Efficiency of institutional spending and investment rules

Schumacher, J.M.

Taylor & Francis
Taylor & Francis Group

Efficiency of institutional spending and investment rules

Johannes M. Schumacher

Section Quantitative Economics, Faculty of Economics and Business, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands

ABSTRACT

Endowment funds and similar institutions aim to generate a benefit stream of unlimited duration on the basis of an initially donated capital. Towards this purpose, responsible trustees need to design a spending policy as well as an investment policy. A combined spending and investment policy is said to be *efficient* if the total net present value of benefits that are paid according to the policy is equal to the initial capital, and *inefficient* if the total net present value is less than that. For several strategies, analytical expressions are given for the total net present value of benefits under the Black-Scholes assumptions. One of the strategies considered is the combination of a fixed-mix investment policy with a benefit policy that pays inflation-indexed benefits as long as ruin does not occur. This strategy is shown to be inefficient in many cases; the effective loss of capital can range from 5% to 15% under realistic parameter values. The inefficiency can be removed by adapting the investment policy and raising the benefits, without increasing the probability of ruin.

1. Introduction

The question of how much to spend now and how much to save for later has been studied extensively in the economic literature. In the present paper, the situation is considered of a fund that is intended to generate a stream of benefit payments that will continue for an indefinite period of time into the future. Examples of such funds include collective pension funds, university endowments, and trust funds such as the Nobel Foundation. The assets held by funds of this type are usually partly invested in risky assets, so that the returns on investments are uncertain. The trustees of the fund have to decide each year which percentage of available assets will be paid out as benefit.[1] Clearly, if capital is spent too quickly, then the danger of exhaustion of assets looms. On the other hand, policies that are too conservative may keep assets in reserve forever. Such policies are *inefficient*, in the sense that they do not make full use of available capital. The focus of the present paper is on this type of inefficiency, which is sometimes referred to as 'oversaving'.

There is an extensive literature on decumulation strategies and endowment spending rules; for surveys, one may consult Cejnek et al. (2014) and Bernhardt & Connelly (2018). Most of the literature focuses on investment and spending rules for retired individuals, rather than for long-lived institutions. While there are clear analogies between the two cases, the differences are equally clear. The main themes in the literature on endowment funds are intergenerational equity and sustainability;

CONTACT Johannes M. Schumacher ✉ j.m.schumacher@uva.nl Section Quantitative Economics, Faculty of Economics and Business, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands

[1] The term 'benefit' will be used in this paper as a generic name for the payments made by a long-term fund, covering also instances such as the salary of a professor occupying an endowed chair, or prize money in the case of the Nobel Foundation. Other terms used in the literature are 'withdrawal' or 'disbursement'.

little attention has been paid to the efficiency of spending and investment policies in the sense of making full use of available capital.

In the context of financial planning for retirement, the notion of oversaving is used by practitioners without a precise definition, and often refers to the accumulation phase rather than the decumulation phase. The scientific literature on spending/investment policies in the decumulation phase largely ignores the efficiency issue, with the notable exception of Scott et al. (2009).[2] In that paper, the authors obtain numerical results by Monte Carlo simulation in the finite-horizon (retirement) case for the strategy that keeps a constant percentage of capital in risky assets and that keeps spending on a constant level in real terms. In the present paper, the efficiency of the same strategy will be studied by analytic methods in the infinite-horizon (institutional) case, along with several other strategies that allow analytical expressions for the total net present value of benefits. The results indicate that efficiency should be an important concern not only for individuals but also for institutions.

The spirit of the present paper is similar to that of Blume (2010), Pye (2017), and Lindset & Matsen (2018). It is not a normative paper seeking to determine optimal policies given an objective function, nor an empirical paper such as Brown et al. (2014) describing the actual behavior of endowment funds. Rather, it is a policy evaluation paper that investigates certain performance characteristics of explicitly stated spending/investment policies. The underlying point of view is that the choice of a spending/investment policy is essentially a multi-objective problem, which cannot readily be translated into a problem with a single reasonably tractable aggregate utility function; instead, it is a worthwhile goal of analysis just to lay out clearly the tradeoffs between several objectives, so as to facilitate decision making by responsible trustees. Various important performance characteristics are discussed in the three papers cited above, but efficiency of use of capital is not covered.

There is a long tradition of academic research on consumption/savings problems; for historical sources one may refer to Ramsey (1928), Phelps (1962), Mirrlees (1974), Merton (1975), to name just a few, and textbook treatments can be found for instance in Turnovsky (2000) and Chang (2004). The typical approach in this literature is based on optimization of a non-saturating aggregate objective function. Efficiency is then not a concern, since it is strictly suboptimal to spend less than the full amount of available capital. In practice, however, benefit policies of long-term funds are usually not designed by optimizing a single aggregated objective function, but rather by a process in which parameters within a chosen policy framework are adjusted until a good compromise has been found between various competing objectives. When the latter approach is followed, there is in general no built-in guarantee for efficiency.

To make analysis possible, it has to be assumed that benefit and investment policies have been formulated explicitly. Many long-term funds do state explicit investment policies, often in the form of fixed percentages of capital to be invested in various asset categories. However, benefit policies are frequently only stated for situations in which asset returns are as expected, while deviations from the stated policy in other cases are left to the discretion of the board. Trends towards more transparency and better accountability, however, may stimulate the development of more explicit guidelines which also cover situations in which returns are higher or lower than foreseen.

The monetary units in this paper will be supposed to be inflation-adjusted, since the use of nominal units is not very meaningful when long horizons are considered. The standard Black-Scholes model that will be used to describe the economic environment will be interpreted in real terms, so that all growth rate parameters represent real growth rates. As a consequence of this interpretation, it becomes more important to consider cases in which the interest rate parameter is zero or even negative. Taking into account the possibility of zero or negative real interest rates is the main technical novelty in the treatment of the constant-mix constant-benefit strategy below, which otherwise relies mostly on results that are essentially already known; see Section 5.1 for detailed references to the literature. Some known results are presented below, however, with derivations that are believed to be

---

[2] Scott et al. (2009) also discuss a second source of inefficiency which is related to agents' objective functions and which will not be analyzed in the present paper.

new and simplified. For simplicity, it is assumed that the institution under study operates purely on the basis of initially available capital, without additional subsidies later on.[3]

The paper is organized as follows. Following the present introduction there is a brief section that specifies the economic setting and introduces a few notational conventions. Section 3 describes the spending and investment policies that will be analyzed in this paper. Efficiency conditions for two of these policies are given in Section 4 by means of direct calculation of the total net present value of benefits. A more indirect method is used in Section 5 to find an explicit formula for the total net present value of benefits in the case of the constant-mix constant-benefit strategy. The amount of inefficiency is determined by parameters that also influence the probability of finite-time ruin, so that there is a tradeoff between the two. A discussion of the tradeoff is given in Section 6, and it is shown how to escape from the unnecessarily tight conditions imposed by the fixed parametric framework of the constant-mix constant-benefit scheme. Conclusions follow in Section 7. There is a brief appendix providing an auxiliary result and details of the proof of one of the claims in the paper.

2. Economic setting and notational conventions

The economic setting of this paper is the standard Black-Scholes economy, characterized by the equations

$$dS_t = \mu_S S_t \, dt + \sigma_S S_t \, dW_t \tag{1a}$$

$$dB_t = rB_t \, dt, \tag{1b}$$

where W_t is standard Brownian motion. The riskless interest rate r, the expected growth rate μ_S of the risky asset,[4] and the risky asset volatility σ_S are constants. The *market price of risk* associated to the model (1) is $\lambda := (\mu_S - r)/\sigma_S$.[5]

As the present paper is concerned with long-term policies, all monetary variables will be taken to be expressed in term of a numéraire of the form $N(t) = e^{r_i t} N_0$, where r_i is a constant representing a required rate of growth of benefits. Technically speaking, the constant r_i does not necessarily have to be equal to the rate of inflation; nevertheless, it will be referred to below as such, for ease of expression and also because this is the typical situation. The parameters μ_S and r then denote the *real* expected rate of return and the *real* interest rate, respectively. Given the interpretation of the interest rate r as the real interest rate, or more generally as the difference between the nominal interest rate and a specified rate of growth of the nominal benefits, it is not inconceivable that the parameter r could take the value 0 or could even be negative. Positivity of r will therefore *not* be assumed.

Extensive use will be made below of the gamma function and its relatives. The following notation will be used:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt, \quad \gamma(\alpha, z) = \int_0^z t^{\alpha-1} e^{-t} \, dt, \quad \Gamma(\alpha, z) = \int_z^\infty t^{\alpha-1} e^{-t} \, dt,$$

$$P(\alpha, z) = \frac{\gamma(\alpha, z)}{\Gamma(\alpha)}, \quad Q(\alpha, z) = \frac{\Gamma(\alpha, z)}{\Gamma(\alpha)}, \quad E_1(z) = \Gamma(0, z) = \int_z^\infty t^{-1} e^{-t} \, dt. \tag{2}$$

The use of the letters P and Q to denote the regularized lower and upper incomplete gamma functions is standard in the literature on special functions. No confusion should arise with the use of the same letters to denote probability. The definition of the gamma function $\Gamma(\alpha)$ can be extended to all $\alpha \neq 0, -1, -2, \ldots$; in particular it will be used below not only for positive values of α, but also for $-1 <$

---

[3] Under complete market assumptions, a future stream of income for the fund, even if uncertain, may be replaced by an equivalent cash value at time 0. Such a reformulation is used for instance in Gollier (2008).

[4] The term 'expected growth rate' or 'expected rate of return' will be used in this paper to refer to the quantity $(\log(ES_T/S_0))/T$, while $E(\log(S_T/S_0))/T$ is called the 'expected geometric rate of return'.

[5] The market price of risk λ is also referred to as the *Sharpe ratio*.

$\alpha < 0$. The regularized upper incomplete gamma function $Q(\alpha, z)$ can likewise be used for this range of parameter values. As defined above, the unregularized lower incomplete gamma function $\gamma(\alpha, z)$ and its regularized version $P(\alpha, z)$ can be used only for $\alpha > 0$, although their definitions might be extended as well via the relationships $P(\alpha, z) = 1 - Q(\alpha, z)$ and $\gamma(\alpha, z) = \Gamma(\alpha)P(\alpha, z)$.

3. Spending and investment policies

Imagine an institution that at time 0 starts with an initial capital A_0 and that at later times does not receive any additional funding. The purpose of the institution is to pay a stream of benefits to support a long-lived goal, such as a university or an annual prize. The size of the institution's capital at time t, measured in terms of the numéraire $N(t)$, is denoted by A_t. The payment of benefits stops when the institution runs out of assets, i.e. when (if ever) the value A_t of the institution's capital reaches 0. The time at which this happens is called the *time of ruin* and is denoted by τ; if ruin never occurs, τ is taken to be equal to ∞. Under the Black-Scholes assumptions, the evolution of the institution's capital A_t is given by

$$\mathrm{d}A_t = \big(r + \alpha_t(\mu_S - r)\big)A_t\,\mathrm{d}t + \alpha_t\sigma_S A_t\,\mathrm{d}W_t - c_t\,\mathrm{d}t, \tag{3}$$

where α_t is the fraction of capital invested in risky assets at time t, and c_t is the spending level at time t.[6] The evolution of capital may also be described under the risk-neutral measure Q, as follows:

$$\mathrm{d}A_t = rA_t\,\mathrm{d}t + \alpha_t\sigma_S A_t\,\mathrm{d}W_t^Q - c_t\,\mathrm{d}t, \quad (t \le \tau) \tag{4}$$

where $W_t^Q = W_t + \lambda t$ is a Brownian motion under Q, and $\lambda = (\mu_S - r)/\sigma_S$ is the price of risk determined by the Black-Scholes model (1). This representation is especially convenient for purposes of valuation by means of the computation of expectations under the risk-neutral measure.

Defining an investment policy and a spending policy means specifying the processes α_t and c_t, for instance as functions of state variables. When this is done, one can define the *value* of the benefit stream, in the sense of the market value under the assumptions of the Black-Scholes economy, as the total time-0 value of benefits paid. This quantity, as a function of initial capital, is given by[7]

$$v(x) = E^Q\Big[\int_0^\tau e^{-rt}c_t\,\mathrm{d}t\Big|A_0 = x\Big], \quad \tau = \min\{t \ge 0 | A_t = 0\}. \tag{5}$$

The notion of value as used here needs to be clearly distinguished from the notion of value used for instance in the theory of optimal dividend policies (see Albrecher & Thonhauser 2009 for an extensive review). The latter notion is typically interpreted as representing a utility value, rather than a financial value; its definition is similar to (5), but real-world expectation is applied instead of expectation under the risk-neutral measure, and a subjective discount factor is used that does not need to be equal to the riskless interest rate. The definition (5) is also different from the notion of value of a stochastic perpetuity as it has been used for instance in Boyle (1976), Milevsky (1997), and Dufresne (2007). When $\alpha_t = \alpha$ and $c_t = c$ for all t, this notion of value is defined as $\int_0^\infty \exp(-(\mu t + \sigma W_t))\,\mathrm{d}t$ where $\mu = r + \alpha(\mu_S - r)$ and $\sigma = \alpha\sigma_S$. It is a random variable whose distribution function provides information concerning the probability of finite-time ruin for a given initial capital, and which is therefore useful for instance for reserving. As noted explicitly by Dufresne (2007, p. 137), it is *not* a 'price' in any sense of the word. The value as defined in (5), however, is a deterministic quantity, and it does represent the fair price of the stream of benefits as understood in mathematical finance.

---

[6] In other words, the inflation-indexed amount of benefit paid between t and $t + \Delta t$ is, to first order, equal to $c_t\Delta t$. The letter c is chosen as standing for 'consumption'.

[7] The expression for $v(x)$ as given in (5) is to be read as $\lim_{T \to \infty} E^Q[\int_0^{T \wedge \tau} e^{-rt}c_t\,\mathrm{d}t | A_0 = x]$, namely as the limit, as T tends to infinity, of the value of benefits received up to time T.

The main combination of spending and investment policies that will be considered in this paper is what will be called the *constant-mix constant-benefit* strategy, in which both the percentage α_t of capital invested in risky assets and the spending level c_t are constants.[8] In the context of retirement planning, this is by far the strategy that is most frequently advised in practice, according to Scott et al. (2009).[9] Providing a stream of benefits that is at least approximately constant in real terms is also often stated as a goal by long-lived institutions.[10] Analytical expressions for various performance characteristics of the constant-mix constant-benefit policy in the institutional case have been obtained by Ho et al. (2010) and by Lindset & Matsen (2018); however, these papers do not investigate efficiency.

In addition to the constant-mix constant-benefit strategy, many other combinations of spending and investment rules have been proposed in the literature, both in the case of individuals and in the case of institutions. Variations have been proposed in particular with respect to spending policies, while the constant-mix investment strategy is often maintained. For instance Lindset & Matsen (2018) discuss three additional spending policies, all in combination with the constant-mix investment strategy. The ratcheting-down strategy of Pye (2000) is also proposed in combination with fixed-mix investments. A policy that has constant benefits and a variable investment mix was obtained by Browne (1997) as the optimal solution for a problem in which a fixed level of benefits is imposed and the aim is to maximize (in a suitable sense) the time to bankruptcy, when the initial capital is less than the value of a perpetuity at the required level. An example of a policy that uses neither constant-mix investments nor constant benefits is provided by the ratcheting-up strategy of Dybvig (1999).

In this paper, attention will be focused in particular on three strategies that allow for analytical treatment. All of these are also analyzed in Lindset & Matsen (2018), but not with respect to their efficiency properties. The first one is the constant-mix constant-benefit strategy, as already mentioned. The second one is the constant-mix constant-rate strategy, in which the benefit paid is always a fixed proportion of the available assets. The third strategy that will be considered is the mean-reverting policy proposed in Lindset & Matsen (2018), in which the spending rate (paid benefit as a proportion of available assets) is not constant but moves around a 'normal' level, driven by an exogenous variable that represents economic conditions. In the cases of the constant and the mean-reverting spending rate, a direct calculation of the net present value of benefits is possible; these will be discussed first. The case of the constant-mix constant-benefit will be handled by means of a PDE method, which in this instance actually reduces to an ODE method.

4. Constant and mean-reverting spending rate

Consider first the scheme with constant investment mix and constant spending rate. Under this scheme, there exist constants α and s such that $\alpha_t = \alpha$ and $c_t = sA_t$ for all t. The evolution of fund capital then follows a geometric Brownian motion, which in particular implies that ruin never occurs, i.e. $\tau = \infty$. To compute the total net present value of benefits, it is convenient to describe the evolution of the institution's assets under the risk-neutral measure:

$$\mathrm{d}A_t = (r - s)A_t\,\mathrm{d}t + \sigma A_t\,\mathrm{d}W_t^Q \tag{6}$$

where $\sigma = \alpha\sigma_S$. From the above it follows that $E^Q[A_t] = \mathrm{e}^{(r-s)t}A_0$. Therefore, it is easy to establish the following.

---

[8] In relation to benefits, the term 'constant' should be read as 'constant in real terms'.

[9] Specifically, the usual advice, as originally developed in Bengen (1994), is to hold an investment portfolio with 60% in stocks and 40% in bonds, and to withdraw 4% of initial capital each year, with adjustment for inflation. The strategy is commonly known as the '4% rule'.

[10] See for instance www.nobelprize.org/uploads/2019/04/annual-report-2018.pdf.

Proposition 4.1: *The total net present value of benefits paid by the policy with constant investment mix and constant spending rate is equal to the initial value of capital. In other words, this policy is efficient.*

Proof: We have[11]

$$E^Q \left[\int_0^\infty e^{-rt} s A_t \, dt \right] = s \int_0^\infty e^{-rt} E^Q[A_t] \, dt = s A_0 \int_0^\infty e^{-st} \, dt = A_0. \qquad \blacksquare$$

The constant-mix constant-spending-rate policy therefore does well both on the criterion of avoiding ruin and on the criterion of being efficient. However, when the parameter α that determines the fraction of capital in risky assets is large, the volatility of benefits is large as well, since the policy transmits asset shocks immediately to benefit shocks. When α is given a low value, then, in order to prevent the expected growth rate of the fund's assets from becoming negative, the spending rate must be chosen close to the real interest rate. Trustees may not find this attractive either and may therefore be looking for alternative policies.

The mean-reverting spending rate policy proposed in Lindset & Matsen (2018) offers a way to smooth benefits by allowing the spending rate to vary. In this policy, the spending rate is a process s_t that satisfies an SDE of the form

$$ds_t = \kappa(\bar{s} - s_t) \, dt + \sigma_m \, dW_t^m \tag{7}$$

where $\kappa > 0$ is a mean reversion parameter, σ_m is a volatility parameter, and W_t^m is a standard Brownian motion that has correlation ρ with the Brownian motion W_t that drives the evolution of the risky asset. All three of κ, σ_m, and ρ are design parameters; typically, the correlation parameter ρ would be chosen to be negative, so that the spending rate is likely to be reduced when a positive asset shock occurs, and vice versa. It should be noted that the Equation (7) allows the spending rate to be negative, which implies that benefits can be negative as well.

To determine the total net present value of benefits determined by the model (3)–(7) with α_t equal to a constant α and $c_t = s_t A_t$, an assumption needs to be made concerning the prices of risk associated to the sources of uncertainty in the model. We can write $W_t^m = \rho W_t + \sqrt{1 - \rho^2} \, \hat{W}_t$ where W_t and $\hat{W}_t$ are independent Brownian motions. It will be assumed that the risk associated to $\hat{W}_t$ is not priced,[12] whereas the price of risk associated to W_t is λ as is the standard assumption in the Black-Scholes model. Writing $W_t = W_t^Q - \lambda t$, one obtains $W_t^m = \rho W_t^Q - \rho \lambda t + \sqrt{1 - \rho^2} \, \hat{W}_t$ where W_t^Q and $\hat{W}_t$ are independent Brownian motions under the pricing measure Q. The model (3)–(7) can then be rewritten under the risk-neutral measure in the form

$$dA_t = (r - s_t) A_t \, dt + \sigma A_t \, dW_t^Q \tag{8}$$

$$ds_t = \kappa(\tilde{s} - s_t) \, dt + \sigma_m \, d\tilde{W}_t^m \tag{9}$$

where $\tilde{W}_t^m = \rho W_t^Q + \sqrt{1 - \rho^2} \, \hat{W}_t$, $\sigma = \alpha \sigma_S$ is the portfolio volatility, and

$$\tilde{s} = \bar{s} - \rho \lambda \frac{\sigma_m}{\kappa} \tag{10}$$

is a risk-adjusted return level. This form of the model is useful for for value calculations.

---

[11] The interchange of expectation and integration is justified by Tonelli's theorem.

[12] Assuming a constant price of risk for $\hat{W}_t$ would introduce an additional parameter, but would not change the nature of the calculations.

To compute the total net present value of benefits when the spending policy is given by $c_t = s_t A_t$, we need an expression for $E^Q[s_t A_t]$. A standard calculation shows that

$$s_t = e^{-\kappa t} s_0 + (1 - e^{-\kappa t})\tilde{s} + \sigma_m \int_0^t e^{-\kappa(t-u)} \, d\tilde{W}_u^m. \tag{11}$$

Noting that

$$d\left(\log A_t - \frac{s_t}{\kappa}\right) = (r - \tilde{s} - \tfrac{1}{2}\sigma^2) \, dt + \sigma \, dW_t^Q - \frac{\sigma_m}{\kappa} \, d\tilde{W}_t^m$$

one finds, using (11),

$$A_t = A_0 \exp\left[\frac{1}{\kappa}\left(e^{-\kappa t} s_0 + (1 - e^{-\kappa t})\tilde{s}\right) + \frac{\sigma_m}{\kappa} \int_0^t e^{-\kappa(t-u)} \, d\tilde{W}_u^m \right.$$
$$\left. + \left(r - \tilde{s} - \frac{1}{2}\sigma^2\right) t + \sigma W_t^Q - \frac{\sigma_m}{\kappa} \tilde{W}_t^m - \frac{s_0}{\kappa}\right]. \tag{12}$$

From the above, it is seen that s_t and $\log A_t$ are jointly normally distributed. One may therefore use the following rule (see the Appendix for a derivation): if X and Y are jointly normal variables, then

$$E[Xe^Y] = [EX + \text{Cov}(X, Y)] \exp\left(EY + \frac{1}{2}\text{Var}(Y)\right). \tag{13}$$

To apply (13), the following expressions are required:[13]

$$Es_t = e^{-\kappa t} s_0 + (1 - e^{-\kappa t})\tilde{s} \tag{14}$$

$$\text{Cov}(s_t, \log A_t) = \frac{\sigma_m^2}{\kappa} \int_0^t e^{-2\kappa(t-u)} \, du + \rho\sigma\sigma_m \int_0^t e^{-\kappa(t-u)} \, du - \frac{\sigma_m^2}{\kappa} \int_0^t e^{-\kappa(t-u)} \, du$$
$$= \frac{\sigma_m^2}{\kappa} \frac{1 - e^{-2\kappa t}}{2\kappa} + \left(\rho\sigma\sigma_m - \frac{\sigma_m^2}{\kappa}\right) \frac{1 - e^{-\kappa t}}{\kappa} \tag{15}$$

$$E[\log A_t] = \log A_0 + \frac{1 - e^{-\kappa t}}{\kappa}(\tilde{s} - s_0) + \left(r - \tilde{s} - \frac{1}{2}\sigma^2\right) t \tag{16}$$

$$\text{Var}(\log A_t) = \frac{\sigma_m^2}{\kappa^2} \int_0^t e^{-2\kappa(t-u)} \, du + \sigma^2 t + \frac{\sigma_m^2}{\kappa^2} t$$
$$+ 2\rho \frac{\sigma_m}{\kappa} \sigma \int_0^t e^{-\kappa(t-u)} \, du - 2\frac{\sigma_m^2}{\kappa^2} \int_0^t e^{-\kappa(t-u)} \, du - 2\rho\sigma \frac{\sigma_m}{\kappa} t$$
$$= \frac{\sigma_m^2}{\kappa^2} \frac{1 - e^{-2\kappa t}}{2\kappa} + \left(\sigma^2 + \frac{\sigma_m^2}{\kappa^2} - 2\rho\sigma \frac{\sigma_m}{\kappa}\right) t + 2\left(\rho\sigma \frac{\sigma_m}{\kappa} - \frac{\sigma_m^2}{\kappa^2}\right) \frac{1 - e^{-\kappa t}}{\kappa}. \tag{17}$$

---

[13] To simplify the notation, reference to the risk-neutral measure Q will be suppressed in the calculation leading up to (20).

Define the function of time $F(t)$ by

$$F(t) = \left(\frac{\sigma_m^2}{2\kappa^2} - \rho\sigma \frac{\sigma_m}{\kappa} - \tilde{s} \right) t + \left(\tilde{s} - s_0 + \rho\sigma \frac{\sigma_m}{\kappa} - \frac{\sigma_m^2}{\kappa^2} \right) \frac{1 - e^{-\kappa t}}{\kappa} + \frac{\sigma_m^2}{2\kappa^2} \frac{1 - e^{-2\kappa t}}{2\kappa}. \quad (18)$$

It follows from (16) and (17) that

$$E[\log A_t] + \frac{1}{2}\mathrm{Var}(\log A_t) = \log A_0 + rt + F(t). \quad (19)$$

Moreover, from (14) and (15), it can be verified that

$$E s_t + \mathrm{Cov}(s_t, \log A_t) = -F'(t).$$

Therefore, according to (13), the net present value of benefits up to time T is given, for all $T > 0$, by[14]

$$E^Q\left[\int_0^T e^{-rt} s_t A_t \, dt \right] = -\int_0^T F'(t) A_0 e^{F(t)} \, dt = A_0 \left(1 - \exp(F(T)) \right). \quad (20)$$

In view of the expression (18), it is convenient to introduce the quantity $\hat{s}$ defined by

$$\hat{s} = \tilde{s} + \rho\sigma \frac{\sigma_m}{\kappa} - \frac{\sigma_m^2}{2\kappa^2} = \bar{s} - \rho(\lambda - \sigma) \frac{\sigma_m}{\kappa} - \frac{\sigma_m^2}{2\kappa^2}. \quad (21)$$

The conclusion can be stated in terms of the function $v(x)$ defined by (4)–(5), indicating the total net present value of benefits as a function of initial capital.

Proposition 4.2: *In the case of the policy with constant-mix investments and mean-reverting spending rates, the total net present value function $v(x)$ defined by (4) and (5) takes the following form:*

$$v(x) = \begin{cases} x & \text{if } \hat{s} > 0 \\ \left(1 - \exp\left(-\left(s_0 + \frac{1}{4}\sigma_m^2/\kappa^2 \right)/\kappa \right) \right) x & \text{if } \hat{s} = 0 \\ -\infty & \text{if } \hat{s} < 0. \end{cases} \quad (22)$$

Consequently, the policy is efficient if and only if the quantity $\hat{s}$ defined in (21) is positive.

Proof: Under the mean-reverting spending rate scheme, ruin does not occur so that the stopping time τ appearing in the definition (5) is equal to infinity. From the expression (18) it is seen that, as t tends to infinity, $F(t)$ tends to $-\infty$ if $\hat{s} > 0$, and to ∞ if $\hat{s} < 0$. In the boundary case $\hat{s} = 0$, we have

$$\lim_{t \to \infty} F(t) = \log A_0 + \left(\tilde{s} - s_0 + \rho\sigma \frac{\sigma_m}{\kappa} - \frac{\sigma_m^2}{\kappa^2} \right) \frac{1}{\kappa} + \frac{\sigma_m^2}{2\kappa^2} \frac{1}{2\kappa} = \log A_0 - \frac{1}{\kappa} \left(s_0 + \frac{\sigma_m^2}{4\kappa^2} \right).$$

The results shown in (22) are obtained from (20) by using these limit values, taking note of the interpretation of the total net present value function as given in footnote 7. ∎

The value $-\infty$ that appears in case $\hat{s} < 0$ is possible because, as mentioned above, the scheme allows negative benefits. The outcome in this case may be related to scenarios in which the spending rate becomes negative, causing the asset value A_t to increase so that the benefit $s_t A_t$ becomes even more negative, which leads to even stronger growth of A_t, and so on.

---

[14] The validity of the interchange of expectation and integration follows from Fubini's theorem by noting that $\int_0^T e^{-rt} E|s_t A_t| \, dt$ is finite. The finiteness of the integral can be seen for instance by using the Cauchy-Schwarz inequality.

5. Constant mix/constant benefits

5.1. *Interpretations of the scheme*

Under the assumptions of Section 2, the evolution of the institution's assets in the scheme with a constant proportion α of capital invested in risky assets and a constant benefit $c > 0$ per unit of time is given by

$$dA_t = \mu A_t \, dt + \sigma A_t \, dW_t - c \, dt \quad (t \leq \tau) \tag{23}$$

where $\sigma := \alpha \sigma_S$ is the portfolio volatility, $\mu = r + \lambda \alpha \sigma_S$ is the expected growth rate, and τ is, as usual, the time of ruin. The benefit stream that is defined by the constant-mix constant-benefit scheme may be described as a *risky perpetuity*, as opposed to a guaranteed perpetuity which provides a constant (indexed) benefit stream forever. The total net present value of benefits can be described simply as the *value* of the risky perpetuity. Specializing the expression (5) to the present case, we have

$$v(x) = E^Q \left[\int_0^\tau e^{-rt} c \, dt \, \Big| \, A_0 = x \right], \qquad \tau = \min\{t \geq 0 | A_t = 0\}. \tag{24}$$

The integral $\int_0^\tau e^{-rt} c \, dt$ is equal to $c(1 - e^{-r\tau})/r$ for $r \neq 0$ and to $c\tau$ for $r = 0$, so that we can also write

$$v(x) = \frac{c}{r} \left(1 - E_x^Q[e^{-r\tau}]\right) \quad (r \neq 0), \quad v(x) = c E_x^Q[\tau] \quad (r = 0) \tag{25}$$

where the subindex x in the expectation symbol refers to conditioning on $A_0 = x$.

A risky perpetuity can also be viewed as a defaultable perpetual bond paying constant coupons. A closed-form formula for the value of such a bond has been provided by Merton (1974, p. 468), based on a similar formula for perpetual American options on stock paying constant dividends (Merton 1973, p. 172). The derivation of the valuation formula starts from a suitable adaptation of the Black-Scholes differential equation to reflect the constant dividend payments. Since the value of a perpetual option does not depend on time, the Black-Scholes partial differential equation is reduced to an ordinary differential equation. By a judicious change of variables, Merton (1973) transforms this equation into Kummer's equation, the solution of which is written in terms of the confluent hypergeometric function. In Section 5.3 below, a more elementary derivation of the valuation formula will be shown. Several alternative expressions will be given, which can be useful for particular purposes. Special attention will be given to the formulation of appropriate boundary conditions in case the real interest rate r is zero or negative.

It is well known (see for instance Milevsky (1997), Metzler (2013), or Section 6.1 below) that the time of ruin of the process (23) can alternatively be described as the time at which a certain integrated geometric Brownian motion process reaches a fixed value.[15] To find a formula for the Laplace transform of the hitting time distribution of integrated geometric Brownian motion is therefore a generalization of the valuation problem for risky perpetuities in the Black-Scholes framework. Such a formula was obtained by Metzler (2013), using essentially the same techniques as Merton (1973). An alternative (also non-elementary) derivation can be found in Cui & Nguyen (2017); see also Cui et al. (2019) for duality relationships that can be of help in more general situations. The process determined by the stochastic differential equation (23) is sometimes called *geometric Brownian motion with affine drift*. Processes of this form arise in a number of applications; see for instance Shiryaev (1961), Wong (1964), Linetsky (2004), and Peskir (2006). A wide array of advanced analytical and numerical techniques has been developed for various problems associated to this process. The point of Section 5.3 below, however, that an elementary solution method suffices for the particular problem of determining the total net present value of benefits. Note that we need to compute $E_x^Q[e^{-r\tau}]$ for a

---

[15] For generalizations of this connection, see Salminen & Yor (2005).

process $\{A_t\}$ that has drift $rA_t - c$ and constant volatility; the double appearance of the parameter r enables the shortcut.

Ruin models with investment income and various forms of the surplus process have been studied extensively in the literature; see Paulsen (2008) for a survey. In particular, in Paulsen & Gjessing (1997) the time of ruin is studied for a model of the form

$$dA_t = \mu A_t \, dt + \sigma A_t \, dW_t^R + p \, dt + \sigma_P \, dW_t^P \tag{26}$$

in which W_t^R and W_t^P are independent Brownian motions. The authors provide a formula for the Laplace transform of the time of ruin (Equation (3.7) in the cited paper). Direct application to the process (23) is precluded, however, by the fact that the formula depends crucially on the assumption $\sigma_P > 0$. Of course, that assumption is natural if p is positive, which in turn is natural when p is interpreted, in the classical way, as a premium.

Models with negative deterministic drift were already discussed by Seal (1969, p. 116 ff.), who notes that such 'dual' models can be motivated in the context of an annuity business, and who also points out the connection to queueing theory. Indeed, one might look at (23) as a netput process in a queueing model where arrivals are modeled by a diffusion process and the service rate is constant. The use of the geometric Brownian motion in the queueing context, however, would seem fairly unusual.

5.2. Differential equation and boundary conditions

A straightforward modification of the standard Black-Scholes equation for a time-independent contingent claim (see Merton 1973, or, from an actuarial perspective, Paulsen & Gjessing (1997)) leads to the following ordinary differential equation for the claim value $v(x)$:

$$\frac{1}{2}\sigma^2 x^2 v''(x) + (rx - c)v'(x) - rv(x) + c = 0 \tag{27}$$

where the variable x refers to the current value of the underlying process (23). The same equation is satisfied by any claim that does not depend on time; the specific nature of the claim is expressed through boundary conditions. When the interest rate r is positive, the standard boundary conditions are $v(0) = 0$ and $\lim_{x\to\infty} v(x) = c/r$. The latter condition is motivated by the fact that c/r is the value of a non-defaultable perpetuity. However, as mentioned above, positivity of the real interest rate r is not assumed in this paper. We therefore turn to an alternative characterization.

Under absence of arbitrage, the value of the benefits received from the constant-mix constant-benefit scheme must be bounded above by the amount of capital that is required at time 0 in order to construct a portfolio strategy that replicates the payments resulting from the scheme. A description of such portfolio strategies is stated in the following lemma.

Lemma 5.1: *Suppose that $v(x)$ is a solution of (27) that satisfies $v(0) = 0$ and $v(x) > 0$ for $x > 0$. Then there exists a portfolio strategy with initial capital $v(A_0)$ that replicates the benefits of the constant-mix constant-benefit scheme with initial capital A_0.*

Proof: Define a process H_t by $H_t = v(A_t)$. By Itô's rule and (27), we have

$$
\begin{aligned}
dH_t &= v'(A_t) \, dA_t + \frac{1}{2}\sigma^2 A_t^2 \, dW_t^Q \\
&= \left(v'(A_t)(rA_t - c) + \frac{1}{2}\sigma^2 A_t^2\right) dt + v'(A_t)\sigma A_t \, dW_t^Q \\
&= (rv(A_t) - c) \, dt + v'(A_t)\sigma A_t \, dW_t^Q \\
&= (rH_t - c) \, dt + v'(A_t)\sigma A_t \, dW_t^Q.
\end{aligned} \tag{28}
$$

The fact that the drift term under the risk-neutral measure Q is of the form $rH_t - c$ shows that the process H_t is the value process of a self-financing portfolio strategy with consumption at the constant rate c. Because the Black-Scholes economy is complete, a strategy can be constructed that realizes this portfolio. Since $H_t > 0$ when $A_t > 0$, and $H_t = 0$ when $A_t = 0$, the process H_t reaches 0 at exactly the same time as the process A_t does, so that the payments from the constant-mix constant-benefit scheme are indeed replicated. The required initial capital for the strategy is $H_0 = v(A_0)$. ∎

To construct the replicating strategy explicitly, one needs to specify portfolio holdings processes h_t^S and h_t^B, representing the number of units held in the risky asset S and in the riskless asset B respectively, such that the following equations are satisfied (see for instance Björk 1998, Ch. 5):

$$H_t = h_t^S S_t + h_t^B B_t \tag{29a}$$

$$dH_t = h_t^S \, dS_t + h_t^B \, dB_t - c \, dt. \tag{29b}$$

To this end, define h_t^S and h_t^B as follows (recall that the constant α represents the fraction of capital invested in the risky asset S according to the fixed-mix policy):

$$h_t^S = \frac{\alpha v'(A_t) A_t}{S_t}, \quad h_t^B = \frac{H_t - \alpha v'(A_t) A_t}{B_t}. \tag{30}$$

Note that the variable A_t is used here as a state variable, rather than as the value of an actually constructed portfolio. The condition (29a) is immediately seen to be satisfied. Moreover, we have

$$h_t^S \, dS_t + h_t^B \, dB_t = \frac{\alpha v'(A_t) A_t}{S_t} \left(rS_t \, dt + \sigma_S S_t \, dW_t^Q \right) + \frac{H_t - \alpha v'(A_t) A_t}{B_t} rB_t \, dt$$

$$= rH_t \, dt + v'(A_t) \sigma A_t \, dW_t^Q = dH_t + c \, dt \tag{31}$$

by (28), so that (29b) is satisfied as well. On the basis of the lemma, the following proposition can be stated.

Proposition 5.2: *The function $v(x)$ defined for $x \geq 0$ by (23) and (24) satisfies the following properties:*

   (i)   *$v(0) = 0$, and $v(x) > 0$ for $x > 0$;*
  (ii) *$v(x)$ is a solution of the differential equation (27);*
 (iii) *if $\tilde{v}(x)$ is a function that satisfies (i) and (ii), then $v(x) \leq \tilde{v}(x)$ for all $x \geq 0$.*

In other words, the value function $v(x)$ is pointwise minimal within the class of functions satisfying (i) and (ii).

It will be shown below that the properties stated in the proposition above are sufficient to identify the value function uniquely.

5.3. Solving the differential equation

Since $v(x) = x$ is a particular solution of the inhomogeneous differential equation (27), the general solution of the equation can be written as $v(x) = x - y(x)$, where $y(x)$ is the general solution of the

corresponding homogeneous equation

$$\frac{1}{2}\sigma^2 x^2 y''(x) + (rx - c)y'(x) - ry(x) = 0. \tag{32}$$

To satisfy (i) in Proposition 5.2, we need

$$y(0) = 0, \quad y(x) < x \text{ for all } x > 0. \tag{33}$$

Let $y_1(x)$ be a nonzero solution of (32) that takes the value 0 at 0; then the general solution of (32) satisfying $y(0) = 0$ can be written in the form $y(x) = c_1 y_1(x)$ with $c_1 \in \mathbb{R}$. Under the assumption (which will be justified below) that $y_1(x) > 0$ for all $x > 0$, the inequality condition in (33) can be written as

$$c_1 \leq \inf_{x>0} \frac{x}{y_1(x)}. \tag{34}$$

The general form of functions satisfying (i) and (ii) of Proposition 5.2 is given by $v(x) = x - c_1 y_1(x)$, where c_1 satisfies (34). Since we are looking for the minimal one among these functions, the constant c_1 should be taken as large as possible. Therefore, the value function is given by

$$v(x) = x - c_1 y_1(x) \qquad \text{with} \quad c_1 = \inf_{x>0} \frac{x}{y_1(x)} = \left(\sup_{x>0} \frac{y_1(x)}{x} \right)^{-1}. \tag{35}$$

The strict inequality in (33) is satisfied if the infimum in (34) (or, equivalently, the supremum in (35)) is not reached for finite x.

It remains to solve the homogeneous differential equation (32). One can reduce the number of parameters from three to two by defining (assuming $\sigma > 0$)

$$a = 2r/\sigma^2, \quad b = 2c/\sigma^2. \tag{36}$$

The equation (32) then can be rewritten as

$$x^2 y''(x) + (ax - b)y'(x) - ay(x) = 0. \tag{37}$$

While it is possible to solve this via a transformation to Kummer's equation, as in Merton (1973) and Metzler (2013), a route that seems shorter is the following. Applying differentiation to (37), one finds that any solution of the second-order equation (32) must also satisfy

$$x^2 y'''(x) + \big((a+2)x - b\big)y''(x) = 0. \tag{38}$$

This equation, although of third order, is easy to solve since it can be read as a first-order equation in $y''(x)$. By direct integration, one finds that the general solution of (38) is

$$y(x) = c_1 y_1(x) + c_2 x + c_3, \qquad \text{with} \quad y_1(x) = \int_0^x \int_0^u z^{-a-2} e^{-b/z} \, dz \, du. \tag{39}$$

To satisfy the condition $y(0) = 0$, one needs to take $c_3 = 0$. Noting that $y_1(0) = y_1'(0) = y_1''(0) = 0$, one sees that the condition $c_2 = 0$ is necessary and sufficient to ensure that $y(x)$ is in fact a solution of the second-order equation (32). Therefore, the value function we are looking for is given by (35), with $y_1(x)$ as specified in (39).

From (39), it is clear that $y_1(x) > 0$ for all $x > 0$. It is also seen that $y_1(x)$ is strictly convex, which, in combination with the fact that $y_1(0) = 0$, proves that the ratio $y_1(x)/x$ is strictly increasing in x. By l'Hôpital's rule, one has

$$\sup_{x>0} \frac{y_1(x)}{x} = \lim_{x \to \infty} \frac{y_1(x)}{x} = \int_0^\infty z^{-a-2} e^{-b/z} \, dz. \tag{40}$$

Applying the substitution $t = b/z$, one finds that the integral is equal to $b^{-(a+1)}\Gamma(a+1)$ if $a > -1$. For $a \leq -1$, the integral is infinite, so that in this case the number c_1 defined in (35) is 0.

In conclusion, the total time-0 value of benefits as defined in (23)–(24) is given in explicit form by the formula

$$v(x) = x - \frac{b^{a+1}}{\Gamma(a+1)} \int_0^x \int_0^u z^{-a-2} e^{-b/z} \, dz \, du \quad (a > -1), \qquad v(x) = x \quad (a \le -1). \quad (41)$$

Some rewriting may still be applied. From integration by parts, one finds

$$y_1(x) = \int_0^x \int_0^u z^{-a-2} e^{-b/z} \, dz \, du = x \int_0^x z^{-a-2} e^{-b/z} \, dt - \int_0^x z^{-a-1} e^{-b/z} \, dz. \quad (42)$$

The substitution $t = b/z$ leads to (see (2) for notation)

$$y_1(x) = x b^{-a-1} \int_{b/x}^\infty t^a e^{-t} \, dt - b^{-a} \int_{b/x}^\infty t^{a-1} e^{-t} \, dt = \frac{x\Gamma(a+1, b/x) - b\Gamma(a, b/x)}{b^{a+1}}. \quad (43)$$

The expression (41) may therefore be rewritten as follows, for $a > -1$ and $a \ne 0$:

$$v(x) = x - \frac{x\Gamma(a+1, b/x) - b\Gamma(a, b/x)}{\Gamma(a+1)} = xP(a+1, b/x) + k\,Q(a, b/x), \quad k = \frac{b}{a} = \frac{c}{r}. \quad (44)$$

The formula above does not apply to the case $a = 0$, because $Q(a, b/x)$ is not defined in that case. Noting that $\Gamma(1, z) = e^{-z}$, and writing the exponential integral $\Gamma(0, z) = \int_z^\infty t^{-1} e^{-t} \, dt$ in the more usual way as $E_1(z)$, we find from (41) and (42) that, for $a = 0$,

$$v(x) = x(1 - e^{-b/x}) + bE_1(b/x). \quad (45)$$

The following proposition summarizes the results of this section. For completeness, the deterministic case ($\sigma = 0$) is also included; the statements in the proposition concerning this situation are easily verified.

Proposition 5.3: *In the case of the policy with constant-mix investments and constant benefits, the total net present value function $v(x)$ defined by (4) and (5) takes the following form, where $\sigma = \alpha\sigma_S$ is the portfolio volatility and c is the spending level:*

$$v(x) = \begin{cases} x & \text{if } r \le -\frac{1}{2}\sigma^2 \\ \min(x, c/r) & \text{if } r > 0 \text{ and } \sigma = 0 \\ x\left(1 - \exp\left(\frac{-2c}{\sigma^2 x}\right)\right) + \frac{2c}{\sigma^2} E_1\left(\frac{2c}{\sigma^2 x}\right) & \text{if } r = 0 \text{ and } \sigma > 0 \\ xP\left(\frac{2r}{\sigma^2} + 1, \frac{2c}{\sigma^2 x}\right) + \frac{c}{r} Q\left(\frac{2r}{\sigma^2}, \frac{2c}{\sigma^2 x}\right) & \text{if } r > -\frac{1}{2}\sigma^2, r \ne 0, \text{ and } \sigma > 0. \end{cases} \quad (46)$$

In particular, the policy is efficient if and only if either the real interest rate is positive, nothing is invested in risky assets, and the ratio of spending level to initial capital is at least equal to the real interest rate, or the real interest rate is zero or negative, and the portfolio volatility σ is at most equal to $\sqrt{-2r}$.

A plot of the total time-0 value of benefits as a function of initial capital is shown in Figure 1, for several values of the real interest rate r.
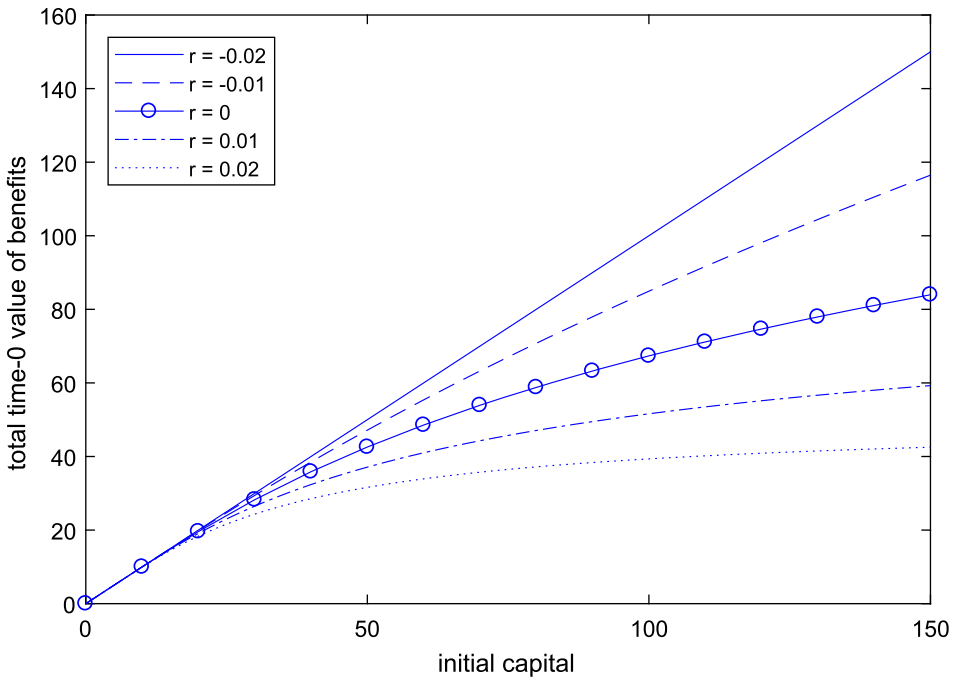
Figure 1. The plot shows the total time-0 value of benefits obtained from a constant-mix constant-benefit scheme under the Black–Scholes assumptions, for different values of the real interest rate as indicated in the legend. The benefit level is normalized to 1, and the assumed value of portfolio volatility σ is 20%.

5.4. Alternative expressions

The expression for the value of a risky perpetuity as given by Merton (1974, p. 468), based on the formula from Merton (1973, p. 172) for the value of a perpetual American option written on a stock paying constant dividends, is

$$v(x) = \frac{b}{a} \left(1 - \frac{(b/x)^a}{\Gamma(a+2)} M(a, a+2, -b/x) \right) \tag{47}$$

where M denotes the confluent hypergeometric function. The same expression is obtained by applying the expression given by Metzler (2013) for the Laplace transform of the hitting time of integrated geometric Brownian motion to the case at hand. Although it may not be obvious at first sight, one can verify, using known properties of the confluent hypergeometric function, that this expression is indeed the same as (44).[16]

Besides the representation in terms of the hypergeometric function, there are several other ways of representing the value of a risky perpetuity. One of these is (for $a > -1, a \neq 0$)

$$v(x) = \int_0^\infty \min \left(x \frac{t^a e^{-t}}{\Gamma(a+1)}, k \frac{t^{a-1} e^{-t}}{\Gamma(a)} \right) dt. \tag{48}$$

To see that this is the same as (44), note that the minimum is determined by the first expression between the brackets when $t < b/x$, and by the second expression when $t > b/x$. The analogous

---

[16] More specifically, this can be done on the basis of relations 13.4.2, 13.6.10, 13.6.12, and 6.5.21 in Abramowitz & Stegun (1972).

representation in case $r = 0$ is

$$v(x) = \int_0^\infty e^{-t} \min(x, b/t)\, \mathrm{d}t. \tag{49}$$

The expected properties $v(x) \le x$ and (for $a > 0$) $v(x) \le k = c/r$, which do not seem immediately apparent from a representation such as (47), are placed into evidence by (48) and (49).

It is seen from (41) that the function $v(x)$ is concave, so that there is a variational representation of $v(x)$ by means of its Fenchel-Legendre transform. The following expression is related to this by a change of variables:

$$v(x) = \min_{u > 0}\left[xP(a + 1, u) + kQ(a, u) \right] \quad (a > -1,\ a \ne 0) \tag{50a}$$

$$v(x) = \min_{u > 0}\left[x(1 - e^{-u}) + bE_1(u) \right] \quad (a = 0). \tag{50b}$$

A proof of this representation is given in the appendix. In both expressions above, the minimum is reached at the point $u = b/x$. The representation (50) is similar to the variational form of the Black-Scholes formula that was found by Décamps & Rochet (1997).

The variational representation is for some purposes more convenient than the explicit form. In particular, it allows for a quick calculation of the derivative of $v(x)$ by means of the envelope theorem. This theorem (see for instance Corbae et al. 2009, Theorem 5.9.8) implies that $v'(x)$ may be found by computing the *partial* derivative of the function appearing within the minimization operation in (50) with respect to x, and evaluating the result at the value of u at which the minimum is achieved. Using the envelope theorem, one immediately finds from (50):

$$v'(x) = P(a + 1, b/x) \quad (a > -1,\ a \ne 0), \qquad v'(x) = 1 - e^{-b/x} \quad (a = 0). \tag{51}$$

This is the equivalent in the present context of the 'delta' (the derivative of the claim price with respect to the value of the underlying) of option pricing theory. Like the delta, the expression (51) plays an important role in replication, as seen from (30).

6. Tradeoffs in the constant-mix/constant-benefit scheme

The primary performance measures within the constant-mix/constant-benefit scheme are the ruin probability and the ratio of the total time-0 value of benefits to the initial capital. The present section focuses on the tradeoffs between these quantities.

6.1. The probability of ruin

The probability of ruin for the model (23) can be derived directly from well known results, as also shown in Milevsky & Robinson (2000). The solution of the stochastic differential equation (23) is given by (see for instance Mao 2007, Chapter 3, Equation (4.3))

$$A_t = \exp\left(\left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \right) \left(A_0 - \int_0^t \exp\left[-\left(\left(\mu - \frac{1}{2}\sigma^2 \right) s + \sigma W_s \right) \right] c\, \mathrm{d}s \right). \tag{52}$$

From this, it follows that finite-time ruin occurs in the event that

$$\int_0^\infty \exp\left[-\left(\left(\mu - \frac{1}{2}\sigma^2 \right) s + \sigma W_s \right) \right] \mathrm{d}s > A_0/c. \tag{53}$$

It is shown by Dufresne (1990, Proposition 4.4.4)[17] that, if $\mu - \frac{1}{2}\sigma^2 > 0$ (in other words, the expected real geometric rate of return is positive), the reciprocal of the random variable that appears on the

---

[17] An alternative proof is given by Milevsky (1997). For additional references, see also Bertoin & Yor (2005) and Dufresne (2007).

left hand side of the inequality above follows a gamma distribution with parameters $2\mu/\sigma^2 - 1$ and $\frac{1}{2}\sigma^2$. If $\mu - \frac{1}{2}\sigma^2 \leq 0$, then the value of the left hand side is ∞, so that finite-time ruin occurs with probability 1. The probability of ruin is therefore given by

$$P(\tau < \infty) = P\left(\frac{2\mu}{\sigma^2} - 1, \frac{2c}{\sigma^2 x}\right), \quad A_0 = x, \quad \mu > \frac{1}{2}\sigma^2. \tag{54}$$

6.2. Selection of the asset mix

As discussed above, the condition $\mu - \frac{1}{2}\sigma^2 > 0$ is necessary and sufficient in the model (23) to avoid certainty of ruin. Within the inflation-adjusted Black-Scholes model, we have $\mu = r + \sigma\lambda$ where r is the real interest rate and λ is the market price of risk. Therefore, to avoid certain ruin, the volatility parameter σ must be chosen such that $r + \sigma\lambda - \frac{1}{2}\sigma^2 > 0$. It is easily verified that this condition holds true if and only if $r + \frac{1}{2}\lambda^2 > 0$ and

$$\lambda - \sqrt{2r + \lambda^2} < \sigma < \lambda + \sqrt{2r + \lambda^2}. \tag{55}$$

From this it is seen that, to avoid certain ruin, the volatility should not be chosen too large. If the real interest is negative, it should not be chosen too small either. If the real rate is equal to $-\frac{1}{2}\lambda^2$ or less, then ruin becomes unavoidable within the chosen scheme. For instance, if $\lambda = 0.3$, then the critical level of the real rate is -4.5%, a value that seems well below realistic levels.

The explicit expression (54) makes it easy to plot the probability of ruin within the model (23) as a function of parameters such as the spending level c and the portfolio volatility σ. A graph of this type is shown in Figure 2.[18] The variable on the horizontal scale is the fraction in risky assets α, which determines the volatility of the fund's portfolio through $\sigma = \alpha\sigma_S$ where σ_S is the volatility of the risky asset in the Black-Scholes model (taken to be equal to 0.2 in the figure).[19] Different curves are drawn, relating to different values of the ratio c/A_0 of initial benefit level to initial capital. This ratio is referred to as the *expenditure ratio*.[20] If the expenditure ratio is less than the real interest rate, then the lowest probabilities of ruin are obtained for riskless portfolios. However, if the expenditure ratio exceeds the real interest rate, then it becomes necessary to choose risky portfolios in order to avoid certainty of ruin. For instance, the plot indicates that, when the expenditure ratio is 2.5%, the lowest probability of ruin is obtained by choosing a portfolio composition with 58% of capital invested in risky assets, resulting in 11.6% volatility of the investments. By doing so, the fund managers have an 87% chance of being able to maintain indexed payments for all time. If this probability is considered too low for the purposes of the fund, then, within the confinements of the given scheme, the only option is to reduce the expenditure ratio.

Numerical values for a number of cases are given in Table 1, which also gives an indication of the sensitivity with respect to the assumed value of the Sharpe ratio. As a first step to determining a suitable investment mix, a board of trustees may use the table to strike a suitable compromise between expenditure ratio on the one hand and the probability of ruin on the other. Then, the corresponding value of the fraction in risky assets can be chosen. It is seen from the table that uncertainty concerning the true value of the Sharpe ratio (or, equivalently, the equity premium) has a strong impact on the minimal probability of ruin, but less so on the corresponding fraction of capital in risky assets.

---

[18] Blume (2010) gives analogous graphs for a somewhat different spending policy in the institutional case (but limited to a 50-year horizon), using historical simulation. Results in tabular form, both for the individual and institutional case and using exact as well as approximate analytic expressions, can be found in Milevsky & Robinson (2005). See also Scott et al. (2009, Table 1) for ruin probabilities for the case of individuals, computed by Monte Carlo, and analogous results in Pye (2000) for a modified spending policy.

[19] This form of presentation is chosen to show the results in terms that are commonly used in practice.

[20] To be precise, c/A_0 is the *initial* expenditure ratio. Since the benefits are subject to deterministic indexation whereas capital growth is affected by random returns, the ratio of annual benefits to available capital is likely to change in the course of time. The term 'expenditure ratio' is nevertheless used in this paper instead of 'initial expenditure ratio', in the interest of brevity.
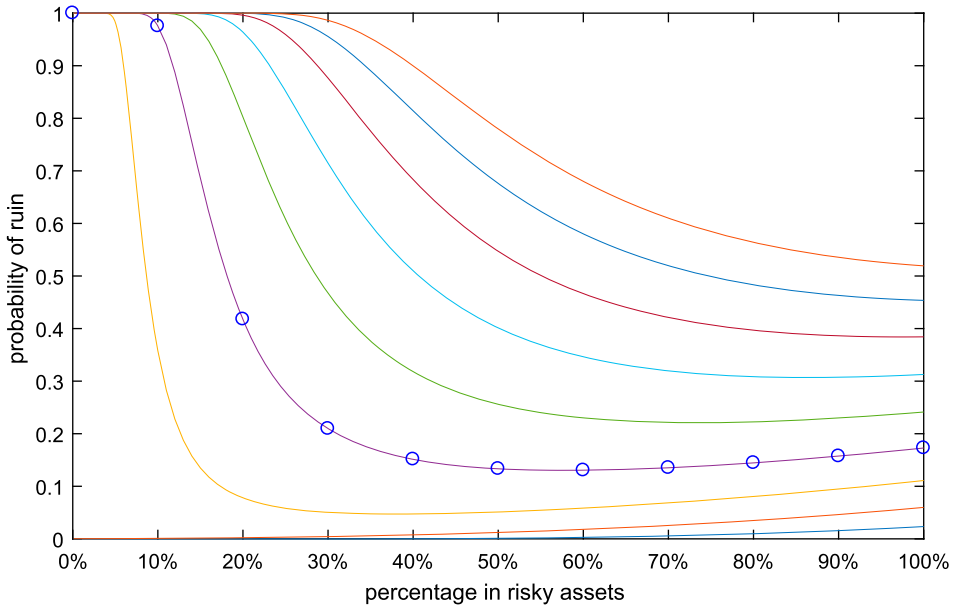
Figure 2. The plot shows the probability of finite-time ruin of a trust fund paying deterministic benefits at a constant rate of index-ation. The probabilities of ruin are shown as functions of the fraction of capital that is invested in risky assets. Different curves correspond to different expenditure ratios (initial level of benefits relative to initial capital). The curves shown correspond to levels of the expenditure ratio ranging from 1% to 5%, with step size equal to 0.5 percentage points. The curve corresponding to the 2.5% level is marked by circles. The following parameter values are used: volatility of risky asset σ_S: 20%, price of risk λ: 0.3; real interest rate r: 1.5%.

Table 1. The table gives the minimal probability of ruin (MPR) that can be achieved in a constant-mix constant-benefit scheme by adapting the portfolio composition, for various values of the initial expenditure ratio and of the Sharpe ratio. The minimizing value of the fraction in risky assets (FRA) is shown for each case as well.

Expenditure ratio	EP = 4%		EP = 6%		EP = 8%	
	MPR	FRA	MPR	FRA	MPR	FRA
2%	21.5%	50.0%	4.7%	38.0%	0.6%	33.6%
2.5%	37.3%	61.4%	13.0%	57.9%	3.2%	52.6%
3%	49.8%	75.6%	22.1%	73.4%	7.3%	68.3%

Notes: The acronym EP stands for equity premium, defined as the product of Sharpe ratio and risky asset volatility. The following parameter values are assumed: real interest rate: 1.5%; risky asset volatility: 20% . The Sharpe ratios (market prices of risk) that correspond to the equity premia 4%, 6%, and 8% are therefore 0.2, 0.3, and 0.4.

6.3. Tradeoff between efficiency and probability of ruin

The tradeoff analysis in Figure 2 and Table 1 focuses on the three-way relationship between the benefit level, the asset mix, and the ruin probability. However, within the constant-mix constant-benefit scheme, the benefit level and the asset mix have an impact also on the total time-0 value of benefits, as shown by (46). In cases where $v(x) < x$, there is a loss of capital due to oversaving. The amount of loss can be expressed through the ratio $(x - v(x))/x = 1 - v(x)/x$, which will be called the *fraction of unused capital*.

For a fixed choice of the asset mix, Figure 3 illustrates the three-way relationship between the fraction of unused capital, the ruin probability, and the level of benefits as expressed through the expenditure ratio. The volatility of the fund's portfolio is taken to be equal to 12%, which corresponds to a 60/40 asset mix if it is assumed that the risky asset volatility in the Black-Scholes model is 20%. The real interest rate is assumed to be equal to 1.5%. Numerical values corresponding to the graph are shown in Table 2. To assess the robustness of outcomes with respect to the assumed value of the
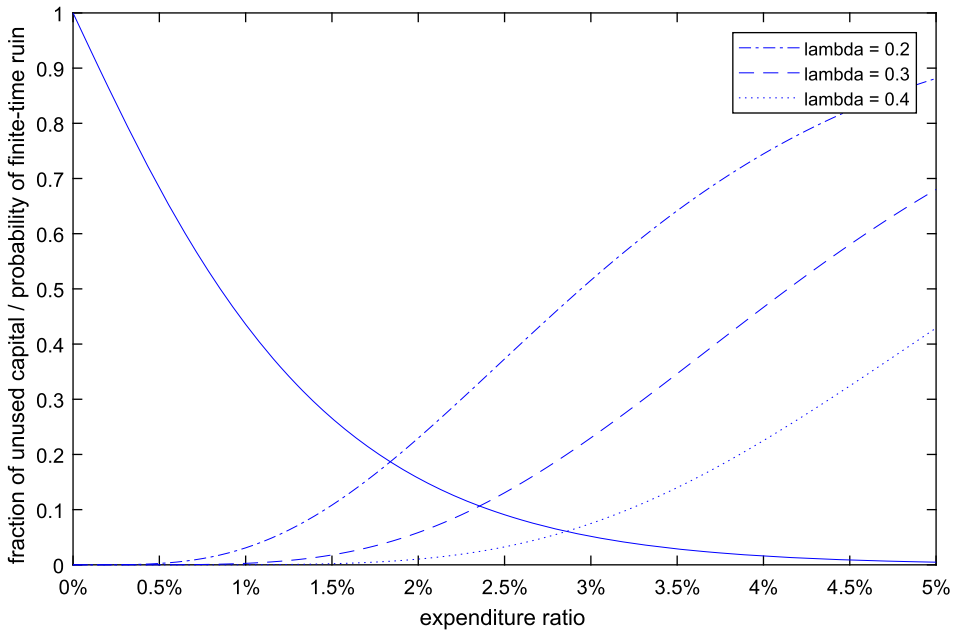
Figure 3. The plot shows the tradeoff between the fraction of unused capital and the probability of finite-time ruin for a constant-mix constant-benefit fund. The following parameter values are used: real interest rate $r = 1.5\%$, portfolio volatility $\sigma = 12\%$. The drawn curve shows the fraction of unused capital. The dashed/dotted curves show the probability of finite-time ruin for several values of the market price of risk λ, as indicated in the legend.

Table 2. The table shows the maximal expenditure ratio that can be achieved within a constant-mix constant-benefit scheme for a given probability of finite-time ruin, and the corresponding fraction of unused capital.

	EP = 4%		EP = 6%		EP = 8%	
Ruin probability	10%	20%	10%	20%	10%	20%
Expenditure ratio	1.46%	1.89%	2.31%	2.86%	3.21%	3.86%
Unused capital	27.7%	17.7%	11.2%	6.1%	4.0%	1.9%

Note: The assumptions on parameter values are the same as in Figure 3; the acronym EP is used in the same way as in Table 1.

Sharpe ratio (the market price of risk), three values of the parameter λ are used, corresponding to equity premia at the levels 4%, 6%, and 8%.

For expenditure ratios less than the real interest rate, it is not rational to use investments in risky assets, since the policy of placing all of capital into riskless assets allows an expenditure ratio that is equal to the real interest rate, while eliminating the risk of finite-time ruin. Investing in risky assets must therefore be motivated by the desire to set the expenditure ratio higher than the real interest rate. It is seen in Figure 3 that the probability of ruin goes up rather quickly when the expenditure ratio is increased, especially at more conservative values of the Sharpe ratio. The table shows that, to make sure that the ruin probability for the fixed-mix strategy with 12% volatility is not more than 20%, the expenditure ratio cannot exceed 2.86% if the 6% value for the equity premium is assumed. Under this policy, the fraction of unused capital is 6.1%. Of course, 2.86% is a substantial improvement with respect to the value 1.5% that would be allowed by the riskless policy, but it does require acceptance of a 20% probability of finite-time ruin. If one wants to limit the ruin probability to 10%, then under the same assumptions as before the expenditure ratio must be limited to 2.31%, and the fraction of unused capital goes up to 11.2%. Broadly speaking, the figure suggests that under moderately optimistic policy choices, the fraction of unused capital typically lies somewhere between 5% and 15%.
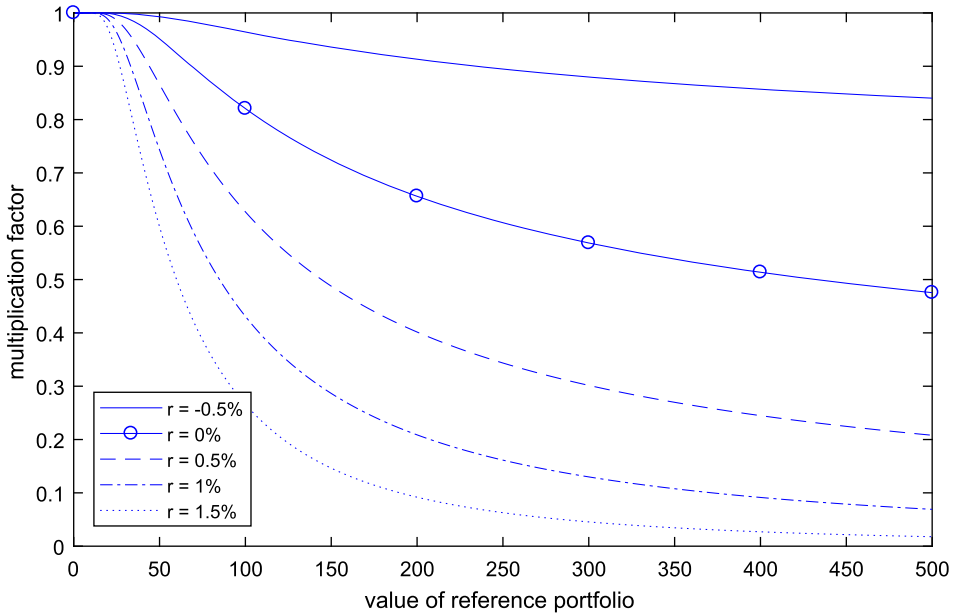
Figure 4. The figure shows the amount of derisking implied by the replication strategy, as a function of the value of the reference (fixed-mix) portfolio. Different curves correspond to different values of the real interest rate, as indicated in the legend. The benefit level is normalized to 1, and portfolio volatility σ is taken to be 12%.

From the figure and the table, the tradeoff between finite-time ruin on the one hand and unused capital on the other may appear quite severe. However, this tradeoff is entirely due to the limitations of the constant-mix constant-benefit scheme. While the tension between benefit level and ruin probability, given a fixed initial capital, is intrinsic and cannot be removed, it is possible to eliminate loss of capital without any effect on the probability of ruin. The capital that is made available in this way can be used for instance to raise the level of benefits. To achieve this, one needs to depart from the fixed-mix policy, as detailed below.

6.4. Elimination of inefficiency

Suppose that the board of a long-term fund has arrived, after careful consideration, at a combination of expenditure ratio and asset mix which in their opinion leads to an acceptable distribution of the time of ruin. Instead of implementing the fixed-mix policy as such, they may choose to implement the replication policy defined by (30). The fixed-mix portfolio is still used in the replication strategy, but only as a reference, rather than as an actual portfolio. The fraction of capital in risky assets as used in the replication scheme is given by (see (30))

$$\frac{h_t^S S_t}{H_t} = \frac{\alpha v'(A_t) A_t}{v(A_t)}, \tag{56}$$

where A_t is the value at time t of the reference portfolio, $v(x)$ is the total value of time-0 benefits as a function of initial capital. This differs from the fixed-mix fraction by the factor $v'(A_t)A_t/v(A_t)$. Since $v(x)$ is a concave function, we have, for $x > 0$, $v'(x) \leq (v(x) - v(0))/x = v(x)/x$, so that $v'(x)x/v(x) \leq 1$ for all $x > 0$. The inequality is strict whenever $v(x)$ is strictly concave; as seen from (46), this happens in all cases in which the portfolio volatility σ is positive, and the real interest rate exceeds $-\frac{1}{2}\sigma^2$. Therefore, in all of these cases, the replication policy is more conservative (lower percentage in risky assets) than the fixed-mix policy, although the difference is small for low values of

the reference portfolio. The multiplication factor is shown as a function of the value of the reference portfolio in Figure 4, for several values of the real interest rate. As can be expected, the factor tends to zero more quickly for higher values of the interest rate.

Instead of reducing initial capital, the replication strategy can also be used, when initial capital is fixed, to increase the level of benefits. The initial value of the reference portfolio should then be taken equal to $\tilde{A}_0 = (A_0/v(A_0))A_0$, and the evolution of the reference portfolio should be given by (23) (or, equivalently, (4)) with c replaced by $\tilde{c} := (A_0/v(A_0))c$. The number of units to be held in the replication portfolio (i.e. the actual portfolio held by the fund) is given by $\tilde{h}_t^S := \alpha\tilde{v}'(\tilde{A}_t)\tilde{A}_t/S_t$ instead of (30), where $\tilde{v}'(x) = P(\tilde{b}/x, a+1)$ and $\tilde{b} = 2\tilde{c}/\sigma^2$. Since the new reference portfolio is just a rescaling of the old one, the time of ruin is still the same in every asset return scenario. Consequently, the same statement holds for the replication portfolio. In this way, the level of benefits is increased with respect to the original fixed-mix strategy, while benefits are provided, in every scenario separately, for just as long as they would be provided by the fixed-mix scheme. For instance, the expenditure ratio corresponding to 10% ruin probability and market price of risk $\lambda = 0.3$, which is 2.31% for the fixed-mix scheme as shown in Table 2, can be raised to 2.60% without any effect on the distribution of the time of ruin. This is possible by eliminating the underuse of capital that is inherent in the constant-mix constant-benefit scheme.

7. Conclusions

The main point of this paper has been to argue that, in the analysis of investment-benefit policies of long-term funds, it is important to pay attention to possible inefficiency in the use of capital: wealth may be shifted indefinitely into the future by a given scheme. If such an effect is present, then the scheme can be improved 'for free', either by reducing the amount of initial capital or by raising the level of benefits that are provided. In the particular situation of the constant-mix constant-benefit strategy, it was shown that the capital left unused by the scheme can, under quite normal conditions, amount to 10% or more. Therefore, the warning against inefficiency that was expressed by Scott et al. (2009) for individuals is also important for institutions.

The presence of inefficiency in the constant-mix constant-benefit scheme can be attributed to the fact that benefits within the scheme are never raised beyond the level that is prescribed by the deterministic indexation. In a sense, there is a mismatch between, on the one hand, the upward potential that is generated by the fixed-mix investment scheme, and, on the other hand, the insensitivity of the benefit policy with respect to capital reserves. Depending on the purpose of a given long-term fund, it may be reasonable to assume that, in the event of abundant investment returns, extra benefits will be paid, so that the effect of shifting wealth forward will be mitigated. Such policies are for instance implied by optimization of an objective given by a non-saturating utility function. However, the present paper is motivated by the situation of nonprofit institutions for which the objective of being able to sustain payments for the long term is much more important than the possibility of raising benefits beyond the level that is required to keep up with inflation. Governors of such an institution would probably decide to set surpluses aside for later use in bad times. However, times so bad that the surpluses will be fully used may in fact never come, as indicated by the calculations above.

A general method to ensure full use of capital has been described in a companion paper (Schumacher 2019). The method is based on allocation of the initial capital across the time-0 values of all future payments. This approach calls for the specification of a notion of 'neutrality' or 'fairness' that can be used to guide the allocation. The concept of fairness may give rise to debate; still it may be easier to agree on such a notion than on the formulation of a utility function aggregating the preferences of all future benefit recipients, which classically serves as a device to prevent underuse of capital.

This paper has focused on fairly simple schemes that can be analyzed by analytic methods. An analytic approach may still be feasible for the 'survival' schemes generated by Browne (1997). For other schemes that have been proposed in the literature (for instance the hybrid policy discussed in

Lindset & Matsen (2018), the ratcheting-up policy of Dybvig (1999), and the ratcheting-down strategy of Pye (2000)), a numerical approach may be required. Application of Monte Carlo techniques could be precarious, especially when the variability of benefits increases with time, since the computation of a limit as the time horizon tends to infinity is called for. An alternative may be provided by a numerical version of the PDE technique that has been used in this paper to analyze the constant-mix constant-benefit scheme. Naturally, it would be of interest to investigate the extent of inefficiency in the policies of long-term funds as they exist in practice. The analysis in this paper suggests that inefficiency may also affect more elaborate investment/benefit policies, if these are not expressly designed to avoid oversaving. Further studies should also concentrate on extending the analysis beyond the basic environment of the Black-Scholes model. In particular, it would be interesting to consider situations in which interest rates and the rate of inflation are variable, as for instance in Brown & Scholz (2019).

Acknowledgements

Disclosure statement

No potential conflict of interest was reported by the author.

References

Abramowitz M. & Stegun I. A. (1972). *Handbook of mathematical functions*. New York: Dover.
Albrecher H. & Thonhauser S. (2009). Optimality results for dividend problems in insurance. *RACSAM – Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas* **103**, 295–320.
Bengen W. P. (1994). Determining withdrawal rates using historical data. *Journal of Financial Planning* **7**(4), 171–180.
Bernhardt T. & Connelly C. (2018). Pension decumulation strategies: a state-of-the-art report. Technical Report 1, Risk Insight Lab, Heriot-Watt University, UK.
Bertoin J. & Yor M. (2005). Exponential functionals of Lévy processes. *Probability Surveys* **2**, 191–212.
Björk T. (1998). *Arbitrage theory in continuous time*. Oxford, UK: Oxford University Press.
Blume M. E. (2010). Endowment spending in volatile markets: what should fiduciaries do? *Review of Quantitative Finance and Accounting* **35**(2), 163–178.
Boyle P. P. (1976). Rates of return as random variables. *Journal of Risk and Insurance* **43**, 693–713.
Brown D. & Scholz D. (2019). Discount rates and asset returns: implications for endowment strategies. ssrn.com/abstract = 3426197.
Brown J. R., Dimmock S. G., Kang J.-K. & Weisbenner S. J. (2014). How university endowments respond to financial market shocks: evidence and implications. *American Economic Review* **104**(3), 931–962.
Browne S. (1997). Survival and growth with a liability: optimal portfolio strategies in continuous time. *Mathematics of Operations Research* **22**, 468–493.
Cejnek G., Franz R., Randl O. & Stoughton N. (2014). A survey of university endowment management research. *Journal of Investment Management* **12**(3), 90–117.
Chang F.-R. (2004). *Stochastic optimization in continuous time*. Cambridge: Cambridge University Press.
Corbae D., Stinchcombe M. B. & Zeman J. (2009). *An introduction to mathematical analysis for economic theory and econometrics*. Princeton, NJ: Princeton University Press.
Cui Z. & Nguyen D. (2017). First hitting time of integral diffusions and applications. *Stochastic Models* **33**, 376–391.
Cui Z., Lee C., Liu Y. & Zhu L. (2019). Transform analysis for Markov processes and applications: an operator-based approach. Stevens Institute of Technology School of Business Research Paper. ssrn.com/abstract = 3336238.
Décamps J.-P. & Rochet J.-C. (1997). A variational approach for pricing options and corporate bonds. *Economic Theory* **9**, 557–569.
Denuit M., Dhaene J., Goovaerts M. & Kaas R. (2006). *Actuarial theory for dependent risks: measures, orders and models*. Chichester, UK: Wiley.
Dufresne D. (1990). The distribution of a perpetuity, with applications to risk theory and pension funding. *Scandinavian Actuarial Journal* **1990**, 39–79.
Dufresne D. (2007). Stochastic life annuities. *North American Actuarial Journal* **11**, 136–157.
Dybvig P. H. (1999). Using asset allocation to protect spending. *Financial Analysts Journal* **55**(1), 49–62.
Gollier C. (2008). Intergenerational risk-sharing and risk-taking of a pension fund. *Journal of Public Economics* **92**, 1463–1485.
Ho G. P., Mozes H. A. & Greenfield P. (2010). The sustainability of endowment spending levels: A wake-up call for university endowments. *Journal of Portfolio Management* **37**(1), 133–146.

Lindset S. & Matsen E. (2018). Institutional spending policies: Implications for future asset values and spending. *Financial Markets and Portfolio Management* **32**(1), 53–76.

Linetsky V. (2004). Spectral expansions for Asian (average price) options. *Operations Research* **52**, 856–867.

Mao X. (2007). *Stochastic differential equations and applications*, 2nd ed. Chichester, UK.Horwood:

Merton R. C. (1973). Theory of rational option pricing. *Bell Journal of Economics and Management Science* **4**, 141–183.

Merton R. C. (1974). On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance* **29**, 449–470.

Merton R. C. (1975). An asymptotic theory of growth under uncertainty. *Review of Economic Studies* **42**, 375–393.

Metzler A. (2013). The Laplace transform of hitting times of integrated geometric Brownian motion. *Journal of Applied Probability* **50**, 295–299.

Milevsky M. A. (1997). The present value of a stochastic perpetuity and the Gamma distribution. *Insurance: Mathematics and Economics* **20**, 243–250.

Milevsky M. A. & Robinson C. (2000). Self-annuitization and ruin in retirement. *North American Actuarial Journal* **4**(4), 112–124.

Milevsky M. A. & Robinson C. (2005). A sustainable spending rate without simulation. *Financial Analysts Journal* **61**(6), 89–100.

Mirrlees J. A. (1974). Optimum accumulation under uncertainty: the case of stationary returns to investment. In Jacques H. Drèze, editor, Allocation Under Uncertainty: Equilibrium and Optimality. London: Palgrave Macmillan, P. 36–50.

Paulsen J. (2008). Ruin models with investment income. *Probability Surveys* **5**, 416–434.

Paulsen J. & Gjessing H. K. (1997). Ruin theory with stochastic return on investments. *Advances in Applied Probability* **29**, 965–985.

Peskir G. (2006). On the fundamental solution of the Kolmogorov-Shiryaev equation. In Yu. Kabanov, R. Liptser, and J. Stoyanov, editors, From Stochastic Calculus to Mathematical Finance: The Shiryaev Festschrift. Berlin: Springer. P. 535–546.

Phelps E. S. (1962). The accumulation of risky capital: a sequential utility analysis. *Econometrica* **30**, 729–743.

Pye G. B. (2000). Sustainable investment withdrawals. *Journal of Portfolio Management* **26**(4), 73–83.

Pye G. B. (2017). Disbursements. *Journal of Portfolio Management* **43**(3), 136–151.

Ramsey F. P. (1928). A mathematical theory of saving. *Economic Journal* **38**, 543–559.

Salminen P. & Yor M. (2005). Perpetual integral functionals as hitting and occupation times. *Electronic Journal of Probability* **10**, 371–419.

Schumacher J. M. (2019). A practical policy design method for endowment funds and collective pension funds. ssrn.com/abstract = 3271783.

Scott J. S., Sharpe W. F. & Watson J. G. (2009). The 4% rule – at what price? *Journal of Investment Management* **7**(3), 31–48.

Seal H. L. (1969). *Stochastic theory of a risk business*. New York: Wiley.

Shiryaev A. N. (1961). The problem of the most rapid detection of a disturbance in a stationary process. *Soviet Mathematics–Doklady* **2**, 795–799.

Turnovsky S. J. (2000). *Methods of macroeconomic dynamics*. Cambridge, MA: MIT Press.

Wang S. S. (2000). A class of distortion operators for pricing financial and insurance risks. *Journal of Risk and Insurance* **67**, 15–36.

Wong E. (1964). The construction of a class of stationary Markoff processes. In Richard Bellman, editor, Sixteenth Symposium in Applied Mathematics – Stochastic Processes in Mathematical Physics and Engineering. Providence, RI: American Mathematical Society. P. 264–276.

Appendix

A.1 Derivation of (13)

The derivation is based on the standard formula for the expectation of a lognormal variable: $E[e^Y] = \exp(EY + \frac{1}{2}\text{Var}(Y))$ when Y is normally distributed. To derive (13), first note that, if $Z \sim N(0,1)$, then $E[Ze^{\sigma Z}] = (d/d\sigma)E[e^{\sigma Z}] = \sigma e^{\frac{1}{2}\sigma^2}$, so that $E[Ye^Y] = \text{Var}(Y)\exp(\frac{1}{2}\text{Var}(Y))$ if Y is normal with expectation 0. From this it immediately follows that $E[Ye^Y] = (EY + \text{Var}(Y))\exp(EY + \frac{1}{2}\text{Var}(Y))$ for normal Y. Finally, if X and Y are jointly normal, then $X - \alpha Y$ and Y are uncorrelated and hence independent where $\alpha := \text{Cov}(X, Y)/\text{Var}(Y)$, so that

$$E[Xe^Y] = E[(X - \alpha Y)e^Y] + \alpha E[Ye^Y] = (EX - \alpha EY)E[e^Y] + \alpha(EY + \text{Var}(Y))E[e^Y]$$

$$= [EX + \text{Cov}(X, Y)]\exp\left(EY + \frac{1}{2}\text{Var}(Y)\right).$$

A.2   Minimum property of pricing formula

The purpose of this section is to provide a proof of the variational formulation (50). First consider the case $r \neq 0$. Let parameter values $x > 0$, $a > 0$, and $b > 0$ be given, and define the function $f(u)$ for $u > 0$ by

$$f(u) = xP(a + 1, u) + \frac{b}{a} Q(a, u). \tag{A1}$$

It is easy to verify that the derivative $f'(u)$ has a single root at $u = b/x$. To show that this extremum is in fact a global minimum, it is convenient to reparametrize the independent variable as follows. Define $u(\theta)$, for $0 < \theta < 1$, by

$$P(a + 1, u(\theta)) = \theta. \tag{A2}$$

The definition is valid since $P(a + 1, u)$ is strictly increasing in u and takes all values between 0 and 1 as its second argument varies from 0 to ∞. Define $g(\theta) = f(u(\theta))$. We have

$$g'(\theta) = x - \frac{b}{a\Gamma(a)} u(\theta)^{a-1} e^{-u(\theta)} u'(\theta).$$

From the definition $P(a + 1, u(\theta)) = \theta$, it follows by differentiation with respect to θ that

$$\frac{1}{\Gamma(a + 1)} u(\theta)^a e^{-u(\theta)} u'(\theta) = 1.$$

From this it can be concluded that $g'(\theta) = x - b/u(\theta)$. It follows that $g'(\theta)$ is strictly monotonically increasing in θ. Consequently, the function $g(\theta)$ is strictly convex. We already know that this function has a single extremum; by convexity, the extremum must be a global minimum. Since g and f are monotonic reparametrizations of each other, it follows that the same conclusion holds for f.

The reasoning is similar in case $r = 0$. Define $f(u)$ for $u > 0$ by $f(u) = x(1 - e^{-u}) + bE_1(u)$. Furthermore, define $u(\theta)$ for $0 < \theta < 1$ by $1 - e^{-u(\theta)} = \theta$, i.e. $u(\theta) = -\log(1 - \theta)$. Write $g(\theta) = f(u(\theta))$. It is easily verified that we have $g'(\theta) = x - b/u(\theta)$ just as in the case above, and the reasoning is completed in the same way.

The connection to the Fenchel-Legendre transform follows from the above. In the case $r \neq 0$, we have

$$v(x) = \min_{0 \leq \theta < 1} \left[x\theta + \frac{b}{a} Q(a, u(\theta)) \right]. \tag{A3}$$

Therefore, the convex function $v^*(\theta)$ defined for $0 < \theta \leq 1$ by

$$v^*(\theta) = \frac{b}{a} Q(a, u(\theta)) \tag{A4}$$

(with value 0 at $\theta = 1$) is the Fenchel-Legendre transform of the concave function $v(x)$ defined on $[0, \infty)$. An analogous formula holds in case $r = 0$.

For $a > 0$, we have $\lim_{\theta \downarrow 0} v^*(\theta) = b/a$; in case $a \leq 0$, the limit is not finite. It may be noted that the function h defined for $a > 0$ and $0 \leq \theta \leq 1$ by $h(\theta) = 1 - Q(a, u(\theta))$ is an increasing concave function that takes the value 0 at 0 and the value 1 at 1. This function could be used as a distortion function to define a risk measure, as discussed for instance in Denuit et al. (2006, § 2.6). There is an analogy with the Wang transform (Wang 2000), which uses the normal distribution.