives, Corollary 4.10 tells us that $\bigcap T \vdash \beta$ for some resolution $\beta$ of $\psi$, which by Lemma A.19 amounts to $\beta \in \bigcap T$. So, for any $\Gamma \in T$ we have $\beta \in \Gamma$ and thus also $\alpha \to \beta \in \Gamma$, since $\Gamma$ is closed under deduction of declaratives and $\beta \vdash \alpha \to \beta$. Now consider any $\Gamma \in S - T$: this means that $\alpha \notin \Gamma$; then since $\Gamma$ is complete we have $\neg\alpha \in \Gamma$, whence $\alpha \to \beta \in \Gamma$, because $\Gamma$ is closed under deduction of declaratives and $\neg\alpha \vdash \alpha \to \beta$. We have thus shown that $\alpha \to \beta \in \Gamma$ for any $\Gamma \in S$, whether $\Gamma \in T$ or $\Gamma \in S - T$. We can then conclude $\alpha \to \beta \in \bigcap S$, whence $\bigcap S, \alpha \vdash \beta$. And since $\beta$ is a resolution of $\psi$ we also have $\bigcap S, \alpha \vdash \psi$. But this is a contradiction since by assumption $\alpha$ is such that $\bigcap S, \alpha \not\vdash \psi$.

**Question mark** If $S \models ?\{\alpha_1, \ldots, \alpha_n\}$, then $S \models \alpha_i$ for some $i$, so by induction hypothesis we have $\bigcap S \vdash \alpha_i$ and by ?-introduction also $\bigcap S \vdash ?\{\alpha_1, \ldots, \alpha_n\}$. Conversely, suppose $\bigcap S \vdash ?\{\alpha_1, \ldots, \alpha_n\}$. Since $\bigcap S$ is a set of declaratives, it follows from Corollary 4.10 that $\bigcap S \vdash \alpha_i$ for some $1 \leq i \leq n$. By induction hypothesis we then have $S \models \alpha_i$, and thus also $S \models ?\{\alpha_1, \ldots, \alpha_n\}$.

$E_a$ **modality** Suppose $\bigcap S \vdash E_a\varphi$. Now consider any $\Gamma \in S$. Since $\bigcap S \subseteq \Gamma$, we have $\Gamma \vdash E_a\varphi$, and since $\Gamma$ is closed under deduction of declaratives, $E_a\varphi \in \Gamma$. By definition of $\Sigma_a^c$, then, for any $T \in \Sigma_a^c(\Gamma)$ we must have $\bigcap T \vdash \varphi$, which by induction hypothesis entails $T \models \varphi$. Since this is true for any $\Gamma \in S$ and any $T \in \Sigma_a^c(\Gamma)$, it follows that $S \models E_a\varphi$.

For the converse, suppose $\bigcap S \nvdash E_a\varphi$. Then $E_a\varphi \notin \bigcap S$, which means that $E_a\varphi \notin \Gamma$ for some $\Gamma \in S$. Then, Lemma A.20 ensures that there exists a state $T \in \Sigma_a^c(\Gamma)$ such that $\bigcap T \nvdash \psi$, that is, by induction hypothesis, such that $T \not\models \psi$. Therefore, we do not have $T \models \varphi$ for every $\Gamma \in S$ and $T \in \Sigma_a^c(\Gamma)$, which means that $S \not\models E_a\varphi$.

$K_a$ **modality** Suppose $\bigcap S \vdash K_a\varphi$, which by Lemma A.19 implies $K_a\varphi \in \bigcap S$. Since $K_a\varphi \dashv\vdash \bigvee_{\alpha\in\mathcal{R}(\varphi)} E_a\alpha$ (Lemma 4.8), we have $\bigvee_{\alpha\in\mathcal{R}(\varphi)} E_a\alpha \in \bigcap S$. Now consider any $\Gamma \in S$. Since $\bigvee_{\alpha\in\mathcal{R}(\varphi)} E_a\alpha \in \bigcap S$, also $\bigvee_{\alpha\in\mathcal{R}(\varphi)} E_a\alpha \in \Gamma$. Since complete theories have the disjunction property, $E_a\alpha \in \Gamma$ for some $\alpha \in \mathcal{R}(\varphi)$. Since $E_a\alpha \in \Gamma$, Lemma 4.15 tells us that $\alpha \in \Delta$ for any $\Delta \in \sigma_a^c(\Gamma)$, so $\alpha \in \bigcap \sigma_a^c(\Gamma)$. Since $\alpha \vdash \varphi$ (Corollary 4.5) we then have $\sigma_a^c(\Gamma) \vdash \varphi$, which by induction hypothesis means that $\sigma_a^c(\Gamma) \models \varphi$. Summing up, for any $\Gamma \in S$ we have $\sigma_a^c(\Gamma) \models \varphi$, and so $S \models K_a\varphi$.

Conversely, suppose $\bigcap S \nvdash K_a\varphi$. Then obviously $K_a\varphi \notin \bigcap S$, so there is a $\Gamma \in S$ such that $K_a\varphi \notin \Gamma$. But then, Lemma A.21 establishes that $\bigcap \sigma_a^c(\Gamma) \nvdash \varphi$, which by induction hypothesis amounts to $\sigma_a^c(\Gamma) \not\models \varphi$. So, it is not the case that $\sigma_a^c(\Gamma) \models \varphi$ for every $\Gamma \in S$, which means that $S \not\models K_a\varphi$.□

## References


[1] van Benthem, J., "Logical dynamics of information and interaction," Cambridge University Press, 2011.

[2] Ciardelli, I., *Interrogative dependencies and the constructive content of inquisitive proofs* (2014), in the proceedings of *WoLLIC 21*.

[3] Ciardelli, I., J. Groenendijk and F. Roelofsen, *Inquisitive semantics: a new notion of meaning*, Language and Linguistics Compass **7** (2013), pp. 459–476.

[4] Ciardelli, I., J. Groenendijk and F. Roelofsen, *On the semantics and logic of declaratives and interrogatives*, Synthese (2013).

[5] Ciardelli, I. and F. Roelofsen, *Inquisitive logic*, Journal of Philosophical Logic **40** (2011), pp. 55–94.

[6] Ciardelli, I. and F. Roelofsen, *Inquisitive dynamic epistemic logic*, Synthese (2014).

[7] van Ditmarsch, H., W. van der Hoek and B. Kooi, "Dynamic Epistemic Logic," Springer, 2007.

[8] Hintikka, J., "Inquiry as inquiry: A logic of scientific discovery," Kluwer Academic Publishers, 1999.

[9] Hintikka, J., "Socratic epistemology: explorations of knowledge-seeking by questioning," Cambridge University Press, 2007.

[10] Väänänen, J., "Dependence Logic: A New Approach to Independence Friendly Logic," Cambridge University Press, Cambridge, 2007.

[11] Wiśniewski, A., *The logic of questions as a theory of erotetic arguments*, Synthese **109** (1996), pp. 1–25.

[12] Yang, F., "On extensions and variants of dependence logic: A study of intuitionistic connectives in the team semantics setting," Ph.D. thesis, University of Helsinki (2014).