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METAPHORICAL OBJECTS IN ADVANCED MATHEMATICAL THINKING

In their critique of "object as a central metaphor in advanced mathematical thinking", Confrey and Costa (1996) describes members of the Advanced Mathematical Thinking Group, including myself, as "reification theorists". By selective quation they attibute theores largely developed independently by Dubinsky and Sfard as being broadly shared. Whilst it is true that many share an interest in the relationship of process and object and the mediating role of symbols, the notion of "reification" is only part of the domain of discourse. In the book "Advanced Mathematical Thinking" to which Confrey and Costa refer, only two chapters out of thirteen can be considered as "reificationist" — a chapter by Harel and Kaput which focuses on the notion of "conceptual entity" as part of a wider theory and a chapter by Dubinsky on "Reflective Abstraction". Quite different persepectives are also presented, for instance the first chapter lays out a much broader vista, and later chapters include Hanna's philosophical chapter on "proof", and Vinner's chapter on "concept definition and concept image: and the work of the French school. In the final chapter (in a theory further elaborated in Tall 1995) attention is drawn to the considerable difference in development between visually-based theories (eg Van Hiele) and symbol-based theories (involving symbol as process and object) and how advanced mathematical thinking as a theory builds on both of them by creative problem-solving and logical theory building. By selectively choosing quotations relating only to process and object and remaining silent on everything else, Confrey and Costa misrepresent my theoretical position and. I suspect, that of other researchers in advanced mathematical thinking.

Although their title focuses on the use of "object" as a central metaphor in advanced mathematicical thinking, their argument seems more directed at the idea of first carrying out a process then reifying it as an object. To understand the nature of sucha a theory. One must understand the meaning of the terms used. Dubinsky asserts that by "process" he means "cognitive process" and includes, for example, the construction of a "poermantnet objecvt" by "encapsulating the process of performing transformation in space which do not destroy the physical object" (Dubinsky et al. 1988: p. 45). It also means that, given, say the axioms for a group, then cognitive processes can be used to prove theorems about groups which can be construed as constructing properties of a mentla object called "a group". I have no difficulty with this viewpoint, however, I do see differetn types of process giving different types of cognitive devemlopment and prefer to separate these out.

In describing a "procept" as an amalgam of process and concept, Gray and Tall (1994) refer to a process as a special kind of mathematical process, such as "addition of two whole numbers" or "evaluation of an algebraic expression" which can usually be performed by one or more specific procedures. A group is

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therefore not a procept because the symbol G does not represent both a process and a concept (although elements of a transformation group are procepts, but that is another story).

Dubinsky's greater generality allows him a more all-encompassing theory including the notions of "permanent object", "encapsulation of process as object" and "defined object" in formal theories.

Because I see these three cognitive activities as very different, I choose to treat them separately as constructing meaning for perceived objects (through teasing out their properties), constructing meaning for mathematical processes (through carrying them out and reflecting on the symbols used), and constructing meaning for defined objects (through proving theorems which construct their properties). Sfard's notion of "operational" and "structura;" appear to have something in common with these ideas but there is work to do to relate the notion of "structural: to the status of perceived figural properties of visual objects and deduced properties of defined objects.

So where does this place the critique of Confrey and Costa". If their criticism is of the notion of "object" itself, then there is nothing to stop this "object" having various aspects, including visual, figural, verbal, symbolic, kinasthetic, and so on. Even formal concepts in advanced mathematics are inspired by all kinds of imagery. It is this viewpoint I take in my "graphic approach to the calculus" which links visual properties of locally straight graphs to corresponding numeric and symbolic aspects leading to quantified definitions in real analysis or infinitesimal concepts in non-standard analysis. My own suggested approach (through dynamic visual global gradients of graphs) is quite distinct from the usual strict hierarchy which proceeds logically from the formal limit concept (Sheath & Tall, 1983; Tall 1985). The very first criticism that "mathematics is strictly hierarchically organised" simply does not follow from the use of the term "object".

Nor does it follow from the use of the term "procept". Even though many traditional approaches develop mathematical processes before concepts, the cognitive theory I apply to the calculus sees function, derivative and integral as procepts but does not have to follow a specific strict hierarchy. I proposed a "principle of selective construction" (Tall, 1993) to den the primacy of (mathematical) process over concept. Using computers to carry out procedures implicitly and display the results explicitly allows the resulting concepts to be studied before, or at the same time as, the corresponding mathematical processes. This uses alternate aspects of the concept, such as dynamic visualisation. For instance, in the British SMP 16-19 syllabus, the structural stability of the Newton-Raphson process is explored visually on the computer before the numerical formula for the procedure is introduced.

Even if the criticism of "object" is aimed at "reification theory: it need not apply to those who see "process" as any *cognitive* process, because this can involve any desired kind of cognitive activity. The criticism can possibly apply to those who assert a strict “mathematical process to object reification” as may be the case with Sfard’s assertion that “operational must always precede structural”.

I leave Anna Sfard and Ed Dubinsky to speak for themselves. I content myself by addressing a few matters of detail and one matter referring to the paper as a whole.

In using quotations from my publications,Confrey and Costa state:

> The implications of this overall agenda for mathematical instruction included a number of assumptions: (a) history of geometry and number can be intellectually isolated from one another. (Tall, 1995: p. 62)

The actual written text is:

> It is interesting to note that these developments can occur quite independently. The Ancient Greeks developed a theory of geometry (including geometric constructions of arithmetic) without any symbolism for algebra and arithmetic, and it is possible to develop arithmetic and algebra without any reference to geometry. However, many useful links have been made between visual and manipulative symbolic methods and it is clearly opportune to take advantage of them to develop a versatile approach which uses each to its best advantage.

I did say geometry and number can be developed separately, but I did not say that they are “intellectually isolated”. I did not refer to the history of algebra at all, or to the history of arithmetic, except in the context of Greek geometry where I referred to the related geometric arithmetic.

They go on to mention a further assumption:

> c) set-theoretic approaches, whilst still imperfect, represent our betst thinking. (Tall, 1995).

I said no such thing. I didsay that “[set-theory] is also flawed” and go on:

> Advanced mathematical thinking today involves using cognitive structures produced by a wide range of mathematical activities to construct new ideas that build on and extend an ever-growing system of established theorems.

The ways in which this thinking can depend on different focus onvisuospatial aspects and symbolic aspects leading to proof is beautifully illustrated by Saunders Maclane:

> In the fall of 1982, Riyadh, Saudi Arabia … we all mounted to the roof … to sit at ease in the starlight. Atiyah and MacLane fell into a discussion, as suited the occasion, about how mathematical research is done. For MacLane it meant getting and understanding the needed definitions, working with them to see what could be calculated and what might be true, to finally come up with new “structure” theorems. For Atiyah, it meant thinking hard about a somewhat vague and uncertain situation, trying to guess what might be found out, and only then finally reaching definitions and the definitive theorems and proofs. This story indicates the ways of doing mathematics can vary sharply, as in this case between the fields of algebra and geometry, while at the end there was full agreement on the final goal: theorems with proofs. Thus differently oriented mathematicians have sharply different ways of thought, but also common standards as to the result. (Maclane, 1994, p. 190–191.)

As hew clearly indicates, even though proof is the final systematisation of advanced mathematics, and the framework on which further theories are developeed, advanced mathematical thinking is so much more:

> We often hear that mathematics consists mainly in "proving theorem" . Is a writer's job mainly that of "writing sentences"? A mathematician's work is mostly a tangle of guesswork, analogy, wishful th9inking and frustration, and proof, far from being the core of discovery, is mjore often than not a way of making sure that our minds are not playing tricks. (Gian-Carl-Rota, 1980: p,xviii.)

Such a view seems close to that of Confrey and Costa, if only itwere used for students as well as researchers. But having used the cited paper Tall (1995) to set me firmly as a "reification theorist" they carefully omit any reference to the final part of the paper where I discuss the value of cooperative problem-solving and how this leads to mathematical attitudes desired by mathematicians, but not achieved by them through their standard teaching methods (Yusof & Tall, 1994). By selective quotations they set up a dichotomy between my apparent view and theirs where I feel we have more in common than divides us.

One final overall matter remains for me a matter of concern. This critique of "object as a central metaphor in advanced mathematical thinking" has relatively few references to specifics of advanced mathematical thinking in the broader sense I have sketched above. The major counterexample they cite relates to the sequence of learning about fractions. There are no specific reference to any topics in university mathematics or to the construction of mathematical objects through definition and proof. I very much share Confrey and Costa's concern about the horrendous difficulties faced by students taking university mathematics courses and the implications that strict "reification theory" may lead to inappropriate sequences of learning by not accounting for alternative aspects of the enterprise. But one cannot change the system without taking into account the mathematics that mathematicians do.

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