Discrepancy and large dense monochromatic subsets

Kang, R.J.; Patel, V.; Regts, G.

DOI
10.4310/JOC.2019.v10.n1.a4
Publication date
2019
Document Version
Submitted manuscript
Published in
Journal of Algebraic Combinatorics

Discrepancy and large dense monochromatic subsets

Ross J. Kang* Viresh Patel† Guus Regts‡

October 8, 2018

Abstract

Erdős and Pach (1983) introduced the natural degree-based generalisations of Ramsey numbers, where instead of seeking large monochromatic cliques in a 2-edge coloured complete graph, we seek monochromatic subgraphs of high minimum or average degree. Here we expand the study of these so-called quasi-Ramsey numbers in a few ways, in particular, to multiple colours and to uniform hypergraphs.

Quasi-Ramsey numbers are known to exhibit a certain unique phase transition and we show that this is also the case across the settings we consider. Our results depend on a density-biased notion of hypergraph discrepancy optimised over sets of bounded size, which may be of independent interest.

Keywords. Ramsey theory, quasi-Ramsey numbers, hypergraph discrepancy, probabilistic method

AMS subject classifications. Primary, 05C55; Secondary, 05D10, 05D40

1 Introduction

Frank Plumpton Ramsey [22] originally addressed the following question. Fixing $q, r \geq 2$, for any k, is there always a finite n such that in any assignment of q colours to the r-element subsets of $[n] = \{1, \ldots, n\}$, there is guaranteed to be a k-element subset of $[n]$ all of whose r-element subsets have the same colour? Ramsey's Theorem states that the answer is yes. The search for the smallest values $R_q^{(r)}(k)$ of n in this question (the Ramsey numbers) is a central part of combinatorial mathematics. This search was begun in seminal papers by Erdős and Szekeres [12] and Erdős [6] for the case $q = r = 2$ showing that

$$\sqrt{2}^k \leq R_2^{(2)}(k) \leq 4^k.$$

After decades, these remain very near to the best known bounds for this parameter.

When $q > 2$ or $r > 2$, our knowledge of the situation is even worse. If $q > 2$ and $r = 2$, then $R_q^{(2)}(k)$ is exponential in k, but the best known bounds on the constants in the base of the exponential are weaker for larger q. More significantly, if $r > 2$ and $q = 2$, then $R_2^{(r)}(k)$ is known to grow like a tower of exponentials in k [8], but the height of this tower is unknown and is subject to a \$500 Erdős prize. On the other hand, note that Erdős and Hajnal (cf. [14]) have shown that $\ln \ln R_4^{(3)}(k) = \Theta(k)$ as $k \to \infty$.

*Department of Mathematics, IMAPP, Radboud University Nijmegen. Email: ross.kang@gmail.com. This work was initiated while this author was supported by a Veni grant from the Netherlands Organisation for Scientific Research (NWO).

†Korteweg de Vries Institute for Mathematics, University of Amsterdam. Email: vpatel@uva.nl. Supported by NWO through the Gravitation Programme Networks (024.002.003).

‡Korteweg de Vries Institute for Mathematics, University of Amsterdam. Email: guusregts@gmail.com. Supported by a personal NWO Veni grant.

For $q = r = 2$, Erdős and Pach [9] formulated a natural degree-based generalisation of the Ramsey numbers. Given $c \in [0,1]$, the basic question is as follows: for any k, what is the smallest $n := R_c(k)$ such that for any graph $G = (V, E)$ on n vertices there exists a subset $S \subseteq V$ of size ℓ at least k such that either $G[S]$ or its complement $\overline{G}[S]$ has minimum degree at least $c(\ell - 1)$? We may also ask this question with average degree instead of minimum degree and denote the corresponding number $\overline{R}_c(k)$. Clearly $\overline{R}_c(k) \leq R_c(k)$ always. We refer to $\overline{R}_c(k)$ and $R_c(k)$ as *quasi-Ramsey numbers*. Of course by taking $c = 1$ we recover the classical two-colour Ramsey numbers for graphs.

Erdős and Pach [9] found that the quasi-Ramsey numbers undergo a dramatic change in growth in k in a narrow window around $c = 1/2$: if $c < 1/2$ then they have linear growth, while if $c > 1/2$ they have singly exponential growth. They developed a fairly precise understanding of the transition at the point $c = 1/2$ — the present authors together with Pach [17] and with Long [15] have recently refined this.

Our purpose in the present paper is to extend the study of quasi-Ramsey numbers to multiple colours and uniform hypergraphs (as was initially considered by Ramsey).

We have been able to show that the precise transition behaviour at the point $c = 1/2$ for $r = q = 2$ is present in a similar way when $r > 2$ or $q > 2$. The proofs of our results rely critically on a density-biased notion of hypergraph discrepancy which is optimised only over those vertex subsets up to a certain size. In fact the most difficult part of the paper is devoted to proving a bound for this type of discrepancy, cf. Theorem 4 below, which we believe to be of independent interest.

1.1 Multi-colour quasi-Ramsey for graphs

For the case $r = 2$, we would first like to study the behaviour of quasi-Ramsey numbers if rather than two colours (namely the graph and its complement) there are $q \geq 2$ colours assigned to the edges of K_n. Motivated partly by related recent work by Falgas-Ravry, Markström, and Verstraëte [13], we treat an even more general setting where each of the q colours has an associated "degree share". Based on Theorem 4 below, we prove the following in Section 3.

Theorem 1. *Fix $q \geq 2$, $(\rho_1, \ldots, \rho_q) \in (0,1)^q$ such that $\sum_{i=1}^{q} \rho_i = 1$, and $\nu \geq 0$. Then there exists a constant $C = C(\nu) > 0$ such that for each k large enough and any q-colouring of the edges of the complete graph on at least $Ck \ln k$ vertices, there exists a colour j and a subset S of the vertices of size $\ell \geq k$ such that the subgraph induced by S in colour j has minimum degree at least $\rho_j(\ell - 1) + \nu\sqrt{\ell - 1}$.*

By a clever weighted random construction (cf. also [17]), Erdős and Pach proved that $R_{1/2}(k) = \Omega(k \ln k / \ln \ln k)$, which means that the quasi-Ramsey number bound for $c = 1/2$ implicit in Theorem 1 is sharp up to a $\ln \ln k$ factor.

We remark that Theorem 1 gives progress on a question posed by Falgas-Ravry, Markström, and Verstraëte [13]. Given a graph G on n vertices with edge density p, they asked for the largest integer $m = g(G)$ such that G contains an induced subgraph on at least m vertices with minimum degree at least $p(m-1)$ (what they called a *full subgraph*) or with maximum degree at most $p(m-1)$ (a *co-full subgraph*). In an earlier version of [13], the authors showed that if $p(1-p) \geq 1/n$ then $g(G) = \Omega(n/(\ln n)^2)$ for all graphs G, and asked whether this bound could be improved to $\Omega(n/\ln n)$. In the latest version of [13], they show $g(G) = \Omega(n/(\ln n))$ and no longer require the condition $p(1-p) \geq 1/n$ (see Theorem 4). Here (addressing the question in the earlier version) we obtain (a strengthening and generalisation of) the same result via Theorem 1 and Corollary 8. Indeed, in the case where the edge density p is fixed, Theorem 1 is a strengthening since we can guarantee slightly higher degree than required by taking $q = 2$ and $\rho_1 = p$. It is a generalisation in the sense of allowing more colours. In Section 3, we show that this $\Omega(n/\ln n)$ bound is also valid for non-constant p, cf. Corollary 8.

1.2 Multi-colour quasi-Ramsey for hypergraphs

The multicolour quasi-Ramsey investigation above naturally extends also to r-uniform hypergraphs, where we consider colourings of the hyperedges of the complete r-uniform hypergraph $K_n^{(r)}$ on n vertices. The *degree* of a vertex is the number of hyperedges incident with the vertex.

As Ramsey numbers for hypergraphs are even less well understood than for graphs, despite a long history, one might expect the hypergraph quasi-Ramsey problem to put up significant resistance. To the contrary, we have found that the precise threshold in quasi-Ramsey numbers for graphs established in [17] is present in an analogous way for hypergraphs. Based on Theorem 4 below and a standard random construction we establish the following result in Section 4.

Theorem 2. *Let $r \geq 2$. Fix $q \geq 2$ and $(\rho_1, \dots, \rho_q) \in (0,1)^q$ with $\sum_{i=1}^q \rho_i = 1$.*

(i) Let $\nu \geq 0$. Then there exists a constant $C > 0$ such that for each $\varepsilon > 0$ and k large enough, for any q-colouring of the edges of the complete r-uniform hypergraph on at least $k^{\nu^2 C^2(1+\varepsilon)+2r/(r+1)}$ vertices there exists a colour j and a subset S of the vertices of size $\ell \geq k$ such that the subhypergraph induced by S in colour j has minimum degree at least

$$\rho_j \binom{\ell-1}{r-1} + \nu\sqrt{\ell^{r-1}\ln \ell}.$$

(ii) There is a constant $C > 0$ such that, if $\nu(\cdot)$ is a non-decreasing non-negative function, then for each k large enough there is a q-colouring of the edges of the complete r-uniform hypergraph on $Ck^{\nu(k)^2+1}$ vertices such that the following holds. For any colour j and any subset S of the vertices of size $\ell \geq k$, the subhypergraph induced by S in colour j has average degree less than

$$\rho_j \binom{\ell-1}{r-1} + \nu(\ell)\sqrt{r\binom{\ell-1}{r-1}\ln \ell}.$$

We note that if we wish to find induced subgraphs with exactly (rather than at least) k vertices, then the following applies for for $\sum_{i=1}^q \rho_i < 1$. The proof appears in Section 4.

Proposition 3. *Let $r \geq 2$. Fix $q \geq 2$ and $(\rho_1, \dots, \rho_q) \in [0,1]^q$ with $\sum_{i=1}^q \rho_i < 1$. Then there exists a constant $C > 0$ such that for each k large enough, for any q-colouring of the edges of the complete r-uniform hypergraph on at most Ck vertices there exists a colour j and a subset S of the vertices of size k such that the subhypergraph induced by S in colour j has minimum degree at least $\rho_i\binom{k-1}{r-1}$.*

The situation could be more nuanced if $\sum_{i=1}^q \rho_i > 1$. It is of course unknown what precisely happens when $\sum_{i=1}^q \rho_i = q$, that is, the regime of the classical hypergraph Ramsey numbers, but we also do not know much for $1 < \sum_{i=1}^q \rho_i < q$. We will elaborate on this and state some open questions in Section 5.

Organisation. As mentioned above the proofs of our main results rely on a discrepancy result, which we state and prove in the next section. In Section 3 we prove Theorem 1 and in Section 4 we prove Theorem 2 and Proposition 3. We conclude with some remarks and open questions in Section 5.

2 Discrepancy over sets of bounded size

In this section we introduce our main tool, a p-discrepancy result for bounded sets in uniform hypergraphs, cf. Theorem 4 below.

Let $r \geq 2$ and let $H = (V, E)$ be an r-uniform hypergraph. For $p \in [0,1]$ and $S \subseteq V$, the *p-discrepancy* of S is defined as

$$D_p(S) := e(S) - p\binom{|S|}{r},$$

the number of hyperedges in the subhypergraph induced by S less a p proportion of the total possible number of hyperedges on S. For several r-uniform hypergraphs defined on the same vertex set, we specify $D_{p,H}(S)$. The *p-discrepancy* of H is defined as

$$D_p(H) := \max_{S \subseteq V} |D_p(S)|. \tag{1}$$

For the classic choice $p = 1/2$, we usually refer to this just as discrepancy. If p is chosen as $|E|/\binom{|V|}{r}$, the hyperedge density of H, then the p-discrepancy measures how uniformly the hyperedges are distributed over the vertices.

A well-known result of Erdős and Spencer [10] states that there exists $C = C(r) > 0$ such that, provided n is large enough, the discrepancy of any r-uniform hypergraph $H = (V, E)$ on n vertices satisfies

$$D_{1/2}(H) \geq Cn^{(r+1)/2}. \tag{2}$$

This is sharp up to the choice of the constant C. The same statement for p-discrepancy with $p = |E|/\binom{|V|}{r}$ was shown by Erdős, Goldberg, Pach and Spencer [7] for $r = 2$ and by Bollobás and Scott [1] for $r > 2$ (where the constant C depends on p).

It is natural to wonder what happens when the sets over which the maximum is taken in (1) all have a bounded number t of vertices. Clearly, one can obtain a constant times $t^{(r+1)/2}$, but in fact one gains a little more. Below we prove the following, generalising the results of Erdős and Spencer [10, 11], of Erdős, Goldberg, Pach and Spencer [7] and, partially, of Bollobás and Scott [1].

Theorem 4. *Let $r \geq 2$. There exist constants $C, D > 0$ such that for any $p \in (0,1)$ the following holds. For each n large enough and all $(\ln n)/D \leq t \leq n$, we have that any r-uniform hypergraph $H = (V, E)$ on n vertices satisfies*

$$\max_{S \subseteq V, |S| \leq t} |D_p(S)| \geq C \min\{p, 1-p\} t^{(r+1)/2} \sqrt{\ln(n/t)}. \tag{3}$$

Note that p in Theorem 4 is not assumed to be the density of the hypergraph. We also note that the case $p = 1/2$ of Theorem 4 was proved for $r = 2$ (i.e. graphs) and shown to be tight up to the choice of the constant C by Erdős and Spencer [11, Theorem 7.1]. A slightly stronger form for the hypergraph case (for $p = 1/2$) was announced and its proof left as a "difficult" exercise in [11, Chapter 7]. To the best of our knowledge no proof has been published. Although Theorem 4 suffices for our purposes, for p varying as a function of n, there is still room for potential improvement in the bound, since a random r-uniform hypergraph with edge density p supplies an upper bound example with instead the factor $\min\{\sqrt{p}, \sqrt{1-p}\}$ in (3).

2.1 Proof of Theorem 4

Our proof may be viewed as an extension of the proof of Erdős and Spencer [10] of (2). We will first prove several lemmas extending lemmas from [10]. We start with the following adaptation of [10, Lemma 2].

Lemma 5. *Fix $c > 0$. Then, for all m, all $y \geq 2$ such that $\ln y \leq cm/4$, and any choice of real numbers $x_1, \ldots, x_m$ satisfying $|x_i| \geq 1$ for at least cm of the $i \in [m]$, we have*

$$\left| \sum_{i \in V} x_i \right| \geq 4^{-1} \sqrt{cm \ln(y)}, \tag{4}$$

for at least $(8y)^{-1} 2^m$ choices of $V \subseteq [m]$.

Proof. For simplicity, let us assume that $cm \in \mathbb{N}$ and that $x_1, \ldots, x_{cm}$ all have absolute value at least 1. For $V \subseteq [m]$ set $\phi(V) := \sum_{i \in V} x_i$, $V_1 := V \cap [cm]$ and $V_2 := V \setminus V_1$. Then $\phi(V) = \phi(V_1) + \phi(V_2)$. Set $c_1 = 16^{-1} cm \ln y$. Then (4) does not hold if and only if $\phi(V_1) \in (-\phi(V_2) - \sqrt{c_1}, -\phi(V_2) + \sqrt{c_1})$. By a result of Erdős [5] this holds for fixed V_2 for at most

$$\sum_{r \,:\, |r - \frac{cm}{2}| \leq \sqrt{c_1}} \binom{cm}{r}$$

choices of $V_1 \subseteq [cm]$. Since $\sqrt{c_1} \leq cm/8$ by assumption on y, it follows from elementary arguments, cf. Proposition 7.3.2 in the lecture notes of Matoušek and Vondrák [19], that

$$2^{cm} - \sum_{r \,:\, |r - \frac{cm}{2}| \leq \sqrt{c_1}} \binom{cm}{r} \geq \frac{2^{cm+1}}{15} \exp\left(\frac{-16c_1}{cm}\right) = \frac{2^{cm+1}}{15y} > \frac{2^{cm}}{8y}.$$

In other words, for fixed V_2, we have for at least $(8y)^{-1} 2^{cm}$ choices of V_1 that (4) holds. Now summing over all possible V_2 proves the lemma. □

We will next prove a result about r-partite r-uniform hypergraphs, or (r,r)-graphs for short. Recall that an r-uniform hypergraph $H = (V,E)$ is said to be *r-partite* if there exists a partition of V into r sets $V_1, \dots, V_r$ such that every hyperedge e intersects all of the V_i exactly once. In this case we sometimes say H is an (r,r)-graph on $V_1 \cup \cdots \cup V_r$. For an r-uniform hypergraph $H = (V,E)$ and pairwise disjoint subsets $S_1, \dots, S_r \subseteq V$, define $e(S_1, \dots, S_r)$ to be the number of hyperedges of H that have exactly one endpoint in S_i for $i \in [r]$. Then define

$$D_p(S_1, \dots, S_r) := e(S_1, \dots, S_r) - p \prod_{i=1}^{r} |S_i|.$$

The next lemma extends [10, Lemma 1].

Lemma 6. *Let $r \geq 2$ and $p \in (0,1)$. There exists constants $c_r, d_r > 0$ such that for all t large enough, all $y \geq 2$ such that $\ln y \leq d_r t$, and any (r,r)-graph H on $A_1 \cup \cdots \cup A_r$, with $|A_i| = t$ for each i, we have $|D_p(B_1, \dots, B_r)| \geq \min\{p, 1-p\} c_r t^{r/2} \sqrt{\ln y}$ for at least $y^{-1} d_r 2^{tr}$ choices of subsets $B_i \subseteq A_i$, $i \in [r]$.*

Proof. The proof is by induction on r. In case $r = 1$, we have for a $(1,1)$-graph $H = (V,E)$ and $S \subseteq V$ that $D_p(S) = \sum_{i \in S} x_i$, with $x_i = 1-p$ if $i \in E$ and $x_i = -p$ if $i \notin E$. Let $\hat{p} = \min\{p, 1-p\}$. Then $|x_i/\hat{p}| \geq 1$ for all $i \in V$. So by Lemma 5 it follows that for at least $(8y)^{-1}2^t$ choices of $S \subseteq V$ we have $|D_p(S)| \geq 4^{-1}\hat{p}\sqrt{t \ln y}$. The base case holds with $d_1 = 8^{-1}$ and $c_1 = 4^{-1}$.

Now assume $r > 1$. For any fixed $a \in A_r$ we can form a $(r-1, r-1)$-graph H_a on $A_1 \cup \cdots \cup A_{r-1}$ by letting $e \in E(H_a)$ if and only if $e \cup \{a\} \in E(H)$. Define

$$Y := \{(B_1, \dots, B_{r-1}, a) \mid B_j \subseteq A_j, a \in A_r, |D_{p,H_a}(B_1, \dots, B_{r-1})| \geq \hat{p} c_{r-1} t^{(r-1)/2}\}$$

By induction, for t large enough, we know that for any $a \in A_r$

$$\left| \{(B_1, \dots, B_{r-1}) \mid B_j \subseteq A_j, |D_{p,H_a}(B_1, \dots, B_{r-1})| \geq \hat{p} c_{r-1} t^{(r-1)/2}\} \right| \geq d_{r-1} 2^{t(r-1)}/e.$$

(Here, we have applied the statement for $(r-1, r-1)$-graphs with $y = e$.) Let us write $d = e^{-1} d_{r-1}$. So $|Y| \geq dt 2^{t(r-1)}$. This implies that out of the $2^{t(r-1)}$ choices of $(B_1, \dots, B_{r-1})$ at least $\frac{1}{2} d 2^{t(r-1)}$ of them satisfy that $|\{a \in A_r : |D_{p,H_a}(B_1, \dots, B_{r-1})| \geq \hat{p} c_{r-1} t^{(r-1)/2}\}| \geq dt/2$. (Otherwise, $|Y| < \frac{1}{2} d 2^{t(r-1)} |A_r| + 2^{t(r-1)} dt/2 < dt 2^{t(r-1)}$, a contradiction.) Fix such a $(B_1, \dots, B_{r-1})$ and define for $a \in A_r$

$$x_a = \frac{D_{p,H_a}(B_1, \dots, B_{r-1})}{\hat{p} c_{r-1} t^{(r-1)/2}}.$$

Then $|x_a| \geq 1$ for at least $dt/2$ of the a in A_r. By Lemma 5, we have, for $\ln y \leq dt/8$, for at least $(8y)^{-1}2^t$ choices of $B_r \subseteq A_r$ that

$$\begin{aligned} |D_p(B_1, \dots, B_{r-1}, B_r)| &= \left| \sum_{a \in B_r} e_{H_a}(B_1, \dots, B_{r-1}) - p \prod_{i=1}^{r} |B_i| \right| \\ &= \left| \sum_{a \in B_r} D_{p,H_a}(B_1, \dots, B_{r-1}) \right| = \hat{p} c_{r-1} t^{(r-1)/2} \left| \sum_{a \in B_r} x_a \right| \\ &\geq t^{r/2} \hat{p} c_{r-1} \sqrt{32^{-1} d \ln y}. \end{aligned} \tag{5}$$

As this holds for at least $\frac{1}{2} d 2^{t(r-1)}$ choices of $(B_1, \dots, B_{r-1})$, it follows that (5) holds for at least $d(16y)^{-1}2^{tr} = d_r y^{-1} 2^{tr}$ choices of $(B_1, \dots, B_r)$. So setting, $d_r = d/16$ and $c_r = c_{r-1}\sqrt{32^{-1}d}$, the proof is finished. □

Lemma 7. *Let $r \geq 2$ and $p \in (0,1)$. There exists constants $c'_r > 0, d'_r > 0$ such that, for each n large enough and any t satisfying $(\ln n)/d'_r \leq t \leq n/2$, each r-uniform hypergraph $H = (V,E)$ on n vertices has pairwise disjoint subsets $B_1, \dots, B_r \subseteq V$ with $|B_i| \leq t/r$ for all i such that*

$$|D_p(B_1, \dots, B_r)| \geq \min\{p, 1-p\} c'_r t^{(r+1)/2} \sqrt{\ln(n/t)}.$$

The proof of this lemma is based on ideas from [11, Chapter 7].

Proof. Write $\hat{p} = \min\{p, 1-p\}$ and write $\alpha = \lfloor t/r \rfloor^{(r-1)/2}\sqrt{\ln(n/t)}$. Now partition V into r pairwise disjoint sets $A_1, \dots, A_r$ with $A_1, \dots, A_{r-1}$ each of size $\lfloor t/r \rfloor$ and A_r of size $n - (r-1)\lfloor t/r \rfloor \geq n/2$. For $a \in A_r$, let H_a be the $(r-1, r-1)$-graph on $A_1 \cup \cdots \cup A_{r-1}$ with $e \in E(H_a)$ if $e \cup \{a\} \in E(H)$.

Let $c = c_{r-1}$ and $d = d_{r-1} > 0$ be the constants from Lemma 6. Setting $d'_r = \frac{d}{r+1}$, we see that we may apply Lemma 6 to H_a with $y = n/t$ to find that when selecting $\mathbf{B}_i \subseteq A_i$ independently and uniformly at random for $i \in [r-1]$, then $|D_{p,H_a}(\mathbf{B}_1, \dots, \mathbf{B}_{r-1})| > \hat{p}c\alpha$ with probability at least dt/n for each $a \in A_r$. We may assume that $d \leq 8/r$. For convenience write $\mathbf{B} = (\mathbf{B}_1, \dots, \mathbf{B}_{r-1})$ and let

$$X(\mathbf{B}) := |\{a \in A_r \mid |D_{p,H_a}(\mathbf{B})| > \hat{p}c\alpha\}|.$$

Then, as $|A_r| \geq n/2$, $\mathbb{E}(X(\mathbf{B})) \geq dt/2$. This implies that there exists $B = (B_1, \dots, B_{r-1})$ with $B_i \subseteq A_i$ for $i \in [r-1]$ such that $X(B) \geq dt/2$. By symmetry we may assume that $|\{a \in A_r \mid D_{p,H_a}(B) > \hat{p}c\alpha\}| \geq dt/4$. Now fix $B_r \subseteq A_r$ of size $dt/4 \leq t/r$ such that $D_{p,H_a}(B) > \hat{p}c\alpha$ for each $a \in B_r$. Then

$$D_p(B_1, \dots, B_r) = \sum_{a \in B_r} D_{p,H_a}(B) \geq \hat{p}c'_r t^{(r+1)/2}\sqrt{\ln(n/t)} \tag{6}$$

for n large enough, with $c'_r = cd/(4r^{(r-1)/2} + 1)$. This finishes the proof. □

We can now prove Theorem 4.

Proof of Theorem 4. Let D be the constant d'_r from Lemma 7. The cases $t > n/2$ follow from the case $t = n/2$. So we may assume $(\ln n)/D \leq t \leq n/2$. By the previous lemma, there is a constant $c > 0$ and sets $B_1, \dots, B_r$ of size at most t/r such that $|D_p(B_1, \dots, B_r)| \geq c\min\{p, 1-p\}t^{(r+1)/2}\sqrt{\log(n/t)}$. Now we claim that

$$\sum_{S \subseteq [r]} (-1)^{|S|} D_p(\bigcup_{i \in S} B_i) = (-1)^r D_p(B_1, B_2, \dots, B_r), \tag{7}$$

which we will prove shortly. Let us first observe that it implies, for at least one of the $2^r - 1$ nonempty subsets S of $[r]$, we have

$$|D_p(\bigcup_{i \in S} B_i)| \geq 2^{-r} D_p(B_1 \dots B_r) \geq 2^{-r} c\min\{p, 1-p\}t^{(\kappa+1)/2}\sqrt{\log(n/t)}.$$

As $|\bigcup_{i \in S} B_i)| \leq t$, setting $C = 2^{-r}c$, this finishes the proof of the theorem.

To prove (7), let us define for a subset $U = \{i_1, \dots, i_m\} \subseteq [r]$ and $\alpha \in \mathbb{Z}^m_{\geq 0}$ such that $\sum_{i=1}^m \alpha_i = r$, $e(B_{i_1}^{\alpha_1}, \dots, B_{i_m}^{\alpha_m})$ to be the number of hyperedges of H that have α_j endpoints in B_{i_j} and define

$$D_p(B_{i_1}^{\alpha_1}, \dots, B_{i_m}^{\alpha_m}) = e(B_{i_1}^{\alpha_1} \dots B_{i_m}^{\alpha_m}) - p\prod_{j=1}^m \binom{|B_{i_j}|}{\alpha_j}.$$

Then for any $U = \{i_1, \dots, i_m\} \subseteq [r]$ we have

$$D_p(\bigcup_{i \in U} B_i) = \sum_{\substack{\alpha \in \mathbb{Z}^m_{\geq 0} \\ \sum_{i=1}^m \alpha_i = r}} D_p(B_{i_1}^{\alpha_1}, \dots, B_{i_m}^{\alpha_m}). \tag{8}$$

We substitute (8) into the left hand side of (7) and examine the various contributions. Let us fix $U = \{i_1, \dots, i_m\} \subseteq [r]$ and $\alpha \in \mathbb{Z}^m$ such that $\sum_{i=1}^m \alpha_i = r$ and such that each $\alpha_i > 0$ and look at the contribution of $D_p(B_{i_1}^{\alpha_1}, \dots, B_{i_m}^{\alpha_m})$ to (7). Clearly, there is a contribution if and only if S contains U. For $m' \geq m$ there are exactly $\binom{r-m}{m'-m}$ sets S that give a contribution of $(-1)^{m'} D_p(B_{i_1}^{\alpha_1}, \dots, B_{i_m}^{\alpha_m})$. So the contribution of the pair U, α to (7) is given by

$$\sum_{i=0}^{r-m} (-1)^{i+m}\binom{r-m}{i} = (-1)^m \sum_{i=0}^{r-m}\binom{r-m}{i} = \begin{cases} 0 & \text{if } m < r \\ (-1)^r & \text{if } m = r. \end{cases}$$

This proves (7). □

3 Multi-colour quasi-Ramsey results for graphs

Here we give a proof of Theorem 1 and discuss some consequences of it. Our proof of Theorem 1 is based on the proof of [15, Theorem 2], which in turn is inspired by a method of Erdős and Pach [9].

Proof of Theorem 1. Let $\phi : \binom{[n]}{2} \to [q]$ be a colouring of the edges of the complete graph on $n \geq Ck \ln k$ vertices. Let us write $G_j = ([n], \phi^{-1}(\{j\}))$, the graph given by colour j for $j \in [q]$. For a set $S \subseteq V$ and $j \in [q]$, we define the following form of *skew-discrepancy*

$$D_{j,\nu}(S) := D_{\rho_j, G_j}(S) - \nu |S|^{3/2}.$$

By $D_j(S)$ we mean $D_{j,0}(S)$ and we refer to $\nu|S|^{3/2}$ as the *skew factor* of the set S.

Let us construct a sequence of graphs as follows. We define $V_0 := [n]$. For $i > 0$, suppose X_{i-1} and V_{i-1} are given. Then amongst all choices of (S, j) where $S \subseteq V_{i-1}$ and j is a colour, let $(X(i), c(i))$ maximize $D_{j,\nu}(S)$ and set $V_i := V_{i-1} \setminus X_i$. Note that by Theorem 4 we alway have that $D_{c(i),\nu}(X_i) > 0$. We stop at step t, the first time that $|V_t| < n/2$. Define for $j \in [q]$, $I_j := \{i \in [t] \mid c(i) = j\}$.

Claim 1. *For each $j \in [q]$ and each $i \in I_j$,*

$$\delta(G_j[X_i]) \geq \rho_j(|X_i| - 1) + \nu(|X_i| - 1)^{1/2}.$$

Proof. Suppose there exists a vertex $x \in X_i$ with strictly smaller minimum degree. Write $n_i := |X_i|$. We may of course assume that $n_i \geq 2$. Set $X_i' := X_i \setminus \{x\}$. Then $e(X_i') = e(X_i) - \deg_{G_j}(x) > e(X_i) - \rho_j(n_i - 1) - \nu(n_i - 1)^{1/2}$. So it follows that

$$\begin{aligned} D_{j,\nu}(X_i') &> e(X_i) - \rho_j \binom{n_i - 1}{2} - \nu(n_i - 1)^{3/2} - \rho_j(n_i - 1) - \nu(n_i - 1)^{1/2} \\ &= e(X_i) - \rho_j \binom{n_i}{2} - \nu((n_i - 1)^{(3/2} + (n_i - 1)^{1/2}). \end{aligned} \tag{9}$$

Now note that $n_i^{3/2} = (n_i - 1 + 1)n_i^{1/2} > (n_i - 1)^{3/2} + (n_i - 1)^{1/2}$. This implies by (9) that $D_{j,\nu}(X_i') > D_{j,\nu}(X_i)$, contradicting the maximality of $D_{j,\nu}(X_i)$. ◊

By Claim 1 we may assume that $|X_i| \leq k - 1$ for all $i \in I_j$ and $j \in [q]$, for else we are done. By symmetry among the colours, we may assume that

$$\sum_{i \in I_1} |X_i| \geq \frac{n}{2q}. \tag{10}$$

Writing $I_1 := \{i_1, \ldots, i_m\}$, we have that for each $s \in [m - q - 1]$:

$$\left(\sum_{j=1}^{q+1} |X_{i_{s+j}}| \right)^{3/2} \leq (q+1)^{3/2}(k-1)^{3/2}. \tag{11}$$

We next show the following:

Claim 2. *For each $s \in [m - q - 1]$, $D_1(X_{i_{s+q+1}}) \leq \frac{5}{2(q+1)} D_1(X_{i_s})$.*

Proof. For any $i \neq j \in I_1$ define

$$D_1(X_i, X_j) := e_{G_1}(X_i, X_j) - \rho_1 |X_i||X_j|,$$

where $e_{G_1}(X_i, X_j)$ denotes the number of edges between X_i and X_j in the graph G_1. Then $D_1(X_i \cup X_j) = D_1(X_i) + D_1(X_j) + D_1(X_i, X_j)$. Let $s \in [m - 1]$. Then, by maximality of $D_{1,\nu}(X_{i_s})$, we have $D_{1,\nu}(X_{i_s}) \geq D_{1,\nu}(X_{i_s} \cup X_{i_{s+1}})$, which implies that

$$D_1(X_{i_{s+1}}) \leq -D_1(X_{i_s}, X_{i_{s+1}}) + \nu |X_{i_s} \cup X_{i_{s+1}}|^{3/2}.$$

Using the obvious fact that $\nu|X|^{3/2} \le \nu|Y|^{3/2}$ if $|X| \le |Y|$, this implies

$$\sum_{t=1}^{q+1} tD_1(X_{i_{s+t}}) \le -\sum_{0\le j<l\le q+1} D_1(X_{i_{s+j}}, X_{i_{s+l}}) + \binom{q+1}{2}\nu\left|\cup_{j=0}^{q+1} X_{i_{s+j}}\right|^{3/2}. \tag{12}$$

Let us now fix $s \in [m-q-1]$. Then, since $D_1(X) = -\sum_{c=2}^{q} D_c(X)$ for any set X, it follows that

$$\begin{aligned}-D_1(\cup_{j=0}^{q+1} X_{i_{s+j}}) = \sum_{c=2}^{q} D_c(\cup_{j=0}^{q+1} X_{i_{s+j}}) &= \sum_{c=2}^{q} D_{c,\nu}(\cup_{j=0}^{q+1} X_{i_{s+j}}) + (q-1)\nu\left|\cup_{j=0}^{q+1} X_{i_{s+j}}\right|^{3/2} \\ &\le (q-1)\left(D_1(X_{i_s}) + \nu\left|\cup_{j=0}^{q+1} X_{i_{s+j}}\right|^{3/2}\right),\end{aligned}$$

by maximality of $D_{1,\nu}(X_{i_s})$. This clearly implies

$$-\sum_{j=1}^{q+1} D_1(X_{i_{s+j}}) - \sum_{0\le j<l\le q+1} D_1(X_{i_{s+j}}, X_{i_{s+l}}) \le qD_1(X_{i_s}) + (q-1)\nu\left|\cup_{j=0}^{q+1} X_{i_{s+j}}\right|^{3/2}. \tag{13}$$

Combining (13) and (12) we obtain

$$\sum_{t=2}^{q+1}(t-1)D_1(X_{i_{s+t}}) \le qD_1(X_{i_s}) + \left(\binom{q+1}{2} + q - 1\right)\nu\left|\cup_{j=0}^{q+1} X_{i_{s+j}}\right|^{3/2},$$

from which it follows, using maximality of $D_{1,\nu}(X_{i_{s+t}})$, that

$$\frac{q(q+1)}{2}D_1(X_{i_{s+q+1}}) \le qD_1(X_{i_s}) + (q(q+1)+q-1))\nu\left|\cup_{j=0}^{q+1} X_{i_{s+j}}\right|^{3/2}.$$

So by (11)

$$D_1(X_{i_{s+q+1}}) \le \frac{2}{q+1}D_1(X_{i_s}) + 3(q+1)^{3/2}\nu(k-1)^{3/2}. \tag{14}$$

As $|V_i| \ge n/2$ for all $i < t$ we know by Theorem 4 that there exists a set $X \subseteq V_{i_s}$ of size at most k whose ρ_j-discrepancy satisfies

$$D_{\rho_j,G_j}(X) \ge \max_{j'\in[q]} \min\{\rho_{j'}, 1-\rho_{j'}\}Ck^{3/2}\sqrt{\ln(C(\nu)\ln k)}, \tag{15}$$

for some $j \in [q]$. Since the skew factor of this set X is at most $\nu k^{3/2}$, it follows that if k (or $C(\nu)$) is large enough, then $D_1(X_{i_s}) \ge 6(q+1)^{5/2}\nu(k-1)^{3/2}$. Combining this with (14) finishes the proof of the claim. ♢

Claim 2 now implies that for $s = m-q-1$ we have $D_1(X_{i_s}) \le \left(\frac{5}{2(q+1)}\right)^{s/(q+1)} D_1(X_{i_1})$ (where for simplicity we have assumed that $s \equiv 0 \pmod{(q+1)}$). Note that $D_1(X_{i_s}) \ge 6(q+1)^{5/2}\nu(k-1)^{3/2}$ (by the proof of Claim 2) and that this is at least 1 when k is large enough. From this we deduce that m is bounded by

$$\frac{(q+1)\ln(D(X_{i_1}))}{\ln(\frac{2}{5}(q+1))} + (q+1) \le \frac{2(q+1)(1+\ln(\frac{2}{5}(q+1))}{\ln(\frac{2}{5}(q+1))}\ln(k-1) =: c(q)\ln(k-1).$$

So by (10) we deduce that at least one of the m sets X_i with $i \in I_1$ satisfies $|X_i| \ge \frac{C(\nu)k}{2qc(q)}$, which for $C(\nu)$ large enough contradicts the fact that $|X_i| \le k-1$ for all $i \in I_1$. This proves the theorem. □

In case $\nu = 0$ in Theorem 1 the proof shows that the statement can actually be strengthened to the case that $(\rho_1, \dots, \rho_q)$ are not constant. Indeed, from (14) we can directly argue that m is bounded by a constant depending on q times $k-1$. This means we do not need (15), which requires that ρ_i is not too small in terms of k or $C(\nu)$. So we have the following corollary, which in particular implies that for any graph G on n vertices, $g(G) = \Omega(n/\ln n)$, partly answering the question of Falgas-Ravry, Markström, and Verstraëte [13].

Corollary 8. *For any $q \ge 2$ there exists a constant C such that for any $k \in \mathbb{N}$, any q-colouring of the edges of the complete graph on $n = Ck\ln k$ vertices and any $(\rho_1, \dots, \rho_q) \in (0,1)^q$ such that $\sum_{i=1}^{q}\rho_i = 1$, there exists a colour $j \in [q]$ and set of vertices S of size ℓ at least k such that the graph induced by S in colour j has minimum degree at least $\rho_j(\ell-1)$.*

Remark. By adapting some results in [15], which are based on discrepancy results of Spencer [23] and Lovász, Spencer and Vesztergombi [18], one can deduce from Theorem 1 that there is a set S of size *exactly* k which has minimum degree at least $\rho_i(k-1)$ plus a constant times $\sqrt{(k-1)/\ln k}$. We leave the details to the reader.

4 A precise threshold for uniform hypergraphs

In this section we prove Proposition 3 and Theorem 2.

4.1 The linear regime

We prove Proposition 3 by combining a greedy deletion argument together with probabilistic thinning, similar to what was done for graphs in [17]. We require the following concentration inequality [20, Corollary 6.10].

Theorem 9 (McDiarmid [20])**.** *Let $Z_1, \dots, Z_n$ be random variables with Z_i taking values in a set A_i and let $Z = (Z_1, \dots, Z_n)$. Let $f : \prod A_i \to \mathbb{R}$ be measurable. Suppose there exist constants $c_1, \dots, c_n$ such that for each $k = 1, \dots, n$*

$$\left|\mathbb{E}(f(Z) \mid Z_i = z_i, i \in [k-1], Z_k = z_k) - \mathbb{E}(f(Z) \mid Z_i = z_i, i \in [k-1], Z_k = z_k')\right| \leq c_k$$

for all $(z_1, \dots, z_{k-1}) \in \prod_{i=1}^{k-1} A_i$ and $z_k, z_k' \in A_k$. Then for all $t > 0$ we have

$$\mathbb{P}(|f(Z) - \mathbb{E}(f(Z))| > t) \leq \exp\left(-2t^2 \Big/ \sum_{i=1}^{n} c_i^2\right).$$

Using this result, we can prove the following lemma, which is a standard application of martingale inequalities, but we spell out the details for completeness.

Lemma 10. *Let $H = (V, E)$ be an r-uniform hypergraph with N vertices and $p\binom{N}{r}$ edges. If $S \subseteq V$ is a uniformly random subset of n distinct vertices, then for any $p > \varepsilon > 0$,*

$$\mathbb{P}\left(e(H[S]) \leq (p-\varepsilon)\binom{n}{r}\right) < \exp\left(\frac{-2\varepsilon^2(n-2(r-1))}{r^2}\right).$$

Proof. We formulate the setup to apply Theorem 9. Pick the random subset S by picking its vertices one at a time uniformly at random from the pool of remaining vertices, and let $Z_1, \dots, Z_n$ be the vertices picked, and let $Z = (Z_1, \dots, Z_n)$.

For $\mathbf{v} = (v_1, \dots, v_n) \in V^n$, write $H[\mathbf{v}] := H[\{v_1, \dots v_n\}]$. Let $f : V^n \to \mathbb{N}$ be defined by setting $f(v_1, \dots, v_n)$ to be the number of edges in $H[\mathbf{v}]$. Note that

$$\mathbb{P}\left(e(H[S]) \leq (p-\varepsilon)\binom{n}{r}\right) = \mathbb{P}\left(f(Z) \leq (p-\varepsilon)\binom{n}{r}\right).$$

Write $V^{(k)}$ for the set of k-component vectors in which all components are distinct. Furthermore, given $\mathbf{z} = (z_1, \dots, z_k) \in V^k$, write $V^{(n)}|\mathbf{z}$ for the set of vectors in $V^{(n)}$ whose first k components are $(z_1, \dots, z_k)$.

Given two vectors $\mathbf{z} = (z_1, \dots, z_{i-1}, z_i)$ and $\mathbf{z}' = (z_1, \dots, z_{i-1}, z_i') \in V^{(i)}$, we define a function $g : V^{(n)}|\mathbf{z} \to V^{(n)}|\mathbf{z}'$ such that g fixes $\mathbf{v}$ if z_i' occurs as a component of $\mathbf{v}$ and replaces z_i with z_i' in $\mathbf{v}$ if z_i' does not occur as a component in $\mathbf{v}$. It is easy to see that g is a bijection.

Now we check the bounded difference condition in Theorem 9. Note first that for $\mathbf{z} = (z_1, \dots, z_{i-1}, z_i) \in V^{(i)}$,

$$\mathbb{E}\left(f(Z) \mid (Z_1, \dots, Z_i) = \mathbf{z}\right) = \sum_{\mathbf{v} \in V^{(n)}|\mathbf{z}} \binom{N-i}{n-i}^{-1} e(H[\mathbf{v}]).$$

Taking $\mathbf{z}' = (z_1, \dots, z_{i-1}, z_i') \in V^{(i)}$, we have

$$\begin{aligned}
&\left|\mathbb{E}\left(f(Z) \mid (Z_1, \dots, Z_i) = \mathbf{z}\right) - \mathbb{E}\left(f(Z) \mid (Z_1, \dots, Z_i) = \mathbf{z}'\right)\right| \\
&= \left| \sum_{\mathbf{v} \in V^{(n)}|_{\mathbf{z}}} \binom{N-i}{n-i}^{-1} e(H[\mathbf{v}]) - \sum_{\mathbf{v} \in V^{(n)}|_{\mathbf{z}'}} \binom{N-i}{n-i}^{-1} e(H[\mathbf{v}]) \right| \\
&= \left| \sum_{\mathbf{v} \in V^{(n)}|_{\mathbf{z}}} \binom{N-i}{n-i}^{-1} (e(H[\mathbf{v}]) - e(H[g(\mathbf{v})]) \right| \\
&\leq \max_{\mathbf{v} \in V^{(n)}|_{\mathbf{z}}} |e(H[\mathbf{v}]) - e(H[g(\mathbf{v})])| \leq \binom{n}{r-1}.
\end{aligned}$$

The last quantity is bounded above by $\binom{n}{r-1}$ because $H[\mathbf{v}]$ and $H[g(\mathbf{v})]$ are two hypergraphs that differ in at most one vertex. Now observing that $\mathbb{E}(f(Z)) = p\binom{n}{r}$ and applying Theorem 9 with $c_k = \binom{n}{r-1}$ for all k yields the result. □

We can now give a proof of Proposition 3.

Proof of Proposition 3. Assume $\sum_{i=1}^{q} \rho_i < 1 - \varepsilon$ for some $\varepsilon > 0$ and let $N = Ck$, where C is to be determined later. Given any q-colouring of the edges of the complete r-uniform hypergraph on N vertices, let H_i be the subhypergraph consisting of edges coloured i and let p_i be the edge density of H_i. Then for some i, we must have that $p_i > \rho_i + \varepsilon/q$. Set $\varepsilon' = \varepsilon/q$. We may assume without loss of generality that $p_1 > \rho_1 + \varepsilon'$.

Now, starting with H_1 and $n = N$, we repeatedly remove an arbitrary vertex of degree less than $[\rho_1 + (\varepsilon'/2)]\binom{n-1}{r-1}$. If we continue for t iterations, then we have removed at most

$$\sum_{i=1}^{t} [\rho_1 + (\varepsilon'/2)] \binom{N-i}{r-1} = [\rho_1 + (\varepsilon'/2)] \left(\binom{N}{r} - \binom{N-t}{r} \right)$$

vertices from H_1. So after t iterations the number of vertices is $n = N - t$ and the number of edges remaining in the hypergraph is at least

$$(\varepsilon'/2)\binom{N}{r}.$$

Since this hypergraph can only have at most $\binom{n}{r}$ edges, it follows that

$$(\varepsilon'/2)\binom{N}{r} \leq \binom{N-t}{r}.$$

It is easy to see that there exists $c = c(\varepsilon', r)$ such that for $n = N - t \leq cN$, this inequality fails. Hence we find a set T of vertices such that with $|T| = n = cN$ vertices such that every vertex in $H_1[T]$ has degree at least $[\rho_1 + (\varepsilon'/2)]\binom{n-1}{r-1}$ in $H_1[T]$. We want that $|T| \geq k$, so it suffices to take $C \geq 1/c$.

Finally we pick $S \subseteq T$ uniformly at random such that $|S| = k$. Let $v \in T$. Then, conditional on $v \in S$, the set $S \setminus \{v\}$ is a uniformly random set $S' \subseteq T \setminus \{v\}$ such that $|S'| = k - 1$. Now, if H' denotes the $(r-1)$-uniform hypergraph H' on $T \setminus \{v\}$ induced by the at least $[\rho_1 + (\varepsilon'/2)]\binom{n-1}{r-1}$ edges incident with v, then the degree of v is the same as $e(H'[S'])$. So it follows from the previous lemma that, conditional on $v \in S$, the probability that the degree of v is at most $[\rho_1 + (\varepsilon'/4)]\binom{k-1}{r-1}$ is exponentially small in k. Since the probability that $v \in S$ is k/n, we have that, unconditionally, the probability that there exists $v \in T$ with degree at most $[\rho_1 + (\varepsilon'/4)]\binom{k-1}{r-1}$ is at most $n \cdot \frac{k}{n} \exp(-\Omega(k)) \to 0$ as $k \to \infty$. We conclude that for large enough k there exists S of size k such that $H_1[S]$ has minimum degree at least $[\rho_1 + (\varepsilon'/4)]\binom{k-1}{r-1}$, as required. □

4.2 From polynomial to super-polynomial growth

Although we treat the significantly more general situation of hypergraphs and multiple biased colours, our proof of Theorem 2 has strong similarities to that of [17, Theorem 3].

Proof of Theorem 2(i). Let c be the constant from Theorem 4 and let $C := \max_{j\in[q]}(c\rho_j)^{-1}$. Define for $\nu \geq 0$ and $j \in [q]$ the following form of skew discrepancy for a set $S \subseteq V$:

$$D_{\nu,j}(S) := D_{\rho_j,H_j}(S) - \nu|S|^{(r+1)/2}\sqrt{\ln|S|}.$$

Let $X \subseteq V$ attain the maximum skew discrepancy over all subsets of V and $j \in [n]$. By symmetry we may assume that it is attained at colour 1. Using that $\binom{\ell-1}{r} + \binom{\ell-1}{r-1} = \binom{\ell}{r}$, we find by a similar argument as in the proof of Claim 1, that

$$\delta(H_1[X]) \geq \rho_1\binom{|X|-1}{r-1} + |X|^{(r-1)/2}\nu\sqrt{\ln|X|}.$$

So it now suffices to show that $|X| \geq k$. By Theorem 4 there exists a set $Y \subseteq V$ of size at most $k^{2r/(r+1)}$ such that $D_{\rho_1,H_1}(Y) \geq c\min\{\rho_1, 1-\rho_1\}k^r\nu C\sqrt{(1+\varepsilon)\ln k} \geq k^r\nu\sqrt{(1+\varepsilon)\ln k}$. As $\varepsilon > 0$, the skew factor of X is dominated by $\nu k^r\sqrt{(1+\varepsilon)\ln k}$ and hence for k large enough we know that $D_{\nu,j}(X) \geq k^r$. This clearly implies that $|X| \geq k$ and finishes the proof. □

For the proof of Theorem 2(ii), first we describe the expected behaviour of what we refer to here as *t-dense sets* — vertex subsets that induce average degree $\overline{\deg}$ at least t — in the random r-uniform hypergraph $H^{(r)}_{n,\rho}$ with vertex set $[n] = \{1,\dots,n\}$ and hyperedge probability ρ. For this, we need a result best stated with large deviations notation, cf. [4]. For $\rho \in (0,1)$, let

$$\Lambda^*_\rho(x) = \begin{cases} x\ln\frac{x}{\rho} + (1-x)\ln\frac{1-x}{1-\rho} & \text{for } x \in [0,1] \\ \infty & \text{otherwise} \end{cases}$$

(where $\Lambda^*_\rho(0) = -\ln(1-\rho)$ and $\Lambda^*_\rho(1) = -\ln\rho$). This is the Fenchel-Legendre transform of the logarithmic moment generating function associated with the Bernoulli distribution with probability ρ (cf. Exercise 2.2.23(b) of [4]). Some calculus checks that $\Lambda^*_\rho(x)$ has a global minimum of 0 at $x = \rho$, is strictly decreasing on $[0,\rho)$ and strictly increasing on $(\rho,1]$. The following is a straightforward adaptation of Lemma 2.2(i) in [16] and bounds the probability that a given subset of k vertices in $H^{(r)}_{n,\rho}$ is t-dense.

Lemma 11. *Given t, k with $t \geq \rho\binom{k-1}{r-1}$,*

$$\Pr\left(\overline{\deg}\left(H^{(r)}_{k,\rho}\right) \geq t\right) \leq \exp\left(-\binom{k}{r}\Lambda^*_\rho\left(t\Big/\binom{k-1}{r-1}\right)\right).$$

Proof of Theorem 2(ii). For any $\eta > 1$ let

$$n = \left\lfloor \frac{k^{\nu(k)^2+1}}{\eta e} \right\rfloor,$$

where k is some large enough integer. For each $i \in [q]$ we write

$$f_i(\ell) = \rho_i\binom{\ell-1}{r-1} + \nu(\ell)\sqrt{r\binom{\ell-1}{r-1}\ln\ell}.$$

Let us consider a random q-colouring of $\binom{[n]}{r}$, the hyperedges of $K^{(r)}_n$, where independently and uniformly each hyperedge is assigned the colour i with probability ρ_i. So, writing H_i for the subhypergraph induced on $[n]$ by the hyperedges of colour i, we see that H_i is distributed as the random r-uniform hypergraph $H^{(r)}_{n,\rho_i}$.

Given a subset $S \subseteq [n]$ of $\ell \geq k$ vertices, let A_S be the event that $\overline{\deg}(H_i[S]) \geq f_i(\ell)$ for some $i \in [r]$. Since $f_i(\ell) \geq \rho_i\binom{\ell-1}{r-1}$ for each i, we have by Lemma 11 that

$$\begin{aligned}\Pr(A_S) &\leq \sum_{i=1}^{q}\exp\left(-\binom{\ell}{r}\Lambda^*_{\rho_i}\left(f_i(\ell)\Big/\binom{\ell-1}{r-1}\right)\right) \\ &\leq \sum_{i=1}^{q}\exp\left(-\binom{\ell}{r}\Lambda^*_{\rho_i}\left(\rho_i + \nu(\ell)\sqrt{r\ln\ell\Big/\binom{\ell-1}{r-1}}\right)\right).\end{aligned}$$

Now, writing

$$\varepsilon = \varepsilon(\ell) = \nu(\ell)\sqrt{r\ln\ell \Big/ \binom{\ell-1}{r-1}},$$

we have by Taylor expansion of $\Lambda^*_{\rho_i}$ (assuming $\varepsilon < \min\{\rho_i, 1-\rho_i\}$) that

$$\begin{aligned}\Lambda^*_{\rho_i}(\rho_i+\varepsilon) &= (\rho_i+\varepsilon)\ln\left(1+\frac{\varepsilon}{\rho_i}\right) + (1-\rho_i-\varepsilon)\ln\left(1-\frac{\varepsilon}{1-\rho_i}\right)\\ &= \sum_{j=1}^{\infty}\frac{\varepsilon^{2j}}{(2j-1)2j}\left(\frac{1}{{\rho_i}^{2j-1}}+\frac{1}{(1-\rho_i)^{2j-1}}\right)+\sum_{j=1}^{\infty}\frac{\varepsilon^{2j+1}}{2j(2j+1)}\left(\frac{1}{(1-\rho_i)^{2j}}-\frac{1}{{\rho_i}^{2j}}\right)\\ &= \frac{\varepsilon^2}{2\rho_i(1-\rho_i)}+O(\varepsilon^3) \geq \varepsilon^2\end{aligned}$$

for ε small enough (and hence k large enough). So the probability that A_S holds for some set $S \subseteq [n]$ of $\ell \geq k$ vertices is at most

$$\begin{aligned}\sum_{S\subseteq[n],|S|\geq k}\Pr(A_S) &\leq \sum_{\ell=k}^{n}\binom{n}{\ell}\sum_{i=1}^{q}\exp\left(-\binom{\ell}{r}\Lambda^*_{\rho_i}(\rho_i+\varepsilon)\right)\\ &\leq q\sum_{\ell=k}^{n}\left(\frac{en}{\ell}\exp\left(-\frac{1}{r}\binom{\ell-1}{r-1}\varepsilon^2\right)\right)^{\ell} \leq q\sum_{\ell=k}^{n}\eta^{-\ell} < 1,\end{aligned}$$

where in this sequence of inequalities we have used the definition of n, the fact that $\ell \geq k$ and $\eta > 1$, and a choice of k large enough. Thus for k large enough there is a q-colouring of the edges of $K_n^{(r)}$ where for each $i \in [q]$ every vertex subset of size $\ell \geq k$ induces a subhypergraph in colour i with average degree less than $f_i(\ell)$, so the result follows. □

5 Concluding remarks and open questions

Let us introduce some notation to facilitate our discussion. Fix $q \geq 2$ and let $(\rho_i)_{i=1}^q$ be a sequence of q numbers in $[0,1]$. Given a colouring ϕ of the complete r-uniform hypergraph $K_n^{(r)}$ on vertex set $[n]$ that assigns each hyperedge a colour from $[q]$, we let $H_{\phi,j}$ denote the subhypergraph $([n], \phi^{-1}(j))$ induced by all hyperedges of colour j for $j \in [q]$. The basic question now becomes the following: for any k, what is the smallest number $n := R^{(r)}_{(\rho_i)_i}(k)$ such that, for any q-colouring ϕ of the hyperedges of $K_n^{(r)}$, there is guaranteed to be a subset $S \subseteq [n]$ of size ℓ at least k such that the subhypergraph $H_{\phi,j}[S]$ induced on S in colour j has minimum degree at least $\rho_j\binom{\ell-1}{r-1}$ for some $j \in [q]$? We may also ask this question with average degree instead of minimum degree and denote the corresponding number $\overline{R}^{(r)}_{(\rho_i)_i}(k)$. Clearly $\overline{R}^{(r)}_{(\rho_i)_i}(k) \leq R^{(r)}_{(\rho_i)_i}(k)$ always. We refer to $R^{(r)}_{(\rho_i)_i}(k)$ and $\overline{R}^{(r)}_{(\rho_i)_i}(k)$ as *q-colour hypergraph quasi-Ramsey numbers*. Note that when $\sum_{i=1}^q \rho_i = q$ we retrieve the ordinary hypergraph Ramsey-numbers.

With this notation we see that for $\sum_{i=1}^q \rho_i < 1$, Proposition 3 shows that $R^{(r)}_{(\rho_i)_i}(k)$, and hence $\overline{R}^{(r)}_{(\rho_i)_i}(k)$, has linear growth in k; Theorem 2 precisely describes the transition from polynomial to super-polynomial growth of the q-colour hypergraph quasi-Ramsey numbers. In particular, for $\sum_{i=1}^q \rho_i > 1$, Theorem 2(ii) implies that $\overline{R}^{(r)}_{(\rho_i)_i}(k)$, and hence $R^{(r)}_{(\rho_i)_i}(k)$, is at least singly exponential in k. This implies for hypergraph quasi-Ramsey numbers that, irrespective of a well-known conjecture of Erdős, Hajnal and Rado [8] (concerning the case $\sum_{i=1}^q \rho_i = q$), there must be a transition for r-uniform hypergraphs with $r \geq 4$ from singly exponential to doubly exponential (or higher) growth in k that takes place for $1 < \sum_{i=1}^q \rho_i \leq q$. It would be an interesting challenge to understand the nature of this transition.

We note that if all the ρ_i are uniformly bounded below 1, Conlon, Fox and Sudakov [2, 3] proved results that imply $R^{(r)}_{(\rho_i)_i}(k)$, and hence $\overline{R}^{(r)}_{(\rho_i)_i}(k)$, has growth that is at most singly exponential in k:

Proposition 12 (Conlon, Fox and Sudakov [2, 3]). *Let $r \geq 2$. Fix $q \geq 2$ and $\varepsilon > 0$ and let $\rho_1 = \cdots = \rho_q = 1 - \varepsilon$. Then*

$$R^{(r)}_{(\rho_i)_i}(k) = \begin{cases} 2^{O(k^2)} & \text{if } q = 3 \text{ [3, Theorem 2] and} \\ 2^{O(k^D)} & \text{if } q \geq 4 \text{ [2, Proposition 6.3]} \end{cases}$$

where $D > 0$ is a fixed constant that depends on r, q and ε.

Along these lines, a first question to resolve is perhaps whether a strengthening of Proposition 12 holds: given $r \geq 3$, $q \geq 2$ and $\varepsilon > 0$, is there some D such that $\ln R^{(r)}_{(\rho_i)_i}(k) = O(k^D)$ if $\sum_{i=1}^{q} \rho_i < q - \varepsilon$? Or could it instead be the case, say, that, given $r \geq 4$, $q \geq 2$ and $\varepsilon > 0$, there is some $D > 0$ such that $\ln \ln \overline{R}^{(r)}_{(\rho_i)_i}(k) = \Omega(k^D)$ if $\rho_1 = 1$ and $\rho_i = \varepsilon/(q-1)$ for $i \in \{2, \ldots, q\}$? These questions can be considered part of a refinement of a problem of Erdős (cf. [21, pp. 21–22]), a problem he described as "interesting and mysterious" and for whose solution he offered \$500. Borrowing his intuition, it might be more natural to believe that the answer to the first question is 'yes' and to the second 'no'. On the other hand, in light of a result of Erdős and Hajnal that was mentioned in the introduction, the answers could depend on r and q.

Acknowledgement

We would like to thank Maria Axenovich for suggesting the study of multi-colour quasi-Ramsey numbers. We are moreover grateful to an anonymous referee for valuable comments, significantly improving the presentation of the paper.

References

[1] B. Bollobás and A. D. Scott. Discrepancy in graphs and hypergraphs. In *More sets, graphs and numbers*, volume 15 of *Bolyai Soc. Math. Stud.*, pages 33–56. Springer, Berlin, 2006.

[2] D. Conlon, J. Fox, and B. Sudakov. Hypergraph Ramsey numbers. *J. Amer. Math. Soc.*, 23(1):247–266, 2010.

[3] D. Conlon, J. Fox, and B. Sudakov. Large almost monochromatic subsets in hypergraphs. *Israel J. Math.*, 181:423–432, 2011.

[4] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*, volume 38 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 2nd edition, 1998.

[5] P. Erdős. On a lemma of Littlewood and Offord. *Bull. Amer. Math. Soc.*, 51:898–902, 1945.

[6] P. Erdős. Some remarks on the theory of graphs. *Bull. Amer. Math. Soc.*, 53:292–294, 1947.

[7] P. Erdős, M. Goldberg, J. Pach, and J. Spencer. Cutting a graph into two dissimilar halves. *J. Graph Theory*, 12(1):121–131, 1988.

[8] P. Erdős, A. Hajnal, and R. Rado. Partition relations for cardinal numbers. *Acta Math. Acad. Sci. Hungar.*, 16:93–196, 1965.

[9] P. Erdős and J. Pach. On a quasi-Ramsey problem. *J. Graph Theory*, 7(1):137–147, 1983.

[10] P. Erdős and J. Spencer. Imbalances in k-colorations. *Networks*, 1:379–385, 1971/72.

[11] P. Erdős and J. Spencer. *Probabilistic Methods in Combinatorics*. Probability and Mathematical Statistics, A Series of Monographs and Textbooks. Academic Press, New York and London, 1974.

[12] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

[13] V. Falgas-Ravry, K. Markström, and J. Verstraëte. Full subgraphs. *Journal of Graph Theory*, to appear.

[14] R. L. Graham, B. L. Rothschild, and J. H. Spencer. *Ramsey theory*. John Wiley & Sons, Inc., New York, 1980. Wiley-Interscience Series in Discrete Mathematics, A Wiley-Interscience Publication.

[15] R. J. Kang, E. Long, V. Patel, and G. Regts. On a Ramsey-type problem of Erdős and Pach. *Bulletin of the London Mathematical Society*, to appear.

[16] R. J. Kang and C. McDiarmid. The t-improper chromatic number of random graphs. *Combin. Probab. Comput.*, 19(1):87–98, 2010.

[17] R. J. Kang, J. Pach, V. Patel, and G. Regts. A precise threshold for quasi-Ramsey numbers. *SIAM J. Discrete Math.*, 29(3):1670–1682, 2015.

[18] L. Lovász, J. Spencer, and K. Vesztergombi. Discrepancy of set-systems and matrices. *European J. Combin.*, 7(2):151–160, 1986.

[19] J. Matoušek and J. Vondrák. The Probabilistic Method. Downloaded from `http://kam.mff.cuni.cz/~matousek/lectnotes.html`, March 2008.

[20] C. McDiarmid. On the method of bounded differences. *Surveys in combinatorics*, 141(1):148–188, 1989.

[21] J. Nešetřil and V. Rödl, editors. *Mathematics of Ramsey theory*, volume 5 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 1990.

[22] F. P. Ramsey. On a problem of formal logic. *Proc. of the London Math. Soc. Ser. 2*, 30(1):264–286, 1930.

[23] J. Spencer. Six standard deviations suffice. *Trans. Amer. Math. Soc.*, 289(2):679–706, 1985.