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ON MULTIPLICITIES OF IRREDUCIBLES IN LARGE TENSOR PRODUCT OF REPRESENTATIONS OF SIMPLE LIE ALGEBRAS.

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ABSTRACT. In this paper we study the asymptotic of multiplicities of irreducible representations in large tensor products of finite dimensional representations of simple Lie algebras and their statistics with respect to Plancherel and character probability measures. We derive the asymptotic distribution of irreducible components for the Plancherel measure, generalizing results of Biane and Tate and Zelditch. We also derive the asymptotic of the character measure for generic parameters and an intermediate scaling in the vicinity of the Plancherel measure. It is interesting that the asymptotic measure is universal and after suitable renormalization does not depend on which representations were multiplied but depends significantly on the degeneracy of the parameter in the character distribution.

1. INTRODUCTION

1.1. The study of the statistics of irreducible components in "large" natural representations goes back to works [12][17][18]. An example of such large representation is the left regular representation of the symmetric group S_N for large N. The Plancherel measure provides a natural probability measure. The statistics of irreducible components in the left regular representation of S_N with respect to the Plancherel measure was exactly the focus of [12][17][18],[5], see also [6][13] and references therein.

The Schur-Weyl duality identifies multiplicities of irreducible components of the N-th tensor power of the vector representation of sl_{n+1} with dimensions of irreducible representations of S_N. The statistics of irreducible sl_{n+1}-components in this tensor products with respect to the Plancherel measure[1] (8) for large N was studied by Kerov in [9]. Kerov [9] discovered that as $N \to \infty$ the discrete measure (8) on the space of highest weights $\mathbb{R}^n$ of sl_{n+1} converges weakly to the measure on the main Weyl chamber given by the radial part of the natural invariant measure on sl_{n+1}^*. Kerov's proof was based upon the hook formula for dimensions of irreducible representations of S_N.

Among more recent results involving similar computations are papers [1],[2] where similar asymptotics were computed for Lie superalgebras $gl(n|m)$, the paper [14], where the asymptotic of multiplicities of irreducible $so(2n + 1)$-modules in the N-th tensor powers of the spinor representation was studied in the limit $N \to \infty$ and [6]. In [4] the asymptotic of multiplicities of irreducible subrepresentations was computed for $V^{\otimes N}$ when $N \to \infty$. These results were extended and strengthened in [16], see also [15].

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[1]If W is a finite dimensional representation of a simple Lie algebra and $W \simeq \oplus_i V_i^{\oplus m_i}$ is its decomposition into irreducibles, then $p_i = \frac{\dim V_i m_i}{\dim W}$ is a probability measure on the set of irreducible components of W. We call it the Plancherel measure by the analogy with the Plancherel measure on the left regular representation of a finite group, or compact Lie group.

Problems of this type are known as *asymptotic representation theory*. A survey of this direction in representation theory can be found in the books [10], [19].

In this paper we derive the asymptotic formula for multiplicities of irreducible representations in tensor powers of finite dimensional representations of simple Lie algebras when the number of factors tends to infinity. The formula generalizes slightly the one obtained in [16],[4]. Here we derive it using a different method, somewhat heuristically. The complete proof is a straightforward extension of results from [16]. The large deviation rate function was also computed in [8] and the convergence of character measures was studied in [?].

1.2. Let $\mathfrak{g}$ be a simple Lie algebra, $V_i, \quad i = 1 \ldots m$ be its finite dimensional representations and $N_k \geq 0$ be integers. Any finite dimensional representation of a simple Lie algebra is completely reducible and therefore:

$$
(1) \qquad \bigotimes_{k=1}^{m} V_k^{\otimes N_k} \cong \bigoplus_{\lambda} W_{\lambda}(\{V_k\}, \{N_k\}) \otimes V_{\lambda},
$$

where the sum is taken over irreducible components of the tensor product, V_{λ} is the irreducible finite dimensional $\mathfrak{g}$-module with the highest weight λ and $W_{\lambda}(\{V_k\}, \{N_k\})$ is the "space of multiplicities":

$$
W_{\lambda}(\{V_k\}, \{N_k\}) \simeq \mathrm{Hom}_{\mathfrak{g}}(\bigotimes_{k=1}^{m} V_k^{\otimes N_k}, V_{\lambda}).
$$

Its dimension $m_{\lambda}(\{V_k\}, \{N_k\})$ is the multiplicity of V_{λ} in the tensor product and we have isomorphism of vector spaces.

Starting from here we assume $V_k = V_{\nu_k}$, otherwise $V_k \simeq \bigoplus V_{\nu}^{\oplus m_{\nu,k}}$. We will use notations $\chi_{\lambda}(\mathfrak{g}) = tr_{V_{\lambda}(\mathfrak{g})}$ for characters of irreducible representations.

Choose a Borel subalgebra $\in \mathfrak{g}$ and let Δ_+ be corresponding positive roots, $\mathfrak{h}$ be the corresponding Cartan subalgebra and $\alpha_1, \ldots, \alpha_r$ be enumerated fundamental roots. Here $r = rank(\mathfrak{g}) = dim(\mathfrak{h})$ is the rank of the Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_{\mathbb{R}}$ be the split real form of $\mathfrak{g}$ and $\mathfrak{h}_{\mathbb{R}}$ be its Cartan subalgebra. When $g = e^t, \quad t \in \subset \mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ the characters $\chi_{\nu_k}(e^t)$ and $\chi_{\lambda}(e^t)$ are positive and as a consequence of the tensor product decomposition we have the identity

$$
\prod_{k} \chi_{\nu_k}(e^t)^{N_k} = \sum_{\lambda} m_{\lambda}(\{V_k\}, \{N_k\}) \chi_{\lambda}(e^t).
$$

Therefore

$$
(2) \qquad p_{\lambda}^{(N)}(t) = \frac{m_{\lambda}(\{V_k\}, \{N_k\}) \chi_{\lambda}(e^t)}{\prod_{k} \chi_{\nu_k}(e^t)^{N_k}}
$$

is a natural probability measure on irreducible components of tensor product: $p_{\lambda}^{(N)}(t) \geq 0$ and $\sum_{\lambda} p_{\lambda}^{(N)}(t) = 1$. We will call it *the character measure*.

We will also assume[2] that components of t in the basis of simple roots are nonnegative, i.e. $t \geq 0$.

In this paper we study the asymptotical behavior of multiplicities m_λ in the limit when $N_k \to \infty$ and $\lambda \to \infty$ such that $N_k = \tau_k/\epsilon$ and $\lambda = \xi/\epsilon$, $\epsilon \to 0$, where $\tau_k \in \mathbb{R}_{\geq 0}$ and $\xi \in \mathfrak{h}_{\geq 0}^*$. As a consequence we will describe the large deviation type asymptotic of the probability measure $\{p_\lambda\}$.

1.3. To state main results of the paper we need a few definitions. First, define

$$f(\tau, t) = \sum_k \tau_k \ln(\chi_{\nu_k}(e^t)).$$

It is a strictly convex function of t. Let $S(\tau, \xi)$ be the Legendre transform of $f(\tau, y)$

(3) $$S(\tau, \xi) = \min_y \left(f(\tau, y) - (y, \xi) \right) = f(\tau, x) - (x, \xi),$$

where (y, ξ) is the Killing form. In the basis of simple roots $\xi = \sum_a \xi_a \alpha_a, y = \sum_a y_a \alpha_a$ and $(y, \xi) = \sum_{ab} y_a B_{ab} \xi_b$, where $B_{ab} = d_a C_{ab} = (\alpha_a, \alpha_b)$ is the symmetrized Cartan matrix and x is the critical point where the minimum is achieved. The critical point x is the unique solution to the equation:

(4) $$\frac{\partial}{\partial x_a} f(\tau, x) = \sum_b B_{ab} \xi_b.$$

Define the matrix K as

$$K_{ac} = -\frac{\partial^2 S(\tau, \xi)}{\partial \xi_a \partial \xi_b},$$

we will see that

$$K_{ac} = \sum_{b,d} B_{ad} (D^{-1})_{db} B_{bc},$$

where

$$D_{ab} = \frac{\partial^2}{\partial y_a \partial y_b} f(\tau, y)|_{y=x},$$

where x is as above.

Recall that $y \in \mathfrak{h}$ is called *regular* if the stabilizer of y in the Weyl group W acting on $\mathfrak{h}$ is trivial. The following theorem extends the result of [4],[16] where it was proven for $m = 1$.

Theorem 1. *If $\xi = \epsilon \lambda$ remain regular as $\epsilon \to 0$ the asymptotic of the multiplicity of V_λ in (1) has the following form*

$$m_\lambda(\{V_k\}, \{N_k\}) = \epsilon^{\frac{r}{2}} \frac{\sqrt{\det K}}{(2\pi)^{\frac{r}{2}}} \Delta(x) e^{-(\rho, x)} e^{\frac{1}{\epsilon} S(\tau, \xi)} (1 + O(\epsilon))$$

---

[2]This is a convenience assumption, the irreducible characters restricted to the Cartan subgroup are invariant with respect to the action of the Weyl group, so we can always choose a representative of the orbit of W through t which is positive.

Here $x \in \mathfrak{h}$ is the Legendre image of $\xi \in \mathfrak{h}^*$, the functions S and the matrix K are as above and $\Delta(x)$ is the denominator in the Weyl formula for characters:

$$\Delta(x) = \prod_{\alpha \in \Delta_+} (e^{\frac{(x,\alpha)}{2}} - e^{-\frac{(x,\alpha)}{2}})$$

Note that the asymptotic is different when ξ is not regular. In this paper we give a heuristic argument why the theorem holds. A rigorous proof, based on the application of the steepest descent method to the integral of characters can be found in [16] for $m = 1$ and for $m > 1$ it is completely parallel.

The next theorem is the description of the statistics of irreducible components with respect to the character measure (2).

Assume $\xi = \epsilon\lambda$ remain regular as $\epsilon \to 0$ and t is regular. Taking into account the asymptotic of the multiplicity and the asymptotic of the charachter $\mathrm{ch}_\lambda(e^t)$ in this limit, we obtain the following asymptotic of the probability $p_\lambda^{(N)}(t)$ as $\epsilon \to 0$ is:

$$(5) \qquad p_\lambda^{(N)}(t) = \epsilon^{\frac{r}{2}} \frac{\sqrt{\det K}}{(2\pi)^{\frac{r}{2}}} \frac{\Delta(x)}{\Delta(t)} e^{-(\rho, x-t)} e^{\frac{1}{\epsilon}\widetilde{S}(\tau,\xi)}(1 + O(\epsilon))$$

where $\widetilde{S}(\tau,\xi) = S(\tau,\xi) - f(\tau, t) + (t, \xi)$. The exponent has maximum at η which is the Legendre image of t and $\widetilde{S}(\tau, \eta) = 0$. Computing this expression in a neighborhood of the critical point η we derive the following statement.

Theorem 2. *If t is regular, the asymptotic character probability distribution is localized at point η with a Gaussian distribution around this point. If we rescale random variable $\xi = \sum_a \xi_a \alpha_a$ near the critical point η as $\xi_l = \eta_l + \sqrt{\epsilon}a_l$, then in the limit $\epsilon \to 0$ the random variable $a \in \mathfrak{h}_\mathbb{R}^* \simeq \mathbb{R}^{r3}$ is distributed with the probability measure $p(a)da_1 \dots da_n$ where da is the Euclidean measure on $\mathbb{R}^r$ and*

$$(6) \qquad p(a,t) = \frac{\sqrt{\det K}}{(2\pi)^{\frac{r}{2}}} e^{-\frac{1}{2}\sum_{bc} a_b K_{bc} a_c}.$$

In other words, the probability measure $\{p_\lambda\}$ weakly converges to the probability measure $p(a)da$

Let Λ be the weight lattice of the Lie algebra $\mathfrak{g}$ and ϵ_n be a positive sequence such that $\epsilon_n \to 0$ when $n \to \infty$. Define $\Lambda_{\epsilon_n} \subset \mathbb{R}^r$ as the image of Λ with respect to the embedding $\lambda = \sum_i \lambda_i \omega_i \mapsto \sum_i \epsilon_n \lambda_i \omega_i \in \mathbb{R}^r$. Define the subset $D \subset \Lambda$ as the set of weights that appear in the decomposition of the tensor product (1) and the subset $D_{\epsilon_n} \subset \Lambda_{\epsilon_n} \subset \mathbb{R}^r$. Note that elements of D have the form $\sum_k N_k \nu_k - \sum_{b=1}^r a_b \alpha_b$ where $a_b \geq 0$ are integers and α_a are simple roots. The weak convergence of measures in the context of theorem 2 means the following. For any bounded continuous function h on $\mathbb{R}^r$, a sequence $\epsilon_n \to 0$, and the condition $\tau_k = \epsilon_n N_k$ are fixed for all k in the tensor product

$$(7) \qquad \sum_{\lambda \in D} h(\sqrt{\epsilon_n}(\lambda - \frac{\eta}{\epsilon_n}))p_\lambda^{(N)}(t) \to \int_{\mathbb{R}^r} h(a)p(a,t)da_1 \dots da_r,$$

---

[3]Here we use the basis of simple roots.

The extreme non regular case is when $t = 0$. In this case the probability distribution is given by

$$p_\lambda^{(N)}(0) = \frac{m_\lambda(\{V_k\}, \{N_k\})dim(V_\lambda)}{\prod_k dim(V_{\nu_k})^{N_k}}.$$

(8)

Theorem 3. *As $\epsilon \to 0$ the Plancherel measure (8) weakly converges to the probability measure $p(a)da$ on $\mathfrak{h}_{\geq 0}^* \simeq \mathbb{R}^r$ where da is the Euclidean measure on $\mathbb{R}^r$ and the density function is*

$$p(a) = \frac{\sqrt{detB}}{(2\pi)^{\frac{r}{2}}} \prod_{\alpha > 0} \frac{(a, \alpha)^2}{(\rho, \alpha)} e^{-\frac{1}{2}\sum_{b,c} a_b B_{bc} a_c},$$

(9)

where random variables λ and a are related as $\lambda = \sqrt{\frac{1}{x\epsilon}}a$ and $x = \frac{1}{dim(\mathfrak{g})}\sum_{k=1}^m \tau_k c_2(\nu_k)$. Here $c_2(\nu)$ is the value of the Casimir element on the irreducible representation V_ν.

This result generalizes slightly convergence results for $m = 1$ from [4][16].

Now consider again the character measure on irreducible components of the tensor product but assume that $t = \sqrt{\frac{\epsilon}{x}}u$ as $\epsilon \to 0$. In this case $p_\lambda^{(N)}(t)$ converges to a natural deformation of the measure (9).

Theorem 4. *As $\epsilon \to 0$, $t = \sqrt{\frac{\epsilon}{x}}u$ and u is fixed, the character measure (2) weakly converges to the measure $p(b, u)db_1 \dots db_r$ on $\mathbb{R}_{\geq 0}^r$ with density*

$$p(b, u) = \frac{\sqrt{\det B}}{(2\pi)^{\frac{r}{2}}} \frac{\prod_{\alpha \in \Delta_+}(b, \alpha)}{\prod_{\alpha \in \Delta_+}(u, \alpha)} \sum_{w \in W}(-1)^w e^{(b, w(u))} e^{-\frac{(b,b)}{2} - \frac{(u,u)}{2}}$$

(10)

Here random variables λ and b as related as $\lambda = \sqrt{\frac{1}{x\epsilon}}b$.

1.4. The paper is organized as follows. In section 3 we derive the asymptotic of the multiplicity function from the character identities. In section 4 we derive the asymptotic distribution with respect to the character measure (2), assuming that t is regular. We also show there that the trivial representation is the most probable with respect to the Plancherel measure and derive the asymptotic statistics around it. Section 5 contains an outline of the derivation of the asymptotic of the multiplicity function using the hook formula for N-th tensor power of the vector representation of sl_{n+1} and the corresponding asymptotic distribution. In section 6 we compute the same asymptotic from theorem 1. We derive intermediate asymptotic and outline the proof of Theorem 3 in section 7. Nonlinear PDE's for the rate function are derived in section 8. In the conclusion we outline some perspectives.

2. PLANCHEREL AND CHARACTER MEASURES AS DENSITY MATRICES

Let W be a finite dimensional representation of a simple Lie algebra $\mathfrak{g}$. Let

$$W \simeq \oplus V_i^{\oplus m_i}$$

be the decomposition of W into irreducibles where m_i is the multiplicity of V_i. By analogy with the usual Plancherel measure for left regular representations we will say that *the Plancherel measure on W* is the probability measure on the set of irreducible components of W with the probability of V_i being

$$p_i = \frac{dim(V_i)m_i}{dim(W)}.$$

This probability measure can be regarded as a quantum probabilistic state on $End(W)$ which assigns to an operator $A : W \to W$ its expectation value $tr(A)/dim(W)$. In other words, this is the state with the density matrix

$$\rho = \frac{I}{dim(W)},$$

where $I : W \to W$ is the identity operator acting in W. This density matrix can be interpreted as the quantum equilibrium state

$$\rho = \frac{e^{-\frac{H}{T}}}{tr(e^{-\frac{H}{T}})}$$

with zero Hamiltonian, $H = 0$, i.e. the topological quantum mechanics with the space of states W.

The character measure (2) can be interpreted in a similar way. It corresponds to the Hamiltonian being semisimple element of $\mathfrak{g}$, i.e. by a conjugation such an H can be brought to an element of the Cartan subalgebra. In notations used in the introduction $t = -H/T$.

3. DERIVATION OF THE ASYMPTOTIC OF MULTIPLICITIES

In this section we give arguments justifying conjecture 1 for the asymptotic of the multiplicity function for tensor product of representations and show how to obtain exponential and sub exponential terms of this asymptotic from their characters. From now on we assume that λ is regular.

3.1. **Large deviation rate function** $S(\tau, \xi)$. We expect[4] that in the limit $N = \tau/\epsilon$ and $\lambda = \xi/\epsilon$, $\epsilon \to 0$ with ν_k and x fixed, the multiplicity function m_λ^N has the following asymptotic:

$$(11) \qquad\qquad m_\lambda^N = e^{\frac{1}{\epsilon}S(\tau,\xi)}\mu(\tau,\xi)(1 + O(\epsilon)).$$

Here $S(\tau, \xi)$ is the large deviation rate function and $\mu(\tau, \xi)$ is a function of τ, ξ which is proportional to a power of ϵ. We will now derive explicit expressions for these functions

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[4]This is proven in [16] for $m = 1$. The proof is based on the application of the steepest descent method and on the orthogonality of characters of irreducible representations of corresponding compact Lie group. For $m > 1$ the proof is completely similar.

via character identities. Note that here λ is in the main Weyl chamber and therefore $\xi \in \mathfrak{h}_{\geq 0}^*$ where ≥ 0 means $\xi = \sum_{a=1}^r \xi_a \alpha_a$ has nonnegative coordinates ξ_a.

From the tensor product decomposition we have:

$$(12) \qquad \prod_k \chi_{\nu_k}(e^t)^{N_k} = \sum_{\lambda \in D} m_\lambda(\{V_k\}, \{N_k\})\chi_\lambda(e^t).$$

where $\chi_\lambda(e^t) = tr_{V_\lambda}(e^t)$ is the character of module V_λ evaluated at $e^t \in G$ and $D \subset \Lambda$ is as in (7).

Now let us substitute (11) into (12) and replace the summation over λ by an integral over ξ:

$$(13) \qquad e^{\frac{f(\tau,t)}{\epsilon}} = \sum_\lambda m_\lambda^N \chi_\lambda(e^t) \simeq \epsilon^{-r} \int_{D_0} e^{\frac{1}{\epsilon}S(\tau,\xi)} \mu(\tau,\xi)\chi_{\frac{\xi}{\epsilon}}(e^t)d\xi,$$

Here r is the rank of Lie algebra $\mathfrak{g}$, $D_0 \subset \mathbb{R}^r$ is the limiting set of D_ϵ as $\epsilon \to 0$. D_0 is a convex polytope in the positive chamber $\mathbb{R}_{\geq 0}^r$ with $\sum_k \tau_k \nu_k$ being the extreme point which is farthest from the origin. The measure $d\xi$ is the Euclidean measure $d\xi_1 \cdots d\xi_r$.

Let us use the Weyl character formula

$$\chi_\lambda(e^t) = \frac{\sum_w e^{(w(\lambda+\rho),t)}(-1)^{l(w)}}{\Delta(t)},$$

where $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$, and $l(w)$ is the length of w (in terms of simple reflections). When t is regular and is in the positive Weyl chamber and $\lambda = \xi/\epsilon$ where ξ is fixed and is also in the positive Weyl chamber, and $\epsilon \to 0$, the term with $w = e$ gives the leading asymptotic and we have:

$$(14) \qquad \chi_{\frac{\xi}{\epsilon}}(e^t) = e^{\frac{1}{\epsilon}(\xi,t)}\frac{e^{(\rho,t)}}{\Delta(t)}(1 + o(1)).$$

From now on we will identify $\mathfrak{h}_{\geq 0}^*$ with $\mathbb{R}_{\geq 0}^r$ using the basis of simple roots.

Substituting (14) into (13) we obtain:

$$(15) \qquad e^{\frac{f(\tau,t)}{\epsilon}} \simeq \epsilon^{-r} \int_{D_0} e^{\frac{1}{\epsilon}S(\tau,\xi)}\mu(\tau,\xi)\chi_{\frac{\xi}{\epsilon}}(x)d\xi \simeq \epsilon^{-r} \int_{\mathbb{R}_{\geq 0}^r} e^{\frac{1}{\epsilon}(S(\tau,\xi)+(\xi,t))}\mu(\tau,\xi)\frac{e^{(\rho,t)}}{\Delta(t)}d\xi.$$

where $d\xi = d\xi_1 \dots d\xi_r$.

Assume that $S(\tau,\xi)$ is strictly concave in ξ. This implies that $S(\tau,\xi) + (\xi,t)$ has unique maximum at some point η. Then the leading contribution to the integral comes from the vicinity of η. It is given by the Gaussian integral over $a = (\xi - \eta)/\sqrt{\epsilon}$ and determined by the second Taylor coefficient of $S(\tau,\xi)$:

$$(16) \qquad e^{\frac{f(\tau,t)}{\epsilon}} = e^{\frac{1}{\epsilon}(S(\tau,\eta)+(t,\eta))}\mu(\tau,\eta)\epsilon^{-r}\frac{e^{(\rho,t)}}{\Delta(t)}\epsilon^{\frac{r}{2}}\int_{\mathbb{R}^r} e^{\frac{1}{2}(a,S^{(2)}a)}d^r a$$

where $S_{ab}^{(2)} = \frac{\partial^2}{\partial\xi_a \partial\xi_b}S(\tau,\xi)|_\eta$.

Comparing most singular terms in this equation we can identify $S(\tau,\xi)$ with the Legendre transform of $f(\tau,y)$ in y:

$$(17) \qquad S(\tau,\xi) = \min_y f(\tau,y) - (y,\xi)) = f(\tau,x) - (x,\xi),$$

Because the characters are strictly convex, the function $f(\tau, x)$ is strictly convex in x. This implies that $S(\tau, \xi)$ is strictly concave in x, which agrees with the assumption made earlier. Note that for the inverse Legendre transform we have

$$(18) \qquad f(\tau, x) = \max_\xi(S(\tau, \xi) + (x, \xi)) = S(\tau, \eta) + (x, \eta),$$

where x and η are related as

$$(19) \qquad \tau \frac{\partial}{\partial x_a} \ln \chi_\omega(x) = \sum_b B_{ab} \eta_b,$$

$$(20) \qquad \frac{\partial S(\tau, \xi)}{\partial \xi_a}|_{\xi=\eta} = -\sum_b B_{ab} x_b.$$

Here B_{ab} is the symmetrized Cartan matrix and $(\eta, \xi) = \sum_a \eta_a B_{ab} \xi_b$.

Since $f(\tau, x))$ is a convex function of x with minimum at $x = 0$, the function $S(\tau, \xi)$ defined in (17) is a concave function of ξ with the unique maximum at $\xi = 0$.

3.2. Subexponential terms.
Now after deriving most singular exponential factors, the equation (16) is reduced to

$$(21) \qquad 1 = \mu(\tau, \eta) \frac{e^{(\rho, t)}}{\Delta(t)} \epsilon^{-\frac{r}{2}} \int_{\mathbb{R}^r} e^{\frac{1}{2}(a, S^{(2)}a)} d^r a.$$

Taking into account that the Gaussian integral can be computed as

$$\int_{\mathbb{R}^r} e^{-\frac{1}{2}(a, Ka)} d^r a = (2\pi)^{r/2} \frac{1}{\sqrt{\det K}}.$$

the equation(21) gives:

$$\mu(\tau, \eta) \frac{e^{(\rho, x)}}{\Delta(x)} \epsilon^{-\frac{r}{2}} (2\pi)^{r/2} \frac{1}{\sqrt{\det K}} = 1.$$

where $K_{ab} = -\frac{\partial^2 S}{\partial \xi_a \partial \xi_b}(\xi)|_{\xi=\eta}$ and ξ_0 is the Legendre image of x.

Therefore, we derived the unknown function $\mu(\tau, \xi)$ as

$$\mu(\tau, \eta) = \epsilon^{r/2} \frac{\sqrt{\det K}}{(2\pi)^{r/2}} \Delta(x) e^{-(\rho, x)}.$$

where η and x are related as in (19).

To calculate partial K explicitly we will use (20) and (19). Differentiating (20) with respect to x_d we get

$$\frac{\partial^2 S}{\partial \xi_a \partial \xi_c}(\xi)|_{\xi=\eta} \frac{\partial(\eta)_c}{\partial x_d} = -B_{ad},$$

then

$$\frac{\partial^2 S}{\partial \xi_a \partial \xi_c}(\xi)|_{\xi=\eta} = -B_{ad} \frac{\partial x_d}{\partial \eta_c}.$$

To obtain $\frac{\partial x_d}{\partial \eta_c}$ we will differentiate (19) with respect to η_c:

$$\frac{\partial^2}{\partial x_a \partial x_b} f(\tau, x) \frac{\partial x_b}{\partial \eta_c} = B_{ac}.$$

Here x is regarded as the Legendre image of η. Denote

$$(22) \qquad D_{ab} = \frac{\partial^2}{\partial x_a \partial x_b} f(\tau, x),$$

then

$$D_{ab} \frac{\partial x_b}{\partial \xi_c} = B_{ac},$$

and therefore

$$\frac{\partial x_b}{\partial \xi_c} = (D^{-1})_{ba} B_{ac}.$$

Finally, we derive the formula for K in terms of f:

$$K_{ab} = B_{ac}(D^{-1})_{cd} B_{db}.$$

4. PROBABILITY DISTRIBUTIONS

Here we will derive probability distributions for two cases, for the character measure and for the Plancherel measure, i.e. we will outline proofs of Theorems 2 and 3.

4.1. **Character distribution.** Here we outline the proof of Theorem 2. The function

$$\widetilde{S}(\tau, \xi) = S(\tau, \xi) - f(\tau, t) + (t, \xi)$$

in the formula (5) is strictly concave in ξ and therefore it has unique maximum at some point η, which is the Legendre image of t with respect to the function $S(\tau, \xi)$. Because $S(\tau, \xi)$ is invariant with respect to the action of the Weyl group, the point η is regular if t is regular. In the vicinity of η we have

$$\widetilde{S}(\tau, \xi) = -\frac{\epsilon}{2} \sum_{a,b} a_a D_{ab} a_b + O(\epsilon^{3/2}),$$

where $a_a = (\xi_a - \eta_a)/\sqrt{\epsilon}$ are the coordinates in the basis of simple roots. Taking this into account and passing to the limit $\epsilon \to 0$ in (5) we obtain the Gaussian distribution (6).

4.2. **Plancherel distribution.** Here we outline the proof of Theorem 3 under the assumption that the asymptotic of multiplicities is uniform in ξ in D_0 including boundary strata[5]. In other words the asymptotic along the boundary can be obtained by taking corresponding limit in ξ.

For the Plancherel probability distribution we have the following asymptotic when $\epsilon \to 0$ and $\xi = \epsilon\lambda$ is finite and regular:

$$(23) \qquad p_\lambda^{(N)}(t) = \epsilon^{\frac{r}{2}+|\Delta_+|} \frac{\sqrt{\det K}}{(2\pi)^{\frac{r}{2}}} \frac{\prod_{\alpha>0}(\xi, \alpha)}{\prod_{\alpha>0}(\rho, \alpha)} \Delta(x) e^{-(\rho, x)} e^{\frac{1}{\epsilon}S(\tau, \xi)}(1 + O(\epsilon)).$$

We used the Weyl formula for the dimension of V_λ:

$$dim(V_\lambda) = \prod_{\alpha \in \Delta_+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}$$

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[5]For a rigorous proof see [4] and [16]

The function $S(\tau, \xi)$ attains its maximum at $\xi = 0$. So, we shall find the asymptotic of the formula (23) near this point. The convexity of the function f implies that for small ξ, its Legendre dual x is also small, therefore for small ξ

$$\sum_b \frac{\partial^2 f(\tau, 0)}{\partial x_a \partial x_b} x_b = \sum_b B_{ab} \xi_b + O(\xi^2).$$

Now let us compute the matrix of second derivative

$$(24) \qquad \frac{\partial^2 f(\tau, 0)}{\partial x_a \partial x_b} = \frac{\partial^2}{\partial x_a \partial x_b} \sum_k \tau_k \ln(tr_{V_{\nu_k}}(e^{\sum_a x_a H_a}))|_{x=0}.$$

Here $\{H_a\}$ is a basis of simple roots in the Cartan subalgebra.

Lemma 1. *The following equality holds:*

$$\frac{\partial^2}{\partial x_a \partial x_b}(\ln tr_{V_\nu}(e^x))|_{x=0} = \frac{c_2(\nu)}{dim(\mathfrak{g})} B_{ab},$$

where B_{ab} is the symmetrized Cartan matrix and c_2 is the Casimir element (see below).

Proof.

$$\frac{\partial^2}{\partial x_a \partial x_b}(\ln tr_{V_\nu}(e^x))|_{x=0} = \frac{\partial}{\partial x_a} \frac{\frac{\partial}{\partial x_b} tr_{V_\nu}(e^x)}{tr_{V_\nu}(e^x)}|_{x=0} =$$

$$= \frac{\frac{\partial^2}{\partial x_a \partial x_b}(tr_{V_\nu}(e^x)) tr_{V_\nu}(e^x) - \frac{\partial}{\partial x_a}(tr_{V_\nu}(e^x)) \frac{\partial}{\partial x_b}(tr_{V_\nu}(e^x))}{(tr_{V_\nu}(e^x))^2}|_{x=0} =$$

$$= \frac{tr_{V_\nu}(H_a H_b e^x) tr_{V_\nu}(e^x) - tr_{V_\nu}(H_a e^x) tr_{V_\nu}(H_b e^x)}{(tr_{V_\nu}(e^x))^2}|_{x=0} =$$

$$(25) \qquad = \frac{tr_{V_\nu}(H_a H_b) tr_{V_\nu}(1)}{(tr_{V_\nu}(1))^2} = \frac{tr_{V_\nu}(H_a H_b)}{\dim V_\nu}.$$

To find the numerator choose a basis e_i in $\mathfrak{g}$. If $(\,.\,,\,.\,)$ is the Killing form and

$$Q_{ij} = (e_i, e_j) = tr_{ad}(e_i e_j), \quad Q^{ij} Q_{jk} = \delta_k^i.$$

then for the Casimir element we have

$$c_2 = Q^{ij} e_i e_j,$$

By Schur's lemma

$$(26) \qquad\qquad tr_{V_\nu}(e_i e_j) = \tilde{x}_\nu Q_{ij}.$$

for some $\tilde{x}_\nu$. This constant is easy to find taking the trace:

$$tr_{V_\nu}(B^{ij} e_i e_j) = c_2(\nu) tr_{V_\nu}(1) = c_2(\nu) \dim(V_\nu),$$

which gives

$$\tilde{x}_\nu = \frac{c_2(\nu) dim(V_\nu)}{\dim \mathfrak{g}},$$

Assume that H_a is part of the basis e_i, then

$$tr_{V_\nu}(H_a H_b) = \frac{c_2(\nu)dim(V_\nu)}{\dim \mathfrak{g}} tr_{ad}(H_a H_b) = \frac{c_2(\nu)dim(V_\nu)}{\dim \mathfrak{g}} d_a C_{ab}.$$

Finally,

$$\frac{\partial^2}{\partial x_a \partial x_b}(\ln tr_{V_\nu}(e^x))|_{x=0} = \frac{c_2(\nu)}{\dim \mathfrak{g}} d_a C_{ab}.$$

$\square$

Denote $x = \frac{1}{dim(\mathfrak{g})} \sum_{k=1}^m \tau_k c_2(\nu_k)$. As a consequence of the lemma above and the asymptotical formula for x_a in terms of ξ_a we have

$$x_a = \frac{\xi_a}{x} + O(\xi^2).$$

Thus, for small ξ we have

$$\prod_{\alpha>0}(x,\alpha) = x^{-|\Delta_+|}\Delta(\xi)(1+o(1)).$$

Recall that the matrix K is defined as

$$K_{ab} = -\frac{\partial^2 S(\tau,\xi)}{\partial \xi_a \partial \xi_b} = \sum_{bd} B_{ad}(D^{-1})_{dc} B_{cb}.$$

Lemma 1 implies that

$$D_{ab}|_{\xi=0} = \frac{\partial^2 f(\tau,y)}{\partial y_a \partial y_b}|_{y=0} = x B_{ab}.$$

Thus, $K = B/x$ and for small ξ we have

$$S(\tau,\xi) = S(\tau,0) - \frac{1}{2x}\xi_a B_{ab}\xi_b + O(\xi^3).$$

Rescaling random variable ξ as $\xi_b = \sqrt{\epsilon x}a_b$ we obtain the following asymptotic for the probabilities in Plancherel distribution:

$$(27) \qquad p_\lambda^{(N)} = \left(\frac{\epsilon}{x}\right)^{\frac{r}{2}} \frac{\sqrt{detB}}{(2\pi)^{\frac{r}{2}}} \prod_{\alpha>0} \frac{(a,\alpha)^2}{(\rho,\alpha)} e^{-\frac{1}{2}\sum_{b,c} a_b B_{bc} a_c}(1+o(1)).$$

Let h be a continuous bounded function on $\mathbb{R}^r_{\geq 0}$. The asymptotic (27) implies the convergence

$$\sum_\lambda h(\sqrt{\frac{\epsilon}{x}}\lambda)p_\lambda^{(N)} \to \int_{\mathbb{R}^r_{\geq 0}} h(a)p(a)da_1 \ldots da_r,$$

where the density function $p(a)$ is given in (9). In other words, the Plancherel measure weakly converges to $p(a)da_1 \ldots da_r$.

5. THE ASYMPTOTIC FOR $\mathfrak{g} = sl_{n+1}$, $m = 1$, $\nu = \omega_1$ VIA THE HOOK LENGTH FORMULA.

In this section we recall the derivation of the pointwise asymptotic of multiplicities and of the Plancherel probability measure using the Schur-Weyl duality and the hook length formula from [9].

5.1. **The asymptotic of multiplicity.** Let us first review known results on multiplicities of irreducible sl_{n+1} modules in the N-th tensor power of the vector representation. Due to the Schur-Weyl duality we have

$$(\mathbb{C}^{n+1})^{\otimes N} \simeq \oplus_\lambda W_\lambda^{(N)} \otimes V_\lambda^{(n+1)},$$

where $V_\lambda^{(n+1)}$ is the irreducible sl_{n+1} module with the highest weight λ corresponding to the partition $l_1 \geq \cdots \geq l_{n+1} \geq 0$, $\sum_{i=1}^{n+1} l_i = N$ and in the simple root basis $\lambda_a = l_a - l_{a+1}$. The space $W_\lambda^{(N)}$ is an irreducible S_N module corresponding to the partition l; $m_\lambda^{(N)} = dim(W_\lambda^{(N)})$ is the multiplicity of $V_\lambda \hookrightarrow (\mathbb{C}^{n+1})^{\otimes N}$. Here the Lie algebra sl_{n+1} acts diagonally and the symmetric group S_N permutes the factors.

The multiplicity function $m_\lambda^{(N)}$ (or the dimension of $W_\lambda^{(N)}$) is determined by the hook length formula, see, for example [10]:

$$m_\lambda^{(N)} = N! \frac{\prod_{i<j} (l_i - l_j - i + j)}{\prod_{i=1}^{n+1} (l_i + n + 1 - i)!}.$$

Using the Stirling formula

$$N! = \sqrt{2\pi N} e^{N \ln N - N} \left(1 + O\left(\frac{1}{N}\right)\right),$$

we obtain the following asymptotics for multiplicities for large N and l_i:

$$m_\lambda^{(N)} = \frac{\sqrt{2\pi N} e^{N \ln N - N} \prod_{i<j} (l_i - l_j)}{(\sqrt{2\pi})^{n+1} \prod_{i=1}^{n+1} l_i^{n+1-i+1/2} e^{l_i \ln l_i - l_i}} \left(1 + O\left(\frac{1}{N}\right)\right).$$

Now, assume $N = \tau/\epsilon$, $l_i = \sigma_i/\epsilon$, where τ, σ_i are finite and $\epsilon \to 0$. Note that τ, σ_i satisfy the condition $\sum_{i=1}^{n+1} \sigma_i = \tau$, $\tau \geq \sigma_1 \geq \cdots \geq \sigma_{n+1} \geq 0$.

Then in the limit $\epsilon \to 0$ we obtain the asymptotic of the multiplicity function:

$$(28) \qquad m_\lambda^{(N)} = \left(\frac{\epsilon}{2\pi}\right)^{\frac{n}{2}} \tau^{\frac{1}{2}} \prod_{i<j} (\sigma_i - \sigma_j) \prod_{i=1}^{n+1} \sigma_i^{-n+i-3/2} e^{\frac{1}{\epsilon} S(\tau,\sigma)} (1 + O(\epsilon)),$$

where

$$(29) \qquad S(\tau, \sigma) = \tau \ln \tau - \sum_{i=1}^{n+1} \sigma_i \ln \sigma_i.$$

In the section 6 we will show that this formula matches the one from Theorem 1.

5.2. **The asymptotic of the Plancherel probability distribution.** Now let us find the asymptotical probability distribution for the Plancherel probability measure which was first studied by Kerov in [9].

$$p_\lambda^{(N)} = \frac{m_\lambda^{(N)} \dim V^\lambda}{(n+1)^N},$$

5.2.1. Dimensions of irreducible sl_{n+1}-modules are given by the Weyl formula

$$\dim V^\lambda = \frac{\prod_{i \leq j}(l_i - l_j)}{\prod_{k=1}^{n+1} k!}.$$

First, let us find the asymptotic of p_λ when $N = \tau/\epsilon$, $l_i = \sigma_i/\epsilon$, $\epsilon \to 0$ and τ, σ_i remain finite with $\sum_{i=1}^{n+1} \sigma_i = \tau$, $\tau \geq \sigma_1 \geq \cdots \geq \sigma_{n+1} \geq 0$. When $\epsilon \to 0$ we have

$$\frac{\prod_{i \leq j}(l_i - l_j)}{\prod_{k=1}^{n+1} k!} \simeq \epsilon^{\frac{(n+1)^2 - n - 1}{2}} \frac{\prod_{i \leq j}(\sigma_i - \sigma_j)}{\prod_{k=1}^{n+1} k!}.$$

Combining this with the asymptotic of the multiplicity we obtain the following pointwise asymptotic of p_λ:

$$p_\lambda = \left(\frac{\epsilon}{2\pi}\right)^{\frac{n^2+2n}{2}} \frac{\prod_{i<j}(\sigma_i - \sigma_j)^2}{\prod_{k=1}^{n} k!} \prod_{i=1}^{n+1} \sigma_i^{-n+i-3/2} e^{\frac{1}{\epsilon}(S(\tau,\sigma) - \tau \ln(n+1))} (1 + O\mathcal{O}(\epsilon)),$$

5.2.2. Now, let us first find the maximum of the large deviation rate function $S(\tau, \sigma)$ on the hypersurface $\sum_{i=1}^{n+1} \sigma_i = \tau$. For this we can use the method of Lagrange multipliers and consider

$$S_\alpha(\tau, \sigma) = S(\tau, \sigma) + \alpha(\sum_{i=1}^{n+1} \sigma_i - \tau).$$

Critical points of $S(\tau, \sigma)$ are solutions to $\frac{\partial S_\alpha}{\partial \sigma_i} = 0$ and $\frac{\partial S_\alpha}{\partial \alpha} = 0$ which gives:

$$(30) \qquad -(\ln \sigma_i + 1) + \alpha = 0, \quad \sum_{i=1}^{n+1} \sigma_i = \tau$$

This system has unique solution

$$\sigma_i = \frac{\tau}{n+1}.$$

Now let us study the behaviour of m_λ^N in the vicinity of this critical point. For this rescale random variables σ_i as:

$$(31) \qquad \sigma_i = \frac{\tau}{n+1} + \sqrt{\epsilon} a_i.$$

Since σ_i are constrained (30), we should have $\sum_{i=1}^{n+1} a_i = 0$ and since $\sigma_i \geq \sigma_{i+1} \geq 0$ we should have $a_i \geq a_{i+1}$. Expanding $S(\tau, \sigma)$ in the Taylor series around $\sigma_i = \tau/(n+1)$ we have

$$S(\tau, \sigma) = S(\tau, \tau/(n+1)) - \frac{n+1}{\tau} \sum_{i=1}^{n+1} \frac{\epsilon a_i^2}{2} + O(\epsilon^{3/2}).$$

For the pointwise asymptotic of the asymptotic Plancherel probability distribution in the vicinity of the critical point we obtain

$$(32)$$
$$p_\lambda^{(N)} = \left(\frac{\epsilon}{2\pi}\right)^{\frac{n}{2}} \frac{1}{1! \cdot 2! \cdot \ldots \cdot n!} \tau^{\frac{-(n+1)^2+1}{2}} (n+1)^{\frac{(n+1)^2}{2}} \prod_{i<j}(a_i - a_j)^2 e^{-\frac{1}{2}\frac{n+1}{\tau}\sum_{i+1}^{n+1} a_i^2} (1 + O(\epsilon)).$$

This implies that the probability measure p_λ converges weakly to the probability distribution of the hyperplane $\sum_i a_i = 0$ in $\mathbb{R}^{n+1}$ with the density function

$$p(a) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \frac{1}{1! \cdot 2! \cdot \ldots n!} \tau^{\frac{-(n+1)^2+1}{2}} (n+1)^{\frac{(n+1)^2}{2}} \prod_{i<j} (a_i - a_j)^2 \, e^{-\frac{1}{2}\frac{n+1}{\tau} \sum_{i+1}^{n+1} a_i^2} (1 + O(\epsilon))$$

This is exactly the result of [9].

6. THE ASYMPTOTIC OF MULTIPLICITIES FOR $\mathfrak{g} = sl_{n+1}$, $m = 1, \nu = \omega_1$ VIA CHARACTER IDENTITIES.

Here we will show that for N-the tensor power of the vector representation of sl_{n+1} the asymptotic of the multiplicity m_λ^N derived from Theorem 1

$$m_\lambda^N = \epsilon^{\frac{r}{2}} \frac{\sqrt{\det K}}{(2\pi)^{\frac{r}{2}}} \Delta(x) e^{-(\rho,x)} e^{\frac{1}{\epsilon}S(\tau,\xi)} (1 + O(\epsilon))$$

coincides with the asymptotic obtained from the hook length formula.

6.1. **The large deviation rate function** $S(\tau, \xi)$. For sl_{n+1} the simple roots are $\alpha_i = e_i - e_{i+1}$, $i = 1 \ldots n$. The first fundamental weight is $\omega_1 = \frac{1}{n+1}(ne_1 - e_2 - e_3 - \cdots - e_n)$. Weights of the first fundamental representation are given by

$$\mu_k = \frac{1}{n+1}(e_1 - e_2 - \cdots - e_{k-1} + ne_k - e_{k+1} - \cdots - e_n), \quad k = 1 \ldots n+1,$$

where $\mu_1 = \omega_1$ is the highest weight.

The character of the first fundamental module:

$$\chi_{\omega_1}(e^y) = e^{(y,\mu_1)} + e^{(y,\mu_2)} + \cdots + e^{(y,\mu_{n+1})} = e^{y_1} + e^{-y_2+y_3} + e^{-y_3+y_4} + \cdots + e^{-y_{n-1}+x_n} + e^{-y_n},$$

where (y, μ_i) is the scalar product in $\mathbb{R}^{n+1}$ and $y = y_1\alpha_1 + y_2\alpha_2 + \cdots + y_n\alpha_n$.

According to the proposed method, the large deviation rate function for the first fundamental representation is given by the following expression:

$$(33) \qquad\qquad S(\tau, \xi) = \tau \ln(\chi_{\omega_1}(x)) - B_{ab}x_a, \xi_b,$$

where x is determined from the system of equations (19):

$$(34) \qquad\qquad \tau \frac{\partial}{\partial x_a} \ln \chi_{\omega_1}(x) = \sum_b B_{ab}\xi_b.$$

Lemma 2. *The solution of system (34) is determined by*

$$\begin{cases} x_1 = \ln\left(\frac{\chi}{\tau}\right) + \ln\left(\frac{\tau}{n+1} + \xi_1\right) \\ \ldots \\ x_{i+1} = (i+1)\ln\left(\frac{\chi}{\tau}\right) + \ln\left(\frac{\tau}{n+1} + \xi_1\right) + \ln\left(\frac{\tau}{n+1} - \xi_1 + \xi_2\right) + \cdots + \ln\left(\frac{\tau}{n+1} - \xi_i + \xi_{i+1}\right) \\ \ldots \\ x_n = -\ln\left(\frac{\chi}{\tau}\right) - \ln\left(\frac{\tau}{n+1} - \xi_n\right) \end{cases},$$

where $\chi = \chi_{\omega_1}$ denotes the character of the first fundamental module.

Proof. Firstly, we write the equations of (34) explicitly:

$$(35) \qquad \begin{cases} \frac{\tau}{\chi}\left(e^{x_1} - e^{-x_1+x_2}\right) = 2\xi_1 - \xi_2 \\ \frac{\tau}{\chi}\left(e^{-x_{i-1}+x_i} - e^{-x_i+x_{i+1}}\right) = -\xi_{i-1} + 2\xi_i - \xi_{i+1}, \quad i = 2\ldots n-1 \\ \frac{\tau}{\chi}\left(e^{-x_{n-1}+x_n} - e^{-x_n}\right) = -\xi_{n-1} + 2\xi_n \end{cases}.$$

To solve (35) let us introduce the variables $z_i = e^{x_i}$. So (35) can be written as:

$$\begin{cases} z_1 - z_1^{-1}z_2 = \frac{\chi}{\tau}(2\xi_1 - \xi_2) \\ z_{i-1}^{-1}z_i - z_i^{-1}z_{i+1} = \frac{\chi}{\tau}(-\xi_{i-1} + 2\xi_i - \xi_{i+1}), \quad i = 2\ldots n-1 \\ z_{n-1}^{-1}z_n - z_n^{-1} = \frac{\chi}{\tau}(-\xi_{n-1} + 2\xi_n) \end{cases}.$$

From the first equation we can express $z_1^{-1}z_2$ in terms of z_1 and then substitute it into the next equation. Step by step, we can express $z_i^{-1}z_{i+1}$ in terms of z_1:

$$(36) \qquad \begin{cases} z_1^{-1}z_2 = z_1 - \frac{\chi}{\tau}(2\xi_1 - \xi_2) \\ z_i^{-1}z_{i+1} = z_1 - \frac{\chi}{\tau}(\xi_1 + \xi_i - \xi_{i+1}), \quad i = 2\ldots n-1 \\ z_n^{-1} = z_1 - \frac{\chi}{\tau}(\xi_1 + \xi_n) \end{cases}.$$

We then need to express z_1 in terms of χ. This can be obtained by noting that

$$(37) \qquad \chi = z_1 + z_1^{-1}z_2 + z_2^{-1}z_3 + \cdots + z_{n-1}^{-1}z_n + z_n^{-1}.$$

Substiting $z_i^{-1}z_{i+1}$ from (36) into (37) we get

$$z_1 = \frac{\chi}{\tau}\left(\frac{\tau}{n+1} + \xi_1\right).$$

Now, we can solve (36) in terms of χ and ξ_i:

$$(38) \qquad \begin{cases} z_1 = \frac{\chi}{\tau}\left(\frac{\tau}{n+1} + \xi_1\right) \\ z_{i+1} = \left(\frac{\chi}{\tau}\right)^{i+1}\left(\frac{\tau}{n+1} + \xi_1\right)\left(\frac{\tau}{n+1} - \xi_1 + \xi_2\right)\ldots\left(\frac{\tau}{n+1} - \xi_i + \xi_{i+1}\right) \\ z_n^{-1} = \frac{\chi}{\tau}\left(\frac{\tau}{n+1} - \xi_n\right) \end{cases}.$$

Finally, we can solve (35)in terms of χ and ξ_i:
(39)
$$\begin{cases} x_1 = \ln\left(\frac{\chi}{\tau}\right) + \ln\left(\frac{\tau}{n+1} + \xi_1\right) \\ x_{i+1} = (i+1)\ln\left(\frac{\chi}{\tau}\right) + \ln\left(\frac{\tau}{n+1} + \xi_1\right) + \ln\left(\frac{\tau}{n+1} - \xi_1 + \xi_2\right) + \cdots + \ln\left(\frac{\tau}{n+1} - \xi_i + \xi_{i+1}\right) \\ x_n = -\ln\left(\frac{\chi}{\tau}\right) - \ln\left(\frac{\tau}{n+1} - \xi_n\right) \end{cases}.$$

$\square$

Lemma 3. *The large deviation rate function (33) is given by*

$$(40) \qquad S(\tau, \sigma) = \tau \ln \tau - \sum_{i=1}^{n+1} \sigma_i \ln \sigma_i.$$

Proof. The large deviation rate function in terms of χ and ξ_i:

$$S(\tau, \xi) = \tau \ln(\chi) - C_{ab} x_a \xi_b = \tau \ln(\chi) - \ln\left(\frac{\chi}{\tau}\right)((n+1)\xi_{n-1} - (n+1)\xi_n) -$$

$$- \ln\left(\frac{\tau}{n+1} + \xi_1\right)(\xi_1 + \xi_{n-1} - \xi_n) - \ln\left(\frac{\tau}{n+1} - \xi_1 + \xi_2\right)(-\xi_1 + \xi_2 + \xi_{n-1} - \xi_n) - \ldots$$

$$\ldots - \ln\left(\frac{\tau}{n+1} - \xi_{n-2} + \xi_{n-1}\right)(-\xi_{n-2} + 2\xi_{n-1} - \xi_n) - \ln\left(\frac{\tau}{n+1} - \xi_n\right).$$

To obtain expression for $\ln \chi$ in terms of ξ_i we substitute solutions of (39) into (37):

$$\chi = \frac{\chi}{\tau}\left(\frac{\tau}{n+1} + \xi_1\right) + \frac{\chi}{\tau}\left(\frac{\tau}{n+1} - \xi_1 + \xi_2\right) + \cdots + \frac{\chi}{\tau}\left(\frac{\tau}{n+1} - \xi_{n-2} + \xi_{n-1}\right) +$$

$$+ \frac{1}{\left(\frac{\chi}{\tau}\right)^n \left(\frac{\tau}{n+1} + \xi_1\right)\left(\cdots + \frac{\tau}{n+1} - \xi_1 + \xi_2\right)\ldots\left(\frac{\tau}{n+1} - \xi_{n-2} + \xi_{n-1}\right)\left(\frac{\tau}{n+1} - \xi_n\right)} + \frac{\chi}{\tau}\left(\frac{\tau}{n+1} - \xi_n\right),$$

which yields

$$(41) \qquad \chi^{n+1} = \frac{\tau^{n+1}}{\left(\frac{\tau}{n+1} + \xi_1\right)\left(\frac{\tau}{n+1} - \xi_1 + \xi_2\right)\ldots\left(\frac{\tau}{n+1} - \xi_{n-1} + \xi_n\right)\left(\frac{\tau}{n+1} - \xi_n\right)},$$

and

$$\ln \chi = \frac{1}{n+1}(\ln \tau^{n+1} - \ln\left(\frac{\tau}{n+1} + \xi_1\right) - \ln\left(\frac{\tau}{n+1} - \xi_1 + \xi_2\right) - \ldots$$

$$\ldots - \ln\left(\frac{\tau}{n+1} - \xi_{n-1} + \xi_n\right) - \ln\left(\frac{\tau}{n+1} - \xi_n\right)).$$

Substituting $\ln \chi$ into the large deviation rate function we finally obtain:

$$S(\tau, \xi) = \tau \ln \tau - \left(\frac{\tau}{n+1} + \xi_1\right)\ln\left(\frac{\tau}{n+1} + \xi_1\right) - \left(\frac{\tau}{n+1} - \xi_1 + \xi_2\right)\ln\left(\frac{\tau}{n+1} - \xi_1 + \xi_2\right) - \ldots$$

$$\ldots - \left(\frac{\tau}{n+1} - \xi_{n-1} + \xi_n\right)\ln\left(\frac{\tau}{n+1} - \xi_{n-1} + \xi_n\right) - \left(\frac{\tau}{n+1} - \xi_n\right)\ln\left(\frac{\tau}{n+1} - \xi_n\right) =$$

$$= \tau \ln \tau - \sigma_1 \ln \sigma_1 - \sigma_2 \ln \sigma_2 - \cdots - \sigma_{n+1} \ln \sigma_{n+1},$$

where

$$\sigma_1 = \frac{\tau}{n+1} + \xi_1, \quad \ldots \quad, \quad \sigma_i = \frac{\tau}{n+1} + \xi_i - \xi_{i-1}, \quad \ldots \quad, \quad \sigma_{n+1} = \frac{\tau}{n+1} - \xi_n$$

$$\square$$

The expression (40) coincides with (29) which we obtained from hook length formula.

6.2. **The matrix K and its determinant.** Now let us compute the determinant of $K(x)$ when x is the Legendre dual to ξ.

We have:

$$\frac{\partial S}{\partial \xi_i} = -\ln\left(\frac{\tau}{n+1} - \xi_{i-1} + \xi_i\right) + \ln\left(\frac{\tau}{n+1} - \xi_i + \xi_{i+1}\right),$$

From here we derive

$$\frac{\partial^2 S}{\partial^2 \xi_i} = -\frac{1}{\left(\frac{\tau}{n+1} - \xi_{i-1} + \xi_i\right)} - \frac{1}{\left(\frac{\tau}{n+1} - \xi_i + \xi_{i+1}\right)} = -\frac{1}{\sigma_i} - \frac{1}{\sigma_{i+1}},$$

$$\frac{\partial^2 S}{\partial \xi_i \partial \xi_{i-1}} = \frac{1}{\left(\frac{\tau}{n+1} - \xi_{i-1} + \xi_i\right)} = \frac{1}{\sigma_i},$$

$$\frac{\partial^2 S}{\partial \xi_i \partial \xi_{i+1}} = \frac{1}{\left(\frac{\tau}{n+1} - \xi_i + \xi_{i+1}\right)} = \frac{1}{\sigma_{i+1}},$$

$$\frac{\partial^2 S}{\partial \xi_i \partial \xi_j} = 0 \quad j \neq i, i-1, i+1.$$

From here we derive

$$\det(K) = \frac{\tau}{\sigma_1 \ldots \sigma_{n+1}}$$

6.3. **The factor** $\Delta(x)e^{-(\rho,x)}$**.** Finally, we need to compute $\Delta(x)$ and $e^{-(\rho,x)}$. The Weyl vector ρ is the sum of fundamental weights ω_i:

$$\omega_i = \frac{n+1-i}{n+1}e_1 + \cdots + \frac{n+1-i}{n+1}e_i - \frac{i}{n+1}e_{i+1} \cdots - \frac{i}{n+1}e_{n+1}.$$

which gives

$$\rho = \frac{n}{2}e_1 + \frac{n-2}{2}e_2 + \cdots + \frac{-n+2}{2}e_{n-1} + \frac{-n}{2}e_{n+1},$$

For $x = \sum_{i=1}^{n} x_i \alpha_i$, where $\alpha_i = e_i - e_{i+1}$, we have

$$e^{(\rho,x)} = e^{x_1} e^{x_2} \ldots e^{x_n}.$$

Lemma 4.

(42)
$$e^{(\rho,x)} = \sigma_1^{\frac{n}{2}} \sigma_2^{\frac{n-2}{2}} \sigma_3^{\frac{n-4}{2}} \ldots \quad \sigma_{n+1}^{-\frac{n}{2}}.$$

Indeed, we can use (39) and (42) to rewrite e^{x_i} in terms of τ and ξ_i and then in terms of σ we obtain:

$$e^{x_1} = \sigma_1^{\frac{n}{n+1}} \sigma_2^{\frac{-1}{n+1}} \ldots \ldots \ldots \ldots \ldots \quad \sigma_n^{\frac{-1}{n+1}}$$

$$e^{x_2} = \sigma_1^{\frac{n-1}{n+1}} \sigma_2^{\frac{n-1}{n+1}} \sigma_3^{\frac{-2}{n+1}} \ldots \ldots \ldots \quad \sigma_{n+1}^{\frac{-2}{n+1}}$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$e^{x_{i+1}} = \sigma_1^{\frac{n-i}{n+1}} \ldots \sigma_{i+1}^{\frac{n-i}{n+1}} \sigma_{i+2}^{-\frac{i+1}{n+1}} \ldots \quad \sigma_{n+1}^{-\frac{i+1}{n+1}}$$

$$e^{x_n} = \sigma_1^{\frac{1}{n+1}} \ldots \ldots \ldots \ldots \ldots \sigma_n^{\frac{1}{n+1}} \sigma_{n+2}^{-\frac{n}{n+1}},$$

This proves lemma 4.

Lemma 5.

$$\Delta(x) = \frac{\prod_{i<j} (\sigma_i - \sigma_j)}{\prod_i^{n+1} \sigma_i^{\frac{n}{2}}}$$

Proof. The Weyl denominator $\Delta(x)$ is the sum

$$\Delta(x) = \sum_w (-1)^{l(w)} e^{(w(\rho), x)}$$

where $(-1)^{l(w)}$ is the signature of the permutation $w \in S_{n+1}$. We have:

$$w(\rho) = \sum_{i=1}^{n+1} \frac{n+1-w(i)}{2} e_i,$$

which implies

$$e^{(w(\rho), x)} = \prod_{i=1}^{n+1} \sigma_i^{\frac{n+1-w(i)}{2}}$$

To evaluate Δ we can bring the factor

$$\prod_{i}^{n+1} \sigma_i^{\frac{n}{2}}.$$

out of summation. The remaining polynomial is the Vandermond determinant:

$$\Delta(x) = \sum_w (-1)^{l(w)} e^{(w(\rho), x)} = \frac{\prod_{i<j} (\sigma_i - \sigma_j)}{\prod_i^{n+1} \sigma_i^{\frac{n}{2}}}$$

$\square$

6.4. **The comparison.** Combining the results of the previous sections we conclude that

$$\epsilon^{\frac{n}{2}} \frac{\sqrt{\det K}}{(2\pi)^{\frac{n}{2}}} \Delta(x) e^{-(\rho, x)} e^{\frac{1}{\epsilon} S(\tau, \xi)} (1+o(1)) = \left(\frac{\epsilon}{2\pi}\right)^{\frac{n}{2}} \tau^{\frac{1}{2}} \prod_{i<j} (\sigma_i - \sigma_j) \prod_{i=1}^{n+1} \sigma_i^{-n+i-3/2} e^{\frac{1}{\epsilon} S(\tau, \sigma)} (1+o(1))$$

where the function $S(\tau, \xi)$ on the left side is defined as the Legendre transform of $\tau \ln(\chi_{\omega_1}(e^x))$ in x and on the right it is given by (29) and is derived from the hook formula.

7. INTERMEDIATE SCALING

In this section we consider the behavior of the character probability measure

(43) $$p_\lambda^{(N)} = \frac{m_\lambda(\{V_k\}, \{N_k\}) \chi_\lambda(e^t)}{\prod_k \chi_{\nu_k}(e^t)^{N_k}}$$

near $t = 0$ in the limit when $\epsilon \to 0$, $N_k = \frac{\tau_k}{\epsilon}$, $\lambda = \sqrt{\frac{x}{\epsilon}} b$, and $t = \sqrt{\frac{\epsilon}{x}} u$[6] where $b \in \mathfrak{h}_{\geq 0}^*$, $u \in \mathfrak{h}_{\geq 0}$ are regular.

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[6]There is an analogue of such asymptotic for every non regular value of t (when $t \subset S \in \mathfrak{h}^* \geq 0$ where S is a stratum of the boundary of the cone $\mathfrak{h}_{\geq 0}^*$). In this case transverse directions to S should be scaled as $\sqrt{\epsilon}$.

Theorem 5. *The pointwise asymptotic of $p_\lambda^{(N)}$ when $\epsilon \to 0$ is*

$$p_\lambda^{(N)}(t) = \left(\frac{\epsilon}{x}\right)^{\frac{r}{2}} \frac{\sqrt{\det B}}{(2\pi)^{\frac{r}{2}}} \prod_{\alpha \in \Delta_+} \frac{(b, \alpha)}{(u, \alpha)} \sum_w (-1)^w e^{(b, w(u))} e^{-\frac{(b,b)}{2} - \frac{(u,u)}{2}} (1 + o(1)).$$

Proof. The pointwise asymptotic of the multiplicity function is given by Theorem 1. As it was shown in section 4.2, the variables x_a in the formula for the asymptotic for small ξ_a are

$$x_b = \frac{\xi_b}{x} + O(\xi^2),$$

where $x = \frac{1}{dim(\mathfrak{g})} \sum_{k=1}^m \tau_k c_2(\nu_k)$ is as in section 4.2. In terms of the variable b we have

$$x_a = \sqrt{\frac{\epsilon}{x}} b_a + o(\sqrt{\epsilon}).$$

As a consequence for the Weyl denominator we have

$$\Delta(x) = \prod_{\alpha \in \Delta_+} (e^{\frac{(x,\alpha)}{2}} - e^{-\frac{(x,\alpha)}{2}}) = \prod_{\alpha \in \Delta_+} \left(\sqrt{\frac{\epsilon}{x}}\right)(a, \alpha) = \left(\sqrt{\frac{\epsilon}{x}}\right)^{|\Delta_+|} \prod_{\alpha \in \Delta_+} (a, \alpha).$$

It is also easy to find the asymptotic of the character:

(44) $$\chi_\lambda(e^t) = \frac{\sum_w (-1)^w e^{(\lambda+\rho, w(t))}}{\Delta(t)} = \frac{\sum_w (-1)^w e^{(b, w(u))}}{\left(\sqrt{\frac{\epsilon}{x}}\right)^{|\Delta_+|} \prod_{\alpha \in \Delta_+} (u, \alpha)},$$

For small ξ the large deviation rate function is

$$S(\xi) = -\frac{1}{2x} \xi_b B_{bc} \xi_c + o(\xi^2),$$

In terms of variable b $\xi_a = \sqrt{\epsilon x} b_a$ and we have:

$$S(\xi) = -\frac{\epsilon}{2} b_a B_{ac} b_c + o(\epsilon),$$

For the matrix K_{ab} as in section 4.2 we have:

$$K_{ab} = \frac{B_{ab}}{x} + O(\epsilon),$$

and therefore

$$\det K = x^{-r} \det B(1 + o(1)).$$

Finally, taking into account that $t = \sqrt{\frac{\epsilon}{x}} u$ we have

$$\prod_k \chi_{V_{\nu_k}}(e^t)^{N_k} = \exp(\sum_k \frac{\tau_k}{\epsilon} \ln(dim(V_{\nu_k})) + \frac{1}{2}(u, u)).$$

Here we used the identity

$$tr_V(xy) = \frac{c_2(V) dim(V)}{dim\mathfrak{g}} (x, y).$$

where we assume that V is irreducible, $c_2(V)$ is the value of the second Casimir on V and (x, y) is the scalar product with respect to the Killing form. The identity follows from the Schur lemma.

Now after the substitution of these asymptotical expressions into (43) we obtain the desired asymptotic:

(45)
$$p_\lambda^{(N)}(t) = \left(\frac{\epsilon}{x}\right)^{\frac{r}{2}} \frac{1}{(2\pi)^{\frac{r}{2}}\sqrt{\det B}} \frac{\prod_{\alpha\in\Delta_+}(b,\alpha)}{\prod_{\alpha\in\Delta_+}(u,\alpha)} \sum_{w\in W}(-1)^w e^{(b,w(u))} e^{-\frac{(b,b)}{2}-\frac{(u,u)}{2}}(1+o(1)).$$

$\square$

Let us check now that the function
$$p(b,u) = \frac{\sqrt{\det B}}{(2\pi)^{\frac{r}{2}}} \frac{\prod_{\alpha\in\Delta_+}(b,\alpha)}{\prod_{\alpha\in\Delta_+}(u,\alpha)} \sum_{w\in W}(-1)^w e^{(b,w(u))} e^{-\frac{(b,b)}{2}-\frac{(u,u)}{2}}$$

is the density of a probability measure on $\mathfrak{h}^\star \geq 0$ ($b = \sum_a b_a\alpha_a, b_a \geq 0$). It is clear that $p(b,u) \geq 0$, so we only have to check the normalization. We naturally expect this to be true since $\sum_\lambda p_\lambda^{(N)}(t) = 1$. The following lemma checks it explicitly.

Lemma 6. *The function $p(b,u)$ is the density of a probability measure on $\mathfrak{h}^\star \geq 0$, i.e.*
$$\int_{\mathfrak{h}\mathbb{R}^r\geq 0} p(b,u)db = 1,$$
where $db_1 \ldots db_r$ is the Euclidean measure on $\mathbb{R}^r$.

Proof. Let us calculate the normalization constant N explicitly:
$$N = \frac{\sqrt{\det B}}{(2\pi)^{\frac{r}{2}}} \int_{\mathfrak{h}^\star\geq 0} \prod_{\alpha\in\Delta_+}\frac{(b,\alpha)}{(u,\alpha)} \sum_w (-1)^w e^{(b,w(u))} e^{-\frac{(b,b)}{2}-\frac{(u,u)}{2}} db$$

$$\frac{\sqrt{\det B}}{(2\pi)^{\frac{r}{2}}} \frac{1}{\prod_{\alpha\in\Delta_+}(u,\alpha)} \frac{1}{|W|} \int_{\mathfrak{h}^\star} \prod_{\alpha\in\Delta_+}(b,\alpha) \sum_w (-1)^w e^{(b,w(u))} e^{-\frac{(b,b)}{2}-\frac{(u,u)}{2}} db.$$

It is easy to compute the Gaussian integral
$$\frac{\sqrt{\det B}}{(2\pi)^{\frac{r}{2}}} \int_{\mathfrak{h}^\star} e^{-\frac{(b,b)}{2}+(b,u)} db =$$
$$= \frac{\sqrt{\det B}}{(2\pi)^{\frac{r}{2}}} \int_{\mathfrak{h}^\star} e^{-\frac{(b-u,b-u)}{2}+\frac{(u,u)}{2}} db = e^{\frac{(u,u)}{2}}.$$

We have
$$\prod_{\alpha\in\Delta_+}(b,\alpha)e^{(b,u)} = \prod_{\alpha\in\Delta_+}(b,\alpha)e^{b_a B_{ab} u_b} = \prod_{\alpha\in\Delta_+}(b,\alpha)e^{b_a u^a} = \prod_{\alpha\in\Delta_+}\left(\frac{\partial}{\partial u},\alpha\right)e^{(b,u)} = D(u)e^{(b,u)},$$

where $D_u = \prod_{\alpha\in\Delta_+}\left(\frac{\partial}{\partial u},\alpha\right)$.

The normalization constant N now can be written as
$$N = \frac{1}{|W|\prod_{\alpha\in\Delta_+}(u,\alpha)} \sum_w (-1)^w J(w(u)),$$

where
$$J(u) = e^{-\frac{(u,u)}{2}} D_u e^{\frac{(u,u)}{2}}.$$

The following identity is easy to check:

$$\sum_w (-1)^w J(w(u)) = |W| \prod_{\alpha \in \Delta_+} (u, \alpha).$$

Thus

$$N = 1.$$

$\square$

Note that (45) implies a weak convergence of the sequence of character measures $\{p_\lambda^{(N)}(t)\}$ to $p(b, u)db$.

8. NONLINEAR PDE FOR THE RATE FUNCTION $S(\tau, \xi)$

Let $V = \oplus_\mu V(\mu)$ be the weight decomposition of V and $d_\mu = \dim(V_\mu)$. For the weight λ, which is well inside the main Weyl chamber, we have the following decomposition of the tensor product

$$(46) \qquad V \otimes V_\lambda \simeq \oplus_{\mu \in wt(V)} V_{\lambda+\mu}^{\oplus d_\mu}.$$

were $wt(V)$ is the set of weights which occur in the representation V.

Let $m_\lambda^{(N)}$ be the multiplicities in

$$V^{\otimes N} \simeq \oplus_\lambda V_\lambda^{\oplus m_\lambda^N}.$$

the decomposition (46) gives the difference equation for multiplicities

$$m_\lambda^{(N+1)} = \sum_{\mu \in wt(V)} d_\mu m_{\lambda-\mu}^{(N)},$$

where we assume again that λ is well inside the main Weyl chamber.

Now, passing to the limit $\epsilon \to 0$ in this difference equation with $N = \frac{\tau}{\epsilon}$, $\lambda = \frac{\xi}{\epsilon}$ we derive the following nonlinear PDE for the large deviation rate function S:

$$(47) \qquad e^{\partial_\tau S} = \sum_{\mu \in wt(V)} d_\mu e^{-\sum_a \mu_a \frac{\partial S}{\partial \xi_a}}.$$

Here we used the basis of simple roots in $\mathfrak{h}^*$: $\mu = \sum_a \mu_a \alpha_a$, $\xi = \sum_a \xi_a \alpha_a$.

The following proposition is a direct verification of the equation (47) for $S(\tau, \xi)$ determined by (17) .

Proposition 1. *The function $S(\tau, \xi)$ defined in (3) satisfies the differential equation (47).*

Proof. By definition

$$S(\tau, \xi) = \tau \ln \chi_\nu(e^x) - (x, \xi),$$

where x is the unique solution to the equation:

$$\tau \frac{\partial}{\partial x_a} \chi_\nu(e^x) = \chi_\nu(e^x) \sum_b B_{ab} \xi_b.$$

Differentiating this with respect to τ we have

$$\frac{\partial}{\partial x_a} \ln \chi_\nu(e^x) + \frac{\partial x_b}{\partial \tau} \frac{\partial^2}{\partial x_a \partial x_b} f(\tau, x) = 0.$$

i.e.

$$D_{ab} \frac{\partial x_b}{\partial \tau} = -\frac{1}{\tau} B_{ab} \xi_b.$$

From here we obtain

$$\partial_\tau S = \ln \chi_\nu(e^x) + \tau \frac{\partial x_b}{\partial \tau} \frac{\partial}{\partial x_b} \ln \chi_\nu(e^x) - \frac{\partial x_a}{\partial \tau} B_{ab} \xi_b = \ln \chi_\nu(e^x).$$

On the other hand

$$\frac{\partial S}{\partial \xi_a} = \frac{\partial x_b}{\partial \xi_a} \tau \frac{\partial}{\partial x_b} \ln \chi_\nu(e^x) - \frac{\partial x_b}{\partial \xi_a} B_{bc} \xi_c - x_b B_{ba} = -x_b B_{ba}.$$

i.e.

$$e^{-\mu_a \frac{\partial S}{\partial \xi_a}} = e^{(\mu, x)}$$

Thus, both sides of (47) are equal to $\chi_\nu(e^x)$ and we verified (47).   □

9. Conclusion

This paper demonstrats that in large tensor products the statistic of irreducible components with respect to the character distribution almost does not depend on which representations being multiplied. But it depends significantly on whether the parameter t in the character distribution is generic or special.

Note that one can associate natural Markov process with the decomposition of tensor powers $V^{\otimes N}$ of a finite dimensional $\mathfrak{g}$-module V. The transition probabilities in such a process are

$$M_{\lambda, \mu} = \frac{\chi_\lambda(e^t) b_{\lambda, \mu}}{\chi_\mu(e^t) \chi(e^t)}$$

where $\chi(e^t)$ is the character of the representation V evaluated at e^t and $b_{\lambda, \mu}$ are multiplicities of irreducible components in the tensor product

$$V \otimes V_\lambda \simeq \oplus_\mu V_\mu^{\oplus_\mu b_{\lambda, \mu}}$$

Note that if λ is sufficiently inside the positive Weyl chamber, $b_{\lambda, \lambda - \mu} = d_\mu$ where d_μ is the multiplicity of weight μ in V.

The character probability distribution (2) is a result of the Markov evolution of character measure of the trivial representation:

$$p_\lambda^{(0)}(t) = \delta_{\lambda, 0}$$

i.e.

$$p^{(N)} = M^N p^{(0)}$$

This paper can be regarded as a study of this Markov process[7] . This process also can be regarded as a random walk on a lattice domain. For results in this direction see [3]; also [13] and references therein. From this point of view the results on the week convergence of character measures can be interpreted as follows:

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[7]Or of its slight generalization when we consider $\otimes_k V_{\nu_k}^{\otimes N_k}$ instead of $V^{\otimes N}$.

- For regular t, as $N \to \infty$, the expectation value of λ behave as $\mathbb{E}(\lambda) = N\nabla \ln(\chi(e^t))(1 + o(1))$ where ∇ is the gradient (with respect to the Killing form on the weight space). As $N \to \infty$ the distribution converges to the Gaussian distribution around the expectation value with the dispersion behaving as $\sqrt{N}$.
- For $t = 0$ and $N \to \infty$ the expectation value of λ vanishes and the asymptotical distribution is given by (9) rescaled such that the dispersion is proportional to $\sqrt{N}$.

Similar interpretation can be given for the intermediate scaling.

Below we will outline some of the future directions and problems in the asymptotic representation theory that are naturally related to this paper.

- We studied the asymptotic of dimensions of irreducible components and the corresponding probability distributions when t is either regular or maximally irregular, i.e. $t = 0$. When t is not regular the asymptotical measure is given by a different formula involving the centralizer of the action of W on ξ. These results will be presented in a separate publication.
- Truncated tensor products, also known as fusion products appear in representation theory of quantum groups at roots of unity and plays an important role in constructing modular categories. The latter are instrumental in conformal filed theory and in topological quantum filed theory. The study of statistics of irreducible components for truncated tensor products is an important problem by itself and an important step in understanding the semiclassical limit of corresponding topological quantum field theory. See [7] and references therein for existing results on fusion products and corresponding random walks.

Other interesting problems expanding the results announced here are related to multiplicities of irreducible components in tensor products of Lie superalgebras and of finite groups of Lie type, i.e. groups similar to $SL_n(F_p)$. In these cases, as well as in the case of quantum groups at roots of unity, tensor products of irreducibles have both irreducible components and blocks of irreducibles. The study of statistics of blocks in large tensor products is another interesting problem that is largely open for investigation.

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