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The application of the blackfold effective theory to the perturbative construction of black holes in higher-dimensions is reviewed. Several solutions with non-trivial horizon geometry and topology are described, such as black helicoidal branes and helicoidal black rings. This hints into a very rich phase diagram for higher-dimensional neutral asymptotically flat black holes.

Keywords: Higher-dimensional black holes; blackfold effective theory.

1. Introduction

The interest in higher-dimensional gravity was spurred by the development of theories of (quantum) gravity which required being formulated in higher-dimensional spacetimes, such as string theory. Following this line of thought, the higher-dimensional version of the Kerr black hole - the Myers-Perry black hole - was constructed[1] and later the first example of a black hole with non-trivial horizon topology in five spacetime dimensions was found[2]. It was then realised[3] that by taking the number of spacetime dimensions D as a parameter in Einstein equations, gravity becomes richer, allowing for an elaborate phase space of black hole solutions.

As the number of spacetime dimensions increases, the problem of obtaining higher-dimensional black hole solutions becomes increasingly complicated. In particular, for $D > 5$ there is no integrable sector of Einstein equations for asymptotically flat solutions and therefore, an absence of solution generating techniques. In fact, in asymptotically flat spacetimes, the only known exact analytic uncharged black hole solutions of Einstein equations are the Myers-Perry black hole and the black ring.[a] In asymptotically Anti-de Sitter (AdS) spacetimes, higher-dimensional versions of the Kerr-AdS black hole have been found[4] as well as black branes with hyperbolic or flat horizon and black holes which are topologically spheres but with two punctures[5]. On the other hand, for instance, in vaccum plane wave spacetimes no exact analytic solution is known.

The difficulty of solving Einstein equations in higher-dimensions lead to the development of perturbative methods such as the blackfold approach[6–8]. This approach exploits an interesting property of higher-dimensional black holes, namely, that in certain regimes of parameter space, black hole horizons can be characterized by two or more widely separated length scales. In these situations, the near-horizon

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[a]In $D = 5$ it is has been possible to find black hole solutions with disconnected horizons composed of combinations of a Myers-Perry black hole and several black rings.

geometry becomes brane-like and the near-horizon metric is that of a perturbed boosted black brane metric. For neutral asymptotically flat spacetimes, this typically requires the black hole to be ultraspinning, i.e., its angular momentum per unit mass must be very large. The blackfold effective theory then consists of locally wrapping boosted black branes on arbitrary submanifolds and requiring stationary equilibrium, as to avoid gravitational collapse.

In the remainder of this article, we will provide some of the basic elements for constructing perturbative black holes using the blackfold approach and later we will discuss its applications by analysing specific examples.

2. Elements of blackfold dynamics

In its simplest setting, the blackfold construction consists of bending neutral boosted black branes with local horizon radius r_0 on an arbitrary $(p + 1)$-dimensional submanifold $\mathcal{W}_{p+1}$ with curvature scale R. To leading order, the metric near the horizon, i.e. at distances $r \ll R$, is the boosted black brane metric in spacetime dimensions $D = n + p + 3$,

$$ds^2 = \left(\gamma_{ab} + \frac{r_0^n}{r^n} u_a u_b \right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2 d\Omega_{(n+1)}^2 + \dots \ , \tag{1}$$

where the induced metric γ_{ab} on $\mathcal{W}_{p+1}$, the horizon size r_0 and the boost velocities u^a are all functions of the local coordinates σ^c on $\mathcal{W}_{p+1}$. The *ellipsis* in (1) denote higher-order corrections beyond the leading order and can be obtained by introducing small perturbations and solving Einstein equations. At distances $r \gg r_0$, to leading order the metric $g_{\mu\nu}(x^\alpha)$ can be freely chosen and can be, for instance, Minkowski or AdS. In order to obtain higher-order corrections in flat spacetime in the region $r \gg r_0$, for example, in the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \dots$, one solves the usual linearised equation

$$\Box \bar{h}^{\mu\nu} = 8\pi G \boldsymbol{T}^{\mu\nu}, \tag{2}$$

where $\bar{h}^{\mu\nu} = h^{\mu\nu} - (1/2)\eta_{\mu\nu}\eta^{\lambda\rho}h_{\lambda\rho}$ and $\boldsymbol{T}^{\mu\nu}$ is the spacetime stress tensor of the (wrapped) black brane (1), which can be obtain from (1) in the region $r \gg r_0$. Due to the a priori assumed hierarchy of scales $r_0 \ll R$, the metric near the horizon (1) can be matched to the metric obtained by solving (2) in the region $r_0 \ll r \ll R$ where the two metrics overlap.

From solving Einstein equations using this order-by-order procedure known as a matched asymptotic expansion, one obtains two sets of constraint equations, namely [9,10]

$$\nabla_a T^{ab} = 0 \ , \quad T^{ab} K_{ab}^i = 0, \tag{3}$$

where K_{ab}^i is the extrinsic curvature of the submanifold $\mathcal{W}_{p+1}$ and the worldvolume stress tensor is defined via

$$\boldsymbol{T}^{\mu\nu} = \int_{\mathcal{W}_{p+1}} \sqrt{-\gamma} \ T^{ab} u_a^\mu u_b^\nu \delta^{(D)}(x^\alpha - X^\alpha(\sigma^c)), \tag{4}$$

with γ being the determinant of the induced metric, $X^\alpha(\sigma^c)$ is set of mapping functions describing the position of the the submanifold in the ambient spacetime and $u_a^\mu = \partial_a X^\mu$ is a set of projectors that project onto the tangent space to the worldvolume $\mathcal{W}_{p+1}$. Tangential directions are denoted by indices $a, b, c, \dots$ while transverse directions to $\mathcal{W}_{p+1}$ are denoted by $i, j, k, \dots$.

The constraint equations (3) can also be derived by simply requiring $\nabla_\mu \boldsymbol{T}^{\mu\nu} = 0$ and, if the worldvolume has boundaries, in addition we also obtain the boundary condition

$$T^{ab}\eta_a|_{\partial \mathcal{W}_{p+1}} = 0. \tag{5}$$

It turns out the worldvolume stress tensor for the metric (1) takes the perfect fluid form

$$T^{ab} = P\gamma^{ab} + (\epsilon + P)u^a u^b , \quad \epsilon + P = \mathcal{T}s, \tag{6}$$

where the boost velocities u^a are now interpreted as fluid velocities, P, ϵ, $\mathcal{T}$, s are the fluid pressure, energy density, temperature and entropy respectively. In particular, one has that

$$P = -\frac{\Omega_{(n+1)}}{16\pi G}r_0^n , \quad \epsilon + P = -(n+1)P , \quad \mathcal{T} = \frac{n}{4\pi r_0}, \tag{7}$$

where $\Omega_{(n+1)}$ is the volume of the unit $(n+1)$-sphere. The constraint equations (3), (5) therefore are the equations of motion of a fluid living on a dynamical surface. Focusing on stationary configurations, which give rise to stationary black holes, the first set of equations in (3) is solved by requiring the fluid velocities to be aligned with a worldvolume Killing vector field $\mathbf{k}^a$ with modulus $\mathbf{k}$ and the global temperature T to be the redshift of the local temperature such that $u^a = \frac{\mathbf{k}^a}{\mathbf{k}}$, $T = \mathbf{k}\mathcal{T}$. In this simplified setting, the remaining equations of motion can be encoded in a free energy functional of the form

$$\mathcal{F}[X^i] = -\int_{\mathcal{B}_p} dV_{(p)} R_0 P, \tag{8}$$

where $\mathcal{B}_p$ is the spatial part of the worldvolume and R_0 is the modulus of the worldvolume Killing vector field ∂_τ associated with worldvolume time τ translations. If one is only interested in scanning the phase space of possible black hole solutions, one may simply use (8) instead of the matched asymptotic expansion. The thermodynamic properties of these fluid configurations can be obtained in the usual manner,

$$S = -\frac{\partial \mathcal{F}}{\partial T} , \quad J_a = -\frac{\partial \mathcal{F}}{\partial \Omega^a}, \tag{9}$$

where S is the total entropy and J_a the angular momentum associated with the angular velocity Ω_a. The total energy M can be obtain using the fact that $\mathcal{F} = M - TS - \Omega^a J_a$. We will now discuss some specific configurations.

3. Black hole horizons

We now apply the previous method and construct some fluid configurations which are dual to certain black holes in asymptotically flat space. We note that using (6) in (3) and (5) leads to the equations of motion in the form

$$K^i = nu^a u^b K^i_{ab} \ , \quad \mathbf{k}|_{\partial \mathcal{W}_{p+1}} = 0. \tag{10}$$

The second equation above states that if a neutral fluid has boundaries then it must be moving at the speed of light there.

3.1. *Black discs: Myers-Perry black holes*

The simplest geometry to embedd in flat spacetime is a two-dimensional rotating plane with an extra time-like direction in $D \geq 6$. This is described by the induced line element $\mathbf{ds}^2$ and Killing vector field

$$\mathbf{ds}^2 = -d\tau^2 + d\rho^2 + \rho^2 d\phi^2 \ , \quad \mathbf{k}^a \partial_a = \partial_\tau + \Omega \partial_\phi. \tag{11}$$

Since the plane is a minimal surface and embedded into flat spacetime it has $K^i = K^i_{ab} = 0$ and hence trivially solves (10). A priori, the coordinate ρ lies within the range $0 \leq \rho < \infty$, however, the boundary condition (10) implies that there is an upper bound $\rho_+ = \Omega^{-1}$ where the fluid must move at the speed of light. This cuts the two-dimensional rotating plane into a two-dimensional rotating disc of size Ω^{-1}. Its horizon radius is given by

$$r_0(\rho) = r_+ \sqrt{1 - \rho^2 \Omega^2} \ , \quad r_+ = \frac{n}{4\pi T}. \tag{12}$$

Introducing coordinates $\cos^2 \theta = 1 - \rho^2 \Omega^2$, one realises that r_0 varies from a maximum size at the axis of rotation $\theta = 0$ and decreases to zero at the edges $\theta = \pi$. The disc is non-trivially fibered over the $\mathbb{S}^{(n+1)}$. Therefore, it gives rise to the black holes with horizon topology $\mathbb{S}^{(D-2)}$. Evaluating its free energy (8) we find

$$\mathcal{F} = \frac{\Omega_{(n+1)}}{8G(n+2)} \frac{r_+^n}{\Omega^2}. \tag{13}$$

Indeed, this reproduces the well known properties of ultraspinning Myers-Perry black holes in $D \geq 6$ [11].

3.2. *Black helicoidal branes*

A non-trivial example of a blackfold configuration in $D \geq 6$ is that obtained by considering another well known minimal surface, namely, the helicoidal and adding one extra time-like direction. In this case the induced line element and Killing vector field take the form

$$\mathbf{ds}^2 = -d\tau^2 + d\rho^2 + \left(\lambda^2 + \rho^2\right) d\phi^2 \ , \quad \mathbf{k}^a \partial_a = \partial_\tau + \Omega \partial_\phi, \tag{14}$$

where λ is the pitch of the helicoid. Both coordinates ρ, ϕ lie within the range $-\infty \leq \rho, \phi < \infty$ but the boundary condition (10) implies that there is a lower and upper bound in the coordinate ρ, namely,

$$\rho_\pm = \pm \frac{\sqrt{1 - \Omega^2 \lambda^2}}{\Omega^2}. \tag{15}$$

This geometry reduces to the disc geometry of the previous section when $\lambda \to 0$. This family of solutions have the topology of a black string $\mathbb{R} \times \mathbb{S}^{(D-3)}$ and reduces to the Myers-Perry solution when $\lambda \to 0$. The two solutions are therefore connected by a topology-change transition. The corresponding exact analytic solution is not known. Its free energy takes the form[12]

$$\mathcal{F} = \frac{V_{(n+2)}}{16\pi G} \frac{r_+^n}{\Omega} \int d\phi \lambda (1 - \lambda^2 \Omega^2)^{\frac{n+1}{2}} {}_2\tilde{F}_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{n+3}{2}; 1 - \frac{1}{\lambda^2 \Omega^2}\right), \tag{16}$$

where $V_{(n+2)} = 2\pi^{\frac{n+3}{2}}$, and reduces to the Myers-Perry free energy in the limit $\lambda \to 0$ when making the coordinate ϕ periodic with period 2π.

3.3. *Helicoidal black rings*

The previous geometry is a boosted helicoidal string with boost velocity Ω and can be bent into a helicoidal black ring with topology $\mathbb{S}^1 \times \mathbb{S}^{(D-3)}$. This geometry can be obtained from the helicoidal brane by making the coordinate ϕ periodic and setting $\lambda = R$, with R now being the radius of the ring.[b] The free energy functional is then

$$\mathcal{F}[R] = \frac{V_{(n+2)}}{8G} \frac{r_+^n}{\Omega} \lambda (1 - R^2 \Omega^2)^{\frac{n+1}{2}} {}_2\tilde{F}_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{n+3}{2}; 1 - \frac{1}{R^2 \Omega^2}\right), \tag{17}$$

and the resultant equations of motion are found by varying it with respect to R reads[13]

$$\frac{1 - (n+2)\Omega^2 R^2}{(1 - \Omega^2 R^2)} - \frac{{}_2\tilde{F}_1\left(\frac{1}{2}, \frac{3}{2}; \frac{n+5}{2}; -\frac{1-\Omega^2 R^2}{\Omega^2 R^2}\right)}{2\Omega^2 R^2 {}_2\tilde{F}_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{n+3}{2}; -\frac{1-\Omega^2 R^2}{\Omega^2 R^2}\right)} = 0. \tag{18}$$

This is a transcendental equation and cannot be solved analytically, though it can easily be solved numerically[13]. One observes that helicoidal black rings, for a given dimension, must rotate slower compared to the usual black rings.

4. Discussion

We have given examples that the blackfold approach easily used in order to find novel black hole horizon geometries and topologies. It can moreover be applied to spacetimes with non-trivial asymptotics. This was applied in[12] to vacuum plane wave spacetimes and, despite their being no exact analytic solutions, many configurations were found to exist leading to a very interesting phase diagram. We believe

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[b]Formally, in order to obtain this geometry one has to first integrate out the finite line along the ρ coordinate and then use the resulting effective theory to construct a ring.

that we have only scratched the surface and many more configurations, besides those found in[11–13], can be constructed by, for example, considering higher-dimensional minimal surfaces.

Acknowledgments

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