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Convergence in Hölder norms with applications to Monte Carlo methods in infinite dimensions

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Abstract

We show that if a sequence of piecewise affine linear processes converges in the strong sense with a positive rate to a stochastic process which is strongly Hölder continuous in time, then this sequence converges in the strong sense even with respect to much stronger Hölder norms and the convergence rate is essentially reduced by the Hölder exponent. Our first application hereof establishes pathwise convergence rates for spectral Galerkin approximations of stochastic partial differential equations. Our second application derives strong convergence rates of multilevel Monte Carlo approximations of expectations of Banach space valued stochastic processes.

Contents

1 Introduction

In this article we study convergence rates for general stochastic processes in Hölder norms. In particular, in the main results of this work (see Corollary 2.8 and Corollary 2.9 in Subsection 2.2 below) we reveal estimates for uniform Hölder errors of general stochastic processes. In this introductory section we now sketch these results and thereafter outline several applications of the general estimates, which can be found in subsequent sections of this article (see Corollary 2.11 in Subsection 2.2, Corollary 4.5 in Subsection 4.3, and Corollary 5.15 in Subsection 5.3 below). To illustrate the key results of this work, we consider the following framework throughout this section. Let $T \in (0, \infty)$ be a real number, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space, and for every function $f\colon [0,T] \to E$ and every natural number $N \in \mathbb{N} = \{1, 2, 3, \ldots\}$ let $[f]_N\colon [0,T] \to E$ be the function which satisfies for all $n \in \{0, 1, \ldots, N-1\}$, $t \in \big[\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ that

$$[f]_N(t) = \big(n + 1 - \tfrac{tN}{T}\big) \cdot f\big(\tfrac{nT}{N}\big) + \big(\tfrac{tN}{T} - n\big) \cdot f\big(\tfrac{(n+1)T}{N}\big) \tag{1.1}$$

(the piecewise affine linear interpolation of $f|_{\{0, T/N, 2T/N, \ldots, (N-1)T/N, T\}}$, cf. (1.19) below).

Theorem 1.1. *Assume the above setting. Then for all $p \in (1, \infty)$, $\varepsilon \in (1/p, 1]$, $\alpha \in [0, \varepsilon - 1/p)$ there exists $C \in (0, \infty)$ such that it holds for all $\beta \in [\varepsilon, 1]$, $N \in \mathbb{N}$ and all $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y\colon [0,T] \times \Omega \to E$ with continuous sample paths that*

$$\begin{aligned} &\Big(\mathbb{E}\Big[\|X - [Y]_N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p} \\ &\leq C N^\varepsilon \bigg(\sup_{n \in \{0,1,\ldots,N\}} \big\| X_{\frac{nT}{N}} - Y_{\frac{nT}{N}} \big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} + N^{-\beta} \|X\|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \bigg). \end{aligned} \tag{1.2}$$

The Hölder and $\mathscr{L}^p$-norms in (1.2) are to be understood in the usual sense (see Subsection 1.1 below for details). Theorem 1.1 is a direct consequence of the more general result in Corollary 2.9 in Subsection 2.2 below, which establishes an estimate similar to (1.2) also for the case of non-equidistant time grids. Moreover, Corollary 2.8 in Subsection 2.2 provides an estimate similar to (1.2) but with $\big(\mathbb{E}\big[\|X-Y\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\big]\big)^{1/p}$ instead of $\big(\mathbb{E}\big[\|X-[Y]_N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\big]\big)^{1/p}$ on the left hand side and with an appropriate Hölder norm of Y occurring on the right hand side. Theorem 1.1 has a number of applications in the numerical approximation of stochastic processes, as the next corollary, Corollary 1.2, clarifies. Corollary 1.2 follows immediately from Theorem 1.1.

Corollary 1.2. *Assume the above setting, let $\beta \in (0,1]$, and let $X\colon [0,T]\times\Omega \to E$ and $Y^N\colon [0,T]\times\Omega\to E$, $N\in\mathbb{N}$, be $(\mathscr{F},\|\cdot\|_E)$-strongly measurable stochastic processes with continuous sample paths which satisfy for all $p\in(1,\infty)$ that $\forall\, N\in\mathbb{N}\colon Y^N=[Y^N]_N$ and*

$$\|X\|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}+\sup_{N\in\mathbb{N}}\left[N^\beta \sup_{n\in\{0,1,\dots,N\}}\big\|X_{\frac{nT}{N}}-Y^N_{\frac{nT}{N}}\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\right]<\infty. \tag{1.3}$$

Then it holds for all $p,\varepsilon\in(0,\infty)$ that

$$\sup_{N\in\mathbb{N}}\left[N^{\beta-\varepsilon}\left(\mathbb{E}\left[\sup_{t\in[0,T]}\|X_t-Y^N_t\|^p_E\right]\right)^{1/p}\right]<\infty. \tag{1.4}$$

It is assumed in (1.3) that a sequence of affine linearly interpolated $(\mathscr{F},\|\cdot\|_E)$-strongly measurable stochastic processes $(Y^N)_{N\in\mathbb{N}}$ converges for every $p\in(1,\infty)$ in $\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)$ to an $(\mathscr{F},\|\cdot\|_E)$-strongly measurable stochastic process X with a positive rate uniformly on all grid points and that this process X admits corresponding temporal Hölder regularity. Corollary 1.2 then shows that these assumptions are sufficient to obtain convergence for every $p\in(1,\infty)$ in the uniform $L^p(\mathbb{P};\|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm with essentially the same rate. Corollary 2.11 in Subsection 2.2 below implies this result as a special case and includes the case of non-equidistant time grids. Moreover, Corollary 2.11 proves an analogous conclusion for convergence in uniform Hölder norms, where the obtained convergence rate is reduced by the considered Hölder exponent. Corollary 2.12 below demonstrates how this principle can be applied to Euler-Maryuama approximations for stochastic differential equations (SDEs) with globally Lipschitz coefficients. Arguments related to Corollary 2.11 can be found in Lemma A1 in Bally, Millet, & Sanz-Solé [4] and in the second display on page 325 in [9].

Corollary 1.2 is particularly useful for the study of stochastic partial differential equations (SPDEs). In general, a solution of an SPDE fails to be a semi-martingale. As a consequence, Doob's maximal inequality cannot be applied to obtain estimates with respect to the $L^2(\mathbb{P};\|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm. However, convergence rates with respect to the $C([0,T],\|\cdot\|_{L^2(\mathbb{P};\|\cdot\|_E)})$-norm are often feasible and Corollary 1.2 can then be applied to obtain convergence rates with respect to the $L^2(\mathbb{P};\|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm. Estimates with respect to the $L^2(\mathbb{P};\|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm are useful for using standard localisation arguments in order to extend results for SPDEs with globally Lipschitz continuous nonlinearities to results for SPDEs with nonlinearities that are only Lipschitz continuous on bounded sets. We demonstrate this in Corollary 4.5 in Subsection 4.3 below in the case of pathwise convergence rates for Galerkin approximations. To be more specific, Corollary 4.5 proves essentially sharp pathwise convergence rates for spatial Galerkin and noise approximations for a large class of SPDEs with non-globally Lipschitz continuous nonlinearities. For example, Corollary 4.5 applies to stochastic Burgers, stochastic Ginzburg-Landau, stochastic Kuramoto-Sivashinsky, and Cahn-Hilliard-Cook equations.

Another prominent application of Corollary 1.2 are multilevel Monte Carlo methods in Banach spaces. For a random variable $X \in \mathscr{L}^2(\mathbb{P}; \|\cdot\|_E)$ convergence in $\mathscr{L}^2(\mathbb{P}; \|\cdot\|_E)$ of Monte Carlo approximations of the expectation $\mathbb{E}[X] \in E$ has only been established if E has so-called (Rademacher) type p for some $p \in (1, 2]$ and in this case the convergence rate is given by $1 - 1/p$ (see, e.g., Heinrich [15] or Corollary 5.12 in Subsection 5.2 below). However, the space $C([0, T], E)$ fails to have type p for any $p \in (1, 2]$. If X has more sample path regularity, this problem can nevertheless be bypassed. More precisely, if it holds for some $\alpha \in (0, 1]$, $p \in (1/\alpha, \infty)$ that $X \in \mathscr{L}^2(\mathbb{P}; \|\cdot\|_{\mathcal{W}^{\alpha,p}([0,T],E)})$, then Monte Carlo approximations of $\mathbb{E}[X] \in \mathcal{W}^{\alpha,p}([0, T], E)$ have been shown to converge in $\mathscr{L}^2(\mathbb{P}; \|\cdot\|_{\mathcal{W}^{\alpha,p}([0,T],E)})$ with rate $1 - 1/\min\{2,p\}$ and, by the Sobolev embedding theorem, also converge in $\mathscr{L}^2(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$ with the same rate. Here for any real numbers $\alpha \in (0, 1]$, $p \in (1/\alpha, \infty)$ we denote by $\mathcal{W}^{\alpha,p}([0, T], E)$ the Sobolev space with regularity parameter α and integrability parameter p of continuous functions from $[0, T]$ to E. Informally speaking, in order to gain control over the variances appearing in multilevel Monte Carlo approximations it is therefore sufficient for the approximations to converge with respect to the $L^2(\mathbb{P}; \|\cdot\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)})$-norm for some $\alpha \in (0, 1]$. For more details, we refer the reader to Section 5 and, in particular, to Corollary 5.15, which formalises this approach for the case of multilevel Monte Carlo approximations of expectations of Banach space valued stochastic processes.

Finally, we mention a few results in the literature which employ some findings from this article. In particular, Corollary 2.10 in this article is applied in the proof of Corollary 6.3 in Jentzen & Pušnik [26] to prove uniform convergence in probability for spatial spectral Galerkin approximations of stochastic evolution equations (SEEs) with semi-globally Lipschitz continuous coefficients (see Proposition 6.4 in Jentzen & Pušnik [26]). Moreover, Corollary 4.4 in this article is employed in Subsection 5.2 and Subsection 5.3 in [7] for transferring initial value regularity results for finite-dimensional stochastic differential equations to the case of infinite-dimensional SPDEs using the examples of the stochastic Burgers equation and the Cahn-Hilliard-Cook equation. Furthermore, Corollary 2.11 in this article is used in the proof of Corollary 5.2 in Hutzenthaler, Jentzen, & Salimova [19] to establish essentially sharp uniform strong convergence rates for spatial spectral Galerkin approximations of linear stochastic heat equations.

1.1 Notation

In this subsection we introduce some of the notation which we use throughout this article. For two sets A and B we denote by $\mathbb{M}(A, B)$ the set of all mappings from A to B. For measurable spaces $(\Omega_1, \mathscr{F}_1)$ and $(\Omega_2, \mathscr{F}_2)$ we denote by $\mathscr{M}(\mathscr{F}_1, \mathscr{F}_2)$ the set of all $\mathscr{F}_1/\mathscr{F}_2$-measurable mappings from Ω_1 to Ω_2. For topological spaces $(E, \mathscr{E})$ and $(F, \mathscr{F})$ we denote by $\mathscr{B}(E)$ the Borel σ-algebra on $(E, \mathscr{E})$ and we denote by $C(E, F)$ the set of all continuous functions from E to F. We denote by $|\cdot| : \mathbb{R} \to [0, \infty)$ the absolute value function on $\mathbb{R}$. We denote by $\Gamma\colon (0, \infty) \to (0, \infty)$ the gamma function, that is, we denote by $\Gamma\colon (0, \infty) \to (0, \infty)$ the function which satisfies for all $x \in (0, \infty)$ that $\Gamma(x) = \int_0^\infty t^{(x-1)}\, e^{-t}\, \mathrm{d}t$. We denote by $\mathscr{E}_r\colon [0, \infty) \to [0, \infty)$, $r \in (0, \infty)$, the mappings which satisfy for all $r \in (0, \infty)$, $x \in [0, \infty)$ that

$$\mathscr{E}_r[x] = \left(\sum_{n=0}^{\infty} \frac{x^{2n} (\Gamma(r))^n}{\Gamma(nr + 1)} \right)^{1/2} = \left(1 + \frac{x^2\, \Gamma(r)}{\Gamma(r + 1)} + \frac{x^4 (\Gamma(r))^2}{\Gamma(2r + 1)} + \ \dots \right)^{1/2} \tag{1.5}$$

(cf. Chapter 7 in Henry [17]). As a notational device to condense the statements and proofs of many results in this article in a mathematically rigorous way, we next introduce the notion of an extendedly semi-normed vector space, which, roughly speaking, corresponds to a vector space with a semi-norm-type function that is allowed to attain infinity. For a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, a $\mathbb{K}$-vector space V, and a mapping $\|\cdot\| : V \to [0, \infty]$ which satisfies for all $v, w \in \{u \in V \colon \|u\| < \infty\}$, $\lambda \in \mathbb{K}$ that $\|\lambda v\| = \sqrt{[\mathrm{Re}(\lambda)]^2 + [\mathrm{Im}(\lambda)]^2}\, \|v\|$ and $\|v + w\| \leq \|v\| + \|w\|$ we call $\|\cdot\|$ an extended semi-norm on V and we call $(V, \|\cdot\|)$ an extendedly semi-normed vector space. For a measure space $(\Omega, \mathscr{F}, \mu)$, a measurable space $(S, \mathscr{S})$, a set $R \subseteq S$, and a function $f \colon \Omega \to R$ we denote by $[f]_{\mu,\mathscr{S}}$ the set given by

$$[f]_{\mu,\mathscr{S}} = \big\{g \in \mathscr{M}(\mathscr{F}, \mathscr{S}) \colon (\exists A \in \mathscr{F} \colon \mu(A) = 0 \text{ and } \{\omega \in \Omega \colon f(\omega) \neq g(\omega)\} \subseteq A)\big\}. \tag{1.6}$$

For a measure space $(\Omega, \mathscr{F}, \mu)$, a normed vector space $(V, \|\cdot\|_V)$, and real numbers $p \in [0, \infty)$, $q \in (0, \infty)$ we denote by $\mathscr{L}^0(\mu; \|\cdot\|_V)$ the set given by

$$\mathscr{L}^0(\mu; \|\cdot\|_V) = \big\{f \in \mathbb{M}(\Omega, V) \colon f \text{ is } (\mathscr{F}, \|\cdot\|_V)\text{-strongly measurable}\big\}, \tag{1.7}$$

we denote by $\|\cdot\|_{\mathscr{L}^q(\mu;\|\cdot\|_V)} : \mathscr{L}^0(\mu; \|\cdot\|_V) \to [0, \infty]$ the mapping which satisfies for all $f \in \mathscr{L}^0(\mu; \|\cdot\|_V)$ that

$$\|f\|_{\mathscr{L}^q(\mu;\|\cdot\|_V)} = \left[\int_\Omega \|f(\omega)\|_V^q \, \mu(\mathrm{d}\omega)\right]^{1/q} \in [0, \infty], \tag{1.8}$$

we denote by $\mathscr{L}^q(\mu; \|\cdot\|_V)$ the set given by

$$\mathscr{L}^q(\mu; \|\cdot\|_V) = \Big\{f \in \mathscr{L}^0(\mu; \|\cdot\|_V) \colon \|f\|_{\mathscr{L}^q(\mu;\|\cdot\|_V)} < \infty\Big\}, \tag{1.9}$$

we denote by $L^p(\mu; \|\cdot\|_V)$ the set given by

$$L^p(\mu; \|\cdot\|_V) = \big\{\{g \in \mathscr{L}^0(\mu; \|\cdot\|_V) \colon \mu(f \neq g) = 0\} \subseteq \mathscr{L}^0(\mu; \|\cdot\|_V) \colon f \in \mathscr{L}^p(\mu; \|\cdot\|_V)\big\}, \tag{1.10}$$

and we denote by $\|\cdot\|_{L^q(\mu;\|\cdot\|_V)} : L^0(\mu; \|\cdot\|_V) \to [0, \infty]$ the function which satisfies for all $f \in \mathscr{L}^0(\mu; \|\cdot\|_V)$ that

$$\big\|\{g \in \mathscr{L}^0(\mu; \|\cdot\|_V) \colon \mu(f \neq g) = 0\}\big\|_{L^q(\mu;\|\cdot\|_V)} = \|f\|_{\mathscr{L}^q(\mu;\|\cdot\|_V)} \in [0, \infty]. \tag{1.11}$$

Note that for every $p \in [1, \infty)$, every measure space $(\Omega, \mathscr{F}, \mu)$, and every normed vector space $(V, \|\cdot\|_V)$ it holds that $\big(\mathscr{L}^0(\mu; \|\cdot\|_V), \|\cdot\|_{\mathscr{L}^p(\mu;\|\cdot\|_V)}\big)$ and $\big(L^0(\mu; \|\cdot\|_V), \|\cdot\|_{L^p(\mu;\|\cdot\|_V)}\big)$ are extendedly semi-normed vector spaces. For a real number $T \in [0, \infty)$, a measurable space $(S, \mathscr{S})$, a normed vector space $(V, \|\cdot\|_V)$, and a mapping $X \colon [0, T] \times S \to V$ which satisfies for all $t \in [0, T]$ that $X_t \colon S \to V$ is an $(\mathscr{S}, \|\cdot\|_V)$-strongly measurable mapping we call X an $(\mathscr{S}, \|\cdot\|_V)$-strongly measurable stochastic process. For a metric space (M, d), an extendedly semi-normed vector space $(E, \|\cdot\|)$, a real number $r \in [0, 1]$, and a set $A \subseteq (0, \infty)$ we denote by $|\cdot|_{\mathscr{C}^{r,A}(M,\|\cdot\|)}, |\cdot|_{\mathscr{C}^r(M,\|\cdot\|)}, \|\cdot\|_{C(M,\|\cdot\|)}, \|\cdot\|_{\mathscr{C}^r(M,\|\cdot\|)} : \mathbb{M}(M, E) \to [0, \infty]$ the mappings which satisfy for all $f \in \mathbb{M}(M, E)$ that

$$|f|_{\mathscr{C}^{r,A}(M,\|\cdot\|)} = \sup\Big(\Big\{\tfrac{\|f(e_1) - f(e_2)\|}{|d(e_1,e_2)|^r} \colon e_1, e_2 \in M, d(e_1, e_2) \in A\Big\} \cup \{0\}\Big) \in [0, \infty], \tag{1.12}$$

$$|f|_{\mathscr{C}^r(M,\|\cdot\|)} = |f|_{\mathscr{C}^{r,(0,\infty)}(M,\|\cdot\|)} \in [0, \infty], \tag{1.13}$$

$$\|f\|_{C(M,\|\cdot\|)} = \sup(\{\|f(e)\| : e \in M\} \cup \{0\}) \in [0, \infty], \tag{1.14}$$

$$\|f\|_{\mathscr{C}^r(M,\|\cdot\|)} = \|f\|_{C(M,\|\cdot\|)} + |f|_{\mathscr{C}^r(M,\|\cdot\|)} \in [0, \infty] \tag{1.15}$$

and we denote by $\mathscr{C}^r(M, \|\cdot\|)$ the set given by

$$\mathscr{C}^r(M, \|\cdot\|) = \Big\{ f \in C(M, E) \colon \|f\|_{\mathscr{C}^r(M, \|\cdot\|)} < \infty \Big\}. \tag{1.16}$$

For Hilbert spaces $(H_i, \langle \cdot, \cdot \rangle_{H_i}, \|\cdot\|_{H_i})$, $i \in \{1, 2\}$, we denote by $(\mathrm{HS}(H_1, H_2), \langle \cdot, \cdot \rangle_{\mathrm{HS}(H_1,H_2)}, \|\cdot\|_{\mathrm{HS}(H_1,H_2)})$ the Hilbert space of Hilbert-Schmidt operators from H_1 to H_2. For a real number $T \in (0, \infty)$ we denote by $\mathscr{P}_T$ the set given by

$$\mathscr{P}_T = \big\{ \theta \subseteq [0, T] \colon \{0, T\} \subseteq \theta \text{ and } \#(\theta) < \infty \big\}. \tag{1.17}$$

We denote by $d_{\max}, d_{\min} \colon \cup_{T \in (0,\infty)} \mathscr{P}_T \to \mathbb{R}$ the functions which satisfy for all $\theta = \{\theta_0, \theta_1, \ldots, \theta_{\#(\theta)-1}\} \in \cup_{T \in (0,\infty)} \mathscr{P}_T$ with $\theta_0 < \theta_1 < \ldots < \theta_{\#(\theta)-1}$ that

$$d_{\max}(\theta) = \max_{j \in \{1,2,\ldots,\#(\theta)-1\}} |\theta_j - \theta_{j-1}| \qquad \text{and} \qquad d_{\min}(\theta) = \min_{j \in \{1,2,\ldots,\#(\theta)-1\}} |\theta_j - \theta_{j-1}|. \tag{1.18}$$

For a normed vector space $(E, \|\cdot\|_E)$, an element $\theta = \{\theta_0, \theta_1, \ldots, \theta_{\#(\theta)-1}\} \in \cup_{T \in (0,\infty)} \mathscr{P}_T$ with $\theta_0 < \theta_1 < \ldots < \theta_{\#(\theta)-1}$, and a function $f \colon [0, \theta_{\#(\theta)-1}] \to E$ we denote by $[f]_\theta \colon [0, \theta_{\#(\theta)-1}] \to E$ the piecewise affine linear interpolation of $f|_{\{\theta_0, \theta_1, \ldots, \theta_{\#(\theta)-1}\}}$, that is, we denote by $[f]_\theta \colon [0, \theta_{\#(\theta)-1}] \to E$ the function which satisfies for all $j \in \{1, 2, \ldots, \theta_{\#(\theta)-1}\}$, $s \in [\theta_{j-1}, \theta_j]$ that

$$[f]_\theta(s) = \frac{(\theta_j - s) f(\theta_{j-1})}{(\theta_j - \theta_{j-1})} + \frac{(s - \theta_{j-1}) f(\theta_j)}{(\theta_j - \theta_{j-1})}. \tag{1.19}$$

2 Convergence in Hölder norms for Banach space valued stochastic processes

2.1 Error bounds for the Hölder norm

Lemma 2.1 (An interpolation-type inequality)**.** *Consider the notation in Subsection 1.1, let $(E, \|\cdot\|_E)$ be a normed vector space, let (M, d) be a metric space, let $f \colon M \to E$ be a function, and let $c \in (0, \infty)$, $\alpha, \beta, \gamma \in [0, 1]$ satisfy $\alpha \leq \beta \leq \gamma$. Then*

$$|f|_{\mathscr{C}^\beta(M, \|\cdot\|_E)} \leq \max\Big\{ c^{\alpha-\beta} \, |f|_{\mathscr{C}^{\alpha,(c,\infty)}(M, \|\cdot\|_E)} \, , c^{\gamma-\beta} \, |f|_{\mathscr{C}^{\gamma,(0,c]}(M, \|\cdot\|_E)} \Big\} \tag{2.1}$$

and

$$|f|_{\mathscr{C}^\beta(M, \|\cdot\|_E)} \leq \max\Big\{ c^{\alpha-\beta} \, |f|_{\mathscr{C}^{\alpha,[c,\infty)}(M, \|\cdot\|_E)} \, , c^{\gamma-\beta} \, |f|_{\mathscr{C}^{\gamma,(0,c)}(M, \|\cdot\|_E)} \Big\} \, . \tag{2.2}$$

Proof of Lemma 2.1. First of all, note that it holds for all $e_1, e_2 \in M$ with $d(e_1, e_2) \in (c, \infty)$ that

$$\frac{\|f(e_1) - f(e_2)\|_E}{|d(e_1, e_2)|^\beta} \leq |d(e_1, e_2)|^{\alpha-\beta} \, |f|_{\mathscr{C}^{\alpha,(c,\infty)}(M, \|\cdot\|_E)} \leq c^{\alpha-\beta} \, |f|_{\mathscr{C}^{\alpha,(c,\infty)}(M, \|\cdot\|_E)} \, . \tag{2.3}$$

In addition, observe that it holds for all $e_1, e_2 \in M$ with $d(e_1, e_2) \in (0, c]$ that

$$\frac{\|f(e_1) - f(e_2)\|_E}{|d(e_1, e_2)|^\beta} \leq |d(e_1, e_2)|^{\gamma-\beta} \, |f|_{\mathscr{C}^{\gamma,(0,c]}(M, \|\cdot\|_E)} \leq c^{\gamma-\beta} \, |f|_{\mathscr{C}^{\gamma,(0,c]}(M, \|\cdot\|_E)} \, . \tag{2.4}$$

Combining (2.3) and (2.4) shows (2.1). The proof of (2.2) is analogous. This finishes the proof of Lemma 2.1. ☐

Lemma 2.2 (Approximation error for affine linear interpolation)**.** *Consider the notation in Subsection 1.1, let $T \in (0,\infty)$, $\theta \in \mathscr{P}_T$, $\alpha \in [0,1]$, let $(E, \|\cdot\|_E)$ be a normed vector space, and let $f\colon [0,T] \to E$ be a function. Then*

$$\|f - [f]_\theta\|_{C([0,T],\|\cdot\|_E)} \leq \left|\tfrac{d_{\max}(\theta)}{2}\right|^\alpha |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} . \tag{2.5}$$

Proof of Lemma 2.2. Throughout this proof let $N \in \mathbb{N}$, $\theta_0, \theta_1, \ldots, \theta_N \in [0,T]$ be the real numbers which satisfy $0 = \theta_0 < \theta_1 < \ldots < \theta_N = T$ and $\theta = \{\theta_0, \theta_1, \ldots, \theta_N\}$, let $s \in [0,T] \setminus \theta$, let $j \in \{1, 2, \ldots, N\}$ be the natural number such that $s \in (\theta_{j-1}, \theta_j)$, and let $g\colon [0,1] \to \mathbb{R}$ be the function which satisfies for all $u \in [0,1]$ that $g(u) = (1-u)\,u^\alpha + u\,(1-u)^\alpha$. Observe that the concavity of the function $[0,\infty) \ni x \mapsto x^\alpha \in \mathbb{R}$ shows for all $u \in [0,1]$ that

$$\begin{aligned} 2^\alpha g(u) &= (1-u)\,(2u)^\alpha + u\,(2(1-u))^\alpha \leq ((1-u)\,2u + u\,2(1-u))^\alpha \\ &= (4u(1-u))^\alpha = (1-(2u-1)^2)^\alpha \leq 1. \end{aligned} \tag{2.6}$$

Note that this proves that

$$\begin{aligned} &\|f(s) - [f]_\theta(s)\|_E \leq \tfrac{(\theta_j - s)}{(\theta_j - \theta_{j-1})} \|f(s) - f(\theta_{j-1})\|_E + \tfrac{(s-\theta_{j-1})}{(\theta_j - \theta_{j-1})} \|f(s) - f(\theta_j)\|_E \\ &\leq \tfrac{(\theta_j - s)}{(\theta_j - \theta_{j-1})} \big(s - \theta_{j-1}\big)^\alpha |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} + \tfrac{(s-\theta_{j-1})}{(\theta_j - \theta_{j-1})} \big(\theta_j - s\big)^\alpha |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \\ &= \Big(\tfrac{(\theta_j - s)}{(\theta_j - \theta_{j-1})} \big(\tfrac{(s-\theta_{j-1})}{(\theta_j - \theta_{j-1})}\big)^\alpha + \tfrac{(s-\theta_{j-1})}{(\theta_j - \theta_{j-1})} \big(\tfrac{(\theta_j - s)}{(\theta_j - \theta_{j-1})}\big)^\alpha \Big) \big(\theta_j - \theta_{j-1}\big)^\alpha |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \\ &= g\big(\tfrac{s - \theta_{j-1}}{\theta_j - \theta_{j-1}}\big) \,\big(\theta_j - \theta_{j-1}\big)^\alpha |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \leq \big(\tfrac{\theta_j - \theta_{j-1}}{2}\big)^\alpha |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} . \end{aligned} \tag{2.7}$$

The proof of Lemma 2.2 is thus completed. ☐

The next result, Corollary 2.3, provides estimates for the Hölder norm differences of two functions by using the difference of the two functions on suitable grid points. Corollary 2.3 is a consequence of Lemma 2.1 and Lemma 2.2.

Corollary 2.3. *Consider the notation in Subsection 1.1, let $T \in (0,\infty)$, $\theta \in \mathscr{P}_T$, $\beta \in [0,1]$, $\alpha \in [0,\beta]$, let $(E, \|\cdot\|_E)$ be a normed vector space, and let $f, g\colon [0,T] \to E$ be functions. Then*

$$\begin{aligned} &|f - g|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \\ &\leq \tfrac{2}{|d_{\max}(\theta)|^\alpha} \left[\sup_{t \in \theta} \|f(t) - g(t)\|_E + \tfrac{|d_{\max}(\theta)|^\beta}{2^\beta} \big(|f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + |g|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} \big) \right] \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &\|f - g\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \\ &\leq \left[\tfrac{2}{|d_{\max}(\theta)|^\alpha} + 1 \right] \left[\sup_{t \in \theta} \|f(t) - g(t)\|_E + \tfrac{|d_{\max}(\theta)|^\beta}{2^\beta} \big(|f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + |g|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} \big) \right]. \end{aligned} \tag{2.9}$$

Proof of Corollary 2.3. Lemma 2.1 and the triangle inequality ensure that

$$\begin{aligned} &|f - g|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \\ &\leq \max\Big\{ |d_{\max}(\theta)|^{-\alpha} |f - g|_{\mathscr{C}^{0,(d_{\max}(\theta),\infty)}([0,T],\|\cdot\|_E)}, |d_{\max}(\theta)|^{\beta-\alpha} |f - g|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} \Big\} \\ &\leq \max\big\{ 2\,|d_{\max}(\theta)|^{-\alpha} \|f - g\|_{C([0,T],\|\cdot\|_E)}, |d_{\max}(\theta)|^{\beta-\alpha} \big(|f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + |g|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} \big) \big\} . \end{aligned} \tag{2.10}$$

In addition, observe that Lemma 2.2 and the triangle inequality assure that

$$\begin{aligned}&\|f-g\|_{C([0,T],\|\cdot\|_E)} \leq \big\|f-[f]_\theta\big\|_{C([0,T],\|\cdot\|_E)} + \big\|[f]_\theta-[g]_\theta\big\|_{C([0,T],\|\cdot\|_E)} + \big\|[g]_\theta-g\big\|_{C([0,T],\|\cdot\|_E)}\\ &\leq \sup_{t\in\theta}\|f(t)-g(t)\|_E + \left|\tfrac{d_{\max}(\theta)}{2}\right|^\beta \left(|f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + |g|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)}\right). \end{aligned} \tag{2.11}$$

Inserting (2.11) into (2.10) yields inequality (2.8). Moreover, adding inequality (2.8) and (2.11) results in inequality (2.9). This finishes the proof of Corollary 2.3. □

Lemma 2.4. *Consider the notation in Subsection 1.1, let $(E, \|\cdot\|_E)$ be a normed vector space, let $T, c \in (0,\infty)$, $\alpha \in [0,1]$, $\theta \in \mathscr{P}_T$, $N \in \mathbb{N}$, $\theta_0, \dots, \theta_N \in [0,T]$ satisfy $0 = \theta_0 < \dots < \theta_N = T$ and $\theta = \{\theta_0, \dots, \theta_N\}$, and let $f\colon [0,T] \to E$ be a function. Then*

$$|[f]_\theta|_{\mathscr{C}^{\alpha,(0,c]}([0,T],\|\cdot\|_E)} \leq \tfrac{c^{1-\alpha}}{d_{\min}(\theta)}\big[\sup\nolimits_{j\in\{1,2,\dots,N\}}\|f(\theta_j)-f(\theta_{j-1})\|_E\big]. \tag{2.12}$$

Proof of Lemma 2.4. Observe that it holds for all $s,t \in [0,T]$ with $t-s \in (0,c]$ that

$$\begin{aligned}\frac{\|[f]_\theta(t)-[f]_\theta(s)\|_E}{|t-s|^\alpha} &= \frac{\big\|\int_{(s,t)\setminus\theta}([f]_\theta)'(u)\,\mathrm{d}u\big\|_E}{|t-s|^\alpha}\\ &\leq \frac{|t-s|\,\big[\sup_{u\in(s,t)\setminus\theta}\|([f]_\theta)'(u)\|_E\big]}{|t-s|^\alpha}\\ &\leq |t-s|^{1-\alpha}\left[\sup_{j\in\{1,2,\dots,N\}}\frac{\|f(\theta_j)-f(\theta_{j-1})\|_E}{|\theta_j-\theta_{j-1}|}\right]\\ &\leq \frac{c^{1-\alpha}}{d_{\min}(\theta)}\left[\sup_{j\in\{1,2,\dots,N\}}\|f(\theta_j)-f(\theta_{j-1})\|_E\right].\end{aligned} \tag{2.13}$$

This completes the proof of Lemma 2.4. □

Lemma 2.5. *Consider the notation in Subsection 1.1, let $(E, \|\cdot\|_E)$ be a normed vector space, let $T \in (0,\infty)$, $\alpha \in [0,1]$, $\theta \in \mathscr{P}_T$, and let $f\colon [0,T] \to E$ be a function. Then $|[f]_\theta|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \leq |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}$.*

Proof of Lemma 2.5. Throughout this proof let $N \in \mathbb{N}$, $\theta_0, \theta_1, \dots, \theta_N \in [0,T]$ be the real numbers which satisfy $0 = \theta_0 < \theta_1 < \dots < \theta_N = T$ and $\theta = \{\theta_0, \theta_1, \dots, \theta_N\}$ and let $n\colon [0,T] \to \mathbb{N}$ and $\rho\colon [0,T] \to [0,1]$ be the functions which satisfy for all $t \in [0,T]$ that

$$n(t) = \min\big\{k \in \{1,2,\dots,N\}\colon t \in [\theta_{k-1},\theta_k]\big\} \qquad \text{and} \qquad \rho(t) = \frac{t-\theta_{n(t)-1}}{\theta_{n(t)}-\theta_{n(t)-1}}. \tag{2.14}$$

Note that it holds for all $t \in [0,T]$ that

$$[f]_\theta(t) = (1-\rho(t))\cdot f(\theta_{n(t)-1}) + \rho(t)\cdot f(\theta_{n(t)}) = f(\theta_{n(t)-1}) + \rho(t)\cdot\big(f(\theta_{n(t)})-f(\theta_{n(t)-1})\big). \tag{2.15}$$

Hence, we obtain for all $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $n(t_1) = n(t_2)$ that

$$\begin{aligned} &\|[f]_\theta(t_1) - [f]_\theta(t_2)\|_E = \big\| \big[(1-\rho(t_1)) \cdot f(\theta_{n(t_1)-1}) + \rho(t_1) \cdot f(\theta_{n(t_1)})\big] \\ &\quad - \big[(1-\rho(t_2)) \cdot f(\theta_{n(t_1)-1}) + \rho(t_2) \cdot f(\theta_{n(t_1)})\big] \big\|_E \\ &= \big\|(\rho(t_2) - \rho(t_1)) \cdot f(\theta_{n(t_1)-1}) + (\rho(t_1) - \rho(t_2)) \cdot f(\theta_{n(t_1)})\big\|_E \\ &= |\rho(t_1) - \rho(t_2)| \cdot \big\| f(\theta_{n(t_1)-1}) - f(\theta_{n(t_1)})\big\|_E \\ &\leq |\rho(t_1) - \rho(t_2)| \, |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \, \big|\theta_{n(t_1)-1} - \theta_{n(t_1)}\big|^\alpha \\ &= |\rho(t_1) - \rho(t_2)|^{1-\alpha} \, |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \, \big|(\rho(t_1) - \rho(t_2)) \cdot \big(\theta_{n(t_1)} - \theta_{n(t_1)-1}\big)\big|^\alpha \\ &\leq |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \, \big|(\rho(t_1) - \rho(t_2)) \cdot \big(\theta_{n(t_1)} - \theta_{n(t_1)-1}\big)\big|^\alpha \\ &= |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \, \big|t_1 - \theta_{n(t_1)-1} - \big(t_2 - \theta_{n(t_1)-1}\big)\big|^\alpha \\ &= |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \, |t_1 - t_2|^\alpha . \end{aligned} \tag{2.16}$$

Moreover, (2.15) ensures for all $t_1, t_2 \in [0,T]$ with $n(t_1) < n(t_2)$ that

$$\begin{aligned} &\|[f]_\theta(t_1) - [f]_\theta(t_2)\|_E = \big\| \big[(1-\rho(t_1)) \cdot f(\theta_{n(t_1)-1}) + \rho(t_1) \cdot f(\theta_{n(t_1)})\big] \\ &\quad - \big[(1-\rho(t_2)) \cdot f(\theta_{n(t_2)-1}) + \rho(t_2) \cdot f(\theta_{n(t_2)})\big] \big\|_E \\ &\leq (1-\rho(t_1))\,(1-\rho(t_2)) \, \big\| f(\theta_{n(t_1)-1}) - f(\theta_{n(t_2)-1})\big\|_E + \rho(t_1)\,\rho(t_2)\,\big\| f(\theta_{n(t_1)}) - f(\theta_{n(t_2)})\big\|_E \\ &+ (1-\rho(t_1))\,\rho(t_2)\,\big\| f(\theta_{n(t_1)-1}) - f(\theta_{n(t_2)})\big\|_E + \rho(t_1)\,(1-\rho(t_2))\,\big\| f(\theta_{n(t_1)}) - f(\theta_{n(t_2)-1})\big\|_E \\ &\leq |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \big\{(1-\rho(t_1))\,(1-\rho(t_2))\,|\theta_{n(t_1)-1} - \theta_{n(t_2)-1}|^\alpha + \rho(t_1)\,\rho(t_2)\,|\theta_{n(t_1)} - \theta_{n(t_2)}|^\alpha \\ &\quad + (1-\rho(t_1))\,\rho(t_2)\,|\theta_{n(t_1)-1} - \theta_{n(t_2)}|^\alpha + \rho(t_1)\,(1-\rho(t_2))\,|\theta_{n(t_1)} - \theta_{n(t_2)-1}|^\alpha\big\}. \end{aligned} \tag{2.17}$$

The concavity of the function $(-\infty, 0] \ni x \mapsto |x|^\alpha \in \mathbb{R}$ hence proves for all $t_1, t_2 \in [0,T]$ with $n(t_1) < n(t_2)$ that

$$\begin{aligned} &\|[f]_\theta(t_1) - [f]_\theta(t_2)\|_E \\ &\leq |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \big|(1-\rho(t_1))\,(1-\rho(t_2))\,\big(\theta_{n(t_1)-1} - \theta_{n(t_2)-1}\big) + \rho(t_1)\,\rho(t_2)\,\big(\theta_{n(t_1)} - \theta_{n(t_2)}\big) \\ &\quad + (1-\rho(t_1))\,\rho(t_2)\,\big(\theta_{n(t_1)-1} - \theta_{n(t_2)}\big) + \rho(t_1)\,(1-\rho(t_2))\,\big(\theta_{n(t_1)} - \theta_{n(t_2)-1}\big)\big|^\alpha \\ &= |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \big|(1-\rho(t_1))\,\theta_{n(t_1)-1} + \rho(t_1)\,\theta_{n(t_1)} - (1-\rho(t_2))\,\theta_{n(t_2)-1} - \rho(t_2)\,\theta_{n(t_2)}\big|^\alpha \\ &= |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \\ &\quad \cdot \big|\big\{\theta_{n(t_1)-1} + \rho(t_1)\,\big[\theta_{n(t_1)} - \theta_{n(t_1)-1}\big]\big\} - \big\{\theta_{n(t_2)-1} + \rho(t_2)\,\big[\theta_{n(t_2)} - \theta_{n(t_2)-1}\big]\big\}\big|^\alpha \\ &= |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \, |t_1 - t_2|^\alpha . \end{aligned} \tag{2.18}$$

Combining this and (2.16) completes the proof of Lemma 2.5. □

Lemma 2.6 (Approximations by piecewise affine linear functions)**.** *Consider the notation in Subsection 1.1, let $(E, \|\cdot\|_E)$ be a normed vector space, let $T \in (0,\infty)$, $\alpha \in [0,1]$, $\beta \in [\alpha, 1]$, $\theta \in \mathscr{P}_T$, and let $f, g\colon [0,T] \to E$ be functions. Then*

$$|f - [g]_\theta|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \leq \tfrac{2\,|d_{\max}(\theta)|^{1-\alpha}}{d_{\min}(\theta)} \sup_{t\in\theta} \|f(t) - g(t)\|_E + 2\,|d_{\max}(\theta)|^{\beta-\alpha}\,|f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} \tag{2.19}$$

and

$$\begin{aligned} &\|f - [g]_\theta\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \\ &\leq \Big(\tfrac{2\,|d_{\max}(\theta)|^{1-\alpha}}{d_{\min}(\theta)} + 1\Big) \sup_{t\in\theta} \|f(t) - g(t)\|_E + \Big(\tfrac{2}{|d_{\max}(\theta)|^\alpha} + \tfrac{1}{2^\beta}\Big)\,|d_{\max}(\theta)|^\beta\,|f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} . \end{aligned} \tag{2.20}$$

Proof of Lemma 2.6. Throughout this proof let $N \in \mathbb{N}$, $\theta_0, \theta_1, \dots, \theta_N \in [0,T]$ be the real numbers which satisfy $0 = \theta_0 < \theta_1 < \ldots < \theta_N = T$ and $\theta = \{\theta_0, \theta_1, \dots, \theta_N\}$. Note that Lemma 2.1 implies that

$$|f - [g]_\theta|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \leq \max\Big\{ |d_{\max}(\theta)|^{-\alpha} \, |f - [g]_\theta|_{\mathscr{C}^{0,(d_{\max}(\theta),\infty)}([0,T],\|\cdot\|_E)} , \; |d_{\max}(\theta)|^{\beta-\alpha} \, |f - [g]_\theta|_{\mathscr{C}^{\beta,(0,d_{\max}(\theta)]}([0,T],\|\cdot\|_E)} \Big\}. \tag{2.21}$$

Next note that Lemma 2.2 ensures that

$$\begin{aligned} &|f - [g]_\theta|_{\mathscr{C}^{0,(d_{\max}(\theta),\infty)}([0,T],\|\cdot\|_E)} \leq 2 \, \|f - [g]_\theta\|_{C([0,T],\|\cdot\|_E)} \\ &\leq 2 \, \|f - [f]_\theta\|_{C([0,T],\|\cdot\|_E)} + 2 \, \|[f]_\theta - [g]_\theta\|_{C([0,T],\|\cdot\|_E)} \\ &\leq 2 \left|\tfrac{d_{\max}(\theta)}{2}\right|^\beta |f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + 2 \cdot \sup_{t\in\theta} \|f(t) - g(t)\|_E \\ &\leq 2 \left|d_{\max}(\theta)\right|^\beta |f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + 2 \cdot \frac{d_{\max}(\theta)}{d_{\min}(\theta)} \cdot \sup_{t\in\theta} \|f(t) - g(t)\|_E \,. \end{aligned} \tag{2.22}$$

Moreover, observe that Lemma 2.4 and Lemma 2.5 imply that

$$\begin{aligned} &|f - [g]_\theta|_{\mathscr{C}^{\beta,(0,d_{\max}(\theta)]}([0,T],\|\cdot\|_E)} \leq |f - [f]_\theta|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + |[f - g]_\theta|_{\mathscr{C}^{\beta,(0,d_{\max}(\theta)]}([0,T],\|\cdot\|_E)} \\ &\leq |f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + |[f]_\theta|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} \\ &\quad + \tfrac{|d_{\max}(\theta)|^{1-\beta}}{d_{\min}(\theta)} \left[\sup_{j\in\{1,2,\dots,N\}} \big\| \big[f(\theta_j) - g(\theta_j)\big] - \big[f(\theta_{j-1}) - g(\theta_{j-1})\big] \big\|_E \right] \\ &\leq 2 \, |f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + \frac{2}{|d_{\max}(\theta)|^\beta} \cdot \frac{d_{\max}(\theta)}{d_{\min}(\theta)} \cdot \sup_{t\in\theta} \|f(t) - g(t)\|_E \,. \end{aligned} \tag{2.23}$$

Substituting (2.23) and (2.22) into (2.21) proves (2.19). It thus remains to prove estimate (2.20). For this note that Lemma 2.2 yields that

$$\begin{aligned} \|f - [g]_\theta\|_{C([0,T],\|\cdot\|_E)} &\leq \|f - [f]_\theta\|_{C([0,T],\|\cdot\|_E)} + \|[f]_\theta - [g]_\theta\|_{C([0,T],\|\cdot\|_E)} \\ &\leq \left|\tfrac{d_{\max}(\theta)}{2}\right|^\beta |f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + \sup_{t\in\theta} \|f(t) - g(t)\|_E \,. \end{aligned} \tag{2.24}$$

Combining (2.19) and (2.24) shows (2.20). The proof of Lemma 2.6 is thus completed. □

2.2 Upper error bounds for stochastic processes with Hölder continuous sample paths

We now turn to the result announced in the introduction which provides convergence of stochastic processes in Hölder norms given convergence on the grid points. For this we first recall the Kolmogorov-Chentsov continuity theorem, cf., e.g., Revuz & Yor [35, Theorem I.2.1 and its proof].

Theorem 2.7 (Kolmogorov-Chentsov continuity theorem)**.** *Consider the notation in Subsection 1.1. There exists a function* $\Xi = (\Xi_{T,p,\alpha,\beta})_{T,p,\alpha,\beta\in\mathbb{R}} \colon \mathbb{R}^4 \to \mathbb{R}$ *such that for every* $T \in [0,\infty)$, $p \in (1,\infty)$, $\beta \in (1/p, 1]$, *every Banach space* $(E, \|\cdot\|_E)$, *every probability space* $(\Omega, \mathscr{F}, \mathbb{P})$, *and every* $X \in \mathscr{C}^\beta([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})$ *there exists an* $(\mathscr{F}, \|\cdot\|_E)$*-strongly measurable stochastic process* $Y \colon [0,T] \times \Omega \to E$ *with continuous sample paths such that it holds for every* $\alpha \in [0, \beta - 1/p)$ *that*

$$\begin{aligned} &\Big(\mathbb{E}\Big[\|Y\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p} \leq \Xi_{T,p,\alpha,\beta} \, \|X\|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty \quad \textit{and} \\ &\forall\, t \in [0,T] \colon \mathbb{P}(X_t = Y_t) = 1. \end{aligned} \tag{2.25}$$

The next result, Corollary 2.8, follows directly from Corollary 2.3 (with $T = T$, $\theta = \theta$, $\beta = \gamma$, $\alpha = \beta$, $E = L^p(\mathbb{P}; \|\cdot\|_E)$, $f = ([0,T] \ni t \mapsto \{Z \in \mathscr{L}^0(\mathbb{P}; \|\cdot\|_E) \colon \mathbb{P}(Z \neq X_t - Y_t) = 0\} \in L^p(\mathbb{P}; \|\cdot\|_E))$, $g = 0$ for $p \in [1,\infty)$, $\beta \in [0,1]$, $\gamma \in [\beta,1]$ and $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y \colon [0,T] \times \Omega \to E$ with $\forall\, t \in [0,T] \colon \|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} < \infty$ in the notation of Corollary 2.3) and the Kolmogorov-Chentsov continuity theorem (see Theorem 2.7 above).

Corollary 2.8 (Grid point approximations)**.** *Consider the notation in Subsection 1.1, let $T \in (0,\infty)$, $\theta \in \mathscr{P}_T$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let $(E, \|\cdot\|_E)$ be a Banach space. Then*

(i) it holds for all $p \in [1,\infty)$, $\beta \in [0,1]$, $\gamma \in [\beta,1]$ and all $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y \colon [0,T] \times \Omega \to E$ that

$$\begin{aligned} &\|X - Y\|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \leq \Big(2\,|d_{\max}(\theta)|^{-\beta} + 1\Big) \Big[\sup\nolimits_{t\in\theta} \|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\ &\quad + |d_{\max}(\theta)|^\gamma \left(|X|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \right)\Big] \end{aligned} \tag{2.26}$$

(ii) and it holds for all $p \in (1,\infty)$, $\beta \in (1/p, 1]$, $\alpha \in [0, \beta - 1/p)$, $\gamma \in [\beta,1]$ and all $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y \colon [0,T] \times \Omega \to E$ with continuous sample paths that

$$\begin{aligned} &\Big(\mathbb{E}\Big[\|X - Y\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p} \leq \Xi_{T,p,\alpha,\beta}\, \|X - Y\|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \\ &\leq \Xi_{T,p,\alpha,\beta} \Big(2\,|d_{\max}(\theta)|^{-\beta} + 1\Big) \Big[\sup\nolimits_{t\in\theta} \|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\ &\quad + |d_{\max}(\theta)|^\gamma \left(|X|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \right)\Big]. \end{aligned} \tag{2.27}$$

The next result, Corollary 2.9, follows directly from Lemma 2.6 (with $E = L^p(\mathbb{P}; \|\cdot\|_E)$, $T = T$, $\alpha = \beta$, $\beta = \gamma$, $\theta = \theta$, $f = ([0,T] \ni t \mapsto \{Z \in \mathscr{L}^0(\mathbb{P}; \|\cdot\|_E) \colon \mathbb{P}(Z \neq X_t - X_0) = 0\} \in L^p(\mathbb{P}; \|\cdot\|_E))$, $g = ([0,T] \ni t \mapsto \{Z \in \mathscr{L}^0(\mathbb{P}; \|\cdot\|_E) \colon \mathbb{P}(Z \neq [Y]_\theta(t) - X_0) = 0\} \in L^p(\mathbb{P}; \|\cdot\|_E))$ for $p \in [1,\infty)$, $\beta \in [0,1]$, $\gamma \in [\beta,1]$ and $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y \colon [0,T] \times \Omega \to E$ with $\sup_{t\in\theta} \|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} + |X|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty$ in the notation of Lemma 2.6) and the Kolmogorov-Chentsov continuity theorem (see Theorem 2.7 above).

Corollary 2.9 (Piecewise affine linear stochastic processes)**.** *Consider the notation in Subsection 1.1, let $T \in (0,\infty)$, $\theta \in \mathscr{P}_T$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let $(E, \|\cdot\|_E)$ be a Banach space. Then*

(i) it holds for all $p \in [1,\infty)$, $\beta \in [0,1]$, $\gamma \in [\beta,1]$ and all $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y \colon [0,T] \times \Omega \to E$ that

$$\begin{aligned} &\|X - [Y]_\theta\|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \leq \Big[\tfrac{2\,|d_{\max}(\theta)|^{1-\beta}}{d_{\min}(\theta)} + 1\Big] \sup_{t\in\theta} \|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\ &\quad + \big[2\,|d_{\max}(\theta)|^{-\beta} + 2^{-\gamma}\big]\, |d_{\max}(\theta)|^\gamma\, |X|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \end{aligned} \tag{2.28}$$

(ii) and it holds for all $p \in (1,\infty)$, $\beta \in (1/p, 1]$, $\alpha \in [0, \beta - 1/p)$, $\gamma \in [\beta,1]$ and all $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y \colon [0,T] \times \Omega \to E$ with continuous sample paths that

$$\begin{aligned} &\Big(\mathbb{E}\Big[\|X - [Y]_\theta\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p} \leq \Xi_{T,p,\alpha,\beta} \Big(\Big[\tfrac{2\,|d_{\max}(\theta)|^{1-\beta}}{d_{\min}(\theta)} + 1\Big] \sup_{t\in\theta} \|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\ &\quad + \big[2\,|d_{\max}(\theta)|^{-\beta} + 2^{-\gamma}\big]\, |d_{\max}(\theta)|^\gamma\, |X|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \Big). \end{aligned} \tag{2.29}$$

In (2.29) in Corollary 2.9 we assume beside other assumptions that α is strictly smaller than γ. In general, this assumption cannot be omitted. To give an example, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $W\colon [0,1]\times\Omega \to \mathbb{R}$ be a one-dimensional standard Brownian motion with continuous sample paths. Then it clearly holds for all $p \in [1,\infty)$ that $\|W\|_{\mathscr{C}^{1/2}([0,1],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})} < \infty$. However, the fact that the sample paths of the Brownian motion are $\mathbb{P}$-a.s. not $1/2$-Hölder continuous (cf., e.g., Revuz & Yor [35, Theorem I.2.7] and, e.g., Arcones [3, Corollary 3.1]) ensures that it holds for all $\theta \in \mathscr{P}_1$, $p \in (0,\infty)$ that $\mathbb{E}\big[\|W - [W]_\theta\|^p_{\mathscr{C}^{1/2}([0,1],|\cdot|)}\big] = \infty$. The following corollary is related to Lemma A1 in Bally, Millet, & Sanz-Solé [4].

Corollary 2.10 ($\mathscr{L}^p$-convergence in Hölder norms for a fixed $p \in [1,\infty)$)**.** *Consider the notation in Subsection 1.1, let $T \in (0,\infty)$, $p \in [1,\infty)$, $\beta \in [0,1]$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(E, \|\cdot\|_E)$ be a Banach space, and let $Y^N\colon [0,T]\times\Omega \to E$, $N \in \mathbb{N}_0$, be $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes with continuous sample paths which satisfy $\limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty$ and $\forall\, t \in [0,T]\colon \limsup_{N\to\infty} \|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} = 0$. Then*

(i) it holds that $|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \le \limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty$,

(ii) it holds for all $\alpha \in [0,1]\cap(-\infty,\beta)$ that $\limsup_{N\to\infty} \|Y^0 - Y^N\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} = 0$,

(iii) and it holds for all $\alpha \in [0,1]\cap(-\infty,\beta - 1/p)$ that

$$\limsup_{N\to\infty} \mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big] = 0. \tag{2.30}$$

Proof of Corollary 2.10. Throughout this proof let $\theta^n \in \mathscr{P}_T$, $n \in \mathbb{N}$, be the sequence which satisfies for all $n \in \mathbb{N}$ that $\theta^n = \{0, \frac{T}{n}, \frac{2T}{n}, \ldots, \frac{(n-1)T}{n}, T\} \in \mathscr{P}_T$. Observe that the assumption that $\forall\, t \in [0,T]\colon \limsup_{N\to\infty} \|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} = 0$ and the assumption that $\limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty$ ensure that

$$\begin{aligned}
&|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} = \sup_{\substack{s,t\in[0,T],\\ s\neq t}} \left[\frac{\|Y^0_t - Y^0_s\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}}{|t-s|^\beta}\right]\\
&= \sup_{\substack{s,t\in[0,T],\\ s\neq t}} \left[\frac{\limsup_{N\to\infty} \|(Y^N_t - Y^N_s) + (Y^0_t - Y^N_t) + (Y^N_s - Y^0_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}}{|t-s|^\beta}\right]\\
&\le \sup_{\substack{s,t\in[0,T],\\ s\neq t}} \limsup_{N\to\infty} \left[\frac{\|Y^N_t - Y^N_s\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}}{|t-s|^\beta}\right] \le \limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty.
\end{aligned} \tag{2.31}$$

This establishes Item (i). In the next step we prove Item (ii). We apply Item (i) in Corollary 2.8 to obtain for all $\alpha \in [0,\beta]$, $n, N \in \mathbb{N}$ that

$$\begin{aligned}
&\|Y^0 - Y^N\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \le \big(2\,|d_{\max}(\theta^n)|^{-\alpha} + 1\big)\Big[\sup\nolimits_{t\in\theta^n} \|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\\
&\quad + |d_{\max}(\theta^n)|^\beta \big(|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\big)\Big]\\
&\le \left(\tfrac{2\,T^{-\alpha}}{n^{-\alpha}} + 1\right) \sup\nolimits_{t\in\theta^n} \|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\\
&\quad + \left(\tfrac{2\,T^{\beta-\alpha}}{n^{\beta-\alpha}} + \tfrac{T^\beta}{n^\beta}\right)\big(|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\big).
\end{aligned} \tag{2.32}$$

Item (i) and the assumption that $\forall\, t \in [0,T]\colon \limsup_{N\to\infty} \|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} = 0$ hence imply for all $\alpha \in [0,\beta]$, $n \in \mathbb{N}$ that

$$\begin{aligned}
&\limsup_{N\to\infty} \|Y^0 - Y^N\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \leq \left[\tfrac{2\,T^{-\alpha}}{n^{-\alpha}} + 1\right] \left[\limsup_{N\to\infty} \sup_{t\in\theta^n} \|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\right] \\
&\quad + \left[\tfrac{4\,T^{\beta-\alpha}}{n^{\beta-\alpha}} + \tfrac{2\,T^\beta}{n^\beta}\right] \limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \\
&= \left[\tfrac{4\,T^{\beta-\alpha}}{n^{\beta-\alpha}} + \tfrac{2\,T^\beta}{n^\beta}\right] \limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty.
\end{aligned} \tag{2.33}$$

Hence, we obtain for all $\alpha \in [0,1] \cap (-\infty,\beta)$ that

$$\begin{aligned}
&\limsup_{N\to\infty} \|Y^0 - Y^N\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} = \limsup_{n\to\infty} \limsup_{N\to\infty} \|Y^0 - Y^N\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \\
&\leq \left[\limsup_{n\to\infty} \tfrac{4\,T^{\beta-\alpha}}{n^{\beta-\alpha}} + \limsup_{n\to\infty} \tfrac{2\,T^\beta}{n^\beta}\right] \limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} = 0.
\end{aligned} \tag{2.34}$$

This shows Item (ii). It thus remains to establish Item (iii) to complete the proof of Corollary 2.10. For this we apply the first inequality in Item (ii) in Corollary 2.8 to obtain for all $r \in (1/p,\infty) \cap (-\infty,\beta]$, $\alpha \in [0, r - 1/p)$, $N \in \mathbb{N}$ that

$$\left(\mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\right)^{1/p} \leq \Xi_{T,p,\alpha,r}\, \|Y^0 - Y^N\|_{\mathscr{C}^r([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}. \tag{2.35}$$

This and Item (ii) imply for all $r \in (1/p,\infty) \cap (-\infty,\beta)$, $\alpha \in [0, r - 1/p)$ that

$$\limsup_{N\to\infty} \mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big] \leq (\Xi_{T,p,\alpha,r})^p \limsup_{N\to\infty} \|Y^0 - Y^N\|^p_{\mathscr{C}^r([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} = 0. \tag{2.36}$$

This establishes Item (iii). The proof of Corollary 2.10 is thus completed. □

The next result, Corollary 2.11 below, is a consequence from Corollary 2.8 and Corollary 2.9.

Corollary 2.11 (Convergence rates with respect to Hölder norms)**.** *Consider the notation in Subsection 1.1, let $T \in (0,\infty)$, $p \in (1,\infty)$, $\beta \in (1/p, 1]$, $(\theta^N)_{N\in\mathbb{N}} \subseteq \mathscr{P}_T$ satisfy $\limsup_{N\to\infty} d_{\max}(\theta^N) = 0$, let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space, let $(E,\|\cdot\|_E)$ be a Banach space, let $Y^N\colon [0,T]\times\Omega \to E$, $N \in \mathbb{N}_0$, be $(\mathscr{F},\|\cdot\|_E)$-strongly measurable stochastic processes with continuous sample paths which satisfy $Y^0_0 \in \mathscr{L}^p(\mathbb{P};\|\cdot\|_E)$ and*

$$|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + \sup_{N\in\mathbb{N}} \Big[\big|d_{\max}(\theta^N)\big|^{-\beta} \sup\nolimits_{t\in\theta^N} \|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\Big] < \infty, \tag{2.37}$$

and assume $\big(\big[\sup_{N\in\mathbb{N}} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty\big]$ or $\big[\sup_{N\in\mathbb{N}} {}^{d_{\max}(\theta^N)}/_{d_{\min}(\theta^N)} < \infty$ and $\forall\, N \in \mathbb{N}\colon Y^N = [Y^N]_{\theta^N}\big]\big)$. Then it holds for all $\alpha \in [0, \beta - 1/p)$, $\varepsilon \in (0,\infty)$ that

$$\sup_{N\in\mathbb{N}} \left[\mathbb{E}\Big[\|Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big] + \big|d_{\max}(\theta^N)\big|^{-(\beta-\alpha-1/p-\varepsilon)} \left(\mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\right)^{1/p}\right] < \infty. \tag{2.38}$$

Proof of Corollary 2.11. Throughout this proof let $c_0 \in [0,\infty)$, $c_1, c_2 \in [0,\infty]$ be the extended real numbers given by

$$\begin{aligned}
&c_0 = |Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + \sup_{N\in\mathbb{N}} \Big[\big|d_{\max}(\theta^N)\big|^{-\beta} \sup\nolimits_{t\in\theta^N} \|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\Big], \\
&c_1 = \sup_{N\in\mathbb{N}} \left[\frac{d_{\max}(\theta^N)}{d_{\min}(\theta^N)}\right], \qquad \text{and} \qquad c_2 = \sup_{N\in\mathbb{N}} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}.
\end{aligned} \tag{2.39}$$

Next we observe that Item (ii) in Corollary 2.8 ensures for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$, $N \in \mathbb{N}$ that

$$\begin{aligned}\Big(\mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p} &\leq \Xi_{T,p,\alpha,r}\left(2\left|d_{\max}(\theta^N)\right|^{-r} + 1\right)\Bigg[\sup_{t\in\theta^N}\|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\ &\quad + \left|d_{\max}(\theta^N)\right|^{\beta}\left(|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\right)\Bigg] \\ &\leq \Xi_{T,p,\alpha,r}\left(2\left|d_{\max}(\theta^N)\right|^{(\beta-r)} + \left|d_{\max}(\theta^N)\right|^{\beta}\right)\Big[c_0 + |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\Big] \\ &\leq \Xi_{T,p,\alpha,r}\left(2\left|d_{\max}(\theta^N)\right|^{(\beta-r)} + \left|d_{\max}(\theta^N)\right|^{\beta}\right)[c_0 + c_2] \\ &= \Xi_{T,p,\alpha,r}\left(2 + \left|d_{\max}(\theta^N)\right|^{r}\right)\left|d_{\max}(\theta^N)\right|^{(\beta-r)}[c_0 + c_2].\end{aligned} \tag{2.40}$$

This implies for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$ that

$$\sup_{N\in\mathbb{N}}\left[\left|d_{\max}(\theta^N)\right|^{-(\beta-r)}\Big(\mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p}\right] \leq \Xi_{T,p,\alpha,r}\,(2 + T^r)\,[c_0 + c_2]. \tag{2.41}$$

Hence, we obtain for all $\alpha \in [0, \beta - 1/p)$, $r \in (\alpha + 1/p, \beta]$ that

$$\begin{aligned}&\sup_{N\in\mathbb{N}}\left[\left|d_{\max}(\theta^N)\right|^{-(\beta-\alpha-1/p-[r-\alpha-1/p])}\Big(\mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p}\right] \\ &\leq \Xi_{T,p,\alpha,\alpha+1/p+[r-\alpha-1/p]}\,(3+T)\,(c_0 + c_2).\end{aligned} \tag{2.42}$$

This shows for all $\alpha \in [0, \beta - 1/p)$, $\varepsilon \in (0, \beta - \alpha - 1/p]$ that

$$\begin{aligned}&\sup_{N\in\mathbb{N}}\left[\left|d_{\max}(\theta^N)\right|^{-(\beta-\alpha-1/p-\varepsilon)}\Big(\mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p}\right] \\ &\leq \Xi_{T,p,\alpha,\alpha+1/p+\varepsilon}\,(3+T)\,(c_0 + c_2).\end{aligned} \tag{2.43}$$

In the next step we note that Item (ii) in Corollary 2.9 proves for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$, $N \in \mathbb{N}$ that

$$\begin{aligned}\Big(\mathbb{E}\Big[\big\|Y^0 - [Y^N]_{\theta^N}\big\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p} &\leq \Xi_{T,p,\alpha,r}\Bigg(\Big[\tfrac{2\,|d_{\max}(\theta^N)|^{1-r}}{d_{\min}(\theta^N)} + 1\Big]\sup_{t\in\theta^N}\|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\ &\quad + \big[2\,|d_{\max}(\theta^N)|^{-r} + 2^{-\beta}\big]\left|d_{\max}(\theta^N)\right|^{\beta}|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\Bigg).\end{aligned} \tag{2.44}$$

This implies for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$, $N \in \mathbb{N}$ that

$$\begin{aligned}&\Big(\mathbb{E}\Big[\big\|Y^0 - [Y^N]_{\theta^N}\big\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p} \\ &\leq c_0\left|d_{\max}(\theta^N)\right|^{\beta}\,\Xi_{T,p,\alpha,r}\Big(\tfrac{2\,|d_{\max}(\theta^N)|^{1-r}}{d_{\min}(\theta^N)} + 1 + 2\,|d_{\max}(\theta^N)|^{-r} + 2^{-\beta}\Big) \\ &\leq 2\,c_0\left|d_{\max}(\theta^N)\right|^{\beta}\,\Xi_{T,p,\alpha,r}\left([c_1 + 1]\left|d_{\max}(\theta^N)\right|^{-r} + 1\right).\end{aligned} \tag{2.45}$$

Hence, we obtain for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$ that

$$\begin{aligned}&\sup_{N\in\mathbb{N}}\left[\left|d_{\max}(\theta^N)\right|^{-(\beta-r)}\Big(\mathbb{E}\Big[\big\|Y^0 - [Y^N]_{\theta^N}\big\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p}\right] \\ &\leq 2\,c_0\,\Xi_{T,p,\alpha,r}\,(c_1 + 1 + T^r) \leq 2\,c_0\,\Xi_{T,p,\alpha,r}\,(2 + T + c_1).\end{aligned} \tag{2.46}$$

This shows for all $\alpha \in [0, \beta - 1/p)$, $r \in (\alpha + 1/p, \beta]$ that

$$\begin{aligned} &\sup_{N\in\mathbb{N}} \left[\left|d_{\max}(\theta^N)\right|^{-(\beta-\alpha-1/p-[r-\alpha-1/p])} \left(\mathbb{E}\Big[\big\| Y^0 - [Y^N]_{\theta^N} \big\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \Big] \right)^{1/p} \right] \\ &\leq 2\, c_0\, \Xi_{T,p,\alpha,\alpha+1/p+[r-\alpha-1/p]}\, (2 + T + c_1)\,. \end{aligned} \tag{2.47}$$

This establishes for all $\alpha \in [0, \beta - 1/p)$, $\varepsilon \in (0, \beta - \alpha - 1/p]$ that

$$\begin{aligned} &\sup_{N\in\mathbb{N}} \left[\left|d_{\max}(\theta^N)\right|^{-(\beta-\alpha-1/p-\varepsilon)} \left(\mathbb{E}\Big[\big\| Y^0 - [Y^N]_{\theta^N} \big\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \Big] \right)^{1/p} \right] \\ &\leq 2\, c_0\, \Xi_{T,p,\alpha,\alpha+1/p+\varepsilon}\, (2 + T + c_1)\,. \end{aligned} \tag{2.48}$$

Combining (2.43) and (2.48) assures for all $\alpha \in [0, \beta - 1/p)$, $\varepsilon \in (0, \infty)$ that

$$\sup_{N\in\mathbb{N}} \left[\left|d_{\max}(\theta^N)\right|^{-(\beta-\alpha-1/p-\varepsilon)} \left(\mathbb{E}\Big[\| Y^0 - Y^N \|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \Big] \right)^{1/p} \right] < \infty. \tag{2.49}$$

In addition, note that the assumption that $Y^0_0 \in \mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)$, the assumption that $|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty$, the assumption that Y^0 has continuous sample paths, and Theorem 2.7 ensure for all $\alpha \in [0, \beta - 1/p)$ that $\mathbb{E}\big[\|Y^0\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\big] < \infty$. This and (2.49) complete the proof of Corollary 2.11. □

The next result, Corollary 2.12 below, illustrates Corollary 2.11 through a simple example. For this note that standard results for the Euler-Maruyama method show under suitable hypotheses for every $p \in [2, \infty)$, $\beta \in [0, 1/2]$ that condition (2.37) in Corollary 2.11 with uniform time steps is satisfied (cf., e.g., Section 10.6 in Kloeden & Platen [29]). The convergence rate established in Corollary 2.12 (see (2.52) below) is essentially sharp; see Proposition 2.14 below. Corollary 2.12 is related to Theorem 1.2 in [8] and Theorem 1.1 in [9].

Corollary 2.12 (Euler-Maruyama method)**.** *Consider the notation in Subsection 1.1, let $T \in (0, \infty)$, $d, m \in \mathbb{N}$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$, let $W\colon [0,T] \times \Omega \to \mathbb{R}^m$ be an m-dimensional standard $(\mathscr{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be globally Lipschitz continuous functions, let $X\colon [0,T] \times \Omega \to \mathbb{R}^d$ be an $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(\mathbb{R}^d)$-adapted stochastic process with continuous sample paths which satisfies $\forall\, p \in [1, \infty)\colon \mathbb{E}\big[\|X_0\|^p_{\mathbb{R}^d}\big] < \infty$ and which satisfies for all $t \in [0,T]$ that*

$$[X_t]_{\mathbb{P},\mathscr{B}(\mathbb{R}^d)} = \left[X_0 + \int_0^t \mu(X_s)\, \mathrm{d}s \right]_{\mathbb{P},\mathscr{B}(\mathbb{R}^d)} + \int_0^t \sigma(X_s)\, \mathrm{d}W_s, \tag{2.50}$$

and let $Y^N\colon [0,T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be mappings which satisfy for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N-1\}$, $t \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$ that $Y^N_0 = X_0$ and

$$Y^N_t = Y^N_{\frac{nT}{N}} + \big(t - \tfrac{nT}{N}\big) \cdot \mu\big(Y^N_{\frac{nT}{N}}\big) + \big(\tfrac{tN}{T} - n\big) \cdot \sigma\big(Y^N_{\frac{nT}{N}}\big)\big(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\big). \tag{2.51}$$

Then it holds for all $\alpha \in [0, 1/2)$, $\varepsilon \in (0, \infty)$, $p \in [1, \infty)$ that

$$\sup_{N\in\mathbb{N}} \left[N^{1/2-\alpha-\varepsilon} \left(\mathbb{E}\Big[\| X - Y^N \|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathbb{R}^d})} \Big] \right)^{1/p} \right] < \infty. \tag{2.52}$$

2.3 Lower error bounds for stochastic processes with Hölder continuous sample paths

In this subsection we comment on the optimality of the convergence rate provided by Corollary 2.11 and Corollary 2.12, respectively. In particular, in the setting of Corollary 2.12, Theorem 3 in Müller-Gronbach [32] shows in the case $\alpha = 0$ that there exists a class of SDEs for which the factors $N^{1/2-\varepsilon}$, $N \in \mathbb{N}$, on the left hand side of the estimate (2.52) can at best – up to a constant – be replaced by the factors $\frac{N^{1/2}}{\log(N)}$, $N \in \mathbb{N}$. In Proposition 2.14 below we show for every $\alpha \in [0, 1/2)$ in the simple case of $\mu = 0$ and $\sigma = (\mathbb{R} \ni x \mapsto 1 \in \mathbb{R})$ in Corollary 2.12 that the factors $N^{1/2-\alpha-\varepsilon}$, $N \in \mathbb{N}$, on the left hand side of the estimate (2.52) can at best – up to a constant – be replaced by the factors $N^{1/2-\alpha}$, $N \in \mathbb{N}$. Our proof of Proposition 2.14 uses the following elementary lemma.

Lemma 2.13. *Consider the notation in Subsection 1.1, let $T \in (0,\infty)$, $p \in [1,\infty)$, $\alpha \in [0,1]$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(E, \|\cdot\|_E)$ be a normed vector space, and let $X \colon [0,T] \times \Omega \to E$ be an $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic process with continuous sample paths. Then*

$$\max\Big\{|X|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}, 2^{(1/p-1)}\,\|X\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\Big\} \le \Big(\mathbb{E}\Big[\|X\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p}. \tag{2.53}$$

The proof of Lemma 2.13 is clear. Instead we now present the promised proposition on the optimality of the convergence rate estimate in Corollary 2.12.

Proposition 2.14. *Consider the notation in Subsection 1.1, let $T \in (0,\infty)$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $W \colon [0,T] \times \Omega \to \mathbb{R}$ be a one-dimensional standard Brownian motion with continuous sample paths, and let $W^N \colon [0,T] \times \Omega \to \mathbb{R}$, $N \in \mathbb{N}$, be mappings which satisfy for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$, $t \in \big[\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ that*

$$W_t^N = \big(n + 1 - \tfrac{tN}{T}\big) \cdot W_{\frac{nT}{N}} + \big(\tfrac{tN}{T} - n\big) \cdot W_{\frac{(n+1)T}{N}}. \tag{2.54}$$

Then it holds for all $\alpha \in [0, 1/2]$, $p \in [1,\infty)$, $N \in \{2,3,4,\ldots\}$ that

$$\|W - W^N\|_{C([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})} = \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{2\sqrt{N}}, \tag{2.55}$$

$$\frac{|W-W^N|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})}}{N^{(\alpha-1/2)}\,T^{-\alpha}\,\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}} = \frac{(1/2-\alpha)^{(1/2-\alpha)}}{2^\alpha\,(1-\alpha)^{(1-\alpha)}} \in \Big[\tfrac{1}{\sqrt{2}}, 1\Big], \tag{2.56}$$

$$\frac{\|W-W^N\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})}}{N^{(\alpha-1/2)}\,T^{-\alpha}\,\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}} = \frac{T^\alpha}{2N^\alpha} + \frac{(1/2-\alpha)^{(1/2-\alpha)}}{2^\alpha\,(1-\alpha)^{(1-\alpha)}} \in \Big[\tfrac{1}{\sqrt{2}}, \tfrac{2+T^\alpha}{2}\Big], \tag{2.57}$$

$$\frac{\Big(\mathbb{E}\Big[\|W-W^N\|^p_{\mathscr{C}^\alpha([0,T],|\cdot|)}\Big]\Big)^{1/p}}{N^{(\alpha-1/2)}\,T^{-\alpha}\,\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}} \ge \frac{(1/2-\alpha)^{(1/2-\alpha)}}{2^\alpha\,(1-\alpha)^{(1-\alpha)}} \ge \frac{1}{\sqrt{2}}. \tag{2.58}$$

Proof of Proposition 2.14. Throughout this proof let $f \colon [0, 1/2] \to (0,\infty)$ be the function which satisfies for all $x \in [0, 1/2]$ that $f(x) = \frac{(1/2-x)^{(1/2-x)}}{2^x\,(1-x)^{(1-x)}}$ and let $g_\alpha \colon (0,1]^2 \to \mathbb{R}$, $\alpha \in [0, 1/2]$, be the functions which satisfy for all $x, y \in (0,1]$, $\alpha \in [0, 1/2]$ that

$$g_\alpha(x,y) = \frac{x\,(1-x) + y\,(1-y)}{(x+y)^{2\alpha}}. \tag{2.59}$$

We first prove (2.55). For this observe that it holds for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N-1\}$, $t \in \left[\frac{nT}{N}, \frac{(n+1)T}{N}\right]$ that

$$\begin{aligned} W_t - W_t^N &= W_t - \left(n + 1 - \tfrac{tN}{T}\right) \cdot W_{\frac{nT}{N}} - \left(\tfrac{tN}{T} - n\right) \cdot W_{\frac{(n+1)T}{N}} \\ &= \left(n - \tfrac{tN}{T}\right) \cdot \left(W_{\frac{(n+1)T}{N}} - W_t\right) + \left(n + 1 - \tfrac{tN}{T}\right) \cdot \left(W_t - W_{\frac{nT}{N}}\right). \end{aligned} \tag{2.60}$$

This and the fact that

$$\forall\, N \in \mathbb{N} \colon \forall\, t \in \left(0, \tfrac{T}{N}\right) \colon \forall\, p \in [1, \infty) \colon \left\| \tfrac{W_t - W_t^N}{\|W_t - W_t^N\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)}} \right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} = \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \tag{2.61}$$

imply that it holds for all $N \in \mathbb{N}$, $p \in [1, \infty)$ that

$$\begin{aligned} &\left\|W - W^N\right\|_{C([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})} = \sup\nolimits_{t \in [0,T]} \left\|W_t - W_t^N\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\ &= \sup\nolimits_{t \in [0, \frac{T}{N}]} \left\|W_t - W_t^N\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} = \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left[\sup\nolimits_{t \in [0, \frac{T}{N}]} \left\|W_t - W_t^N\right\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)}\right] \\ &= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left[\sup\nolimits_{t \in [0, \frac{T}{N}]} \left\| \tfrac{tN}{T} \cdot \left(W_t - W_{\frac{T}{N}}\right) + \left(1 - \tfrac{tN}{T}\right) \cdot W_t \right\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)}\right] \\ &= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left[\sup\nolimits_{t \in [0, \frac{T}{N}]} \left[\left(\tfrac{tN}{T}\right)^2 \cdot \left(\tfrac{T}{N} - t\right) + \left(1 - \tfrac{tN}{T}\right)^2 \cdot t\right]^{1/2}\right] \\ &= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{N}} \left[\sup_{t \in [0,1]} \sqrt{\left(t^2 \cdot (1-t) + (1-t)^2 \cdot t\right)}\right] \\ &= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{N}} \left[\sup_{t \in [0,1]} \sqrt{t \cdot (1-t)}\right] = \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{2\sqrt{N}}. \end{aligned} \tag{2.62}$$

This establishes (2.55). In the next step we prove (2.56). For this observe that (2.60) shows for all $N \in \{2, 3, 4, \ldots\}$, $n \in \{1, 2, \ldots, N-1\}$, $t_1 \in [0, \frac{T}{N}]$, $t_2 \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$, $p \in [1, \infty)$ that

$$\begin{aligned} &\left\|\left(W_{t_2} - W_{t_2}^N\right) - \left(W_{t_1} - W_{t_1}^N\right)\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\ &= \left\| \left(n - \tfrac{t_2 N}{T}\right) \cdot \left(W_{\frac{(n+1)T}{N}} - W_{t_2}\right) + \left(n + 1 - \tfrac{t_2 N}{T}\right) \cdot \left(W_{t_2} - W_{\frac{nT}{N}}\right) \right. \\ &\quad \left. + \tfrac{t_1 N}{T} \cdot \left(W_{\frac{T}{N}} - W_{t_1}\right) + \left(\tfrac{t_1 N}{T} - 1\right) \cdot W_{t_1} \right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\ &= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left[\left(n - \tfrac{t_2 N}{T}\right)^2 \left(\tfrac{(n+1)T}{N} - t_2\right) \right. \\ &\qquad \left. + \left(n + 1 - \tfrac{t_2 N}{T}\right)^2 \left(t_2 - \tfrac{nT}{N}\right) + \tfrac{(t_1)^2 N^2}{T^2} \left(\tfrac{T}{N} - t_1\right) + \left(\tfrac{t_1 N}{T} - 1\right)^2 t_1 \right]^{\frac{1}{2}} \\ &= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{N}} \left[\left(\tfrac{t_2 N}{T} - n\right) \left(n + 1 - \tfrac{t_2 N}{T}\right) + \tfrac{t_1 N}{T} \left(1 - \tfrac{t_1 N}{T}\right) \right]^{\frac{1}{2}}. \end{aligned} \tag{2.63}$$

Moreover, (2.54) ensures for all $N \in \mathbb{N}$, $t_1, t_2 \in [0, \frac{T}{N}]$, $p \in [1, \infty)$ with $t_1 < t_2$ that

$$\begin{aligned} &\left\|\left(W_{t_2} - W_{t_2}^N\right) - \left(W_{t_1} - W_{t_1}^N\right)\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\ &= \left\|\left(W_{t_2} - \tfrac{t_2 N}{T} \cdot W_{\frac{T}{N}}\right) - \left(W_{t_1} - \tfrac{t_1 N}{T} \cdot W_{\frac{T}{N}}\right)\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\ &= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left\|W_{t_2} - W_{t_1} + \tfrac{(t_1 - t_2)N}{T} \cdot W_{\frac{T}{N}}\right\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)} \\ &= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \end{aligned} \tag{2.64}$$

$$\cdot \left\| \Big(1 + \tfrac{(t_1-t_2)N}{T}\Big) \cdot (W_{t_2} - W_{t_1}) + \tfrac{(t_1-t_2)N}{T} \cdot \Big(W_{\frac{T}{N}} - W_{t_2}\Big) + \tfrac{(t_1-t_2)N}{T} \cdot W_{t_1} \right\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)}$$

$$= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left[\left|1 + \tfrac{(t_1-t_2)N}{T}\right|^2 \cdot (t_2 - t_1) + \tfrac{|t_1-t_2|^2 N^2}{T^2} \cdot \big(\tfrac{T}{N} - t_2\big) + \tfrac{|t_1-t_2|^2 N^2}{T^2} \cdot t_1 \right]^{1/2}$$

$$= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \cdot \left[1 + \tfrac{2(t_1-t_2)N}{T} + \tfrac{(t_1-t_2)^2 N^2}{T^2} + \tfrac{|t_1-t_2|\, N^2}{T^2} \cdot \big(\tfrac{T}{N} + t_1 - t_2\big)\right]^{1/2} \cdot (t_2 - t_1)^{1/2}$$

$$= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \cdot \Big(1 + \tfrac{(t_1-t_2)N}{T}\Big)^{1/2} \cdot (t_2 - t_1)^{1/2}.$$

Combining (2.63) and (2.64) proves for all $N \in \{2, 3, 4, \ldots\}$, $\alpha \in [0, 1/2]$, $p \in [1, \infty)$ that

$$\big|W - W^N\big|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})} = \sup_{t_1,t_2\in[0,T],\, t_1<t_2} \frac{\big\|\big(W_{t_2} - W^N_{t_2}\big) - \big(W_{t_1} - W^N_{t_1}\big)\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{|t_1 - t_2|^\alpha}$$

$$= \sup_{t_1\in[0,\frac{T}{N}],\, t_2\in[0,T],\, t_1<t_2} \frac{\big\|\big(W_{t_2} - W^N_{t_2}\big) - \big(W_{t_1} - W^N_{t_1}\big)\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{|t_1 - t_2|^\alpha}$$

$$= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left| \max\left\{ \sup_{\substack{t_1,t_2\in[0,\frac{T}{N}],\\ t_1<t_2}} \frac{\big(1+\frac{(t_1-t_2)N}{T}\big)}{(t_2-t_1)^{(2\alpha-1)}},\ \sup_{\substack{t_1\in[0,\frac{T}{N}],\\ t_2\in(\frac{T}{N},\frac{2T}{N}]}} \frac{T\Big[\Big(\frac{t_2 N}{T} - 1\Big)\Big(2 - \frac{t_2 N}{T}\Big) + \frac{t_1 N}{T}\Big(1 - \frac{t_1 N}{T}\Big)\Big]}{N\,(t_2-t_1)^{2\alpha}} \right\} \right|^{\frac{1}{2}}. \tag{2.65}$$

This implies for all $N \in \{2, 3, 4, \ldots\}$, $\alpha \in [0, 1/2]$, $p \in [1, \infty)$ that

$$\big|W - W^N\big|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})}$$

$$= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left| \left|\tfrac{T}{N}\right|^{(1-2\alpha)} \max\left\{ \sup_{x\in(0,1]} \frac{(1-x)}{x^{(2\alpha-1)}},\ \sup_{\substack{x\in[0,1],\\ y\in(1,2]}} \frac{(y-1)\,(2-y) + x\,(1-x)}{(y-x)^{2\alpha}} \right\} \right|^{\frac{1}{2}}$$

$$= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left|\tfrac{T}{N}\right|^{\left(\frac{1}{2}-\alpha\right)} \left[\max\left\{ \sup_{x\in(0,1]} \frac{x\,(1-x)}{x^{2\alpha}},\ \sup_{\substack{x\in[0,1],\\ y\in(0,1]}} \frac{y\,(2-(y+1)) + x\,(1-x)}{([y+1]-[1-x])^{2\alpha}} \right\} \right]^{\frac{1}{2}}. \tag{2.66}$$

Hence, we obtain for all $N \in \{2, 3, 4, \ldots\}$, $\alpha \in [0, 1/2]$, $p \in [1, \infty)$ that

$$\big|W - W^N\big|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})}$$

$$= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left|\tfrac{T}{N}\right|^{\left(\frac{1}{2}-\alpha\right)} \left[\max\left\{ \sup_{y\in(0,1]} \frac{y\,(1-y)}{y^{2\alpha}},\ \sup_{x\in[0,1]} \sup_{y\in(0,1]} \frac{x\,(1-x) + y\,(1-y)}{(x+y)^{2\alpha}} \right\} \right]^{\frac{1}{2}}$$

$$= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left|\tfrac{T}{N}\right|^{\left(\frac{1}{2}-\alpha\right)} \left[\sup_{x\in[0,1]} \sup_{y\in(0,1]} \frac{x\,(1-x) + y\,(1-y)}{(x+y)^{2\alpha}} \right]^{1/2} \tag{2.67}$$

$$= N^{(\alpha-1/2)}\, T^{-\alpha}\, \|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \left[\sup_{x,y\in(0,1]} g_\alpha(x,y) \right]^{1/2}.$$

To complete the proof of (2.56), we study a few properties of the functions g_α, $\alpha \in [0, 1/2]$. Note that it holds for all $x, y \in [0, 1]$ that

$$x(1-x) + y(1-y) = (x+y)\Big(1 - \frac{x+y}{2}\Big) - \frac{(x-y)^2}{2} \le 2\,\Big(\frac{x+y}{2}\Big)\Big(1 - \frac{x+y}{2}\Big). \tag{2.68}$$

In addition, observe that it holds for all $\alpha \in [0, 1/2]$, $z \in (0,1)$ that

$$\tfrac{\partial}{\partial z}\big(z^{(1-2\alpha)}\,(1-z)\big) = (1-2\alpha)\,z^{-2\alpha}\,(1-z) - z^{(1-2\alpha)} = -2\,(1-\alpha)\,\big[z - \tfrac{1/2-\alpha}{1-\alpha}\big]\,z^{-2\alpha}. \quad (2.69)$$

Combining this with (2.68) ensures for all $\alpha \in [0, 1/2]$, $x, y \in (0,1]$ that

$$\begin{aligned} g_\alpha(x,y) &= \frac{x(1-x)+y(1-y)}{(x+y)^{2\alpha}} \le 2^{(1-2\alpha)}\Big(\frac{x+y}{2}\Big)^{1-2\alpha}\Big(1-\frac{x+y}{2}\Big) \\ &\le 2^{(1-2\alpha)} \sup_{z\in(0,1]}\big[z^{(1-2\alpha)}(1-z)\big] = 2^{(1-2\alpha)}\big[\tfrac{1/2-\alpha}{1-\alpha}\big]^{(1-2\alpha)}\big[1-\tfrac{1/2-\alpha}{1-\alpha}\big] \\ &= 2^{-2\alpha}\big[\tfrac{1/2-\alpha}{1-\alpha}\big]^{(1-2\alpha)}\big[\tfrac{1}{1-\alpha}\big] = \Big[\tfrac{(1/2-\alpha)^{(1/2-\alpha)}}{2^\alpha\,(1-\alpha)^{(1-\alpha)}}\Big]^2 = [f(\alpha)]^2. \end{aligned} \quad (2.70)$$

This proves for all $\alpha \in [0, 1/2]$ that

$$[f(\alpha)]^2 = \sup_{z\in(0,1]}\big[(2\,z)^{(1-2\alpha)}(1-z)\big] = \sup_{x\in(0,1]} g_\alpha(x,x) \le \sup_{x,y\in(0,1]} g_\alpha(x,y) \le [f(\alpha)]^2. \quad (2.71)$$

This shows for all $\alpha \in [0, 1/2]$ that

$$\sup_{x,y\in(0,1]} g_\alpha(x,y) = \sup_{x\in(0,1]} g_\alpha(x,x) = [f(\alpha)]^2. \quad (2.72)$$

Next note that it holds for all $\alpha \in (0, 1/2)$ that

$$f(\alpha) = \exp\big(\big(\tfrac{1}{2}-\alpha\big)\cdot\ln\big(\tfrac{1}{2}-\alpha\big) + (\alpha-1)\cdot\ln(1-\alpha) - \alpha\cdot\ln(2)\big). \quad (2.73)$$

Moreover, observe that it holds for all $\alpha \in (0, 1/2)$ that

$$\begin{aligned} &\tfrac{\partial}{\partial\alpha}\big(\big(\tfrac{1}{2}-\alpha\big)\cdot\ln\big(\tfrac{1}{2}-\alpha\big) + (\alpha-1)\cdot\ln(1-\alpha) - \alpha\cdot\ln(2)\big) \\ &= -\ln\big(\tfrac{1}{2}-\alpha\big) - 1 + \ln(1-\alpha) + 1 - \ln(2) = \ln(1-\alpha) - \ln\big(\tfrac{1}{2}-\alpha\big) - \ln(2) \\ &= \ln\big(\tfrac{1-\alpha}{1-2\alpha}\big) > 0. \end{aligned} \quad (2.74)$$

This and (2.73) ensure that f is strictly increasing. Equation (2.72) hence proves for all $\alpha \in [0, 1/2]$ that

$$\sup_{x,y\in(0,1]} g_\alpha(x,y) = \sup_{x\in(0,1]} g_\alpha(x,x) = [f(\alpha)]^2 \in \Big[|f(0)|^2, \big|f(\tfrac{1}{2})\big|^2\Big] = \big[\tfrac{1}{2}, 1\big]. \quad (2.75)$$

Putting this into (2.67) establishes (2.56). Combining (2.55) with (2.56) proves (2.57). Moreover, (2.56) and Lemma 2.13 imply (2.58). The proof of Proposition 2.14 is thus completed. □

3 Basic results for mild solutions of SEEs

In this section we collect a number of elementary results for mild solution processes of SEEs, most of which are well-known.

3.1 Temporal regularity of solutions of SEEs

Proposition 3.1. *Consider the notation in Subsection 1.1, let* $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ *and* $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ *be separable* $\mathbb{R}$*-Hilbert spaces, let* $\mathbb{H} \subseteq H$ *be a non-empty orthonormal basis of* H*, let* $\lambda\colon \mathbb{H} \to \mathbb{R}$ *be a function with* $\sup_{h\in\mathbb{H}} \lambda_h < 0$*, let* $A\colon D(A) \subseteq H \to H$ *be the linear operator which satisfies* $D(A) = \{v \in H\colon \sum_{h\in\mathbb{H}} |\lambda_h \langle h, v\rangle_H|^2 < \infty\}$ *and* $\forall\, v \in D(A)\colon Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$*, let* $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$*,* $r \in \mathbb{R}$*, be a family of interpolation spaces associated to* $-A$ *(cf., e.g., [36, Section 3.7]), let* $T \in (0,\infty)$*,* $p \in [2,\infty)$*,* $\gamma \in \mathbb{R}$*,* $\eta \in [0,1)$*,* $\beta \in [\gamma - \eta/2, \gamma]$*,* $F \in C(H_\gamma, H_{\gamma-\eta})$*,* $B \in C(H_\gamma, \mathrm{HS}(U, H_\beta))$ *satisfy* $|F|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_{\gamma-\eta}})} + |B|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{\mathrm{HS}(U,H_\beta)})} < \infty$*, let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space with a normal filtration* $(\mathscr{F}_t)_{t\in[0,T]}$*, let* $(W_t)_{t\in[0,T]}$ *be an* Id_U*-cylindrical* $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$*-Wiener process, and let* $X\colon [0,T]\times\Omega \to H_\gamma$ *be an* $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$*-predictable stochastic process which satisfies for all* $t \in [0,T]$ *that* $\sup_{s\in[0,T]} \|X_s\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} < \infty$ *and*

$$
\begin{aligned}
[X_t]_{\mathbb{P},\mathscr{B}(H_\gamma)} = \bigg[e^{tA} X_0 + \int_0^t \mathbb{1}_{\{\int_0^t \|e^{(t-u)A} F(X_u)\|_{H_\gamma} \mathrm{d}u < \infty\}}\, e^{(t-s)A} F(X_s)\,\mathrm{d}s \bigg]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\
+ \int_0^t e^{(t-s)A} B(X_s)\,\mathrm{d}W_s.
\end{aligned}
\tag{3.1}
$$

Then it holds for all $r \in [\gamma, \min\{1+\gamma-\eta, 1/2+\beta\})$*,* $\varepsilon \in [0, \min\{1+\gamma-\eta-r, 1/2+\beta-r\})$ *that* $\inf_{s\in(0,T]} \mathbb{P}(X_s \in H_r) = 1$ *and*

$$
\begin{aligned}
&\sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{\|(X_{t_1} - e^{t_1 A}X_0)\,\mathbb{1}_{\{X_{t_1}\in H_r\}} - (X_{t_2} - e^{t_2 A}X_0)\,\mathbb{1}_{\{X_{t_2}\in H_r\}}\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}}{|t_1-t_2|^\varepsilon} \right) \\
&\le \left[\sup_{s\in[0,T]} \|F(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \frac{2\,T^{(1+\gamma-\eta-r-\varepsilon)}}{(1+\gamma-\eta-r-\varepsilon)} \\
&\quad + \left[\sup_{s\in[0,T]} \|B(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_\beta)})} \right] \frac{\sqrt{p\,(p-1)}\,T^{(1/2+\beta-r-\varepsilon)}}{(1+2\beta-2r-2\varepsilon)^{1/2}} < \infty.
\end{aligned}
\tag{3.2}
$$

Proof of Proposition 3.1. Note that the fact that it holds for all $u \in [0,1]$ that

$$
\sup_{t\in(0,T]} t^u \|(-A)^u e^{tA}\|_{L(H)} \le 1 \quad \text{and} \quad \sup_{t\in(0,T]} t^{-u} \|(-A)^{-u}(e^{tA} - \mathrm{Id}_H)\|_{L(H)} \le 1 \tag{3.3}
$$

(cf., e.g., [34, Lemma 12.36]) implies that it holds for all $r \in [\gamma, 1+\gamma-\eta)$, $\varepsilon \in [0, 1+\gamma-\eta-r)$, $t_1 \in [0,T)$, $t_2 \in (t_1, T]$ that

$$
\begin{aligned}
&\bigg\| \int_0^{t_1} \mathbb{1}_{\{\int_0^{t_1} \|e^{(t_1-u)A}F(X_u)\|_{H_r}\mathrm{d}u<\infty\}}\, e^{(t_1-s)A} F(X_s)\,\mathrm{d}s \\
&\qquad - \int_0^{t_2} \mathbb{1}_{\{\int_0^{t_2} \|e^{(t_2-u)A}F(X_u)\|_{H_r}\mathrm{d}u<\infty\}}\, e^{(t_2-s)A} F(X_s)\,\mathrm{d}s \bigg\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})} \\
&\le \int_{t_1}^{t_2} \big\| e^{(t_2-s)A} F(X_s) \big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}\,\mathrm{d}s + \int_0^{t_1} \big\| e^{(t_1-s)A}\big(\mathrm{Id}_{H_{\gamma-\eta}} - e^{(t_2-t_1)A}\big) F(X_s) \big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}\,\mathrm{d}s \\
&\le \left[\sup_{s\in[0,T]} \|F(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \left[\int_{t_1}^{t_2} (t_2-s)^{\gamma-\eta-r}\,\mathrm{d}s + \int_0^{t_1} (t_1-s)^{\gamma-\eta-r-\varepsilon}(t_2-t_1)^\varepsilon\,\mathrm{d}s \right] \\
&= \left[\sup_{s\in[0,T]} \|F(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \left[\frac{(t_2-t_1)^{(1+\gamma-\eta-r)}}{(1+\gamma-\eta-r)} + \frac{(t_2-t_1)^\varepsilon (t_1)^{(1+\gamma-\eta-r-\varepsilon)}}{(1+\gamma-\eta-r-\varepsilon)} \right]
\end{aligned}
\tag{3.4}
$$

$$\leq \left[\sup_{s\in[0,T]} \|F(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})}\right]\left[\frac{2\,T^{(1+\gamma-\eta-r-\varepsilon)}(t_2-t_1)^{\varepsilon}}{(1+\gamma-\eta-r-\varepsilon)}\right].$$

Moreover, (3.3) ensures for all $r \in [\gamma, 1/2+\beta)$, $\varepsilon \in [0, 1/2+\beta-r)$, $t_1 \in [0,T)$, $t_2 \in (t_1, T]$ that

$$\begin{aligned}
&\left\|\int_0^{t_1} e^{(t_1-s)A}B(X_s)\,\mathrm{d}W_s - \int_0^{t_2} e^{(t_2-s)A}B(X_s)\,\mathrm{d}W_s\right\|^2_{L^p(\mathbb{P};\|\cdot\|_{H_r})}\\
&\leq \tfrac{p\,(p-1)}{2}\int_{t_1}^{t_2}\left\|e^{(t_2-s)A}B(X_s)\right\|^2_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_r)})}\,\mathrm{d}s\\
&\quad+\tfrac{p\,(p-1)}{2}\int_0^{t_1}\left\|e^{(t_1-s)A}\big(\mathrm{Id}_{H_\beta}-e^{(t_2-t_1)A}\big)B(X_s)\right\|^2_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_r)})}\,\mathrm{d}s\\
&\leq \tfrac{p\,(p-1)}{2}\left[\sup_{s\in[0,T]}\|B(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_\beta)})}\right]^2\\
&\quad\cdot\left[\int_{t_1}^{t_2}(t_2-s)^{(2\beta-2r)}\,\mathrm{d}s+\int_0^{t_1}(t_1-s)^{(2\beta-2r-2\varepsilon)}(t_2-t_1)^{2\varepsilon}\,\mathrm{d}s\right]\\
&\leq \tfrac{p\,(p-1)}{2}\left[\sup_{s\in[0,T]}\|B(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_\beta)})}\right]^2\left[\frac{2\,T^{(1+2\beta-2r-2\varepsilon)}(t_2-t_1)^{2\varepsilon}}{(1+2\beta-2r-2\varepsilon)}\right].
\end{aligned}\tag{3.5}$$

Combining (3.4) and (3.5) completes the proof of Proposition 3.1. □

Corollary 3.2. *Consider the notation in Subsection 1.1, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of H, let $\lambda\colon \mathbb{H} \to \mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}} \lambda_h < 0$, let $A\colon D(A) \subseteq H \to H$ be the linear operator which satisfies $D(A) = \big\{v \in H\colon \sum_{h\in\mathbb{H}} |\lambda_h\langle h, v\rangle_H|^2 < \infty\big\}$ and $\forall\, v \in D(A)\colon Av = \sum_{h\in\mathbb{H}} \lambda_h\langle h, v\rangle_H h$, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, let $T \in (0,\infty)$, $p \in [2,\infty)$, $\gamma \in \mathbb{R}$, $\eta \in [0,1)$, $\beta \in [\gamma - \eta/2, \gamma]$, $\delta \in [\gamma,\infty)$, $F \in C(H_\gamma, H_{\gamma-\eta})$, $B \in C(H_\gamma, \mathrm{HS}(U, H_\beta))$ satisfy $|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})} + |B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_\beta)})} < \infty$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$-Wiener process, and let $X\colon [0,T]\times\Omega \to H_\gamma$ be an $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-predictable stochastic process which satisfies for all $t \in [0,T]$ that $\sup_{s\in[0,T]} \|X_s\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} < \infty$, $X_0(\Omega) \subseteq H_\delta$, $\mathbb{E}[\|X_0\|^p_{H_\delta}] < \infty$, and*

$$\begin{aligned}
[X_t]_{\mathbb{P},\mathscr{B}(H_\gamma)} &= \left[e^{tA}X_0 + \int_0^t \mathbb{1}_{\{\int_0^t \|e^{(t-u)A}F(X_u)\|_{H_\gamma}\mathrm{d}u<\infty\}}\, e^{(t-s)A}F(X_s)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)}\\
&\quad+\int_0^t e^{(t-s)A}B(X_s)\,\mathrm{d}W_s.
\end{aligned}\tag{3.6}$$

Then it holds for all $r \in [\gamma, \min\{1+\gamma-\eta, 1/2+\beta\})$, $\varepsilon \in [0, \min\{1+\gamma-\eta-r, 1/2+\beta-r\})$ that $\inf_{s\in(0,T]} \mathbb{P}(X_s \in H_r) = 1$ and

$$\begin{aligned}
&\sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}}\left(\frac{|\min\{t_1,t_2\}|^{\max\{r+\varepsilon-\delta,0\}}\,\|\mathbb{1}_{\{X_{t_1}\in H_r\}}X_{t_1}-\mathbb{1}_{\{X_{t_2}\in H_r\}}X_{t_2}\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}}{|t_1-t_2|^{\varepsilon}}\right)\\
&\leq \|X_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\delta,r+\varepsilon\}}})}+\left[\sup_{s\in[0,T]}\|F(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})}\right]\frac{2\,T^{(1+\gamma-\eta-\min\{\delta,r+\varepsilon\})}}{(1+\gamma-\eta-r-\varepsilon)}\\
&\quad+\left[\sup_{s\in[0,T]}\|B(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_\beta)})}\right]\frac{\sqrt{p\,(p-1)}\,T^{(1/2+\beta-\min\{\delta,r+\varepsilon\})}}{(1+2\beta-2r-2\varepsilon)^{1/2}}<\infty.
\end{aligned}\tag{3.7}$$

Proof of Corollary 3.2. The fact that $\forall\, u \in [0,1]\colon \big(\sup_{t\in(0,T]} t^u \|(-A)^u e^{tA}\|_{L(H)} \leq 1$ and $\sup_{t\in(0,T]} t^{-u} \|(-A)^{-u}(e^{tA} - \mathrm{Id}_H)\|_{L(H)} \leq 1\big)$ ensures for all $r \in [\gamma, \min\{1+\gamma-\eta, 1/2+\beta\})$, $\varepsilon \in [0, \min\{1+\gamma-\eta-r, 1/2+\beta-r\})$ that

$$
\begin{aligned}
&\sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{\| |\min\{t_1,t_2\}|^{\max\{r+\varepsilon-\delta,0\}} (e^{t_1 A} X_0 - e^{t_2 A} X_0)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}}{|t_1 - t_2|^{\varepsilon}} \right) \\
&\leq \sup_{\substack{t_1,t_2\in[0,T],\\ t_1< t_2}} \left(\frac{\| |t_1|^{\max\{r+\varepsilon-\delta,0\}} (-A)^{r-\min\{\delta,r+\varepsilon\}} (e^{t_1 A} - e^{t_2 A})\|_{L(H)} \, \|X_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\delta,r+\varepsilon\}}})}}{|t_1 - t_2|^{\varepsilon}} \right) \\
&\leq \sup_{\substack{t_1,t_2\in(0,T],\\ t_1< t_2}} \left(|t_1|^{\max\{r+\varepsilon-\delta,0\}} \|(-A)^{r+\varepsilon-\min\{\delta,r+\varepsilon\}}\, e^{t_1 A}\|_{L(H)} \, \|X_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\delta,r+\varepsilon\}}})} \right) \qquad (3.8) \\
&= \sup_{t_1\in(0,T]} \left(|t_1|^{\max\{r+\varepsilon-\delta,0\}} \|(-A)^{\max\{r+\varepsilon-\delta,0\}}\, e^{t_1 A}\|_{L(H)} \, \|X_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\delta,r+\varepsilon\}}})} \right) \\
&\leq \|X_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\delta,r+\varepsilon\}}})}.
\end{aligned}
$$

Combining this with the triangle inequality and Proposition 3.1 proves for all $r \in [\gamma, \min\{1+\gamma-\eta, 1/2+\beta\})$, $\varepsilon \in [0, \min\{1+\gamma-\eta-r, 1/2+\beta-r\})$ that $\inf_{s\in(0,T]} \mathbb{P}(X_s \in H_r) = 1$ and

$$
\begin{aligned}
&\sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{|\min\{t_1,t_2\}|^{\max\{r+\varepsilon-\delta,0\}} \, \|\mathbb{1}_{\{X_{t_1}\in H_r\}} X_{t_1} - \mathbb{1}_{\{X_{t_2}\in H_r\}} X_{t_2}\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}}{|t_1 - t_2|^{\varepsilon}} \right) \\
&\leq \sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{\| |\min\{t_1,t_2\}|^{\max\{r+\varepsilon-\delta,0\}} (e^{t_1 A} X_0 - e^{t_2 A} X_0)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}}{|t_1 - t_2|^{\varepsilon}} \right) + T^{\max\{r+\varepsilon-\delta,0\}} \\
&\quad \cdot \sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{\|(X_{t_1} - e^{t_1 A} X_0)\, \mathbb{1}_{\{X_{t_1}\in H_r\}} - (X_{t_2} - e^{t_2 A} X_0)\, \mathbb{1}_{\{X_{t_2}\in H_r\}}\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}}{|t_1 - t_2|^{\varepsilon}} \right) \qquad (3.9) \\
&\leq \|X_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\delta,r+\varepsilon\}}})} + \left[\sup_{s\in[0,T]} \|F(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \frac{2\, T^{(1+\gamma-\eta-\min\{\delta,r+\varepsilon\})}}{(1+\gamma-\eta-r-\varepsilon)} \\
&\quad + \left[\sup_{s\in[0,T]} \|B(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_\beta)})} \right] \frac{\sqrt{p\,(p-1)}\, T^{(1/2+\beta-\min\{\delta,r+\varepsilon\})}}{(1+2\beta-2r-2\varepsilon)^{1/2}} < \infty.
\end{aligned}
$$

The proof of Corollary 3.2 is thus completed. □

3.2 A priori bounds for solutions of SEEs

Lemma 3.3. *Consider the notation in Subsection 1.1 and let* $\mathbb{B}\colon (0,\infty)^2 \to (0,\infty)$ *and* $\mathrm{E}_\eta\colon [0,\infty) \to [0,\infty)$, $\eta \in (-\infty,1)$, *be the functions which satisfy for all* $\eta \in (-\infty,1)$, $x,y \in (0,\infty)$, $z \in [0,\infty)$ *that* $\mathbb{B}(x,y) = \int_0^1 t^{(x-1)}(1-t)^{(y-1)}\,\mathrm{d}t$ *and* $\mathrm{E}_\eta(z) = 1 + \sum_{n=1}^{\infty} z^n \prod_{k=0}^{n-1} \mathbb{B}(1-\eta, k(1-\eta)+1)$. *Then it holds for all* $\eta \in (-\infty,1)$, $x \in [0,\infty)$ *that* $\sqrt{\mathrm{E}_\eta(x^2)} = \mathscr{E}_{(1-\eta)}(x)$.

Proof of Lemma 3.3. Note that the fact that $\forall\, x,y \in (0,\infty)\colon \mathbb{B}(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$ implies that it holds for all $\eta \in (-\infty, 1)$, $x \in [0, \infty)$ that

$$\begin{aligned}
\mathrm{E}_\eta(x^2) &= 1 + \sum_{n=1}^{\infty} (x^2)^n \prod_{k=0}^{n-1} \mathbb{B}(1-\eta, k(1-\eta)+1) \\
&= 1 + \sum_{n=1}^{\infty} x^{2n} \prod_{k=0}^{n-1} \frac{\Gamma(1-\eta)\,\Gamma(k(1-\eta)+1)}{\Gamma((k+1)(1-\eta)+1)} = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}[\Gamma(1-\eta)]^n}{\Gamma(n(1-\eta)+1)} \\
&= \sum_{n=0}^{\infty} \frac{x^{2n}[\Gamma(1-\eta)]^n}{\Gamma(n(1-\eta)+1)} = [\mathscr{E}_{(1-\eta)}(x)]^2.
\end{aligned} \tag{3.10}$$

The proof of Lemma 3.3 is thus completed. □

Proposition 3.4 (A priori bounds)**.** *Consider the notation in Subsection 1.1, let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of H, let $\lambda \colon \mathbb{H} \to \mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}} \lambda_h < 0$, let $A \colon D(A) \subseteq H \to H$ be the linear operator which satisfies $D(A) = \{v \in H \colon \sum_{h\in\mathbb{H}} |\lambda_h \langle h, v\rangle_H|^2 < \infty\}$ and $\forall\, v \in D(A) \colon Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$, let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, let $T \in (0, \infty)$, $p \in [2, \infty)$, $\gamma \in \mathbb{R}$, $\eta \in [0, 1)$, $F \in C(H_\gamma, H_{\gamma-\eta})$, $B \in C(H_\gamma, \mathrm{HS}(U, H_{\gamma-\eta/2}))$ satisfy $|F|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_{\gamma-\eta}})} + |B|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{\mathrm{HS}(U, H_{\gamma-\eta/2})})} < \infty$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$-Wiener process, and let $X \colon [0, T] \times \Omega \to H_\gamma$ be an $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-predictable stochastic process which satisfies for all $t \in [0, T]$ that $\sup_{s\in[0,T]} \|X_s\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{H_\gamma})} < \infty$ and*

$$\begin{aligned}
[X_t]_{\mathbb{P}, \mathscr{B}(H_\gamma)} &= \left[e^{tA} X_0 + \int_0^t \mathbb{1}_{\{\int_0^t \|e^{(t-u)A} F(X_u)\|_{H_\gamma} \mathrm{d}u < \infty\}}\, e^{(t-s)A} F(X_s)\, \mathrm{d}s \right]_{\mathbb{P}, \mathscr{B}(H_\gamma)} \\
&\quad + \int_0^t e^{(t-s)A} B(X_s)\, \mathrm{d}W_s.
\end{aligned} \tag{3.11}$$

Then

$$\begin{aligned}
&\sup_{t\in[0,T]} \big\| \max\{1, \|X_t\|_{H_\gamma}\} \big\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)} \leq \sqrt{2}\, \big\| \max\{1, \|X_0\|_{H_\gamma}\} \big\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)} \\
&\cdot \mathscr{E}_{(1-\eta)} \left[\tfrac{T^{1-\eta}\sqrt{2}}{\sqrt{1-\eta}} \left(\sup_{v\in H_\gamma} \tfrac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1, \|v\|_{H_\gamma}\}} \right) + \sqrt{T^{1-\eta} p(p-1)} \left(\sup_{v\in H_\gamma} \tfrac{\|B(v)\|_{\mathrm{HS}(U, H_{\gamma-\eta/2})}}{\max\{1, \|v\|_{H_\gamma}\}} \right) \right] < \infty.
\end{aligned} \tag{3.12}$$

Proof of Proposition 3.4. The Burkholder-Davis-Gundy-type inequality in Lemma 7.7 in Da Prato & Zabczyk [10], the fact that $\forall\, u \in [0, 1] \colon \sup_{t\in(0,T]} t^u \|(-A)^u e^{tA}\|_{L(H)} \leq 1$, and

Hölder's inequality imply that it holds for all $t \in [0,T]$ that

$$
\begin{aligned}
&\big\|\max\{1, \|X_t\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\
&\leq \big\|\max\{1, \|X_0\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} + \int_0^t \big\|e^{(t-s)A}F(X_s)\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\,\mathrm{d}s \\
&\quad + \sqrt{\frac{p\,(p-1)}{2}}\left[\int_0^t \big\|e^{(t-s)A}B(X_s)\big\|^2_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_\gamma)})}\,\mathrm{d}s\right]^{1/2} \\
&\leq \big\|\max\{1, \|X_0\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} + \left[\frac{t^{(1-\eta)}}{(1-\eta)}\int_0^t (t-s)^{-\eta}\,\|F(X_s)\|^2_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})}\,\mathrm{d}s\right]^{1/2} \\
&\quad + \sqrt{\frac{p\,(p-1)}{2}}\left[\int_0^t (t-s)^{-\eta}\,\|B(X_s)\|^2_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})}\,\mathrm{d}s\right]^{1/2} \\
&\leq \big\|\max\{1, \|X_0\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} + \left[\int_0^t (t-s)^{-\eta}\,\big\|\max\{1, \|X_s\|_{H_\gamma}\}\big\|^2_{\mathscr{L}^p(\mathbb{P};|\cdot|)}\,\mathrm{d}s\right]^{1/2} \\
&\quad \cdot \left[\sqrt{\tfrac{T^{(1-\eta)}}{(1-\eta)}}\left(\sup_{v\in H_\gamma}\tfrac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1,\|v\|_{H_\gamma}\}}\right) + \sqrt{\tfrac{p\,(p-1)}{2}}\left(\sup_{v\in H_\gamma}\tfrac{\|B(v)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}}{\max\{1,\|v\|_{H_\gamma}\}}\right)\right].
\end{aligned}
\tag{3.13}
$$

This and the fact that $\forall\, a,b\in\mathbb{R}\colon (a+b)^2 \leq 2\,(a^2+b^2)$ prove for all $t\in[0,T]$ that

$$
\begin{aligned}
&\big\|\max\{1, \|X_t\|_{H_\gamma}\}\big\|^2_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \leq 2\,\big\|\max\{1, \|X_0\|_{H_\gamma}\}\big\|^2_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\
&\quad + \int_0^t (t-s)^{-\eta}\,\big\|\max\{1, \|X_s\|_{H_\gamma}\}\big\|^2_{\mathscr{L}^p(\mathbb{P};|\cdot|)}\,\mathrm{d}s \\
&\quad \cdot \left[\sqrt{\tfrac{2\,T^{(1-\eta)}}{(1-\eta)}}\left(\sup_{v\in H_\gamma}\tfrac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1,\|v\|_{H_\gamma}\}}\right) + \sqrt{p\,(p-1)}\left(\sup_{v\in H_\gamma}\tfrac{\|B(v)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}}{\max\{1,\|v\|_{H_\gamma}\}}\right)\right]^2.
\end{aligned}
\tag{3.14}
$$

E.g., Lemma 2.6 in Andersson, Jentzen, & Kurniawan [2] and Lemma 3.3 hence complete the proof of Proposition 3.4. □

3.3 A strong perturbation estimate for SEEs

Proposition 3.5 (Perturbation estimate). *Consider the notation in Subsection 1.1, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H}\subseteq H$ be a non-empty orthonormal basis of H, let $\lambda\colon \mathbb{H}\to\mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}}\lambda_h < 0$, let $A\colon D(A)\subseteq H\to H$ be the linear operator which satisfies $D(A) = \{v\in H\colon \sum_{h\in\mathbb{H}}|\lambda_h\langle h,v\rangle_H|^2 < \infty\}$ and $\forall\, v\in D(A)\colon Av = \sum_{h\in\mathbb{H}}\lambda_h\langle h,v\rangle_H h$, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r\in\mathbb{R}$, be a family of interpolation spaces associated to $-A$, let $T\in[0,\infty)$, $p\in[2,\infty)$, $\gamma\in\mathbb{R}$, $\eta\in[0,1)$, $F\in C(H_\gamma, H_{\gamma-\eta})$, $B\in C(H_\gamma, \mathrm{HS}(U,H_{\gamma-\eta/2}))$ satisfy $|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})} + |B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})} < \infty$, let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega,\mathscr{F},\mathbb{P},(\mathscr{F}_t)_{t\in[0,T]})$-Wiener process, and let $X^1, X^2\colon [0,T]\times\Omega\to H_\gamma$ be $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-predictable stochastic processes which satisfy $\max_{k\in\{1,2\}}\sup_{s\in[0,T]}\|X^k_s\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} < \infty$. Then*

$$
\begin{aligned}
&\sup_{t\in[0,T]}\big\|X^1_t - X^2_t\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \\
&\leq \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}\,|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}\,|B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})}\right] \\
&\quad \cdot \sqrt{2}\sup_{t\in[0,T]}\left\|\left[X^1_t - \int_0^t \mathbb{1}_{\{\int_0^t \|e^{(t-r)A}F(X^1_r)\|_{H_\gamma}\mathrm{d}r<\infty\}}e^{(t-s)A}F(X^1_s)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)}\right.
\end{aligned}
\tag{3.15}
$$

$$- \int_0^t e^{(t-s)A} B(X_s^1)\,\mathrm{d}W_s - \bigg\{ \Big[X_t^2 - \int_0^t \mathbb{1}_{\{\int_0^t \|e^{(t-r)A}F(X_r^2)\|_{H_\gamma}\,\mathrm{d}r<\infty\}} e^{(t-s)A} F(X_s^2)\,\mathrm{d}s \Big]_{\mathbb{P},\mathscr{B}(H_\gamma)}$$
$$- \int_0^t e^{(t-s)A} B(X_s^2)\,\mathrm{d}W_s \bigg\} \bigg\|_{L^p(\mathbb{P};\|\cdot\|_{H_\gamma})} < \infty.$$

Proof of Proposition 3.5. Throughout this proof we assume w.l.o.g. that $T \neq 0$ and throughout this proof let $\mathscr{A}\colon H_{\gamma+1} \subseteq H_\gamma \to H_\gamma$ be the linear operator which satisfies for all $v \in H_{\gamma+1}$ that $\mathscr{A}v = \sum_{h\in\mathbb{H}} \lambda_h \langle (-\lambda_h)^{-\gamma} h, v\rangle_{H_\gamma} (-\lambda_h)^{-\gamma} h$. Observe that $(H_{r+\gamma}, \langle\cdot,\cdot\rangle_{H_{r+\gamma}}, \|\cdot\|_{H_{r+\gamma}})$, $r \in \mathbb{R}$, is a family of interpolation spaces associated to $-\mathscr{A}$. This, Lemma 3.3, and Proposition 2.7 in Andersson, Jentzen, & Kurniawan [2] show for all $\varepsilon \in (0,\infty)$ that

$$\begin{aligned}
&\sup_{t\in(0,T]} \big\| X_t^1 - X_t^2 \big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \\
&\leq \mathscr{E}_{(1-\eta)} \bigg[\tfrac{T^{1-\eta}\sqrt{2}\,|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} \sup_{t\in(0,T]} t^\eta \|(-\mathscr{A})^\eta e^{t\mathscr{A}}\|_{L(H_\gamma)} \\
&\quad + \sqrt{T^{1-\eta}p(p-1)}\big(|B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})} + \varepsilon\big) \sup_{t\in(0,T]} t^{\eta/2} \|(-\mathscr{A})^{\eta/2} e^{t\mathscr{A}}\|_{L(H_\gamma)} \bigg] \\
&\quad \cdot \sqrt{2} \sup_{t\in(0,T]} \bigg\| \Big[X_t^1 - \int_0^t \mathbb{1}_{\{\int_0^t \|e^{(t-r)\mathscr{A}}F(X_r^1)\|_{H_\gamma}\,\mathrm{d}r<\infty\}} e^{(t-s)\mathscr{A}} F(X_s^1)\,\mathrm{d}s \Big]_{\mathbb{P},\mathscr{B}(H_\gamma)} \qquad (3.16) \\
&\quad - \int_0^t e^{(t-s)\mathscr{A}} B(X_s^1)\,\mathrm{d}W_s - \bigg\{ \Big[X_t^2 - \int_0^t \mathbb{1}_{\{\int_0^t \|e^{(t-r)\mathscr{A}}F(X_r^2)\|_{H_\gamma}\,\mathrm{d}r<\infty\}} e^{(t-s)\mathscr{A}} F(X_s^2)\,\mathrm{d}s \Big]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\
&\quad - \int_0^t e^{(t-s)\mathscr{A}} B(X_s^2)\,\mathrm{d}W_s \bigg\} \bigg\|_{L^p(\mathbb{P};\|\cdot\|_{H_\gamma})}.
\end{aligned}$$

The fact that $\forall\, u \in [0,1]\colon \sup_{t\in(0,T]} t^u \|(-\mathscr{A})^u e^{t\mathscr{A}}\|_{L(H_\gamma)} \leq 1$ hence proves (3.15). The proof of Proposition 3.5 is thus completed. □

3.4 Existence of continuous solutions of SEEs

The next result, Proposition 3.6 below, proves the existence of continuous solution processes of SPDEs (see, e.g., Theorem 7.1 in van Neerven, Veraar, & Weis [37] for a similar result in a more general framework).

Proposition 3.6. *Consider the notation in Subsection 1.1, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of H, let $T \in (0,\infty)$, $p \in [2,\infty)$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$-Wiener process, let $\lambda\colon \mathbb{H} \to \mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}} \lambda_h < 0$, let $A\colon D(A) \subseteq H \to H$ be the linear operator which satisfies $D(A) = \big\{v \in H\colon \sum_{h\in\mathbb{H}} |\lambda_h \langle h, v\rangle_H|^2 < \infty\big\}$ and $\forall\, v \in D(A)\colon Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, and let $\gamma \in \mathbb{R}$, $\eta \in [0,1)$, $F \in C(H_\gamma, H_{\gamma-\eta})$, $B \in C(H_\gamma, \mathrm{HS}(U, H_{\gamma-\eta/2}))$, $\xi \in \mathscr{M}(\mathscr{F}_0, \mathscr{B}(H_\gamma))$ satisfy $|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})} + |B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})} < \infty$. Then there exists an $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-adapted stochastic process $X\colon [0,T]\times\Omega \to H_\gamma$ with continuous sample paths which satisfies for all $t \in [0,T]$ that $[X_t]_{\mathbb{P},\mathscr{B}(H_\gamma)} = \big[e^{tA}\xi + \int_0^t e^{(t-s)A} F(X_s)\,\mathrm{d}s\big]_{\mathbb{P},\mathscr{B}(H_\gamma)} +$*

$\int_0^t e^{(t-s)A} B(X_s)\,\mathrm{d}W_s$ *and*

$$\begin{aligned}&\sup_{t\in[0,T]} \big\|\max\{1, \|X_t\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \leq \sqrt{2}\,\big\|\max\{1,\|\xi\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\ &\cdot \mathscr{E}_{(1-\eta)}\Bigg[\tfrac{T^{1-\eta}\sqrt{2}}{\sqrt{1-\eta}}\left(\sup_{v\in H_\gamma} \tfrac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1,\|v\|_{H_\gamma}\}}\right) + \sqrt{T^{1-\eta}p(p-1)}\left(\sup_{v\in H_\gamma}\tfrac{\|B(v)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}}{\max\{1,\|v\|_{H_\gamma}\}}\right)\Bigg].\end{aligned} \tag{3.17}$$

Proof of Proposition 3.6. Throughout this proof let $\Omega_n \in \mathscr{F}_0$, $n \in \mathbb{N}_0$, be the sets which satisfy for all $n \in \mathbb{N}_0$ that $\Omega_n = \{\|\xi\|_{H_\gamma} < n\}$ and let $\xi_n \colon \Omega \to H_\gamma$, $n \in \mathbb{N}$, be the mappings which satisfy for all $n \in \mathbb{N}$ that $\xi_n = \xi\,\mathbb{1}_{\Omega_n}$. Note that it holds for all $q \in [0,\infty)$, $n \in \mathbb{N}$ that $\mathbb{E}\big[\|\xi_n\|_{H_\gamma}^q\big] \leq n^q < \infty$. E.g., Theorem 5.1 in Jentzen & Kloeden [27], Proposition 3.1, and the Kolmogorov-Chentsov continuity theorem (see Theorem 2.7) hence ensure that there exist $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-adapted stochastic processes with continuous sample paths $X^n \colon [0,T]\times\Omega \to H_\gamma$, $n \in \mathbb{N}$, which satisfy for all $n \in \mathbb{N}$, $t \in [0,T]$ that $\sup_{s\in[0,T]} \mathbb{E}\big[\|X_s^n\|_{H_\gamma}^p\big] < \infty$ and

$$[X_t^n]_{\mathbb{P},\mathscr{B}(H_\gamma)} = \left[e^{tA}\xi_n + \int_0^t e^{(t-s)A}F(X_s^n)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A}B(X_s^n)\,\mathrm{d}W_s. \tag{3.18}$$

Observe that it holds for all $k \in \mathbb{N}$, $n,m \in \{k,k+1,\ldots\}$, $t \in [0,T]$ that

$$\begin{aligned}[(X_t^n - X_t^m)\,\mathbb{1}_{\Omega_k}]_{\mathbb{P},\mathscr{B}(H_\gamma)} &= \left[\int_0^t e^{(t-s)A}\big[F(\mathbb{1}_{\Omega_k}X_s^n) - F(\mathbb{1}_{\Omega_k}X_s^m)\big]\mathbb{1}_{\Omega_k}\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\ &\quad + \int_0^t e^{(t-s)A}\big[B(\mathbb{1}_{\Omega_k}X_s^n) - B(\mathbb{1}_{\Omega_k}X_s^m)\big]\mathbb{1}_{\Omega_k}\,\mathrm{d}W_s.\end{aligned} \tag{3.19}$$

Proposition 2.1 in Jentzen & Kurniawan [25] hence shows for all $k \in \mathbb{N}$, $n,m \in \{k,k+1,\ldots\}$ that

$$\sup_{t\in[0,T]} \|(X_t^n - X_t^m)\,\mathbb{1}_{\Omega_k}\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} = 0. \tag{3.20}$$

This implies that

$$\mathbb{P}\left(\forall\, k \in \mathbb{N} \colon \forall\, n,m \in \{k,k+1,\ldots\} \colon \mathbb{1}_{\Omega_k}\left[\sup_{t\in[0,T]} \|X_t^n - X_t^m\|_{H_\gamma}\right] = 0\right) = 1. \tag{3.21}$$

Next let $Y \colon [0,T]\times\Omega \to H_\gamma$ be the mapping which satisfies for all $(t,\omega) \in [0,T]\times\Omega$ that

$$Y_t(\omega) = \sum_{n=1}^{\infty} X_t^n(\omega)\cdot\mathbb{1}_{\Omega_n\setminus\Omega_{n-1}}(\omega). \tag{3.22}$$

Note that it holds for all $n \in \mathbb{N}$ that

$$\begin{aligned}&\mathbb{1}_{\Omega_n}\sup_{t\in[0,T]}\|Y_t - X_t^n\|_{H_\gamma} = \sup_{t\in[0,T]}\|Y_t\,\mathbb{1}_{\Omega_n} - X_t^n\,\mathbb{1}_{\Omega_n}\|_{H_\gamma} \\ &= \sup_{t\in[0,T]}\left\|\left[\sum_{k=1}^n X_t^k\,\mathbb{1}_{\Omega_k\setminus\Omega_{k-1}}\right] - X_t^n\,\mathbb{1}_{\Omega_n}\right\|_{H_\gamma} = \sup_{t\in[0,T]}\left\|\sum_{k=1}^n (X_t^k - X_t^n)\,\mathbb{1}_{\Omega_k\setminus\Omega_{k-1}}\right\|_{H_\gamma} \\ &= \sum_{k=1}^n\left[\mathbb{1}_{\Omega_k}\sup_{t\in[0,T]}\|X_t^k - X_t^n\|_{H_\gamma}\right]\mathbb{1}_{\Omega_k\setminus\Omega_{k-1}}.\end{aligned} \tag{3.23}$$

This and (3.21) show that

$$\mathbb{P}\bigg(\forall\, n\in\mathbb{N}\colon \mathbb{1}_{\Omega_n}\sup_{t\in[0,T]}\|Y_t-X_t^n\|_{H_\gamma}=0\bigg)=1. \tag{3.24}$$

Hence, we obtain for all $n\in\mathbb{N}$, $t\in[0,T]$ that

$$\begin{aligned}
&[Y_t\,\mathbb{1}_{\Omega_n}]_{\mathbb{P},\mathscr{B}(H_\gamma)}=[X_t^n\,\mathbb{1}_{\Omega_n}]_{\mathbb{P},\mathscr{B}(H_\gamma)}\\
&=\bigg(\bigg[e^{tA}\xi_n+\int_0^t e^{(t-s)A}F(X_s^n)\,\mathrm{d}s\bigg]_{\mathbb{P},\mathscr{B}(H_\gamma)}+\int_0^t e^{(t-s)A}B(X_s^n)\,\mathrm{d}W_s\bigg)\,\mathbb{1}_{\Omega_n}\\
&=\bigg(\bigg[e^{tA}\xi+\int_0^t e^{(t-s)A}\,\mathbb{1}_{\Omega_n}F(X_s^n)\,\mathrm{d}s\bigg]_{\mathbb{P},\mathscr{B}(H_\gamma)}+\int_0^t e^{(t-s)A}\,\mathbb{1}_{\Omega_n}B(X_s^n)\,\mathrm{d}W_s\bigg)\,\mathbb{1}_{\Omega_n}\\
&=\bigg(\bigg[e^{tA}\xi+\int_0^t e^{(t-s)A}F(Y_s)\,\mathrm{d}s\bigg]_{\mathbb{P},\mathscr{B}(H_\gamma)}+\int_0^t e^{(t-s)A}B(Y_s)\,\mathrm{d}W_s\bigg)\,\mathbb{1}_{\Omega_n}.
\end{aligned} \tag{3.25}$$

This implies for all $t\in[0,T]$ that

$$[Y_t]_{\mathbb{P},\mathscr{B}(H_\gamma)}=\bigg[e^{tA}\xi+\int_0^t e^{(t-s)A}F(Y_s)\,\mathrm{d}s\bigg]_{\mathbb{P},\mathscr{B}(H_\gamma)}+\int_0^t e^{(t-s)A}B(Y_s)\,\mathrm{d}W_s. \tag{3.26}$$

Next note that (3.24) and Proposition 3.4 ensure for all $n\in\mathbb{N}$ that

$$\begin{aligned}
&\sup_{t\in[0,T]}\big\|\max\{1,\|Y_t\,\mathbb{1}_{\Omega_n}\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}=\sup_{t\in[0,T]}\big\|\max\{1,\|X_t^n\,\mathbb{1}_{\Omega_n}\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}\\
&\leq\sup_{t\in[0,T]}\big\|\max\{1,\|X_t^n\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}\leq\sqrt{2}\,\big\|\max\{1,\|\xi_n\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}\\
&\cdot\mathscr{E}_{(1-\eta)}\Bigg[\tfrac{T^{1-\eta}\sqrt{2}}{\sqrt{1-\eta}}\bigg(\sup_{v\in H_\gamma}\tfrac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1,\|v\|_{H_\gamma}\}}\bigg)+\sqrt{T^{1-\eta}p(p-1)}\bigg(\sup_{v\in H_\gamma}\tfrac{\|B(v)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}}{\max\{1,\|v\|_{H_\gamma}\}}\bigg)\Bigg].
\end{aligned} \tag{3.27}$$

This and Fatou's lemma imply for all $t\in[0,T]$ that

$$\begin{aligned}
&\big\|\max\{1,\|Y_t\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}=\Big\|\liminf_{n\to\infty}\max\{1,\|Y_t\,\mathbb{1}_{\Omega_n}\|_{H_\gamma}\}\Big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}\\
&\leq\liminf_{n\to\infty}\big\|\max\{1,\|Y_t\,\mathbb{1}_{\Omega_n}\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}\leq\sqrt{2}\,\big\|\max\{1,\|\xi\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}\\
&\cdot\mathscr{E}_{(1-\eta)}\Bigg[\tfrac{T^{1-\eta}\sqrt{2}}{\sqrt{1-\eta}}\bigg(\sup_{v\in H_\gamma}\tfrac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1,\|v\|_{H_\gamma}\}}\bigg)+\sqrt{T^{1-\eta}p(p-1)}\bigg(\sup_{v\in H_\gamma}\tfrac{\|B(v)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}}{\max\{1,\|v\|_{H_\gamma}\}}\bigg)\Bigg].
\end{aligned} \tag{3.28}$$

The proof of Proposition 3.6 is thus completed. □

3.5 Uniqueness of left-continuous solutions of SEEs with semi-globally Lipschitz continuous coefficients

The proof of the next result, Proposition 3.7, is similar to the proof of Theorem 7.4 in Da Prato & Zabczyk [10] (also see, e.g., Lemma 8.2 in van Neerven, Veraar, & Weis [37] for an analogous result in a more general framework).

Proposition 3.7 (Local solutions)**.** *Consider the notation in Subsection 1.1, let* $(H,\langle\cdot,\cdot\rangle_H,\|\cdot\|_H)$ *and* $(U,\langle\cdot,\cdot\rangle_U,\|\cdot\|_U)$ *be separable* $\mathbb{R}$*-Hilbert spaces, let* $\mathbb{H}\subseteq H$ *be a non-empty orthonormal basis of* H*, let* $\lambda\colon\mathbb{H}\to\mathbb{R}$ *be a function with* $\sup_{h\in\mathbb{H}}\lambda_h<0$*, let* $A\colon D(A)\subseteq H\to H$ *be the linear operator which satisfies* $D(A)=\big\{v\in H\colon$

$\sum_{h\in\mathbb{H}} |\lambda_h \langle h, v\rangle_H|^2 < \infty\big\}$ *and* $\forall\, v \in D(A)\colon Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$, *let* $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, *be a family of interpolation spaces associated to* $-A$, *let* $T \in (0,\infty)$, $\gamma \in \mathbb{R}$, $\eta \in [0,1)$, $F \in C(H_\gamma, H_{\gamma-\eta})$, $B \in C(H_\gamma, \mathrm{HS}(U, H_{\gamma-\eta/2}))$ *satisfy for all bounded sets* $E \subseteq H_\gamma$ *that* $|F|_E|_{\mathscr{C}^1(E,\|\cdot\|_{H_{\gamma-\eta}})} + |B|_E|_{\mathscr{C}^1(E,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})} < \infty$, *let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space with a normal filtration* $(\mathscr{F}_t)_{t\in[0,T]}$, *let* $(W_t)_{t\in[0,T]}$ *be an* Id_U*-cylindrical* $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$*-Wiener process, let* $\tau_k\colon \Omega \to [0,T]$, $k \in \{1,2\}$, *be* $(\mathscr{F}_t)_{t\in[0,T]}$*-stopping times, and let* $X^k\colon [0,T]\times\Omega \to H_\gamma$, $k \in \{1,2\}$, *be* $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$*-adapted stochastic processes with left-continuous and bounded sample paths which satisfy for all* $k \in \{1,2\}$, $t \in [0,T]$ *that*

$$\begin{aligned}\big[X_t^k\, \mathbb{1}_{\{t\le\tau_k\}}\big]_{\mathbb{P},\mathscr{B}(H_\gamma)} = \Bigg(\bigg[e^{tA}X_0^k + \int_0^t \mathbb{1}_{\{s<\tau_k\}}\, e^{(t-s)A}F(X_s^k)\,\mathrm{d}s\bigg]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\ + \int_0^t \mathbb{1}_{\{s<\tau_k\}}\, e^{(t-s)A}B(X_s^k)\,\mathrm{d}W_s\Bigg)\mathbb{1}_{\{t\le\tau_k\}}.\end{aligned} \tag{3.29}$$

Then $\mathbb{P}\big(\forall\, t \in [0,T]\colon \mathbb{1}_{\{X_0^1=X_0^2\}}\, X^1_{\min\{t,\tau_1,\tau_2\}} = \mathbb{1}_{\{X_0^1=X_0^2\}}\, X^2_{\min\{t,\tau_1,\tau_2\}}\big) = 1$.

Corollary 3.8 is an immediate consequence from Proposition 3.7.

Corollary 3.8 (Continuous solutions)**.** *Consider the notation in Subsection 1.1, let* $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ *and* $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ *be separable* $\mathbb{R}$*-Hilbert spaces, let* $\mathbb{H} \subseteq H$ *be a non-empty orthonormal basis of* H, *let* $\lambda\colon \mathbb{H} \to \mathbb{R}$ *be a function with* $\sup_{h\in\mathbb{H}} \lambda_h < 0$, *let* $A\colon D(A) \subseteq H \to H$ *be the linear operator which satisfies* $D(A) = \big\{v \in H\colon \sum_{h\in\mathbb{H}} |\lambda_h \langle h, v\rangle_H|^2 < \infty\big\}$ *and* $\forall\, v \in D(A)\colon Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$, *let* $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, *be a family of interpolation spaces associated to* $-A$, *let* $T \in (0,\infty)$, $\gamma \in \mathbb{R}$, $\eta \in [0,1)$, $F \in C(H_\gamma, H_{\gamma-\eta})$, $B \in C(H_\gamma, \mathrm{HS}(U, H_{\gamma-\eta/2}))$ *satisfy for all bounded sets* $E \subseteq H_\gamma$ *that* $|F|_E|_{\mathscr{C}^1(E,\|\cdot\|_{H_{\gamma-\eta}})} + |B|_E|_{\mathscr{C}^1(E,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})} < \infty$, *let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space with a normal filtration* $(\mathscr{F}_t)_{t\in[0,T]}$, *let* $(W_t)_{t\in[0,T]}$ *be an* Id_U*-cylindrical* $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$*-Wiener process, and let* $X^k\colon [0,T]\times\Omega \to H_\gamma$, $k \in \{1,2\}$, *be* $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$*-adapted stochastic processes with continuous sample paths which satisfy for all* $k \in \{1,2\}$, $t \in [0,T]$ *that*

$$[X_t^k]_{\mathbb{P},\mathscr{B}(H_\gamma)} = \bigg[e^{tA}X_0^1 + \int_0^t e^{(t-s)A}F(X_s^k)\,\mathrm{d}s\bigg]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A}B(X_s^k)\,\mathrm{d}W_s. \tag{3.30}$$

Then $\mathbb{P}\big(\forall\, t \in [0,T]\colon X_t^1 = X_t^2\big) = 1$.

4 Convergence in Hölder norms for Galerkin approximations

4.1 Setting

Consider the notation in Subsection 1.1, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of H, let $T, \iota \in (0,\infty)$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$-Wiener process, let $\lambda\colon \mathbb{H} \to \mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}} \lambda_h < 0$, let $A\colon D(A) \subseteq H \to H$ be the linear operator which satisfies

$$D(A) = \left\{v \in H\colon \sum_{h\in\mathbb{H}} |\lambda_h \langle h, v\rangle_H|^2 < \infty\right\} \tag{4.1}$$

and which satisfies for all $v \in D(A)$ that

$$Av = \sum_{h \in \mathbb{H}} \lambda_h \langle h, v \rangle_H h, \tag{4.2}$$

let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, let $\gamma \in \mathbb{R}$, $\alpha \in [0, 1)$, $\beta \in [0, 1/2)$, $\chi \in [\beta, 1/2)$, $F \in C(H_\gamma, H_{\gamma-\alpha})$, $B \in C(H_\gamma, \mathrm{HS}(U, H_{\gamma-\beta}))$ satisfy for all bounded sets $E \subseteq H_\gamma$ that

$$|F|_E|_{\mathscr{C}^1(E, \|\cdot\|_{H_{\gamma-\alpha}})} + |B|_E|_{\mathscr{C}^1(E, \|\cdot\|_{\mathrm{HS}(U, H_{\gamma-\beta})})} < \infty, \tag{4.3}$$

let $\mathbb{H}_N \subseteq \mathbb{H}$, $N \in \mathbb{N}_0$, be sets which satisfy $\mathbb{H}_0 = \mathbb{H}$ and $\sup_{N \in \mathbb{N}} N^\iota \sup\bigl(\{1/|\lambda_h| \colon h \in \mathbb{H} \backslash \mathbb{H}_N\} \cup \{0\}\bigr) < \infty$, let $P_N \in L(H_{\min\{0, \gamma-1\}})$, $N \in \mathbb{N}_0$, and $\mathscr{P}_N \in L(U)$, $N \in \mathbb{N}_0$, be linear operators which satisfy for all $N \in \mathbb{N}_0$, $v \in H$ that

$$P_N(v) = \sum_{h \in \mathbb{H}_N} \langle h, v \rangle_H \, h, \tag{4.4}$$

and let $X^N \colon [0, T] \times \Omega \to H_\gamma$, $N \in \mathbb{N}_0$, be $(\mathscr{F}_t)_{t \in [0,T]}/\mathscr{B}(H_\gamma)$-adapted stochastic processes with continuous sample paths which satisfy for all $N \in \mathbb{N}_0$, $t \in [0, T]$ that

$$\bigl[X_t^N\bigr]_{\mathbb{P}, \mathscr{B}(H_\gamma)} = \left[e^{tA} P_N X_0^0 + \int_0^t e^{(t-s)A} P_N F(X_s^N) \, \mathrm{d}s\right]_{\mathbb{P}, \mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A} P_N B(X_s^N) \mathscr{P}_N \, \mathrm{d}W_s. \tag{4.5}$$

4.2 Strong convergence in Hölder norms for Galerkin approximations of SEEs with globally Lipschitz continuous nonlinearities

The next lemma, Lemma 4.1 below, follows directly from, e.g., Proposition 3.6 and, e.g., Corollary 3.8.

Lemma 4.1. *Assume the setting in Subsection 4.1, let $p \in [2, \infty)$, $\eta \in [\max\{\alpha, 2\beta\}, 1)$, $N \in \mathbb{N}_0$, and assume that*

$$\mathbb{E}\bigl[\|X_0^0\|_{H_\gamma}^p\bigr] + |F|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_{\gamma-\alpha}})} + |B|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{\mathrm{HS}(U, H_{\gamma-\beta})})} < \infty. \tag{4.6}$$

Then

$$\begin{aligned} &\sup_{t \in [0,T]} \bigl\|\max\{1, \|X_t^N\|_{H_\gamma}\}\bigr\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)} \leq \sqrt{2} \, \bigl\|\max\{1, \|X_0^0\|_{H_\gamma}\}\bigr\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)} \\ &\cdot \mathscr{E}_{(1-\eta)}\left[\tfrac{T^{1-\eta}\sqrt{2}}{\sqrt{1-\eta}}\left(\sup_{v \in H_\gamma} \tfrac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1, \|v\|_{H_\gamma}\}}\right) + \sqrt{T^{1-\eta} p(p-1)}\left(\sup_{v \in H_\gamma} \tfrac{\|B(v)\mathscr{P}_N\|_{\mathrm{HS}(U, H_{\gamma-\eta/2})}}{\max\{1, \|v\|_{H_\gamma}\}}\right)\right]. \end{aligned} \tag{4.7}$$

Lemma 4.2. *Assume the setting in Subsection 4.1, let $p \in [2, \infty)$, $\eta \in [\max\{\alpha, 2\beta\}, 1)$, $N \in \mathbb{N}_0$, and assume that*

$$\mathbb{E}\bigl[\|X_0^0\|_{H_\gamma}^p\bigr] + |F|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_{\gamma-\alpha}})} + |B|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{\mathrm{HS}(U, H_{\gamma-\beta})})} < \infty. \tag{4.8}$$

Then

$$\sup_{t\in[0,T]} \left\|X_t^0 - X_t^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \leq \Bigg[\sqrt{2}\sup_{t\in[0,T]} \left\|(P_0 - P_N)X_t^0\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}$$
$$+ \tfrac{T^{1/2-\chi}\sqrt{p\,(p-1)}}{\sqrt{1-2\chi}}\left(1 + \sup_{t\in[0,T]} \|X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\right)\left(\sup_{v\in H_\gamma} \frac{\|B(v)(\mathscr{P}_0 - \mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1 + \|v\|_{H_\gamma}}\right)\Bigg]$$
$$\cdot\, \mathscr{E}_{(1-\eta)}\left[\tfrac{T^{1-\eta}\sqrt{2}\,|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}\,|B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})}\,\|\mathscr{P}_0\|_{L(U)}\right] < \infty. \tag{4.9}$$

Proof of Lemma 4.2. First of all, observe that Lemma 4.1 ensures that

$$\sup_{t\in[0,T]} \max\big\{\|X_t^0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}, \|X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\big\} < \infty. \tag{4.10}$$

We can hence apply Proposition 3.5 to obtain that

$$\sup_{t\in[0,T]} \left\|X_t^0 - X_t^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}$$
$$\leq \mathscr{E}_{(1-\eta)}\left[\tfrac{T^{1-\eta}\sqrt{2}\,|P_N F(\cdot)|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}\,|P_N B(\cdot)\mathscr{P}_0|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})}\right]$$
$$\cdot\sqrt{2}\sup_{t\in[0,T]} \Bigg\|\left[X_t^0 - \int_0^t e^{(t-s)A}P_N F(X_s^0)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} - \int_0^t e^{(t-s)A}P_N B(X_s^0)\mathscr{P}_0\,\mathrm{d}W_s$$
$$+ \left[\int_0^t e^{(t-s)A}P_N F(X_s^N)\,\mathrm{d}s - X_t^N\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A}P_N B(X_s^N)\mathscr{P}_0\,\mathrm{d}W_s\Bigg\|_{L^p(\mathbb{P};\|\cdot\|_{H_\gamma})}. \tag{4.11}$$

This shows that

$$\sup_{t\in[0,T]} \left\|X_t^0 - X_t^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}$$
$$\leq \mathscr{E}_{(1-\eta)}\left[\tfrac{T^{1-\eta}\sqrt{2}\,|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}\,|B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})}\,\|\mathscr{P}_0\|_{L(U)}\right]$$
$$\cdot\sqrt{2}\sup_{t\in[0,T]} \left\|\left[(P_0 - P_N)X_t^0\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A}P_N B(X_s^N)(\mathscr{P}_0 - \mathscr{P}_N)\,\mathrm{d}W_s\right\|_{L^p(\mathbb{P};\|\cdot\|_{H_\gamma})}. \tag{4.12}$$

The Burkholder-Davis-Gundy-type inequality in Lemma 7.7 in Da Prato & Zabczyk [10] hence implies that

$$\sup_{t\in[0,T]} \left\|X_t^0 - X_t^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}$$
$$\leq \mathscr{E}_{(1-\eta)}\left[\tfrac{T^{1-\eta}\sqrt{2}\,|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}\,|B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})}\,\|\mathscr{P}_0\|_{L(U)}\right]$$
$$\cdot\sqrt{2}\Bigg[\sup_{t\in[0,T]} \left\|(P_0 - P_N)X_t^0\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \tag{4.13}$$
$$+ \sup_{s\in[0,T]} \left\|B(X_s^N)[\mathscr{P}_0 - \mathscr{P}_N]\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\chi})})}\sqrt{\tfrac{p\,(p-1)}{2}\sup_{t\in[0,T]}\int_0^t (t-s)^{-2\chi}\,\mathrm{d}s}\,\Bigg].$$

This and (4.10) complete the proof of Lemma 4.2. □

Corollary 4.3. *Assume the setting in Subsection 4.1, let* $\vartheta \in [0, \min\{1-\alpha, 1/2-\beta\})$, $p \in [2,\infty)$, *and assume that* $X_0^0(\Omega) \subseteq H_{\gamma+\vartheta}$ *and*

$$\mathbb{E}\big[\|X_0^0\|_{H_{\gamma+\vartheta}}^p\big] + |F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\alpha}})} + |B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\beta})})} < \infty, \tag{4.14}$$

$$\sup_{N\in\mathbb{N}} \sup_{v\in H_\gamma} \left[\frac{N^{\iota\vartheta}\,\|B(v)(\mathscr{P}_0-\mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\right] < \infty. \tag{4.15}$$

Then it holds that

$$\sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \big(\|F(X_t^N)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\alpha}})} + \|B(X_t^N)\mathscr{P}_N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\chi})})}\big) < \infty \tag{4.16}$$

and

$$\sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \big(N^{\iota\vartheta}\,\|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\big) < \infty. \tag{4.17}$$

Proof of Corollary 4.3. Combining the assumptions that $X_0^0(\Omega) \subseteq H_{\gamma+\vartheta}$ and $\mathbb{E}\big[\|X_0^0\|_{H_{\gamma+\vartheta}}^p\big] < \infty$ with, e.g., Proposition 3.6 and, e.g., Corollary 3.8 ensures that $\forall\, t\in[0,T]\colon \mathbb{P}\big(X_t^0 \in H_{\gamma+\vartheta}\big) = 1$ and $\sup_{t\in[0,T]} \mathbb{E}\big[\|\mathbb{1}_{\{X_t^0\in H_{\gamma+\vartheta}\}} X_t^0\|_{H_{\gamma+\vartheta}}^p\big] < \infty$. This and the assumption that $\sup_{N\in\mathbb{N}} N^\iota \sup\big(\{1/|\lambda_h|\colon h\in\mathbb{H}\backslash\mathbb{H}_N\}\cup\{0\}\big) < \infty$ imply that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \Big[N^{\iota\vartheta}\,\big\|(P_0-P_N)X_t^0\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\Big] \\
&\leq \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \Big[N^{\iota\vartheta}\,\big\|(-A)^{-\vartheta}(P_0|_{H_\gamma} - P_N|_{H_\gamma})\big\|_{L(H_\gamma)}\,\big\|\mathbb{1}_{\{X_t^0\in H_{\gamma+\vartheta}\}} X_t^0\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})}\Big] \\
&\leq \Big[\sup_{N\in\mathbb{N}} N^{\iota\vartheta}\,\big\|(-A)^{-1}(\mathrm{Id}_{H_\gamma} - P_N|_{H_\gamma})\big\|_{L(H_\gamma)}^{\vartheta}\Big]\Big[\sup_{t\in[0,T]} \big\|\mathbb{1}_{\{X_t^0\in H_{\gamma+\vartheta}\}} X_t^0\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})}\Big] \qquad (4.18)\\
&= \Big[\sup_{N\in\mathbb{N}} N^{\iota\vartheta}\,\big[\sup\big(\{1/|\lambda_h|\colon h\in\mathbb{H}\backslash\mathbb{H}_N\}\cup\{0\}\big)\big]^{\vartheta}\Big]\Big[\sup_{t\in[0,T]} \big\|\mathbb{1}_{\{X_t^0\in H_{\gamma+\vartheta}\}} X_t^0\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})}\Big] \\
&< \infty.
\end{aligned}$$

In addition, observe that (4.14), (4.15), and Lemma 4.1 ensure that

$$\sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \|X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} < \infty. \tag{4.19}$$

The triangle inequality and again (4.15) hence prove that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \|B(X_t^N)\mathscr{P}_N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\chi})})} \\
&\leq \Big(1 + \sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \|X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\Big)\Big(\sup_{N\in\mathbb{N}_0} \sup_{v\in H_\gamma} \tfrac{\|B(v)\mathscr{P}_N\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\Big) \\
&\leq \Big(1 + \sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \|X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\Big) \qquad (4.20)\\
&\quad\cdot \Big(\sup_{v\in H_\gamma} \tfrac{\|B(v)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}\|\mathscr{P}_0\|_{L(U)}}{1+\|v\|_{H_\gamma}} + \sup_{N\in\mathbb{N}_0} \sup_{v\in H_\gamma} \tfrac{\|B(v)(\mathscr{P}_0-\mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\Big) < \infty.
\end{aligned}$$

In the next step we combine (4.19), (4.18), and (4.15) with Lemma 4.2 to obtain that

$$\sup\nolimits_{N\in\mathbb{N}_0} \sup\nolimits_{t\in[0,T]} \big(N^{\iota\vartheta}\,\|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\big) < \infty. \tag{4.21}$$

Furthermore, observe that (4.19) assures that $\sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \|F(X_t^N)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\alpha}})} < \infty$. This, (4.20), and (4.21) complete the proof of Corollary 4.3. □

The next result, Corollary 4.4, proves strong convergence rates in Hölder norms for spatial spectral Galerkin approximations of SEEs with globally Lipschitz continuous nonlinearities. Note in the setting of Corollary 4.4 that, e.g., Becker et al. [5, Theorem 1.1 and Lemma 2.6] show in the case $\iota = 2$, $\delta = 0$ that the convergence rate established in (4.23) is essentially sharp (cf., e.g., Conus, Jentzen, & Kurniawan [6, Lemma 7.2]).

Corollary 4.4. *Assume the setting in Subsection 4.1, let $\vartheta \in (0, \min\{1-\alpha, 1/2-\beta\})$, $p \in (1/\vartheta, \infty)$, and assume that $X_0^0(\Omega) \subseteq H_{\gamma+\vartheta}$, $\mathbb{E}\big[\|X_0^0\|_{H_{\gamma+\vartheta}}^p\big] < \infty$, $|F|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_{\gamma-\alpha}})} < \infty$, $|B|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\beta})})} < \infty$, and*

$$\sup_{N\in\mathbb{N}} \sup_{v\in H_\gamma} \left[\frac{\|B(v)\mathscr{P}_N\|_{\mathrm{HS}(U,H_{\gamma-\beta})} + N^{\iota\vartheta}\, \|B(v)(\mathscr{P}_0 - \mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1 + \|v\|_{H_\gamma}} \right] < \infty. \tag{4.22}$$

Then it holds for all $\delta \in [0, \vartheta - 1/p)$, $\varepsilon \in (0,\infty)$ that

$$\sup_{N\in\mathbb{N}} \left[\mathbb{E}\Big[\|X^N\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}^p\Big] + N^{\iota\,(\vartheta-\delta-1/p-\varepsilon)} \Big(\mathbb{E}\Big[\|X^0 - X^N\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}^p\Big]\Big)^{1/p} \right] < \infty. \tag{4.23}$$

Proof of Corollary 4.4. Throughout this proof let $\eta \in \mathbb{R}$ be the real number given by $\eta = \max\{\alpha, 2\beta\}$ and let $\theta^N \in \mathscr{P}_T$, $N \in \mathbb{N}$, be a sequence of sets such that

$$\sup_{N\in\mathbb{N}} \left[\frac{d_{\max}(\theta^N)}{N^{-\iota}} + \frac{N^{-\iota}}{d_{\min}(\theta^N)} \right] < \infty. \tag{4.24}$$

In particular, this ensures that $\limsup_{N\to\infty} d_{\max}(\theta^N) = 0$. In addition, Corollary 4.3 proves that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \left[\big|d_{\max}(\theta^N)\big|^{-\vartheta} \sup_{t\in\theta^N} \|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \right] \\
&\leq \left[\sup_{N\in\mathbb{N}} \tfrac{|d_{\max}(\theta^N)|^{-\vartheta}}{N^{\iota\vartheta}} \right] \left(\sup_{N\in\mathbb{N}} \sup_{t\in\theta^N} N^{\iota\vartheta}\, \|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \right) \\
&\leq \left[\sup_{N\in\mathbb{N}} \tfrac{N^{-\iota}}{d_{\min}(\theta^N)} \right]^\vartheta \left(\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} N^{\iota\vartheta}\, \|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \right) < \infty.
\end{aligned} \tag{4.25}$$

Next note that, e.g., Corollary 3.2 shows for all $N \in \mathbb{N}_0$, $\varepsilon \in (0, \min\{1-\eta, 1/2-\beta\})$ that

$$\begin{aligned}
&\sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{|\min\{t_1,t_2\}|^{\max\{\gamma+\varepsilon-(\gamma+\vartheta),0\}}\, \|X_{t_1}^N - X_{t_2}^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}}{|t_1 - t_2|^\varepsilon} \right) \\
&\leq \|X_0^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\gamma+\vartheta,\gamma+\varepsilon\}}})} + \left[\sup_{s\in[0,T]} \big\|P_N F(X_s^N)\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \frac{2\,T^{(1+\gamma-\eta-\min\{\gamma+\vartheta,\gamma+\varepsilon\})}}{(1-\eta-\varepsilon)} \\
&\quad + \left[\sup_{s\in[0,T]} \big\|P_N B(X_s^N)\mathscr{P}_N\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\beta})})} \right] \frac{\sqrt{p\,(p-1)}\,T^{(1/2+\gamma-\beta-\min\{\gamma+\vartheta,\gamma+\varepsilon\})}}{(1-2\beta-2\varepsilon)^{1/2}} < \infty.
\end{aligned} \tag{4.26}$$

This and the fact that $\min\{1-\eta, 1/2-\beta\} = \min\{1-\max\{\alpha,2\beta\}, 1/2-\beta\} = \min\{1-$

$\alpha, 1/2 - \beta\} > \vartheta > 0$ imply that

$$
\begin{aligned}
&\sup_{N\in\mathbb{N}_0} \sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{\|X^N_{t_1} - X^N_{t_2}\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}}{|t_1 - t_2|^\vartheta} \right) \\
&\leq \sup_{N\in\mathbb{N}_0} \|X^N_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})} + \left[\sup_{N\in\mathbb{N}_0} \sup_{s\in[0,T]} \|F(X^N_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \frac{2\,T^{(1-\eta-\vartheta)}}{(1-\eta-\vartheta)} \\
&\quad + \left[\sup_{N\in\mathbb{N}_0} \sup_{s\in[0,T]} \|B(X^N_s)\mathscr{P}_N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\beta})})} \right] \frac{\sqrt{p\,(p-1)}\,T^{(1/2-\beta-\vartheta)}}{(1-2\beta-2\vartheta)^{1/2}}.
\end{aligned}
\tag{4.27}
$$

Corollary 4.3 and estimate (4.22) hence prove that

$$
\begin{aligned}
&\sup_{N\in\mathbb{N}_0} |X^N|_{\mathscr{C}^\vartheta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})})} \\
&\leq \|X^0_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})} + \left[\sup_{N\in\mathbb{N}_0} \sup_{s\in[0,T]} \|F(X^N_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \frac{2\,T^{(1-\eta-\vartheta)}}{(1-\eta-\vartheta)} \\
&\quad + \left[\sup_{N\in\mathbb{N}_0} \sup_{s\in[0,T]} \|B(X^N_s)\mathscr{P}_N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\beta})})} \right] \frac{\sqrt{p\,(p-1)}\,T^{(1/2-\beta-\vartheta)}}{(1-2\beta-2\vartheta)^{1/2}} < \infty.
\end{aligned}
\tag{4.28}
$$

This, (4.25), and the fact that $\vartheta \in (1/p, 1]$ allow us to apply Corollary 2.11 to obtain for all $\delta \in [0, \vartheta - 1/p)$, $\varepsilon \in (0,\infty)$ that

$$
\begin{aligned}
\sup_{N\in\mathbb{N}} \Bigg[&\mathbb{E}\Big[\|X^N\|^p_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}\Big] \\
&+ \big|d_{\max}(\theta^N)\big|^{-(\vartheta-\delta-1/p-\varepsilon)} \Big(\mathbb{E}\Big[\|X^0 - X^N\|^p_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}\Big] \Big)^{1/p} \Bigg] < \infty.
\end{aligned}
\tag{4.29}
$$

Combining this with the fact that $\sup_{N\in\mathbb{N}}\big[\frac{d_{\max}(\theta^N)}{N^{-\iota}}\big] < \infty$ completes the proof of Corollary 4.4. □

4.3 Almost sure convergence in Hölder norms for Galerkin approximations of SEEs with semi-globally Lipschitz continuous nonlinearities

The proof of the following corollary employs a standard localisation argument; see, e.g., [13, 33].

Corollary 4.5. *Assume the setting in Subsection 4.1, let $\vartheta \in (0, \min\{1-\alpha, 1/2 - \beta\})$, assume that $\mathbb{P}\big(X^0_0 \in H_{\gamma+\vartheta}\big) = 1$, and assume for all non-empty bounded sets $E \subseteq H_\gamma$ that*

$$
\sup_{N\in\mathbb{N}} \sup_{v\in E} \left[\frac{\|B(v)\mathscr{P}_N\|_{\mathrm{HS}(U,H_{\gamma-\beta})} + N^{\iota\vartheta}\,\|B(v)(\mathscr{P}_0 - \mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1 + \|v\|_{H_\gamma}} \right] < \infty. \tag{4.30}
$$

Then it holds for all $\delta \in [0,\vartheta)$, $\varepsilon \in (0,\infty)$ that

$$
\mathbb{P}\bigg(\sup_{N\in\mathbb{N}} \big[N^{\iota(\vartheta-\delta-\varepsilon)}\, \|X^0 - X^N\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} \big] < \infty \bigg) = 1. \tag{4.31}
$$

Proof of Corollary 4.5. Throughout this proof we assume w.l.o.g. that $X_0^0(\Omega) \subseteq H_{\gamma+\vartheta}$, let $\delta \in [0, \vartheta)$, let $\phi_{r,M} \colon H_r \to H_r$, $r \in \mathbb{R}$, $M \in (0,\infty)$, be the mappings which satisfy for all $r \in \mathbb{R}$, $M \in (0,\infty)$, $v \in H_r$ that

$$\phi_{r,M}(v) = v \cdot \min\left\{1, \frac{M+1}{1 + \|v\|_{H_r}}\right\}, \tag{4.32}$$

let $\xi_M \colon \Omega \to H_\gamma$, $M \in \mathbb{N}$, be the mappings which satisfy for all $M \in \mathbb{N}$ that $\xi_M = \phi_{\gamma+\vartheta,M}(X_0^0)$, let $F_M \colon H_\gamma \to H_{\gamma-\alpha}$, $M \in \mathbb{N}$, and $B_M \colon H_\gamma \to \mathrm{HS}(U, H_{\gamma-\beta})$, $M \in \mathbb{N}$, be the mappings which satisfy for all $M \in \mathbb{N}$ that $F_M = F \circ \phi_{\gamma,M}$ and $B_M = B \circ \phi_{\gamma,M}$, and let $S_M \subseteq H_\gamma$, $M \in \mathbb{N}$, be the sets which satisfy for all $M \in \mathbb{N}$ that $S_M = \{v \in H_\gamma \colon \|v\|_{H_\gamma} \leq M+1\}$. Observe that it holds for all $v, w \in H_\gamma$, $M \in \mathbb{N}$ that

$$\begin{aligned}
&\|\phi_{\gamma,M}(v) - \phi_{\gamma,M}(w)\|_{H_\gamma} \\
&= \left\| \frac{v\,(1 + \|w\|_{H_\gamma}) \min\{1 + \|v\|_{H_\gamma}, M+1\} - w\,(1 + \|v\|_{H_\gamma}) \min\{1 + \|w\|_{H_\gamma}, M+1\}}{(1 + \|v\|_{H_\gamma})\,(1 + \|w\|_{H_\gamma})} \right\|_{H_\gamma} \\
&\leq \|v - w\|_{H_\gamma} \\
&+ \left\| \frac{w\,\big[(1 + \|w\|_{H_\gamma}) \min\{1 + \|v\|_{H_\gamma}, M+1\} - (1 + \|v\|_{H_\gamma}) \min\{1 + \|w\|_{H_\gamma}, M+1\}\big]}{(1 + \|v\|_{H_\gamma})\,(1 + \|w\|_{H_\gamma})} \right\|_{H_\gamma} \\
&\leq \|v - w\|_{H_\gamma} \\
&\quad + \frac{\big|(1 + \|w\|_{H_\gamma}) \min\{1 + \|v\|_{H_\gamma}, M+1\} - (1 + \|v\|_{H_\gamma}) \min\{1 + \|w\|_{H_\gamma}, M+1\}\big|}{(1 + \|v\|_{H_\gamma})}.
\end{aligned} \tag{4.33}$$

This ensures for all $v, w \in H_\gamma$, $M \in \mathbb{N}$ that

$$\begin{aligned}
&\|\phi_{\gamma,M}(v) - \phi_{\gamma,M}(w)\|_{H_\gamma} \\
&\leq \|v - w\|_{H_\gamma} + \frac{\big|\|w\|_{H_\gamma} - \|v\|_{H_\gamma}\big| \min\{1 + \|v\|_{H_\gamma}, M+1\}}{(1 + \|v\|_{H_\gamma})} \\
&\quad + \frac{(1 + \|v\|_{H_\gamma})\,\big|\min\{1 + \|v\|_{H_\gamma}, M+1\} - \min\{1 + \|w\|_{H_\gamma}, M+1\}\big|}{(1 + \|v\|_{H_\gamma})} \\
&\leq \|v - w\|_{H_\gamma} + \big|\|w\|_{H_\gamma} - \|v\|_{H_\gamma}\big| \\
&\quad + \big|\min\{1 + \|v\|_{H_\gamma}, M+1\} - \min\{1 + \|w\|_{H_\gamma}, M+1\}\big| \\
&\leq 3\,\|v - w\|_{H_\gamma}.
\end{aligned} \tag{4.34}$$

Hence, we obtain for all $M \in \mathbb{N}$ that $|\phi_{\gamma,M}|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_\gamma})} \leq 3$. This, the fact that $\forall\, M \in \mathbb{N} \colon |F|_{S_M}|_{\mathscr{C}^1(S_M, \|\cdot\|_{H_{\gamma-\alpha}})} + |B|_{S_M}|_{\mathscr{C}^1(S_M, \|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\beta})})} + |\phi_{\gamma,M}|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_\gamma})} < \infty$, and the fact that $\forall\, M \in \mathbb{N} \colon \phi_{\gamma,M}(H_\gamma) \subseteq S_M$ ensure that it holds for all $M \in \mathbb{N}$, $p \in [1,\infty)$ that

$$|F_M|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_{\gamma-\alpha}})} + |B_M|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\beta})})} + \mathbb{E}\big[\|\xi_M\|_{H_{\gamma+\vartheta}}^p\big] < \infty. \tag{4.35}$$

E.g., Proposition 3.6 hence proves that there exist $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-adapted stochastic processes $\mathscr{X}^{N,M} \colon [0,T] \times \Omega \to H_\gamma$, $N \in \mathbb{N}_0$, $M \in \mathbb{N}$, with continuous sample paths such that it holds for all $N \in \mathbb{N}_0$, $M \in \mathbb{N}$, $t \in [0,T]$ that

$$\begin{aligned}
\big[\mathscr{X}_t^{N,M}\big]_{\mathbb{P},\mathscr{B}(H_\gamma)} = \bigg[e^{tA} P_N \xi_M &+ \int_0^t e^{(t-s)A} P_N F_M(\mathscr{X}_s^{N,M})\,\mathrm{d}s \bigg]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\
&+ \int_0^t e^{(t-s)A} P_N B_M(\mathscr{X}_s^{N,M}) \mathscr{P}_N \,\mathrm{d}W_s
\end{aligned} \tag{4.36}$$

(cf., e.g., Theorem 7.1 in van Neerven, Veraar, & Weis [37]). We now introduce a bit more notation. Let $\tau_{N,M} \colon \Omega \to [0,T]$, $M \in \mathbb{N}$, $N \in \mathbb{N}_0$, be the mappings which satisfy for all $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ that

$$\tau_{N,M} = \min\Big\{T\,\mathbb{1}_{\{\|X_0^0\|_{H_{\gamma+\vartheta}} \leq M\}}, \inf\Big(\big\{t \in [0,T] \colon \|\mathscr{X}_t^{N,M}\|_{H_\gamma} \geq M\big\} \cup \{T\}\Big)\Big\}, \tag{4.37}$$

let $\Upsilon \in \mathscr{F}$ be the set given by

$$\begin{aligned} &\Upsilon = \\ &\Big[\cap_{N\in\mathbb{N}_0} \cup_{M\in\mathbb{N}} \cap_{m\in\{M,M+1,\ldots\}} \{\tau_{N,m} = T\}\Big] \cap \Big[\cap_{M\in\mathbb{N},N\in\mathbb{N}_0} \Big\{\|\mathscr{X}^{N,M}\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} < \infty\Big\}\Big] \\ &\cap \Big[\cap_{M\in\mathbb{N},N\in\mathbb{N}_0} \Big(\big\{\|X_0^0\|_{H_{\gamma+\vartheta}} > M\big\} \cup \big\{\forall\, t \in [0,T] \colon \mathscr{X}^{N,M}_{\min\{t,\tau_{N,M}\}} = X^N_{\min\{t,\tau_{N,M}\}}\big\}\Big)\Big] \\ &\cap \Big[\cap_{M,n\in\mathbb{N}} \Big\{\sup\nolimits_{N\in\mathbb{N}} \big(N^{\iota(\vartheta-\delta-1/n)}\, \|\mathscr{X}^{0,M} - \mathscr{X}^{N,M}\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}\big) < \infty\Big\}\Big], \end{aligned} \tag{4.38}$$

let $\mathscr{M} \colon \Upsilon \to \mathbb{N}$ be the mapping which satisfies for all $\omega \in \Upsilon$ that

$$\mathscr{M}(\omega) = \min\big\{M \in \mathbb{N} \cap \big(\|X_0^0(\omega)\|_{H_{\gamma+\vartheta}}, \infty\big) \colon \forall\, m \in \{M, M+1, \ldots\} \colon \tau_{0,m}(\omega) = T\big\}, \tag{4.39}$$

and let $\mathscr{N} \colon \Upsilon \to \mathbb{N}$ be the mapping which satisfies for all

$$\omega \in \Upsilon \subseteq \Big\{\mathfrak{w} \in \Omega \colon \Big[\forall\, M \in \mathbb{N} \colon \limsup\nolimits_{N\to\infty} \|\mathscr{X}^{0,M}(\mathfrak{w}) - \mathscr{X}^{N,M}(\mathfrak{w})\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} = 0\Big]\Big\}$$

that

$$\mathscr{N}(\omega) = \min\Big\{N \in \mathbb{N} \colon \sup\nolimits_{n\in\{N,N+1,\ldots\}} \|\mathscr{X}^{0,2\mathscr{M}(\omega)}(\omega) - \mathscr{X}^{n,2\mathscr{M}(\omega)}(\omega)\|_{C([0,T],\|\cdot\|_{H_\gamma})} < 1\Big\}. \tag{4.40}$$

Observe that (4.38) ensures for all $\omega \in \Upsilon$, $N \in \mathbb{N}_0$, $M \in \mathbb{N}$, $t \in [0, \tau_{N,M}(\omega)]$ with $M \geq \|X_0^0(\omega)\|_{H_{\gamma+\vartheta}}$ that

$$\mathscr{X}_t^{N,M}(\omega) = X_t^N(\omega). \tag{4.41}$$

This, the fact that $\forall\, \omega \in \Upsilon, N \in \mathbb{N}_0 \colon \exists\, M \in \mathbb{N} \colon \forall\, m \in \{M, M+1, \ldots\} \colon \tau_{N,m}(\omega) = T$, and the fact that $\forall\, \omega \in \Upsilon, N \in \mathbb{N}_0, m \in \mathbb{N} \colon \|\mathscr{X}^{N,m}(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} < \infty$ prove that it holds for all $\omega \in \Upsilon$, $N \in \mathbb{N}_0$ that

$$\|X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} < \infty. \tag{4.42}$$

Next note that (4.39) ensures for all $\omega \in \Upsilon$, $M \in \{\mathscr{M}(\omega), \mathscr{M}(\omega)+1, \ldots\}$ that

$$\tau_{0,M}(\omega) = T \qquad \text{and} \qquad M \geq \mathscr{M}(\omega) > \|X_0^0(\omega)\|_{H_{\gamma+\vartheta}}. \tag{4.43}$$

This and (4.41) show for all $\omega \in \Upsilon$, $M \in \{\mathscr{M}(\omega), \mathscr{M}(\omega)+1, \ldots\}$, $t \in [0,T]$ that

$$\mathscr{X}_t^{0,M}(\omega) = X_t^0(\omega) = \mathscr{X}_t^{0,\mathscr{M}(\omega)}(\omega). \tag{4.44}$$

This, (4.43), and (4.37) prove for all $\omega \in \Upsilon$ that

$$\sup\nolimits_{t\in[0,T]} \|\mathscr{X}_t^{0,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} = \sup\nolimits_{t\in[0,T]} \|\mathscr{X}_t^{0,\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} \leq \mathscr{M}(\omega). \tag{4.45}$$

The triangle inequality and (4.40) hence assure for all $\omega \in \Upsilon$, $N \in \{\mathscr{N}(\omega), \mathscr{N}(\omega)+1, \ldots\}$ that

$$\begin{aligned} &\sup\nolimits_{t\in[0,T]} \|\mathscr{X}_t^{N,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} \\ &\leq \sup\nolimits_{t\in[0,T]} \|\mathscr{X}_t^{0,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} + \sup\nolimits_{t\in[0,T]} \|\mathscr{X}_t^{0,2\mathscr{M}(\omega)}(\omega) - \mathscr{X}_t^{N,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} \\ &< \sup\nolimits_{t\in[0,T]} \|\mathscr{X}_t^{0,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} + 1 \leq \mathscr{M}(\omega) + 1 \leq 2\mathscr{M}(\omega). \end{aligned} \tag{4.46}$$

This and the fact that $\forall \omega \in \Upsilon\colon \|X_0^0(\omega)\|_{H_{\gamma+\vartheta}} < \mathscr{M}(\omega) \leq 2\mathscr{M}(\omega)$ prove for all $\omega \in \Upsilon$, $N \in \{\mathscr{N}(\omega), \mathscr{N}(\omega)+1, \ldots\}$ that $\tau_{N,2\mathscr{M}(\omega)}(\omega) = T$. Again the fact that $\forall \omega \in \Upsilon\colon \|X_0^0(\omega)\|_{H_{\gamma+\vartheta}} < \mathscr{M}(\omega) \leq 2\mathscr{M}(\omega)$ and (4.41) hence show for all $\omega \in \Upsilon$, $N \in \{\mathscr{N}(\omega), \mathscr{N}(\omega)+1, \ldots\}$, $t \in [0,T]$ that $\mathscr{X}_t^{N,2\mathscr{M}(\omega)}(\omega) = X_t^N(\omega)$. This and (4.44) prove for all $\omega \in \Upsilon$, $\varepsilon \in (0,\infty)$ that

$$\begin{aligned}
&\sup\nolimits_{N\in\mathbb{N}} N^{\iota(\vartheta-\delta-\varepsilon)} \, \|X^0(\omega) - X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} \\
&\leq \sup\nolimits_{N\in\{1,2,\ldots,\mathscr{N}(\omega)\}} N^{\iota\vartheta} \, \|X^0(\omega) - X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} \\
&\quad + \sup\nolimits_{N\in\{\mathscr{N}(\omega),\mathscr{N}(\omega)+1,\ldots\}} N^{\iota(\vartheta-\delta-\varepsilon)} \, \|X^0(\omega) - X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} \\
&\leq [\mathscr{N}(\omega)]^{\iota\vartheta} \Big[\|X^0(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} + \sup\nolimits_{N\in\{1,2,\ldots,\mathscr{N}(\omega)\}} \|X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}\Big] \\
&\quad + \sup\nolimits_{N\in\{\mathscr{N}(\omega),\mathscr{N}(\omega)+1,\ldots\}} N^{\iota(\vartheta-\delta-\varepsilon)} \, \|\mathscr{X}^{0,2\mathscr{M}(\omega)}(\omega) - \mathscr{X}^{N,2\mathscr{M}(\omega)}(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}.
\end{aligned} \tag{4.47}$$

Combining this with (4.42) and (4.38) ensures for all $\omega \in \Upsilon$, $\varepsilon \in (0,\infty)$ that

$$\begin{aligned}
&\sup\nolimits_{N\in\mathbb{N}} N^{\iota(\vartheta-\delta-\varepsilon)} \, \|X^0(\omega) - X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} \\
&\leq \; [\mathscr{N}(\omega)]^{\iota\vartheta} \textstyle\sum_{N=0}^{\mathscr{N}(\omega)} \|X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} \\
&\quad + \sup\nolimits_{N\in\{\mathscr{N}(\omega),\mathscr{N}(\omega)+1,\ldots\}} N^{\iota(\vartheta-\delta-\varepsilon)} \, \|\mathscr{X}^{0,2\mathscr{M}(\omega)}(\omega) - \mathscr{X}^{N,2\mathscr{M}(\omega)}(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} < \infty.
\end{aligned} \tag{4.48}$$

It thus remains to prove that $\mathbb{P}(\Upsilon) = 1$ to complete the proof of Corollary 4.5. For this observe that the assumption (4.30) shows for all $M \in \mathbb{N}$ that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \sup_{v\in H_\gamma} \left[\frac{\|B_M(v)\mathscr{P}_N\|_{\mathrm{HS}(U,H_{\gamma-\beta})} + N^{\iota\vartheta} \, \|B_M(v)(\mathscr{P}_0 - \mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1 + \|v\|_{H_\gamma}}\right] \\
&\leq \sup_{N\in\mathbb{N}} \sup_{v\in S_M} \left[\frac{\|B(v)\mathscr{P}_N\|_{\mathrm{HS}(U,H_{\gamma-\beta})} + N^{\iota\vartheta} \, \|B(v)(\mathscr{P}_0 - \mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1 + \|v\|_{H_\gamma}}\right] < \infty.
\end{aligned} \tag{4.49}$$

Corollary 4.4 hence proves for all $p \in (1/\vartheta, \infty)$, $r \in [0, \vartheta - 1/p)$, $\varepsilon \in (0,\infty)$, $M \in \mathbb{N}$ that

$$\sup_{N\in\mathbb{N}} \left[\mathbb{E}\Big[\|\mathscr{X}^{N,M}\|^p_{\mathscr{C}^r([0,T],\|\cdot\|_{H_\gamma})}\Big] + N^{\iota\,(\vartheta-r-\varepsilon)}\Big(\mathbb{E}\Big[\|\mathscr{X}^{0,M} - \mathscr{X}^{N,M}\|^p_{\mathscr{C}^r([0,T],\|\cdot\|_{H_\gamma})}\Big]\Big)^{1/p}\right] < \infty. \tag{4.50}$$

A standard Borel-Cantelli-type argument (see, e.g., Lemma 2.1 in Kloeden & Neuenkirch [28]) hence ensures for all $\varepsilon \in (0,\infty)$, $M \in \mathbb{N}$ that

$$\mathbb{P}\Big(\sup\nolimits_{N\in\mathbb{N}} \big(N^{\iota(\vartheta-\delta-\varepsilon)} \|\mathscr{X}^{0,M} - \mathscr{X}^{N,M}\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}\big) < \infty\Big) = 1. \tag{4.51}$$

Hence, we obtain that

$$\mathbb{P}\Big(\forall\, M, n \in \mathbb{N}\colon \sup\nolimits_{N\in\mathbb{N}} \big[N^{\iota(\vartheta-\delta-1/n)} \, \|\mathscr{X}^{0,M} - \mathscr{X}^{N,M}\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}\big] < \infty\Big) = 1. \tag{4.52}$$

In addition, (4.50) proves for all $N \in \mathbb{N}_0$, $M \in \mathbb{N}$ that $\mathbb{P}\big(\mathscr{X}^{N,M} \in \mathscr{C}^\delta([0,T], \|\cdot\|_{H_\gamma})\big) = 1$. This, in turn, ensures that

$$\mathbb{P}\Big(\forall\, M \in \mathbb{N}, N \in \mathbb{N}_0\colon \mathscr{X}^{N,M} \in \mathscr{C}^\delta([0,T], \|\cdot\|_{H_\gamma})\Big) = 1. \tag{4.53}$$

Next observe that it holds for all $t \in [0,T]$, $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ that

$$\big[\mathscr{X}_t^{N,M} - e^{tA} P_N \mathscr{X}_0^{0,M}\big]_{\mathbb{P},\mathscr{B}(H_\gamma)} \mathbb{1}_{\{t\leq\tau_{N,M}\}}$$

$$
\begin{aligned}
&= \bigg(\Big[\int_0^t e^{(t-s)A} P_N F_M(\mathscr{X}_s^{N,M}) \, \mathrm{d}s \Big]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A} P_N B_M(\mathscr{X}_s^{N,M}) \mathscr{P}_N \, \mathrm{d}W_s \bigg) \mathbb{1}_{\{t \le \tau_{N,M}\}} \\
&= \bigg(\Big[\int_0^t \mathbb{1}_{\{s<\tau_{N,M}\}} \, e^{(t-s)A} P_N F_M(\mathscr{X}_s^{N,M}) \, \mathrm{d}s \Big]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\
&\quad + \int_0^t \mathbb{1}_{\{s<\tau_{N,M}\}} e^{(t-s)A} P_N B_M(\mathscr{X}_s^{N,M}) \mathscr{P}_N \, \mathrm{d}W_s \bigg) \mathbb{1}_{\{t \le \tau_{N,M}\}} \\
&= \bigg(\Big[\int_0^t \mathbb{1}_{\{s<\tau_{N,M}\}} \, e^{(t-s)A} P_N F(\mathscr{X}_s^{N,M}) \, \mathrm{d}s \Big]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\
&\quad + \int_0^t \mathbb{1}_{\{s<\tau_{N,M}\}} \, e^{(t-s)A} P_N B(\mathscr{X}_s^{N,M}) \mathscr{P}_N \, \mathrm{d}W_s \bigg) \mathbb{1}_{\{t \le \tau_{N,M}\}}.
\end{aligned}
\tag{4.54}
$$

E.g., Proposition 3.7 hence shows for all $N \in \mathbb{N}_0$, $M \in \mathbb{N}$ that

$$
\mathbb{P}\Big(\forall\, t \in [0,T] \colon \mathbb{1}_{\{\mathscr{X}_0^{N,M} = X_0^N\}} \mathscr{X}_{\min\{t,\tau_{N,M}\}}^{N,M} = \mathbb{1}_{\{\mathscr{X}_0^{N,M} = X_0^N\}} X_{\min\{t,\tau_{N,M}\}}^{N} \Big) = 1 \tag{4.55}
$$

(cf., e.g., Lemma 8.2 in van Neerven, Veraar, & Weis [37]). This implies for all $N \in \mathbb{N}_0$, $M \in \mathbb{N}$ that

$$
\mathbb{P}\big(\{ \|X_0^0\|_{H_{\gamma+\vartheta}} > M \} \cup \big\{ \forall\, t \in [0,T] \colon \mathscr{X}_{\min\{t,\tau_{N,M}\}}^{N,M} = X_{\min\{t,\tau_{N,M}\}}^{N} \big\} \big) = 1. \tag{4.56}
$$

Hence, we obtain that

$$
\mathbb{P}\Big(\cap_{M\in\mathbb{N}, N\in\mathbb{N}_0} \Big[\{ \|X_0^0\|_{H_{\gamma+\vartheta}} > M \} \cup \big\{ \forall\, t \in [0,T] \colon \mathscr{X}_{\min\{t,\tau_{N,M}\}}^{N,M} = X_{\min\{t,\tau_{N,M}\}}^{N} \big\} \Big] \Big) = 1. \tag{4.57}
$$

In the next step we combine this with (4.37) to obtain for all $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ that

$$
\mathbb{P}\Big(\tau_{N,M} = \min\Big\{ T \, \mathbb{1}_{\{\|X_0^0\|_{H_{\gamma+\vartheta}} \le M\}}, \inf\big(\big\{ t \in [0,T] \colon \|X_t^N\|_{H_\gamma} \ge M \big\} \cup \{T\} \big) \Big\} \Big) = 1. \tag{4.58}
$$

This shows for all $N \in \mathbb{N}_0$, $M_1, M_2 \in \mathbb{N}$ with $M_1 \le M_2$ that $\mathbb{P}\big(\tau_{N,M_1} \le \tau_{N,M_2}\big) = 1$. This, (4.58), and the fact that $\forall\, \omega \in \Omega, N \in \mathbb{N}_0 \colon \sup_{t\in[0,T]} \|X_t^N(\omega)\|_{H_\gamma} < \infty$ imply that it holds for all $N \in \mathbb{N}_0$ that $\mathbb{P}\big(\cup_{M\in\mathbb{N}} \cap_{m\in\{M,M+1,\dots\}} \{\tau_{N,m} = T\} \big) = 1$. This, in turn, proves that

$$
\mathbb{P}\big(\cap_{N\in\mathbb{N}_0} \cup_{M\in\mathbb{N}} \cap_{m\in\{M,M+1,\dots\}} \{\tau_{N,m} = T\} \big) = 1. \tag{4.59}
$$

Combining (4.59), (4.53), (4.57), and (4.52) proves that $\mathbb{P}(\varUpsilon) = 1$. The proof of Corollary 4.5 is thus completed. □

5 Cubature methods in Banach spaces

We first discuss in Subsection 5.1 a number of preliminary definitions related to the Monte Carlo method in Banach spaces. In Subsection 5.2 we present an elementary error estimate for the Monte Carlo method in Corollary 5.12. In Subsection 5.3 we then illustrate how expectations of Banach space valued functions of stochastic processes can be approximated.

5.1 Preliminaries

As mentioned in the introduction, the rate of convergence of Monte Carlo approximations in a Banach space depends on the so-called *type* of the Banach space; cf., e.g., Section 9.2 in Ledoux & Talagrand [30]. In order to define the type of a Banach space, we first reconsider a few concepts from the literature.

Definition 5.1. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let J be a set, and let $r_j \colon \Omega \to \{-1, 1\}$, $j \in J$, be a family of independent random variables with $\forall\, j \in J \colon \mathbb{P}(r_j = 1) = \mathbb{P}(r_j = -1)$. Then we say that $(r_j)_{j \in J}$ is a $\mathbb{P}$-Rademacher family.*

Definition 5.2. *Let $p \in (0, \infty)$ and let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space. Then we denote by $\mathscr{T}_p(E) \in [0, \infty]$ the extended real number given by*

$$\mathscr{T}_p(E) = \sup\left(\left\{ r \in [0,\infty) \colon \begin{array}{c} \exists \textit{ probability space } (\Omega, \mathscr{F}, \mathbb{P}) \colon \\ \exists\, \mathbb{P}\textit{-Rademacher family } (r_j)_{j \in \mathbb{N}} \colon \\ \exists\, k \in \mathbb{N} \colon \exists\, x_1, x_2, \ldots, x_k \in E \backslash \{0\} \colon \\ r = \frac{\left(\mathbb{E}\left[\| \sum_{j=1}^{k} r_j x_j \|_E^p\right]\right)^{1/p}}{\left(\sum_{j=1}^{k} \|x_j\|_E^p\right)^{1/p}} \end{array} \right\} \cup \{0\}\right) \tag{5.1}$$

and we call $\mathscr{T}_p(E)$ the type p-constant of E.

Definition 5.3. *Let $p \in (0, \infty)$ and let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space which satisfies $\mathscr{T}_p(E) < \infty$. Then we say that $(E, \|\cdot\|_E)$ has type p (we say that E has type p).*

Note that it holds for all $p \in (0, \infty)$, all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ with type p, all probability spaces $(\Omega, \mathscr{F}, \mathbb{P})$, all $\mathbb{P}$-Rademacher families $(r_j)_{j \in \mathbb{N}}$, and all $k \in \mathbb{N}$, $x_1, x_2, \ldots, x_k \in E$ that

$$\left\| \sum_{j=1}^{k} r_j x_j \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \leq \mathscr{T}_p(E) \left(\sum_{j=1}^{k} \|x_j\|_E^p \right)^{1/p}. \tag{5.2}$$

In addition, observe that it holds for all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$, all probability spaces $(\Omega, \mathscr{F}, \mathbb{P})$, all $\mathbb{P}$-Rademacher families $(r_j)_{j \in \mathbb{N}}$, and all $p \in [2, \infty)$, $k \in \mathbb{N}$, $x \in E \backslash \{0\}$ that

$$\mathscr{T}_p(E) \geq \frac{\left\| \sum_{j=1}^{k} r_j x \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)}}{\left[\sum_{j=1}^{k} \|x\|_E^p \right]^{1/p}} \geq \frac{\|x\|_E \left\| \sum_{j=1}^{k} r_j \right\|_{\mathscr{L}^2(\mathbb{P}; |\cdot|)}}{k^{1/p} \|x\|_E} = \frac{k^{1/2} \|x\|_E}{k^{1/p} \|x\|_E} = k^{(1/2 - 1/p)}. \tag{5.3}$$

In particular, it holds for all $p \in (2, \infty)$ and all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ with $E \neq \{0\}$ that $\mathscr{T}_p(E) = \infty$. Furthermore, observe that Jensen's inequality together with the fact that it holds for all normed $\mathbb{R}$-vector spaces $(E, \|\cdot\|_E)$ and all $p \in (0, \infty)$, $q \in [p, \infty)$, $k \in \mathbb{N}$, $x_1, \ldots, x_k \in E$ that

$$\left(\sum_{j=1}^{k} \|x_j\|_E^q \right)^{1/q} \leq \left(\sum_{j=1}^{k} \|x_j\|_E^p \right)^{1/p} \tag{5.4}$$

assures that it holds for all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ and all $p, q \in (0, \infty)$ with $p \leq q$ that $\mathscr{T}_p(E) \leq \mathscr{T}_q(E)$. Hence, it holds for every $\mathbb{R}$-Banach space $(E, \|\cdot\|_E)$ that the function $(0, \infty) \ni p \mapsto \mathscr{T}_p(E) \in [0, \infty]$ is non-decreasing. This and the triangle inequality ensure for all $p \in (0, 1]$ and all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ with $E \neq \{0\}$ that $\mathscr{T}_p(E) = 1$. In particular, note that it holds for all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ that $\sup_{p \in (0,1]} \mathscr{T}_p(E) \leq 1 < \infty$. Additionally, observe that it holds for all $p \in (0, 2]$ and all $\mathbb{R}$-Hilbert spaces

$(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ with $H \neq \{0\}$ that $\mathscr{T}_p(H) = 1$. Furthermore, we note that it holds for every probability space $(\Omega, \mathscr{F}, \mathbb{P})$, every $p, q \in [1, \infty)$, and every $\mathbb{R}$-Banach space $(E, \|\cdot\|_E)$ with type q that $L^p(\mathbb{P}; \|\cdot\|_E)$ has type $\min\{p, q\}$; cf., e.g., Proposition 7.1.4 in Hytönen et al. [24], Section 9.2 in Ledoux & Talagrand [30], or Theorem 6.2.14 in Albiac & Kalton [1]. In particular, it holds for every $p \in [1, \infty)$ and every probability space $(\Omega, \mathscr{F}, \mathbb{P})$ that $L^p(\mathbb{P}; |\cdot|)$ has type $\min\{p, 2\}$.

Definition 5.4. *Let $p, q \in (0, \infty)$. Then we denote by $\mathscr{K}_{p,q} \in [0, \infty]$ the extended real number given by*

$$\mathscr{K}_{p,q} = \sup\left\{ r \in [0,\infty)\colon \begin{array}{c} \exists\, \mathbb{R}\text{-Banach space } (E, \|\cdot\|_E)\colon \\ \exists\, \text{probability space } (\Omega, \mathscr{F}, \mathbb{P})\colon \\ \exists\, \mathbb{P}\text{-Rademacher family } (r_j)_{j\in\mathbb{N}}\colon \exists\, k \in \mathbb{N}\colon \\ \exists\, x_1, x_2, \ldots, x_k \in E\backslash\{0\}\colon r = \frac{\left(\mathbb{E}\left[\|\sum_{j=1}^k r_j x_j\|_E^p\right]\right)^{1/p}}{\left(\mathbb{E}\left[\|\sum_{j=1}^k r_j x_j\|_E^q\right]\right)^{1/q}} \end{array} \right\} \tag{5.5}$$

and we call $\mathscr{K}_{p,q}$ the (p,q)-Kahane-Khintchine constant.

The celebrated *Kahane-Khintchine inequality* asserts that it holds for all $p, q \in (0, \infty)$ that $\mathscr{K}_{p,q} < \infty$; see, e.g., Theorem 6.2.5 in Albiac & Kalton [1]. Observe that Jensen's inequality ensures for all $p, q \in (0, \infty)$ with $p \leq q$ that $\mathscr{K}_{p,q} = 1$. The nontrivial assertion of the Kahane-Khintchine inequality is the fact that it holds for all $p, q \in (0, \infty)$ with $p > q$ that $\mathscr{K}_{p,q} < \infty$. In our analysis below we also use the following two abbreviations.

Definition 5.5. *Let $p, q \in (0, \infty)$ and let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space. Then we denote by $\Theta_{p,q}(E) \in [0, \infty]$ the extended real number given by $\Theta_{p,q}(E) = 2\mathscr{T}_q(E)\mathscr{K}_{p,q}$.*

Definition 5.6. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $p \in (0, \infty)$, let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space, and let $X \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$. Then we denote by $\sigma_{p,E}(X) \in [0, \infty]$ the extended real number given by $\sigma_{p,E}(X) = \left(\mathbb{E}\left[\|X - \mathbb{E}[X]\|_E^p\right]\right)^{1/p}$.*

5.2 Monte Carlo methods in Banach spaces

In this subsection we collect a few elementary results on sums of random variables with values in Banach spaces. The next result, Lemma 5.7 below, can be found, e.g., in Section 2.2 of Ledoux & Talagrand [30].

Lemma 5.7 (Symmetrisation lemma)**.** *Consider the notation in Subsection 1.1, let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $\xi, \tilde{\xi} \in \mathscr{L}^0(\mathbb{P}; \|\cdot\|_E)$ be independent mappings which satisfy $\mathbb{E}\big[\|\tilde{\xi}\|_E\big] < \infty$ and $\mathbb{E}[\tilde{\xi}] = 0$, and let $\varphi\colon [0, \infty) \to [0, \infty)$ be a convex and non-decreasing function. Then*

$$\mathbb{E}\big[\varphi(\|\xi\|_E)\big] \leq \mathbb{E}\big[\varphi(\|\xi - \tilde{\xi}\|_E)\big]. \tag{5.6}$$

Proof of Lemma 5.7. Jensen's inequality assures that

$$\begin{aligned}
&\mathbb{E}\big[\varphi(\|\xi\|_E)\big] = \mathbb{E}\big[\varphi(\|\xi - \mathbb{E}[\tilde{\xi}]\|_E)\big] = \int_\Omega \varphi\bigg(\bigg\|\int_\Omega \xi(\omega) - \tilde{\xi}(\tilde{\omega})\,\mathbb{P}(\mathrm{d}\tilde{\omega})\bigg\|_E\bigg)\,\mathbb{P}(\mathrm{d}\omega) \\
&\le \int_\Omega \varphi\bigg(\int_\Omega \|\xi(\omega) - \tilde{\xi}(\tilde{\omega})\|_E\,\mathbb{P}(\mathrm{d}\tilde{\omega})\bigg)\,\mathbb{P}(\mathrm{d}\omega) \\
&\le \int_\Omega \int_\Omega \varphi(\|\xi(\omega) - \tilde{\xi}(\tilde{\omega})\|_E)\,\mathbb{P}(\mathrm{d}\tilde{\omega})\,\mathbb{P}(\mathrm{d}\omega) \\
&= \int_\Omega \int_\Omega \mathbb{1}_{\overline{\xi(\Omega)}^E \times \overline{\tilde{\xi}(\Omega)}^E}\big(\xi(\omega), \tilde{\xi}(\tilde{\omega})\big)\,\varphi(\|\xi(\omega) - \tilde{\xi}(\tilde{\omega})\|_E)\,\mathbb{P}(\mathrm{d}\tilde{\omega})\,\mathbb{P}(\mathrm{d}\omega) \\
&= \int_E \int_E \mathbb{1}_{\overline{\xi(\Omega)}^E \times \overline{\tilde{\xi}(\Omega)}^E}(x, y)\,\varphi(\|x - y\|_E)\,\big(\tilde{\xi}(\mathbb{P})\big)(\mathrm{d}y)\big(\xi(\mathbb{P})\big)(\mathrm{d}x) \\
&= \int_{E \times E} \mathbb{1}_{\overline{\xi(\Omega)}^E \times \overline{\tilde{\xi}(\Omega)}^E}(x, y)\,\varphi(\|x - y\|_E)\,\big((\xi, \tilde{\xi})(\mathbb{P})\big)(\mathrm{d}x, \mathrm{d}y) = \mathbb{E}\big[\varphi(\|\xi - \tilde{\xi}\|_E)\big].
\end{aligned} \tag{5.7}$$

This completes the proof of Lemma 5.7. □

Corollary 5.8 (Symmetrisation corollary)**.** *Let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $\xi, \tilde{\xi} \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$ be independent and identically distributed mappings which satisfy $\mathbb{E}[\xi] = 0$, and let $\varphi\colon [0, \infty) \to [0, \infty)$ be a convex and non-decreasing function. Then*

$$\mathbb{E}\big[\varphi(\|\xi\|_E)\big] \le \mathbb{E}\big[\varphi(\|\xi - \tilde{\xi}\|_E)\big] \le \mathbb{E}\big[\varphi(2\,\|\xi\|_E)\big]. \tag{5.8}$$

Proof of Corollary 5.8. Lemma 5.7 shows that

$$\begin{aligned}
\mathbb{E}\big[\varphi(\|\xi\|_E)\big] &\le \mathbb{E}\big[\varphi(\|\xi - \tilde{\xi}\|_E)\big] \le \mathbb{E}\big[\varphi(\|\xi\|_E + \|\tilde{\xi}\|_E)\big] = \mathbb{E}\big[\varphi(\tfrac{1}{2}\,2\,\|\xi\|_E + \tfrac{1}{2}\,2\,\|\tilde{\xi}\|_E)\big] \\
&\le \mathbb{E}\big[\tfrac{1}{2}\,\varphi(2\,\|\xi\|_E) + \tfrac{1}{2}\,\varphi(2\,\|\tilde{\xi}\|_E)\big] = \tfrac{1}{2}\,\mathbb{E}\big[\varphi(2\,\|\xi\|_E)\big] + \tfrac{1}{2}\,\mathbb{E}\big[\varphi(2\,\|\tilde{\xi}\|_E)\big] \\
&= \tfrac{1}{2}\,\mathbb{E}\big[\varphi(2\,\|\xi\|_E)\big] + \tfrac{1}{2}\,\mathbb{E}\big[\varphi(2\,\|\xi\|_E)\big] = \mathbb{E}\big[\varphi(2\,\|\xi\|_E)\big].
\end{aligned} \tag{5.9}$$

The proof of Corollary 5.8 is thus completed. □

As a straightforward application we obtain the following randomisation result, cf., e.g., Lemma 6.3 in Ledoux & Talagrand [30].

Lemma 5.9 (Randomisation)**.** *Let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $k \in \mathbb{N}$, let $\xi_j \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$, $j \in \{1, \ldots, k\}$, satisfy for all $j \in \{1, \ldots, k\}$ that $\mathbb{E}[\xi_j] = 0$, and let $r_j\colon \Omega \to \{-1, 1\}$, $j \in \{1, \ldots, k\}$, be a $\mathbb{P}$-Rademacher family such that $\xi_1, \xi_2, \ldots, \xi_k, r_1, r_2, \ldots, r_k$ are independent. Then it holds for all $p \in [1, \infty)$ that*

$$\bigg\|\sum_{j=1}^{k} \xi_j\bigg\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \le 2\,\bigg\|\sum_{j=1}^{k} r_j \xi_j\bigg\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)}. \tag{5.10}$$

Proof of Lemma 5.9. Throughout this proof let $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P}) = (\Omega \times \Omega, \mathscr{F} \otimes \mathscr{F}, \mathbb{P} \otimes \mathbb{P})$, let $\mathbf{r}_j\colon \boldsymbol{\Omega} \to \{-1, 1\}$, $j \in \{1, \ldots, k\}$, be the mappings which satisfy for all $\boldsymbol{\omega} = (\omega, \tilde{\omega}) \in \boldsymbol{\Omega}$, $j \in \{1, \ldots, k\}$ that $\mathbf{r}_j(\boldsymbol{\omega}) = r_j(\omega)$, and let $\boldsymbol{\xi}_j\colon \boldsymbol{\Omega} \to E$, $j \in \{1, \ldots, k\}$, and $\tilde{\boldsymbol{\xi}}_j\colon \boldsymbol{\Omega} \to E$, $j \in \{1, \ldots, k\}$, be the mappings which satisfy for all $\boldsymbol{\omega} = (\omega, \tilde{\omega}) \in \boldsymbol{\Omega}$, $j \in \{1, \ldots, k\}$ that $\boldsymbol{\xi}_j(\boldsymbol{\omega}) = \xi_j(\omega)$ and $\tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega}) = \xi_j(\tilde{\omega})$. The fact that

$$\{0, 1\} \times \{1, \ldots, k\} \ni (i, j) \mapsto \begin{cases} \boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j & : i = 0 \\ \mathbf{r}_j & : i = 1 \end{cases} \tag{5.11}$$

is a family of independent mappings and the fact that $\forall\, j \in \{1,\dots,k\}\colon (\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j)(\mathbf{P}) = (\tilde{\boldsymbol{\xi}}_j - \boldsymbol{\xi}_j)(\mathbf{P})$ prove for all $p \in [1,\infty)$ that

$$\begin{aligned}
&\int_{\boldsymbol{\Omega}} \left\| \sum_{j=1}^{k} \mathbf{r}_j(\boldsymbol{\omega}) \big[\boldsymbol{\xi}_j(\boldsymbol{\omega}) - \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega})\big] \right\|_E^p \mathbf{P}(\mathrm{d}\boldsymbol{\omega}) \\
&= \int_{\boldsymbol{\Omega}} \left\| \sum_{j=1}^{k} \mathbb{1}^E_{\overline{\cup_{i=0}^1 [(-1)^i(\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j)](\boldsymbol{\Omega})}} \big(\mathbf{r}_j(\boldsymbol{\omega}) \big[\boldsymbol{\xi}_j(\boldsymbol{\omega}) - \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega})\big]\big)\, \mathbf{r}_j(\boldsymbol{\omega}) \big[\boldsymbol{\xi}_j(\boldsymbol{\omega}) - \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega})\big] \right\|_E^p \mathbf{P}(\mathrm{d}\boldsymbol{\omega}) \\
&= \int_{(\{-1,1\}\times E)^k} \left\| \sum_{j=1}^{k} \mathbb{1}^E_{\overline{\cup_{i=0}^1 [(-1)^i(\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j)](\boldsymbol{\Omega})}} (z_j x_j)\, z_j x_j \right\|_E^p \\
&\quad \big((\mathbf{r}_1, \boldsymbol{\xi}_1 - \tilde{\boldsymbol{\xi}}_1, \dots, \mathbf{r}_k, \boldsymbol{\xi}_k - \tilde{\boldsymbol{\xi}}_k)(\mathbf{P})\big)(\mathrm{d}z_1, \mathrm{d}x_1, \dots, \mathrm{d}z_k, \mathrm{d}x_k) \\
&= \int_{\{-1,1\}} \int_E \cdots \int_{\{-1,1\}} \int_E \left\| \sum_{j=1}^{k} \mathbb{1}^E_{\overline{\cup_{i=0}^1 [(-1)^i(\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j)](\boldsymbol{\Omega})}} (z_j x_j)\, z_j x_j \right\|_E^p \\
&\quad \big((\boldsymbol{\xi}_k - \tilde{\boldsymbol{\xi}}_k)(\mathbf{P})\big)(\mathrm{d}x_k) \big((\mathbf{r}_k)(\mathbf{P})\big)(\mathrm{d}z_k)\ \dots\ \big((\boldsymbol{\xi}_1 - \tilde{\boldsymbol{\xi}}_1)(\mathbf{P})\big)(\mathrm{d}x_1) \big((\mathbf{r}_1)(\mathbf{P})\big)(\mathrm{d}z_1) \\
&= \int_E \cdots \int_E \left\| \sum_{j=1}^{k} \mathbb{1}^E_{\overline{\cup_{i=0}^1 [(-1)^i(\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j)](\boldsymbol{\Omega})}} (x_j)\, x_j \right\|_E^p \\
&\quad \big((\boldsymbol{\xi}_k - \tilde{\boldsymbol{\xi}}_k)(\mathbf{P})\big)(\mathrm{d}x_k)\ \dots\ \big((\boldsymbol{\xi}_1 - \tilde{\boldsymbol{\xi}}_1)(\mathbf{P})\big)(\mathrm{d}x_1) \\
&= \int_{E^k} \left\| \sum_{j=1}^{k} \mathbb{1}^E_{\overline{\cup_{i=0}^1 [(-1)^i(\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j)](\boldsymbol{\Omega})}} (x_j)\, x_j \right\|_E^p \big((\boldsymbol{\xi}_1 - \tilde{\boldsymbol{\xi}}_1, \dots, \boldsymbol{\xi}_k - \tilde{\boldsymbol{\xi}}_k)(\mathbf{P})\big)(\mathrm{d}x_1, \dots, \mathrm{d}x_k) \\
&= \int_{\boldsymbol{\Omega}} \left\| \sum_{j=1}^{k} \big[\boldsymbol{\xi}_j(\boldsymbol{\omega}) - \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega})\big] \right\|_E^p \mathbf{P}(\mathrm{d}\boldsymbol{\omega}).
\end{aligned} \tag{5.12}$$

Furthermore, the fact that $\sum_{j=1}^k \boldsymbol{\xi}_j \colon \boldsymbol{\Omega} \to E$ and $\sum_{j=1}^k \tilde{\boldsymbol{\xi}}_j \colon \boldsymbol{\Omega} \to E$ are independent, the facts that $\int_{\boldsymbol{\Omega}} \big\|\sum_{j=1}^k \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega})\big\|_E \mathbf{P}(\mathrm{d}\boldsymbol{\omega}) < \infty$ and $\int_{\boldsymbol{\Omega}} \sum_{j=1}^k \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega})\, \mathbf{P}(\mathrm{d}\boldsymbol{\omega}) = 0$, Lemma 5.7, and (5.12) imply that it holds for all $p \in [1,\infty)$ that

$$\begin{aligned}
&\left\| \sum_{j=1}^{k} \xi_j \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} = \left\| \sum_{j=1}^{k} \boldsymbol{\xi}_j \right\|_{\mathscr{L}^p(\mathbf{P};\|\cdot\|_E)} \le \left\| \sum_{j=1}^{k} (\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j) \right\|_{\mathscr{L}^p(\mathbf{P};\|\cdot\|_E)} \\
&= \left\| \sum_{j=1}^{k} \mathbf{r}_j (\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j) \right\|_{\mathscr{L}^p(\mathbf{P};\|\cdot\|_E)} \le \left\| \sum_{j=1}^{k} \mathbf{r}_j \boldsymbol{\xi}_j \right\|_{\mathscr{L}^p(\mathbf{P};\|\cdot\|_E)} + \left\| \sum_{j=1}^{k} \mathbf{r}_j \tilde{\boldsymbol{\xi}}_j \right\|_{\mathscr{L}^p(\mathbf{P};\|\cdot\|_E)} \\
&= 2 \left\| \sum_{j=1}^{k} r_j \xi_j \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}.
\end{aligned} \tag{5.13}$$

The proof of Lemma 5.9 is thus completed. □

The next result, Proposition 5.10 below, is the key to estimating the statistical error term in the Banach space valued Monte Carlo method in the next subsection. Proposition 5.10 is similar to, e.g., Proposition 9.11 in Ledoux & Talagrand [30].

Proposition 5.10 (Sums of independent, centred, Banach space valued random variables)**.** *Let $k \in \mathbb{N}$, $q \in [1,2]$, let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space with type q, let $(\Omega, \mathscr{F}, \mathbb{P})$*

be a probability space, and let $\xi_j \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$, $j \in \{1, \ldots, k\}$, *be independent mappings which satisfy for all* $j \in \{1, \ldots, k\}$ *that* $\mathbb{E}[\xi_j] = 0$. *Then it holds for all* $p \in [q, \infty)$ *that*

$$\left\| \sum_{j=1}^{k} \xi_j \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \leq \Theta_{p,q}(E) \left(\sum_{j=1}^{k} \|\xi_j\|^q_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \right)^{1/q}. \tag{5.14}$$

Proof of Proposition 5.10. Throughout this proof let $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ be a probability space, let $r_j \colon \tilde{\Omega} \to \{-1, 1\}$, $j \in \{1, \ldots, k\}$, be a $\tilde{\mathbb{P}}$-Rademacher family, and let $\boldsymbol{\xi}_j \colon \Omega \times \tilde{\Omega} \to E$, $j \in \{1, \ldots, k\}$, and $\mathbf{r}_j \colon \Omega \times \tilde{\Omega} \to \{-1, 1\}$, $j \in \{1, \ldots, k\}$, be the mappings which satisfy for all $\boldsymbol{\omega} = (\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}$, $j \in \{1, \ldots, k\}$ that $\boldsymbol{\xi}_j(\boldsymbol{\omega}) = \xi_j(\omega)$ and $\mathbf{r}_j(\boldsymbol{\omega}) = r_j(\tilde{\omega})$. Lemma 5.9 and the triangle inequality show for all $p \in [q, \infty)$ that

$$\begin{aligned}
&\left\| \sum_{j=1}^{k} \xi_j \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} = \left\| \sum_{j=1}^{k} \boldsymbol{\xi}_j \right\|_{\mathscr{L}^p(\mathbb{P} \otimes \tilde{\mathbb{P}}; \|\cdot\|_E)} \leq 2 \left\| \sum_{j=1}^{k} \mathbf{r}_j \boldsymbol{\xi}_j \right\|_{\mathscr{L}^p(\mathbb{P} \otimes \tilde{\mathbb{P}}; \|\cdot\|_E)} \\
&= 2 \left(\int_{\Omega} \left\| \sum_{j=1}^{k} r_j(\cdot) \xi_j(\omega) \right\|^p_{\mathscr{L}^p(\tilde{\mathbb{P}}; \|\cdot\|_E)} \mathbb{P}(\mathrm{d}\omega) \right)^{1/p} \\
&\leq 2 \mathscr{K}_{p,q} \left(\int_{\Omega} \left\| \sum_{j=1}^{k} r_j(\cdot) \xi_j(\omega) \right\|^p_{\mathscr{L}^q(\tilde{\mathbb{P}}; \|\cdot\|_E)} \mathbb{P}(\mathrm{d}\omega) \right)^{1/p} \\
&\leq 2 \mathscr{K}_{p,q} \mathscr{T}_q(E) \left\| \left(\sum_{j=1}^{k} \|\xi_j\|^q_E \right)^{1/q} \right\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)} = 2 \mathscr{K}_{p,q} \mathscr{T}_q(E) \left\| \sum_{j=1}^{k} \|\xi_j\|^q_E \right\|^{1/q}_{\mathscr{L}^{p/q}(\mathbb{P}; |\cdot|)} \\
&\leq 2 \mathscr{K}_{p,q} \mathscr{T}_q(E) \left(\sum_{j=1}^{k} \|\xi_j\|^q_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \right)^{1/q}.
\end{aligned} \tag{5.15}$$

This finishes the proof of Proposition 5.10. □

The result in Corollary 5.11 below is a direct consequence of Proposition 5.10.

Corollary 5.11 (Sums of independent Banach space valued random variables)**.** *Let* $M \in \mathbb{N}$, $q \in [1, 2]$, *let* $(E, \|\cdot\|_E)$ *be an* $\mathbb{R}$*-Banach space with type* q, *let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space, and let* $\xi_j \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$, $j \in \{1, \ldots, M\}$, *be independent. Then it holds for all* $p \in [q, \infty)$ *that*

$$\sigma_{p,E}\left(\sum_{j=1}^{M} \xi_j \right) \leq \Theta_{p,q}(E) \left(\sum_{j=1}^{M} |\sigma_{p,E}(\xi_j)|^q \right)^{1/q}. \tag{5.16}$$

Corollary 5.12 (Monte Carlo methods in Banach spaces)**.** *Let* $M \in \mathbb{N}$, $q \in [1, 2]$, *let* $(E, \|\cdot\|_E)$ *be an* $\mathbb{R}$*-Banach space with type* q, *let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space, and let* $\xi_j \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$, $j \in \{1, \ldots, M\}$, *be independent and identically distributed. Then it holds for all* $p \in [q, \infty)$ *that*

$$\left\| \mathbb{E}[\xi_1] - \frac{1}{M} \sum_{j=1}^{M} \xi_j \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} = \frac{\sigma_{p,E}\left(\sum_{j=1}^{M} \xi_j\right)}{M} \leq \frac{\Theta_{p,q}(E)\, \sigma_{p,E}(\xi_1)}{M^{1-1/q}}. \tag{5.17}$$

Results on lower and upper error bounds related to Corollary 5.12 can be found, e.g., in Theorem 1 in Daun & Heinrich [11] and in Corollary 2 in Heinrich & Hinrichs [16]. Note that Corollary 5.12 does not imply convergence if the underlying Banach space $(E, \|\cdot\|_E)$ has only type 1, in the sense that it holds for all $q \in (1, \infty)$ that $\mathscr{T}_q(E) = \infty$.

5.3 Multilevel Monte Carlo methods in Banach spaces

In many situations the work required to obtain a certain accuracy of an approximation using the Monte Carlo method can be reduced by using a multilevel Monte Carlo method. Heinrich [14, 15] was first to observe this and established multilevel Monte Carlo methods concerning convergence in a Banach (function) space. However, these methods do not apply to SDEs. Then Giles [12] derived the complexity reduction of multilevel Monte Carlo methods for SDEs. The minor contribution of Proposition 5.13 below to the literature on multilevel Monte Carlo methods is to combine the approaches of Heinrich [14] and of Giles [12] into a single result on multilevel Monte Carlo methods in Banach spaces. The useful observation of Proposition 5.13 generalises the discussion in Section 4 of Heinrich [15].

Proposition 5.13 (Abstract multilevel Monte Carlo methods in Banach spaces)**.** *Let $q \in [1,2]$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(V_1, \|\cdot\|_{V_1})$ be an $\mathbb{R}$-Banach space with type q, let $(V_2, \|\cdot\|_{V_2})$ be an $\mathbb{R}$-Banach space with $V_1 \subseteq V_2$ continuously, let $v \in V_2$, $L \in \mathbb{N}$, $M_1, \dots, M_L \in \mathbb{N}$, and for every $\ell \in \{1, \dots, L\}$ let $D_{\ell,k} \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_{V_1})$, $k \in \{1, \dots, M_\ell\}$, be independent and identically distributed. Then it holds for all $p \in [q, \infty)$ that*

$$\begin{aligned}
&\left\| v - \sum_{\ell=1}^{L} \frac{1}{M_\ell} \sum_{k=1}^{M_\ell} D_{\ell,k} \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_2})} \\
&\leq \left\| v - \sum_{\ell=1}^{L} \mathbb{E}\big[D_{\ell,1}\big] \right\|_{V_2} + \|\mathrm{Id}_{V_1}\|_{L(V_1,V_2)}\, \Theta_{p,q}(V_1) \sum_{\ell=1}^{L} \frac{\sigma_{p,V_1}(D_{\ell,1})}{(M_\ell)^{1-1/q}}.
\end{aligned} \tag{5.18}$$

Proof of Proposition 5.13. The triangle inequality and Corollary 5.12 imply for all $p \in [q, \infty)$ that

$$\begin{aligned}
&\left\| v - \sum_{\ell=1}^{L} \frac{1}{M_\ell} \sum_{k=1}^{M_\ell} D_{\ell,k} \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_2})} \\
&\leq \left\| v - \sum_{\ell=1}^{L} \mathbb{E}[D_{\ell,1}] \right\|_{V_2} + \left\| \sum_{\ell=1}^{L} \mathbb{E}[D_{\ell,1}] - \sum_{\ell=1}^{L} \frac{1}{M_\ell} \sum_{k=1}^{M_\ell} D_{\ell,k} \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_2})} \\
&\leq \left\| v - \sum_{\ell=1}^{L} \mathbb{E}[D_{\ell,1}] \right\|_{V_2} + \|\mathrm{Id}_{V_1}\|_{L(V_1,V_2)} \sum_{\ell=1}^{L} \left\| \mathbb{E}[D_{\ell,1}] - \frac{1}{M_\ell} \sum_{k=1}^{M_\ell} D_{\ell,k} \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_1})} \\
&\leq \left\| v - \sum_{\ell=1}^{L} \mathbb{E}[D_{\ell,1}] \right\|_{V_2} + \|\mathrm{Id}_{V_1}\|_{L(V_1,V_2)}\, \Theta_{p,q}(V_1) \sum_{\ell=1}^{L} \frac{\sigma_{p,V_1}(D_{\ell,1})}{(M_\ell)^{1-1/q}}.
\end{aligned} \tag{5.19}$$

This completes the proof of Proposition 5.13. □

Corollary 5.14 (Multilevel Monte Carlo methods in Banach spaces)**.** *Consider the notation in Subsection 1.1, let $q \in [1,2]$, $L \in \mathbb{N}_0$, $M_0, M_1, \dots, M_{L+1}, N_0, N_1, \dots, N_L \in \mathbb{N}$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(V_i, \|\cdot\|_{V_i})$, $i \in \{1,2\}$, be separable $\mathbb{R}$-Banach spaces such that $(V_1, \|\cdot\|_{V_1})$ has type q and such that $V_1 \subseteq V_2$ continuously, let $(V_3, \|\cdot\|_{V_3})$ be an $\mathbb{R}$-Banach space, let $f \in \mathscr{M}(\mathscr{B}(V_3), \mathscr{B}(V_2))$, $g \in \mathscr{M}(\mathscr{B}(V_3), \mathscr{B}(V_1))$, $X \in \mathscr{M}(\mathscr{F}, \mathscr{B}(V_3))$ satisfy $\mathbb{E}\big[\|f(X)\|_{V_2}\big] < \infty$, for every $n \in \mathbb{N}$ let $Y^{n,l,k} \in \mathscr{M}(\mathscr{F}, \mathscr{B}(V_3))$, $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, satisfy $\mathbb{E}\big[\|g(Y^{n,0,1})\|_{V_1}\big] < \infty$, assume that $Y^{N_0,0,k}$, $k \in \mathbb{N}$, are independent and identically* *distributed, and assume for every $\ell \in \mathbb{N} \cap [0, L]$ that $(Y^{N_{(\ell-1)},\mathfrak{l},k}, Y^{N_\ell,\mathfrak{l},k})$, $k \in \mathbb{N}$, $\mathfrak{l} \in \mathbb{N}_0$, are independent and identically distributed. Then it holds for all $p \in [q, \infty)$ that*

$$\begin{aligned}
&\left\| \mathbb{E}[f(X)] - \tfrac{1}{M_0} \sum_{k=1}^{M_0} g(Y^{N_0,0,k}) - \sum_{\ell=1}^{L} \tfrac{1}{M_\ell} \sum_{k=1}^{M_\ell} \left[g(Y^{N_\ell,\ell,k}) - g(Y^{N_{(\ell-1)},\ell,k}) \right] \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_2})} \\
&\leq \left\| \mathbb{E}[f(X)] - \mathbb{E}\big[g(Y^{N_L,0,1})\big] \right\|_{V_2} \qquad (5.20) \\
&\quad + \|\mathrm{Id}_{V_1}\|_{L(V_1,V_2)}\, \Theta_{p,q}(V_1) \left(\tfrac{\sigma_{p,V_1}\left(g(Y^{N_0,0,1})\right)}{(M_0)^{1-1/q}} + \sum_{\ell=1}^{L} \tfrac{\sigma_{p,V_1}\left(g(Y^{N_\ell,0,1}) - g(Y^{N_{(\ell-1)},0,1})\right)}{(M_\ell)^{1-1/q}} \right) \\
&\leq \left\| \mathbb{E}[f(X)] - \mathbb{E}\big[g(Y^{N_L,0,1})\big] \right\|_{V_2} \\
&\quad + \|\mathrm{Id}_{V_1}\|_{L(V_1,V_2)}\, \Theta_{p,q}(V_1) \left(\tfrac{2\, \|g(X)\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_1})}}{(M_0)^{1-1/q}} + \sum_{\ell=0}^{L} \tfrac{4\, \left\|g(Y^{N_\ell,0,1}) - g(X)\right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_1})}}{(\min\{M_\ell, M_{\ell+1}\})^{1-1/q}} \right).
\end{aligned}$$

Proof of Corollary 5.14. Proposition 5.13 and the identity

$$\mathbb{E}\big[g(Y^{N_L,0,1})\big] = \mathbb{E}\big[g(Y^{N_0,0,1})\big] + \sum_{\ell=1}^{L} \mathbb{E}\big[g(Y^{N_\ell,0,1}) - g(Y^{N_{(\ell-1)},0,1})\big] \qquad (5.21)$$

imply the first inequality in (5.20). Next note that the triangle inequality implies for all $\xi \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_{V_1})$, $p \in [q, \infty)$ that $\sigma_{p,V_1}(\xi) \leq 2\, \|\xi\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_1})}$. This and again the triangle inequality show for all $p \in [q, \infty)$ that

$$\begin{aligned}
&\tfrac{\sigma_{p,V_1}\left(g(Y^{N_0,0,1})\right)}{(M_0)^{1-1/q}} + \sum_{\ell=1}^{L} \tfrac{\sigma_{p,V_1}\left(g(Y^{N_\ell,0,1}) - g(Y^{N_{(\ell-1)},0,1})\right)}{(M_\ell)^{1-1/q}} \\
&\leq \tfrac{2\, \left\|g(Y^{N_0,0,1})\right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_1})}}{(M_0)^{1-1/q}} + \sum_{\ell=1}^{L} \tfrac{2\, \left\|g(Y^{N_\ell,0,1}) - g(Y^{N_{(\ell-1)},0,1})\right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_1})}}{(M_\ell)^{1-1/q}} \\
&\leq \tfrac{2\, \|g(X)\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_1})} + 2\, \left\|g(Y^{N_0,0,1}) - g(X)\right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_1})}}{(M_0)^{1-1/q}} \qquad (5.22) \\
&\quad + \sum_{\ell=1}^{L} \tfrac{2\, \left\|g(Y^{N_\ell,0,1}) - g(X)\right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_1})} + 2\, \left\|g(Y^{N_{(\ell-1)},0,1}) - g(X)\right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_1})}}{(M_\ell)^{1-1/q}} \\
&\leq \tfrac{2\, \|g(X)\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_1})}}{(M_0)^{1-1/q}} + \sum_{\ell=0}^{L} \tfrac{4\, \left\|g(Y^{N_\ell,0,1}) - g(X)\right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_1})}}{(\min\{M_\ell, M_{\ell+1}\})^{1-1/q}}.
\end{aligned}$$

This implies the second inequality in (5.20). The proof of Corollary 5.14 is thus completed. □

Corollary 5.15 (Convergence of multilevel Monte Carlo approximations)**.** *Consider the notation in Subsection 1.1, let $T \in (0, \infty)$, $\beta \in (0, 1]$, $\alpha \in (0, \beta)$, $c, r \in [0, \infty)$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(E, \|\cdot\|_E)$ be a separable $\mathbb{R}$-Banach space with type 2, let $X \colon [0, T] \times \Omega \to E$ be a stochastic process with continuous sample paths which satisfies for all $p \in [1, \infty)$, $\gamma \in [0, \beta)$ that $X \in \mathscr{C}^\gamma\big([0, T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)}\big)$, for every $N \in \mathbb{N}$, $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ let $Y^{N,\ell,k} \colon [0, T] \times \Omega \to E$ be a stochastic process which satisfies for all $n \in \{0, 1, \ldots, N-1\}$, $t \in \big[\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $p \in [1, \infty)$, $\rho \in [0, \beta)$ that*

$$Y^{N,\ell,k}_t = \big(n + 1 - \tfrac{tN}{T}\big) \cdot Y^{N,\ell,k}_{\frac{nT}{N}} + \big(\tfrac{tN}{T} - n\big) \cdot Y^{N,\ell,k}_{\frac{(n+1)T}{N}}, \qquad (5.23)$$

$$\sup_{M\in\mathbb{N}}\sup_{m\in\{0,1,\ldots,M\}}\Big(M^\rho\,\big\|X_{\frac{mT}{M}}-Y^{M,0,1}_{\frac{mT}{M}}\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\Big)<\infty, \tag{5.24}$$

assume for every $N_1,N_2\in\mathbb{N}$ *that* $(Y^{N_1,\ell,k},Y^{N_2,\ell,k})$, $k\in\mathbb{N}$, $\ell\in\mathbb{N}_0$, *are independent and identically distributed, and let* $f\colon C([0,T],E)\to C([0,T],E)$ *be a* $\mathscr{B}\big(C([0,T],E)\big)/\mathscr{B}\big(C([0,T],E)\big)$*-measurable function which satisfies for all* $v,w\in\mathscr{C}^\alpha([0,T],\|\cdot\|_E)$ *that* $f(v),f(w)\in\mathscr{C}^\alpha([0,T],\|\cdot\|_E)$ *and*

$$\|f(v)-f(w)\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\le c\Big(1+\|v\|^r_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}+\|w\|^r_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big)\|v-w\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}. \tag{5.25}$$

Then it holds that

$$\mathbb{E}\Big[\|f(X)\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]<\infty, \tag{5.26}$$

it holds for all $p\in[1,\infty)$, $\rho\in[0,\beta-\alpha)$ *that*

$$\sup_{N\in\mathbb{N}}\left[N^\rho\Big(\mathbb{E}\Big[\|f(X)-f(Y^{N,0,1})\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p}\right]<\infty, \tag{5.27}$$

and it holds for all $p\in[1,\infty)$, $\gamma\in[0,\alpha)$, $\rho\in[0,\beta-\alpha)$ *that*

$$\sup_{L\in\mathbb{N}}\left[2^{L\cdot\min\{\rho,1/2\}}L^{-\mathbb{1}_{\{1/2\}}(\rho)}\left|\mathbb{E}\left[\left\|\mathbb{E}[f(X)]-\sum_{k=1}^{2^L}\frac{f(Y^{1,0,k})}{2^L}-\sum_{\ell=1}^{L}\sum_{k=1}^{2^{L-\ell}}\frac{f(Y^{2^\ell,\ell,k})-f(Y^{2^{(\ell-1)},\ell,k})}{2^{L-\ell}}\right\|^p_{\mathscr{C}^\gamma([0,T],\|\cdot\|_E)}\right]\right|^{1/p}\right]<\infty. \tag{5.28}$$

Proof of Corollary 5.15. Throughout this proof let $\gamma\in[0,\alpha)$, $\delta\in(\gamma,\frac{3\gamma+\alpha}{4})$, let $C^1([0,T],E)$ be the $\mathbb{R}$-vector space of continuously Fréchet differentiable functions from $[0,T]$ to E, let $\|\cdot\|_{C^1([0,T],E)}\colon C^1([0,T],E)\to[0,\infty)$ be the function which satisfies for all $v\in C^1([0,T],E)$ that $\|v\|_{C^1([0,T],E)}=\|v\|_{C([0,T],\|\cdot\|_E)}+\|v'\|_{C([0,T],\|\cdot\|_E)}$, let $\mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E)$ be the Sobolev space with regularity parameter $(\alpha+\gamma)/2\in(0,1)$ and integrability parameter $4/(\alpha-\gamma)\in(4,\infty)$ of continuous functions from $[0,T]$ to E, let

$$\|\cdot\|_{\mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E)}\colon\mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E)\to[0,\infty)$$

be the function which satisfies for all $v\in\mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E)$ that

$$\|v\|_{\mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E)}=\left[\int_0^T\|v(t)\|_E^{\frac{4}{\alpha-\gamma}}\,\mathrm{d}t+\int_0^T\int_0^T\frac{\|v(t)-v(s)\|_E^{\frac{4}{\alpha-\gamma}}}{|t-s|^{\frac{3\alpha+\gamma}{\alpha-\gamma}}}\,\mathrm{d}t\,\mathrm{d}s\right]^{\frac{\alpha-\gamma}{4}}, \tag{5.29}$$

let $V_1,V_2\subseteq\mathscr{C}^\gamma([0,T],\|\cdot\|_E)$ be the sets given by $V_1=\mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E)$ and

$$V_2=\left\{v\in\mathscr{C}^\gamma([0,T],\|\cdot\|_E)\colon\limsup_{n\to\infty}\sup_{s,t\in[0,T],\,0<|s-t|<1/n}\frac{\|v(s)-v(t)\|_E}{|s-t|^\gamma}=0\right\} \tag{5.30}$$

(cf., e.g., Lunardi [31, Section 0.2]), let $\|\cdot\|_{V_1}\colon V_1\to[0,\infty)$ be the function given by $\|\cdot\|_{V_1}=\|\cdot\|_{\mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E)}$, let $\|\cdot\|_{V_2}\colon V_2\to[0,\infty)$ be the function which satisfies for all $v\in V_2$ that $\|v\|_{V_2}=\|v\|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_E)}$, let $(V_3,\|\cdot\|_{V_3})$ be the $\mathbb{R}$-Banach space given by

$$(V_3,\|\cdot\|_{V_3})=\Big(C([0,T],E),\|\cdot\|_{C([0,T],\|\cdot\|_E)}\big|_{C([0,T],E)}\Big), \tag{5.31}$$

and let $\mathfrak{f}\colon V_3 \to V_2$ and $g\colon V_3 \to V_1$ be the functions which satisfy for all $v \in V_3$ that $\mathfrak{f}(v) = g(v) = \mathbb{1}_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}(v) f(v)$. Observe that the Kolmogorov-Chentsov continuity theorem (see Theorem 2.7) together with the assumptions that $X \in \cap_{p\in[1,\infty)} \cap_{\eta\in[0,\beta)} \mathscr{C}^\eta\big([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\big)$ and that X has continuous sample paths implies for all $p \in [1,\infty)$ that $\mathbb{E}\big[\|X\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\big] < \infty$. This, assumption (5.25), Hölder's inequality, and Corollary 2.11 show for all $p \in [1,\infty)$, $\rho \in [0, \beta-\alpha)$ that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}}\Big(N^\rho\, \mathbb{E}\Big[\|\mathfrak{f}(X) - g(Y^{N,0,1})\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big) \\
&\leq \sup_{N\in\mathbb{N}}\left[N^\rho\Big(\mathbb{E}\Big[\|f(X) - f(Y^{N,0,1})\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p}\right] \\
&\leq \sup_{N\in\mathbb{N}}\left[N^\rho\Big(\mathbb{E}\Big[\Big(c\,\big(1 + \|X\|^r_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} + \|Y^{N,0,1}\|^r_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\big)\right. \\
&\qquad\qquad\left.\cdot \|X - Y^{N,0,1}\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big)^p\Big]\Big)^{1/p}\right] \qquad (5.32) \\
&\leq c\left[1 + \Big(\mathbb{E}\Big[\|X\|^{2pr}_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/(2p)} + \sup_{N\in\mathbb{N}}\Big(\mathbb{E}\Big[\|Y^{N,0,1}\|^{2pr}_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/(2p)}\right] \\
&\qquad \cdot \sup_{N\in\mathbb{N}}\left[N^\rho\Big(\mathbb{E}\Big[\|X - Y^{N,0,1}\|^{2p}_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/(2p)}\right] < \infty.
\end{aligned}$$

Assumption (5.25) also ensures for all $p \in [1,\infty)$ that

$$\begin{aligned}
&\mathbb{E}\Big[\|f(X)\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big] \leq \Big(\mathbb{E}\Big[\|f(X)\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p} \qquad (5.33) \\
&\leq \|f(0)\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} + c\left[\Big(\mathbb{E}\Big[\|X\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p} + \Big(\mathbb{E}\Big[\|X\|^{(r+1)p}_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p}\right] < \infty.
\end{aligned}$$

Next note that $(V_1, \|\cdot\|_{V_1})$ is a separable $\mathbb{R}$-Banach space with type 2. In addition, the fact that $(C^1([0,T],E), \|\cdot\|_{C^1([0,T],E)})$ is a separable $\mathbb{R}$-Banach space, the fact that $C^1([0,T],E) \subseteq \mathscr{C}^\gamma([0,T], \|\cdot\|_E)$ continuously, and the fact that

$$\overline{C^1([0,T],E)}^{\mathscr{C}^\gamma([0,T],\|\cdot\|_E)} = V_2 \qquad (5.34)$$

(cf., e.g., Lunardi [31, Proposition 0.2.1]) prove that $(V_2, \|\cdot\|_{V_2})$ is a separable $\mathbb{R}$-Banach space. Moreover, the Sobolev embedding theorem proves that $V_1 \subseteq \mathscr{C}^\delta([0,T], \|\cdot\|_E)$ continuously. This and the fact that $\mathscr{C}^\delta([0,T], \|\cdot\|_E) \subseteq V_2$ continuously establish that $V_1 \subseteq V_2$ continuously. Combining (5.33) with (5.32) and the fact that $\mathscr{C}^\alpha([0,T], \|\cdot\|_E) \subseteq V_1$ continuously hence implies for all $p \in [1,\infty)$, $\rho \in [0,\beta-\alpha)$ that $\mathbb{E}\big[\|\mathfrak{f}(X)\|_{V_2}\big] + \sup_{N\in\mathbb{N}} \mathbb{E}\big[\|g(Y^{N,0,1})\|_{V_1}\big] < \infty$, $\|g(X)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})} < \infty$, and

$$\sup_{N\in\mathbb{N}}\Big(N^\rho\, \mathbb{E}\Big[\|\mathfrak{f}(X) - g(Y^{N,0,1})\|_{V_2}\Big]\Big) + \sup_{N\in\mathbb{N}}\Big(N^\rho\, \|g(X) - g(Y^{N,0,1})\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}\Big) < \infty. \qquad (5.35)$$

Furthermore, observe that it holds for all $L \in \mathbb{N}$, $\rho \in [0,\beta-\alpha) \setminus \{\frac{1}{2}\}$ that

$$\sum_{\ell=1}^{L}(2^\ell)^{-\rho}\, 2^{-\frac{1}{2}(L-\ell)} = 2^{-\frac{L}{2}}\sum_{\ell=1}^{L} 2^{(\frac{1}{2}-\rho)\ell} = 2^{-\frac{L}{2}}\,\tfrac{1-2^{(\frac{1}{2}-\rho)L}}{2^{\rho-\frac{1}{2}}-1} = 2^{-L\cdot\min\{\rho,\frac{1}{2}\}}\,\tfrac{1-2^{-|\frac{1}{2}-\rho|L}}{|1-2^{\rho-\frac{1}{2}}|} \leq \tfrac{2^{-L\cdot\min\{\rho,\frac{1}{2}\}}}{|1-2^{\rho-\frac{1}{2}}|} \qquad (5.36)$$

and

$$\sum_{\ell=1}^{L} (2^{\ell})^{-\frac{1}{2}} \, 2^{-\frac{1}{2}(L-\ell)} = 2^{-\frac{L}{2}} L. \tag{5.37}$$

Combining Corollary 5.14 with (5.35), (5.36), and (5.37) implies (5.28). This finishes the proof of Corollary 5.15. □

Corollary 5.15 can be applied to many SDEs. Under general conditions on the coefficient functions of the SDEs (see, e.g., Theorem 1.3 and Subsection 3.1 in [18]), suitable stopped-tamed Euler approximations (cf. (6) in [23] or (10) in [21]) converge in the strong sense with convergence rate $1/2$. We note that the classical Euler-Maruyama approximations do not satisfy condition (5.24) for most SDEs with superlinearly growing coefficients; see Theorem 2.1 in [20] and Theorem 2.1 in [22]. Moreover, under general conditions on the coefficients it holds that the solution process is strongly $1/2$-Hölder continuous in time. In conclusion, provided that a suitable numerical scheme is employed, Corollary 5.15 can be applied to many SDEs with $\beta = 1/2$.

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