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One-form superfluids & magnetohydrodynamics

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Abstract: We use the framework of generalised global symmetries to study various hydrodynamic regimes of hot electromagnetism. We formulate the hydrodynamic theories with an unbroken or a spontaneously broken U(1) one-form symmetry. The latter of these describes a one-form superfluid, which is characterised by a vector Goldstone mode and a two-form superfluid velocity. Two special limits of this theory have been studied in detail: the string fluid limit where the U(1) one-form symmetry is partly restored, and the electric limit in which the symmetry is completely broken. The transport properties of these theories are investigated in depth by studying the constraints arising from the second law of thermodynamics and Onsager's relations at first order in derivatives. We also construct a hydrostatic effective action for the Goldstone modes in these theories and use it to characterise the space of all equilibrium configurations. To make explicit contact with hot electromagnetism, the traditional treatment of magnetohydrodynamics, where the electromagnetic photon is incorporated as dynamical degrees of freedom, is extended to include parity-violating contributions. We argue that the chemical potential and electric fields are not independently dynamical in magnetohydrodynamics, and illustrate how to eliminate these within the hydrodynamic derivative expansion using Maxwell's equations. Additionally, a new hydrodynamic theory of non-conducting, but polarised, plasmas is formulated, focusing primarily on the magnetically dominated sector. Finally, it is shown that the different limits of one-form superfluids formulated in terms of generalised global symmetries are exactly equivalent to magnetohydrodynamics and the hydrodynamics of non-conducting plasmas at the non-linear level.

Keywords: Effective Field Theories, Global Symmetries, Spontaneous Symmetry Breaking, Holography and quark-gluon plasmas

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1 Introduction

Hot electromagnetism is the theory that describes the interaction between electromagnetic and thermal degrees of freedom of matter at finite temperature. At sufficiently long wavelengths and time scales, this theory admits certain hydrodynamic regimes within which these interactions are well approximated by the physics of plasmas. Magnetohydrodynamics (MHD) is one of the most well studied of these regimes, applicable to conducting plasmas for which the electric fields are short range/Debye screened and the plasma is electrically neutral at hydrodynamic length scales [1]. Over the past decades, MHD has developed into a framework capable of describing a wide range of phenomena, from the modelling of accretion disks surrounding astrophysical black holes to the magnetic confinement of hot plasmas at fusion reactors [2].

Despite its historical success as a phenomenological theory, the traditional treatments of MHD have only recently began to incorporate some of the modern developments in

hydrodynamics [3], which have proven to be extremely useful to further our understanding of ordinary fluid and superfluid dynamics [4, 5]. These developments, among many others, include: the understanding of hydrodynamics as an effective field theory [6]; the relevance of hydrostatic partition functions that describe all equilibrium states in hydrodynamics [7, 8]; the role of symmetries and classification schemes in constraining transport properties [9, 10]; the usefulness of black hole physics and holography in the evaluation of transport coefficients [11, 12]; a Lagrangian formulation of dissipative hydrodynamics [13–15]; the incorporation of boundaries/surfaces in hydrodynamic descriptions [16, 17]; a novel understanding of non-relativistic limits [18, 19]; and the application of the framework of generalised global symmetries to reformulate hydrodynamic theories [20–25].

The overarching goal of this work is to further develop the effective hydrodynamic theories of hot electromagnetism under the light of some of these recent developments, and to investigate another of its hydrodynamic regimes besides MHD. In particular, we provide a new formulation of dissipative MHD in terms of a system with higher-form conservation laws, which is better suited for numerical studies, classify all dissipative transport coefficients that appear at first order in a long-wavelength expansion and resolve standing issues related to the definition of hydrostatic equilibrium. Besides providing a new framework for understanding the MHD regime, this work also focuses on a novel formulation of the hydrodynamic description of non-conducting plasmas that can nevertheless be polarised, which we refer to as *bound-charge plasmas*. Physical examples of such systems include a polarised neutral gas of atoms interacting with a bath of photons.

The main tool used throughout this work is the framework of generalised global symmetries [26], which has recently been used in the context of MHD, recasting it as a theory of hydrodynamics with a global U(1) one-form symmetry [20, 21, 24].[1] The traditional treatment of MHD involves incorporating the electromagnetic photon A_μ as a dynamical degree of freedom in the hydrodynamic description, coupled to an external conserved current J^μ_{ext} (see e.g. [3]). On the other hand, the corresponding string fluid formulation, originates from the insight that electromagnetism admits a two-form current $J^{\mu\nu} = \epsilon^{\mu\nu\lambda\rho}\partial_\lambda A_\rho$, where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ is the electromagnetic field strength, that is conserved due to the Bianchi identity $\nabla_{[\mu}F_{\nu\lambda]} = 0$.[2] This two-form current gives rise to a dipole charge that counts the number of magnetic field lines crossing any two-dimensional surface, and couples to an external two-form gauge field $b_{\mu\nu}$. The three-form field strength $H_{\mu\nu\lambda} = 3\partial_{[\mu}b_{\nu\lambda]}$ associated with $b_{\mu\nu}$ is seen as related to the external conserved current as $J^\mu_{\text{ext}} = \epsilon^{\mu\nu\lambda\rho}H_{\nu\lambda\rho}/6$. Both these formulations are developed and extended in this work and, in order to avoid any ambiguity, one of main results obtained here can be summarised as follows:

> *Under the identification* $J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ *and* $J^\mu_{ext} = \frac{1}{6}\epsilon^{\mu\nu\lambda\rho}H_{\nu\lambda\rho}$*, the formulation of MHD in terms of generalised global symmetries is exactly equivalent to the traditional treatment of MHD with a dynamical gauge field.*

[1] Throughout this work, we often refer to to this formulation as the string fluid formulation of MHD.

[2] This process of dualisation is commonly applied in the context of numerical studies of MHD [27]. The conservation of the two-form current splits into what is usually denoted as the induction equation and the no-monopole constraint. However, no formal study of the hydrodynamic properties and expansion in this context had been performed. This is one of the goals of this paper.

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A few remarks are now in order: this equivalence is proven here at the full non-linear level including parity-violating terms; both the formulations make no assumptions regarding the strength of the magnetic fields; and both the formulations are developed using the principles of effective field theory and hydrodynamic expansions. Finally, the traditional treatment as developed here, following [3], is more general than its corresponding formulation in terms of generalised global symmetries, as it is capable of describing plasmas that are not necessarily electrically neutral at the hydrodynamic length scales.[3]

Despite the formulation of MHD in terms of generalised global symmetries, as thus far developed, being less general than the corresponding traditional treatment, it should be noted that there are several important reasons why this different formulation is actually more useful. Most applications of MHD, specially in the context of astrophysics, concern themselves with plasmas that are electrically neutral at the hydrodynamic length scales [1], in which case both of these formulations are equally applicable in general, but the formulation in terms of generalised global symmetries is easier to implement in numerical simulations [27]. Moreover, when expressed in terms of generalised global symmetries, the formulation rests solely on the symmetry principles (and their breaking), without having to incorporate a microscopic dynamical gauge field. Additionally, the chemical potential μ and electric fields E^μ that enter the traditional formulation, but not the string fluid formulation, are superfluous and not independently dynamical in the hydrodynamic regime. As a matter of fact, we show in the course of this work how Maxwell's equations can be exactly solved within a derivative expansion, so as to completely remove these fields from the hydrodynamic description. Finally, within this string fluid formulation, we directly obtain the fluid constitutive relations for the physically observable electromagnetic fields in terms of the background current sources, which allow for a cleaner extraction of the respective correlation functions.

Earlier formulations of MHD within the framework of generalised global symmetries [20, 21] (see also [3]) take the viewpoint that MHD is a theory of long fluctuating strings (i.e. magnetic field lines). The string direction h^μ and their chemical potential ϖ serve as fundamental degrees of freedom in the theory, while assuming that the one-form symmetry is unbroken. As has already been explained in [24], while this treatment is phenomenologically sufficient to understand the hydrodynamic fluctuations around a given initial equilibrium fluid configuration, it does not allow for a precise understanding of the space of allowed equilibrium configurations by means of a hydrostatic effective action (or partition function). This problem can be resolved, as advocated in [24], by carefully breaking the one-form symmetry along the direction of the fluid flow, which leads to the exact same description for string fluids out of equilibrium as presented in [20, 21]. However, it is now possible to properly define *equilibrium* configurations by constructing a hydrostatic effective action for the a magnetic scalar potential φ, which can be understood as the

[3]It may be possible to relax the assumptions of the string fluid formulation in order to be able to describe plasmas that are not electrically neutral. Further comments on this point are left to a more speculative discussion in section 8.

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	Neutral Fluid	Zero-form Ord. Fluid	Zero-form Superfluid	One-form Ord. Fluid	One-form String Fluid	One-form Superfluid
Hydrostatic	0	0	4	1	5	22
Non-hs Non-diss	0	0	11	0	8	66
Dissipative	2	3	15	4	11	78
Total	2	3	30	5	24	166

Table 1. Comparison between counting of independent transport coefficients at one-derivative order for various phases of neutral, zero-form, and one-form charged fluids. CPT and Onsager's relations have not been implemented in this count. Here *Non-hs* refers to non-hydrostatic and *Non-diss* to non-dissipative.

Goldstone scalar associated with the partially broken one-form symmetry.[4] The theory is thus better understood as a theory of one-form superfluidity.

This work introduces a novel framework of one-form superfluids in which the one-form symmetry is completely broken, giving rise to a vector Goldstone mode φ_μ [26, 28, 29]. A specific sector of this theory, where part of the one-form symmetry is restored, describes MHD. In general, however, one-form superfluids characterise many hydrodynamic regimes of hot electromagnetism without any assumption on the relative strength of electric and magnetic fields. As an example, the theory will be used to describe the hydrodynamic regime of magnetically dominated bound-charge plasmas (BCP), whose traditional treatment has also been developed here and shown to be equivalent. Below, the different connections between one-form superfluids and different aspects of hot electromagnetism are described in more detail, together with the organisation of this paper and some of its main results. A comparison between the number of transport coefficient in various phases of neutral, zero-form, and one-form hydrodynamics is given in table 1.

One-form hydrodynamics and hot electromagnetism. One of main purposes of this work is to contribute to a systematic study of one-form hydrodynamics and its applications. As such, this paper begins in section 2 with a discussion on the proper identification of the degrees of freedom in one-form hydrodynamics, motivated from considerations in equilibrium thermal field theories. This section also introduces the general methodology of one-form hydrodynamics (adiabaticity equation, second law of thermodynamics, hydrostatic effective actions, etc) that will be used in later sections to formulate novel theories of hydrodynamics with generalised global symmetries. The identification of the correct degrees of freedom of one-form hydrodynamics leads to a warm up exercise: the formulation of one-form hydrodynamics for which the one-form symmetry is unbroken, in section 3. This theory turns out to be quite different from string fluids as formulated in previous

[4]The conventional formulation of MHD has a massless propagating degree of freedom, namely the photon. However, electric fields in MHD are screened. This means that in the dual formulation of MHD in terms of generalised global symmetries, not all the components of the *dual photon*, which can be seen as the Goldstone of the spontaneously broken one-form symmetry [26, 28, 29], are actually present. However, since the magnetic fields are still unscreened, at least some components of the dual photon must still exist. Therefore, we refer to this phase as a partially broken phase of a one-form symmetry.

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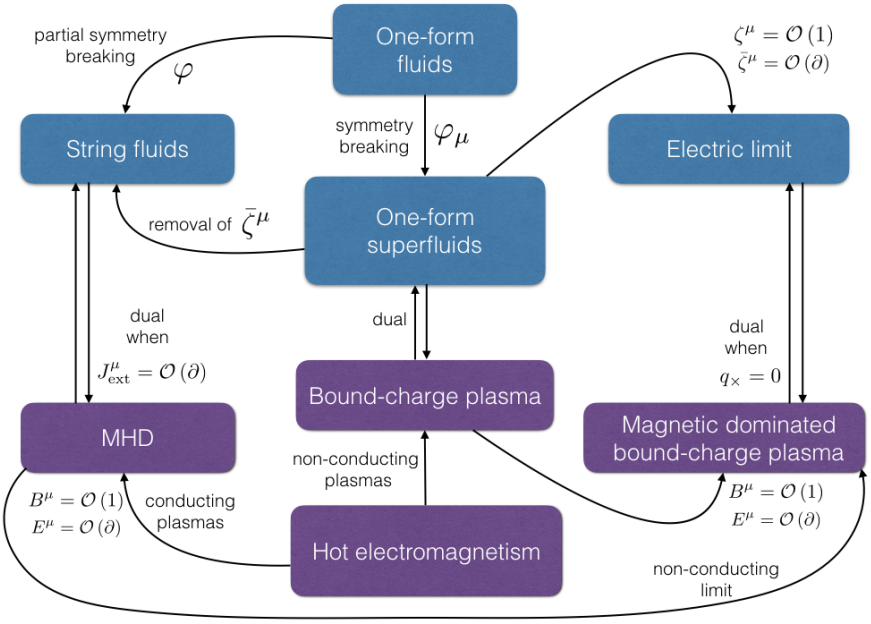

Figure 1. Schematic representation of the connections between one-form (super)fluids and hot electromagnetism

works, which are naively assumed to have the one-form symmetry unbroken, and had not previously been considered in the literature. Having formulated the theory of one-form hydrodynamics, this work progresses by incorporating the vector Goldstone mode φ_μ arising due to the spontaneous breaking the one-form symmetry (see figure 1). This makes up the core of section 4, where a theory of one-form superfluids is developed and its different limits described. This theory introduces a two-form superfluid velocity $\xi_{\mu\nu}$ (the gauge-invariant covariant derivative of φ_μ) which in four spacetime dimensions can be decomposed into two vectors, ζ^μ and $\bar{\zeta}^\mu$. These can be understood as electric and magnetic fields associated with $\xi_{\mu\nu}$, respectively.

We study two limits of one-form superfluids in detail: the string fluid limit and the electric limit. The string fluid limit, discussed in section 5, can be obtained by partially breaking the one-form symmetry along the fluid velocity u^μ, which results in the appearance of a scalar Goldstone mode φ. The same theory can also be obtained directly from one-form superfluids by dropping any dependence on $\bar{\zeta}^\mu$ from the constitutive relations (see figure 1). The scalar Goldstone φ, in this interpretation, is understood as the time component of the vector Goldstone mode, that is $\varphi = u^\mu \varphi_\mu / T$, where T is the fluid temperature. On the other hand, the electric limit taken in section 6 does not switch off the $\bar{\zeta}^\mu$ dependence. Rather, it assumes a derivative hierarchy $\zeta^\mu = \mathcal{O}(1)$ and $\bar{\zeta}^\mu = \mathcal{O}(\partial)$ between the components of $\xi_{\mu\nu}$, rendering $\bar{\zeta}^\mu$ subleading in the hydrodynamic derivative expansion. Though equivalent at ideal order, string fluids and the electric limit of one-form superfluids deviate considerably upon including one-derivative corrections.

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In section 7, the connections between one-form superfluids, including its limits, and different hydrodynamic regimes of hot electromagnetism are discussed. We have specially focused on two regimes: MHD and bound-charge plasmas. The MHD regime, applicable to conducting plasmas for which the magnetic fields are arbitrary $B^\mu = \mathcal{O}(1)$ and electric fields are weak $E^\mu = \mathcal{O}(\partial)$, is shown to be exactly equivalent to string fluids when $J^\mu_{\text{ext}} = \mathcal{O}(\partial)$, as advertised earlier. The full map between the transport coefficients in the two formulations at first order in derivatives is given, together with the solution to the Maxwell's equations that eliminates non-propagating degrees of freedom from the hydrodynamical description. Here, the traditional treatment of MHD is also extended to include all transport coefficients at first order in derivatives, taking into account parity-violating terms. Also in section 7, the traditional treatment of the bound-charge plasma regime is formulated for the first time, and is applicable to non-conducting plasmas (i.e. plasmas with only bound-charges and no free charge carriers). These are argued to be exactly equivalent to one-form superfluids, with the explicit mapping of constitutive relations worked out at ideal order. At first order in derivatives, attention is given to the magnetic dominated bound-charge plasma, where $B^\mu = \mathcal{O}(1)$ and $E^\mu = \mathcal{O}(\partial)$, similarly to MHD. These are shown to be exactly equivalent to the electric limit of one-form superfluids, provided that a certain transport coefficient $q_\times$ is set to zero. These connections have been summarised in figure 1.

Finally, in section 8 a discussion of some of these results is given together with interesting future research directions. Some of the calculational details relevant to this work have been assembled into appendix A. We have provided a comparison of our results with the effective action approach of [30] in appendix B. We also clarify the constraints imposed by discrete symmetries, such as parity and CPT, in various phases of one-form hydrodynamics in appendix C.

Comments on related work. During the completion of this work, we became aware of an upcoming related work that investigates different aspects of magnetohydrodynamics [30], and which has considerable overlap with [24]. We have provided a comparison between our work and that of [30] in appendix B. We also generalised parts of [30] as to construct an ideal order effective Lagrangian for the hydrodynamic theories of section 3 and 4. Additionally, we have also formulated an order parameter that describes the partial breaking of the one-form symmetry required to formulate MHD in the language of generalised global symmetries.

2 The setup of one-form hydrodynamics

In this section we introduce the fundamental degrees of freedom associated with one-form hydrodynamics and the conservation equations that constrain and govern their dynamical evolution, including in the presence of gapless modes. These degrees of freedom are motivated by extending the degrees of freedom characterising thermal equilibrium partition functions into the out-of-equilibrium context. Analogous to the case of usual zero-form charged hydrodynamics, the symmetry properties of the background fields to which these fluids couple to are key guiding principles in the identification of the correct degrees of

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Dynamical field	*Symbol*
Fluid velocity	u^μ with $u^\mu u_\mu = -1$
Temperature	T
Zero-form chemical potential	μ

Table 2. Dynamical fields for zero-form charged fluids.

freedom. The requirement that one-form fluids satisfy the second law of thermodynamics leads to a generalised adiabaticity equation that can be used to constrain the transport properties of one-form fluids. The formalism described here and associated set of tools (hydrostatic effective action, adiabaticity equation, etc) is the point of departure for the construction of novel theories of hydrodynamics with generalised global symmetries that we provide in later sections of this paper.

2.1 Symmetries, conservation, and hydrodynamic variables

The Noether theorem ascertains that any theory that is invariant under global Poincaré transformations and U(1) zero-form transformations must admit a conserved energy-momentum tensor $T^{\mu\nu}$ and charge current J^μ. Coupling the theory to a spacetime background with metric $g_{\mu\nu}$ and gauge field A_μ, the conservation equations associated with these symmetries take the form[5]

$$\nabla_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho \ , \qquad \nabla_\mu J^\mu = 0 \ . \tag{2.1}$$

Here ∇_μ is the covariant derivative associated with $g_{\mu\nu}$ and $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ is the field strength associated with A_μ. Focusing on the case of four spacetime dimensions, eq. (2.1) consists of a system of five conservation equations. Hydrodynamics is the low-energy effective description at finite temperature of such systems and its formulation requires picking an arbitrary set of five dynamical fields, as in table 2, whose dynamics is governed by eq. (2.1). If, besides the hydrodynamic modes, the system admits gapless modes at low energy, collectively represented by Φ, then eq. (2.1) must be supplied with additional equations of motion describing the evolution of Φ. Once the dynamical fields have been chosen and the gapless modes identified, the hydrodynamic theory is obtained by writing down the most generic "constitutive relations" for $T^{\mu\nu}$ and J^μ in terms of u^μ, T, μ, and Φ in a long-wavelength derivative expansion. Empirical physical requirements, such as the second law of thermodynamics and Onsager's relations, impose constrains on these constitutive relations.

The motivation for the choice of hydrodynamic fields as in table 2 originates from considerations in equilibrium thermal field theories, as we now outline. Under a generic infinitesimal symmetry transformation parametrised by $\mathfrak{X} = (\chi^\mu, \Lambda^\chi)$, where χ^μ is associated with diffeomorphisms and Λ^χ with gauge transformations, the background fields transform

[5] In writing eq. (2.1) we have assumed that the symmetries are non-anomalous.

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according to

$$\begin{aligned}\delta_{\mathcal{X}} g_{\mu\nu} &= \pounds_\chi g_{\mu\nu} = 2\nabla_{(\mu}\chi_{\nu)} \ , \\ \delta_{\mathcal{X}} A_\mu &= \pounds_\chi A_\mu + \partial_\mu \Lambda^\chi = \partial_\mu \left(\Lambda^\chi + \chi^\mu A_\mu\right) + \chi^\nu F_{\nu\mu} \ ,\end{aligned} \tag{2.2}$$

while the symmetry parameters themselves transform as[6]

$$\delta_{\mathcal{X}} \mathcal{X}' = [\mathcal{X}, \mathcal{X}'] = \left(\pounds_\chi \chi'^\mu \ , \ \pounds_\chi \Lambda'^\chi - \pounds_{\chi'} \Lambda^\chi\right) \ . \tag{2.3}$$

We assume that the background manifold admits a timelike isometry $\mathcal{K} = (\mathbf{k}^\mu, \Lambda^{\mathbf{k}})$ with $\mathbf{k}^\mu \mathbf{k}_\mu < 0$, i.e. $\delta_{\mathcal{K}} g_{\mu\nu} = \delta_{\mathcal{K}} A_\mu = 0$. On such backgrounds, we can define a global thermal state by the grand-canonical partition function[7]

$$\mathcal{Z}[g_{\mu\nu}, A_\mu] = \mathrm{tr} \exp\left[\int_\Sigma \mathrm{d}\sigma_\mu \left(T^{\mu\nu}\mathbf{k}_\nu + (\Lambda^{\mathbf{k}} + \mathbf{k}^\lambda A_\lambda) J^\mu\right)\right] \ , \tag{2.4}$$

where the trace is taken over all the equilibrium configurations of Φ which satisfy $\delta_{\mathcal{K}}\Phi = 0$. In eq. (2.4), Σ denotes an arbitrary Cauchy slice with volume element $\mathrm{d}\sigma_\mu$. Using eq. (2.1), it may be verified that $\mathcal{Z}$ is independent of the choice of Σ and it is also manifestly invariant under the symmetries of the theory. It is the aim of hydrodynamics to describe slight departures from the global thermal state by replacing the background isometry $\mathcal{K}$ with an arbitrary set of slowly varying dynamical fields $\mathcal{B} = (\beta^\mu, \Lambda^\beta)$, which are related to those in table 2 via

$$\beta^\mu = \frac{u^\mu}{T} \ , \quad \Lambda^\beta + \beta^\mu A_\mu = \frac{\mu}{T} \ . \tag{2.5}$$

This is the more natural way to think of the hydrodynamic degrees of freedom. As detailed below, the identification of the correct degrees of freedom in the case of one-form fluids follows a similar reasoning whose starting point is the equilibrium partition function.

Analogous to systems invariant under zero-form U(1) transformations, physical systems that are invariant under global Poincaré and U(1) one-form transformations admit a conserved energy-momentum tensor $T^{\mu\nu}$ and two-form charge current $J^{\mu\nu}$ such that

$$\nabla_\mu T^{\mu\nu} = \frac{1}{2} H^{\nu\rho\sigma} J_{\rho\sigma} \ , \qquad \nabla_\mu J^{\mu\nu} = 0 \ , \tag{2.6}$$

where $H_{\mu\nu\rho} = 3\partial_{[\mu} b_{\nu\rho]}$ is the field strength associated with a two-form gauge field $b_{\mu\nu}$. In order to describe the effective low-energy hydrodynamic theory for systems with a global U(1) one-form symmetry, a suitable choice of dynamical fields is required. As in the

[6] Symmetry transformations of the background are required to form a Lie algebra such that $[\delta_{\mathcal{X}}, \delta_{\mathcal{X}'}] g_{\mu\nu} = \delta_{[\mathcal{X},\mathcal{X}']} g_{\mu\nu}$ and $[\delta_{\mathcal{X}}, \delta_{\mathcal{X}'}] A_\mu = \delta_{[\mathcal{X},\mathcal{X}']} A_\mu$, which fixes eq. (2.3). Similarly in the case of one-form symmetries, requiring $[\delta_{\mathcal{X}}, \delta_{\mathcal{X}'}] g_{\mu\nu} = \delta_{[\mathcal{X},\mathcal{X}']} g_{\mu\nu}$ and $[\delta_{\mathcal{X}}, \delta_{\mathcal{X}'}] b_{\mu\nu} = \delta_{[\mathcal{X},\mathcal{X}']} b_{\mu\nu}$ fixes the transformation properties eq. (2.8) provided that we require the fields to transform appropriately under diffeomorphisms.

[7] For example, in the standard case of a static fluid coupled to a flat background $g_{\mu\nu} = \eta_{\mu\nu}$ and no external gauge fields $A_\mu = 0$, one works with $\mathcal{K} = (\mathbf{k}^\mu = \delta^\mu_t/T_0, \Lambda^{\mathbf{k}} = \mu_0/T_0)$, where T_0 and μ_0 are the temperature and chemical potential of the global thermal state. In this case we get the conventional expression for the grand-canonical partition function $\mathcal{Z} = \mathrm{tr} \exp\left[-T_0^{-1} \int \mathrm{d}^3x \left(T^{00} - \mu_0 J^0\right)\right]$. Note that we can always perform a gauge transformation to set $\Lambda^{\mathbf{k}} = 0$ at the expense of $A_\mu = \mu_0 \delta^t_\mu$, leading to the same result.

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Dynamical field	*Symbol*
Fluid velocity	u^μ with $u^\mu u_\mu = -1$
Temperature	T
One-form chemical potential	μ_μ

Table 3. Dynamical fields for one-form charged fluids.

case of zero-form fluids, it is noted that under a generic infinitesimal one-form symmetry transformation parametrised by $\mathcal{X} = (\chi^\mu, \Lambda^\chi_\mu)$, with Λ^χ_μ being the parameter associated with one-form gauge transformations, the background fields transform according to

$$\begin{aligned}\delta_\mathcal{X} g_{\mu\nu} &= \pounds_\chi g_{\mu\nu} = 2\nabla_{(\mu}\chi_{\nu)} \ , \\ \delta_\mathcal{X} b_{\mu\nu} &= \pounds_\chi b_{\mu\nu} + 2\partial_{[\mu}\Lambda^\chi_{\nu]} = 2\partial_{[\mu}\left(\Lambda^\chi_{\nu]} + \chi^\lambda b_{\lambda\mu}\right) + \chi^\lambda H_{\lambda\mu\nu} \ ,\end{aligned} \tag{2.7}$$

while the symmetry parameters transform as (see footnote 6)

$$\delta_\mathcal{X}\mathcal{X}' = [\mathcal{X}, \mathcal{X}'] = \left(\pounds_\chi \chi'^\mu, \pounds_\chi \Lambda'^\chi_\mu - \pounds_{\chi'}\Lambda^\chi_\mu\right) \ . \tag{2.8}$$

When coupled to spacetime backgrounds that admit a timelike isometry $\mathcal{K} = (\mathbf{k}^\mu, \Lambda^\mathbf{k}_\mu)$, we can define a global thermal state by means of the grand-canonical partition function

$$\mathcal{Z}[g_{\mu\nu}, b_{\mu\nu}] = \mathrm{tr}\exp\left[\int_\Sigma \mathrm{d}\sigma_\mu \left(T^{\mu\nu}\mathbf{k}_\nu + (\Lambda^\mathbf{k}_\nu + \mathbf{k}^\lambda b_{\lambda\nu})J^{\mu\nu}\right)\right] \ . \tag{2.9}$$

Following the same chain of reasoning as for zero-form symmetries, we are led to the natural choice of hydrodynamic fields for one-form hydrodynamics as $\mathcal{B} = (\beta^\mu, \Lambda^\beta_\mu)$. By defining

$$\beta^\mu = \frac{u^\mu}{T} \ , \quad \Lambda^\beta_\mu + \beta^\nu b_{\nu\mu} = \frac{\mu_\mu}{T} \ , \tag{2.10}$$

these fields can be recast in a more conventional form as in table 3. However, unlike zero-form fluids, μ_μ defined in this way is not gauge invariant. Instead, it transforms akin to a one-form gauge field

$$\delta_\mathcal{X}\frac{\mu_\mu}{T} = \pounds_\chi \frac{\mu_\mu}{T} - \partial_\mu\left(\beta^\nu \Lambda^\chi_\nu\right) \ . \tag{2.11}$$

This should not come as a surprise since the time-component of the one-form conservation equation, $\nabla_\mu J^{\mu t}$, is not a dynamical equation but merely a constraint [22]. Correspondingly, one degree of freedom in μ_μ is rendered unphysical due to the gauge transformation. Note that since μ_μ/T transforms as a one-form gauge field, all of its gauge-invariant physical information can be captured by the antisymmetric derivative $\partial_{[\mu}(\mu_{\nu]}/T)$. Having identified the dynamical fields for one-form fluids, we proceed by defining the hydrostatic partition function and deriving the adiabaticity equation.

2.2 Hydrostatic effective action and the second law of thermodynamics

The hydrostatic effective action is an important cornerstone of hydrodynamics. It describes the entire set of equilibrium configurations admissible by the fluid for a given arrangement

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of the background sources. These configurations can then be used as a starting point for studying deviations away from equilibrium order by order in the derivative expansion (e.g. dispersion relations for Alfvén waves in MHD). In this subsection we introduce the generalities of hydrostatic effective action relevant to this work and illustrate their connection to the second law of thermodynamics.

For the purposes of this paper, it is assumed that the microscopic field theories underlying the hydrodynamic regime are sufficiently well behaved, so that the equilibrium partition function (2.9) can be computed via a Euclidean path integral

$$\mathcal{Z}[g_{\mu\nu}, b_{\mu\nu}] = \int \mathcal{D}\Phi \exp\left(-S^{\rm hs}[g_{\mu\nu}, b_{\mu\nu}; \Phi]\right) \quad . \tag{2.12}$$

$S^{\rm hs}[g_{\mu\nu}, b_{\mu\nu}; \Phi]$, known as the hydrostatic effective action, contains all the possible diffeomorphism and gauge-invariant terms composed of $g_{\mu\nu}$, $b_{\mu\nu}$, and Φ in the presence of a timelike isometry $\mathcal{K}$. The variation of the effective action with respect to the background sources yields the conserved currents

$$T^{\mu\nu}_{\rm hs} = \frac{2}{\sqrt{-g}} \frac{\delta S^{\rm hs}[g_{\mu\nu}, b_{\mu\nu}; \Phi]}{\delta g_{\mu\nu}} \ , \qquad J^{\mu\nu}_{\rm hs} = \frac{2}{\sqrt{-g}} \frac{\delta S^{\rm hs}[g_{\mu\nu}, b_{\mu\nu}; \Phi]}{\delta b_{\mu\nu}} \ , \tag{2.13}$$

while the equilibrium configurations of the gapless modes are obtained by extremising the effective action with respect to Φ leading to

$$K^{\Phi}_{\rm hs} = \frac{\delta S^{\rm hs}[g_{\mu\nu}, b_{\mu\nu}; \Phi]}{\delta \Phi} = 0 \quad . \tag{2.14}$$

As a consistency condition on the general hydrodynamic constitutive relations (including dissipative effects), we require them to match with eq. (2.13) when we revert to the global thermal state by setting $\mathcal{B} = \mathcal{K}$. This requirement yields strict constraints on their form at every derivative order [7, 8].

Schematically, the hydrostatic effective action appearing in (2.12) can be parametrised as

$$S^{\rm hs}[g_{\mu\nu}, b_{\mu\nu}; \Phi] = \int_{\Sigma} {\rm d}\sigma_{\mu} N^{\mu}_{\rm hs} \ , \tag{2.15}$$

where $N^{\mu}_{\rm hs}$ is the hydrostatic free energy current that satisfies $\nabla_{\mu} N^{\mu}_{\rm hs} = 0$. As we leave the global thermal state, the free energy current is no longer conserved. To see this, let us slightly depart from equilibrium by replacing $\mathcal{K}$ with $\mathcal{B}$ and performing a $\mathcal{B}$-variation of $S^{\rm hs}$. We obtain the hydrostatic adiabaticity equation

$$\nabla_{\mu} N^{\mu}_{\rm hs} = \frac{1}{2} T^{\mu\nu}_{\rm hs} \delta_{\mathcal{B}} g_{\mu\nu} + \frac{1}{2} J^{\mu\nu}_{\rm hs} \delta_{\mathcal{B}} b_{\mu\nu} + K^{\Phi}_{\rm hs} \delta_{\mathcal{B}} \Phi \quad . \tag{2.16}$$

Physically, it is equivalent to the statement that entropy is conserved in a hydrostatic configuration. To wit, defining the entropy current as

$$S^{\mu}_{\rm hs} = N^{\mu}_{\rm hs} - \frac{1}{T} T^{\mu\nu}_{\rm hs} u_{\nu} - \frac{1}{T} J^{\mu\nu}_{\rm hs} \mu_{\nu} \ , \tag{2.17}$$

and using the conservation equations (2.6), the adiabaticity equation can be rewritten as $\nabla_{\mu} S^{\mu}_{\rm hs} = 0$. However, in a generic out-of-equilibrium hydrodynamic configuration with

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entropy current S^μ, we expect entropy to be produced, leading to the second law of thermodynamics

$$\nabla_\mu S^\mu = \Delta \geq 0 \quad . \tag{2.18}$$

Here Δ is a non-negative quadratic form which vanishes in a hydrostatic configuration. Correspondingly, the generic adiabaticity equation (2.16) in the out-of-equilibrium context is an extrapolation of its hydrostatic counterpart

$$\nabla_\mu N^\mu = \frac{1}{2} T^{\mu\nu} \delta_{\mathcal{B}} g_{\mu\nu} + \frac{1}{2} J^{\mu\nu} \delta_{\mathcal{B}} b_{\mu\nu} + K^\Phi \delta_{\mathcal{B}} \Phi + \Delta \ , \quad \Delta \geq 0 \quad , \tag{2.19}$$

where the different quantities involved may also include non-hydrostatic contributions, and can be viewed as generalisation of the requirement of a hydrostatic effective action. A set of physical constitutive relations $T^{\mu\nu}$, $J^{\mu\nu}$, K^Φ associated with a hydrodynamic system are required to be accompanied by a free energy current N^μ and a quadratic form Δ such that eq. (2.19) is satisfied for all fluid configurations. Below we will show how the adiabaticity equation can be used to obtain constraints on the hydrodynamic constitutive relations.

Traditionally, one takes a slightly different route with the second law of thermodynamics as the starting point in order to arrive at these constraints [31]. Switching off the extra gapless modes Φ for the moment, one requires that for every set of physical constitutive relations $T^{\mu\nu}$, $J^{\mu\nu}$, the hydrodynamic system in question must admit an associated entropy current S^μ whose divergence is positive semi-definite, $\nabla_\mu S^\mu \geq 0$, on the solutions of the conservation equations. Given such an entropy current, it is always possible to go off-shell and write down an equivalent statement for the second law by introducing arbitrary linear combinations of the conservation equations [32] (see also [33])

$$\nabla_\mu S^\mu + \mathcal{A}_\nu \left(\nabla_\mu T^{\mu\nu} - \frac{1}{2} H^{\nu\rho\sigma} J_{\rho\sigma} \right) + \mathcal{B}_\nu \nabla_\mu J^{\mu\nu} = \Delta \geq 0 \quad . \tag{2.20}$$

Here $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$ are arbitrary multipliers composed of the hydrodynamic and background fields, and introduced as to satisfy this equation offshell. Recall that the hydrodynamic fields u^μ, T, and μ_μ were some arbitrary set of fields chosen to describe the system, and like in any field theory, can admit arbitrary field redefinitions. We can use this freedom to set $\mathcal{A}_\mu = u_\mu/T$ and $\mathcal{B}_\mu = \mu_\mu/T$. Having done that, and using the relation between free-energy and entropy currents, i.e.

$$N^\mu = S^\mu + \frac{1}{T} T^{\mu\nu} u_\nu + \frac{1}{T} J^{\mu\nu} \mu_\nu \ , \tag{2.21}$$

it is easy to see that the offshell second law of thermodynamics (2.20) reduces to the adiabaticity equation (2.19). Hence the constraints imposed by the second law of thermodynamics are equivalent to the ones imposed by the adiabaticity equation. The latter, however, turns out to be functionally advantageous to implement. An entirely analogous argument follows in the presence of additional gapless modes [34].

2.3 Constitutive relations up to first order

In the bulk of this paper, we will derive the constitutive relations allowed by the adiabaticity equation (2.19) up to one-derivative order for several cases of interest. As shall be explained

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Ordinary one-form fluids	
$\mathcal{N}$ (1)	$\epsilon^{\mu\nu\rho\sigma}u_\mu H_{\nu\rho\sigma}$
$\eta^{(\mu\nu)(\rho\sigma)}$ (2)	$P^{\mu\nu}P^{\rho\sigma}$, $P^{\rho(\mu}P^{\nu)\sigma}$
$\chi^{(\mu\nu)[\rho\sigma]}$, $\chi'^{[\rho\sigma](\mu\nu)}$ (2×0)	—
$\sigma^{[\mu\nu][\rho\sigma]}$ (2)	$u^{[\mu}P^{\nu][\rho}u^{\sigma]}$, $P^{\rho[\mu}P^{\nu]\sigma}$
One-form string fluids	
$\mathcal{N}$ (5)	$h^\mu\partial_\mu T$, $h^\mu\partial_\mu\frac{\varpi}{T}$, $\epsilon^{\mu\nu\rho\sigma}u_\mu H_{\nu\rho\sigma}$, $\epsilon^{\mu\nu\rho\sigma}u_\mu h_\nu\partial_\rho u_\sigma$, $\epsilon^{\mu\nu\rho\sigma}u_\mu h_\nu\partial_\rho h_\sigma$
$\eta^{(\mu\nu)(\rho\sigma)}$ (8)	$h^\mu h^\nu h^\rho h^\sigma$, $h^\mu h^\nu\Delta^{\rho\sigma}$, $\Delta^{\mu\nu}h^\rho h^\sigma$, $h^{(\mu}\Delta^{\nu)(\rho}h^{\sigma)}$, $\Delta^{\mu\nu}\Delta^{\rho\sigma}$, $\Delta^{\rho(\mu}\Delta^{\nu)\sigma}$, $h^{(\mu}\epsilon^{\nu)(\rho}h^{\sigma)}$, $\epsilon^{\rho(\mu}\Delta^{\nu)\sigma}$
$\chi^{(\mu\nu)[\rho\sigma]}$, $\chi'^{[\rho\sigma](\mu\nu)}$ (2×4)	$h^{(\mu}\Delta^{\nu)[\rho}h^{\sigma]}$, $h^{(\mu}\epsilon^{\nu)[\rho}h^{\sigma]}$, $h^\mu h^\nu\epsilon^{\rho\sigma}$, $\Delta^{\mu\nu}\epsilon^{\rho\sigma}$
$\sigma^{[\mu\nu][\rho\sigma]}$ (3)	$h^{[\mu}\Delta^{\nu][\rho}h^{\sigma]}$, $h^{[\mu}\epsilon^{\nu][\rho}h^{\sigma]}$, $\epsilon^{\mu\nu}\epsilon^{\rho\sigma}$
One-form superfluids	
$\mathcal{N}$ (22)	$h^\mu_a\partial_\mu T$, $\epsilon^{\mu\nu\rho\sigma}u_\mu h_{a\nu}\partial_\rho u_\sigma$, $\epsilon^{\mu\nu\rho\sigma}u_\mu h_{a\nu}\partial_\rho\zeta_\sigma$, $\epsilon^{\mu\nu\rho\sigma}u_\mu h_{a\nu}\partial_\rho\bar{\zeta}_\sigma$, $h^\mu_a h^\nu_b\nabla_{(\mu}\zeta_{\nu)}$, $h^\mu_a h^\nu_b\nabla_{(\mu}\bar{\zeta}_{\nu)}$
$\eta^{(\mu\nu)(\rho\sigma)}$ (36)	$h^{(\mu}_a h^{\nu)}_b h^{(\rho}_c h^{\sigma)}_d$
$\chi^{(\mu\nu)[\rho\sigma]}$, $\chi'^{[\rho\sigma](\mu\nu)}$ (2×36)	$h^{(\mu}_a h^{\nu)}_b h^{[\rho}_c h^{\sigma]}_d$, $h^{(\mu}_a h^{\nu)}_b h^{[\rho}_c u^{\sigma]}$
$\sigma^{[\mu\nu][\rho\sigma]}$ (36)	$h^{[\mu}_a h^{\nu]}_b h^{[\rho}_c h^{\sigma]}_d$, $u^{[\mu}h^{\nu]}_a h^{[\rho}_b h^{\sigma]}_c$, $h^{[\mu}_a h^{\nu]}_b h^{[\rho}_c u^{\sigma]}$, $u^{[\mu}h^{\nu]}_a h^{[\rho}_b u^{\sigma]}$

Table 4. One-derivative hydrostatic and non-hydrostatic structures for various phases of one-form hydrodynamic constitutive relations. Here $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$. For string fluids, h^μ is the string director field while ϖ is the string chemical potential; $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu - h^\mu h^\nu$, $\epsilon^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}u_\rho h_\sigma$. For one form superfluids, $\xi_{\mu\nu}$ is the superfluid velocity; $\zeta_\mu = \xi_{\mu\nu}u^\nu$, $\bar{\zeta}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu\xi_{\rho\sigma}$. On the other hand, h^μ_a is a set of orthonormal vectors made out of ζ^μ, $\bar{\zeta}^\mu$, and $\epsilon^{\mu\nu\rho\sigma}u_\nu\zeta_\rho\bar{\zeta}_\sigma$. These constitutive relations are *not* generically written in Landau frame, but have been expressed in certain frames convenient for each case respectively.

in the later sections, for all of these cases, the adiabaticity equation (2.19) can be reduced to a simpler version

$$\nabla_\mu N^\mu = \frac{1}{2}T^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu} + \frac{1}{2}J^{\mu\nu}\delta_{\mathcal{B}}b_{\mu\nu} + \Delta \ , \quad \Delta \geq 0 \ , \tag{2.22}$$

where the $\delta_{\mathcal{B}}\Phi$ term has been removed by going onshell and using the available field redefinition freedom. It is possible to broadly classify the constitutive relations satisfying eq. (2.22) into hydrostatic, i.e. constitutive relations that remain independent in a hydrostatic configuration, and non-hydrostatic, i.e. constitutive relations that vanish in a hydrostatic configuration.

The hydrostatic constitutive relations are characterised by a hydrostatic free energy current $N^\mu_{\rm hs} = \mathcal{N}\beta^\mu + \Theta^\mu_{\mathcal{N}}$, where $\mathcal{N}$ is made out of all the independent hydrostatic scalars,

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while $\Theta^\mu_{\mathcal{N}}$ is a non-hydrostatic vector[8] defined via

$$\nabla_\mu\left(\mathcal{N}\beta^\mu\right)=\frac{1}{\sqrt{-g}}\delta_{\mathcal{B}}\left(\sqrt{-g}\,\mathcal{N}\right)=\frac{1}{2}\left(\mathcal{N}g^{\mu\nu}+2\frac{\delta\mathcal{N}}{\delta g_{\mu\nu}}\right)\delta_{\mathcal{B}}g_{\mu\nu}+2\frac{\delta\mathcal{N}}{\delta b_{\mu\nu}}\delta_{\mathcal{B}}b_{\mu\nu}-\nabla_\mu\Theta^\mu_{\mathcal{N}}\ . \tag{2.23}$$

Comparing with eq. (2.22), it is possible to read out the hydrostatic constitutive relations as

$$T^{\mu\nu}_{\rm hs}=\mathcal{N}g^{\mu\nu}+2\frac{\delta\mathcal{N}}{\delta g_{\mu\nu}}\ ,\quad J^{\mu\nu}_{\rm hs}=2\frac{\delta\mathcal{N}}{\delta b_{\mu\nu}}\ . \tag{2.24}$$

In turn, the non-hydrostatic constitutive relations up to first order are simply given as the most generic linear combinations of $\delta_{\mathcal{B}}g_{\mu\nu}$ and $\delta_{\mathcal{B}}b_{\mu\nu}$. To wit

$$\begin{pmatrix}T^{\mu\nu}_{\rm nhs}\\ J^{\mu\nu}_{\rm nhs}\end{pmatrix}=-T\begin{pmatrix}\eta^{(\mu\nu)(\rho\sigma)} & \chi^{(\mu\nu)[\rho\sigma]}\\ \chi'^{[\mu\nu](\rho\sigma)} & \sigma^{[\mu\nu][\rho\sigma]}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma}\\ \frac{1}{2}\delta_{\mathcal{B}}b_{\rho\sigma}\end{pmatrix}\ . \tag{2.25}$$

Here $\eta^{(\mu\nu)(\rho\sigma)}$, $\chi^{(\mu\nu)[\rho\sigma]}$, $\chi'^{(\mu\nu)[\rho\sigma]}$ and $\sigma^{[\mu\nu][\rho\sigma]}$ are the most general zero-derivative structures, with associated arbitrary transport coefficients, composed of the hydrodynamic fields identified in the previous section. In particular, there are no zero-derivative non-hydrostatic constitutive relations. Inserting eq. (2.25) into eq. (2.22) it can be inferred that they satisfy eq. (2.22) with $N^\mu_{\rm nhs}=0$ and

$$\Delta=T\begin{pmatrix}\frac{1}{2}\delta_{\mathcal{B}}g_{\mu\nu} & \frac{1}{2}\delta_{\mathcal{B}}b_{\mu\nu}\end{pmatrix}\begin{pmatrix}\eta^{(\mu\nu)(\rho\sigma)} & \chi^{(\mu\nu)[\rho\sigma]}\\ \chi'^{[\mu\nu](\rho\sigma)} & \sigma^{[\mu\nu][\rho\sigma]}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma}\\ \frac{1}{2}\delta_{\mathcal{B}}b_{\rho\sigma}\end{pmatrix}\geq 0\ . \tag{2.26}$$

It follows that the symmetric part of the non-hydrostatic transport coefficient matrix

$$\frac{1}{2}\begin{pmatrix}\eta^{(\mu\nu)(\rho\sigma)}+\eta^{(\rho\sigma)(\mu\nu)} & \chi^{(\mu\nu)[\rho\sigma]}+\chi'^{[\rho\sigma](\mu\nu)}\\ \chi'^{(\mu\nu)[\rho\sigma]}+\chi'^{[\rho\sigma](\mu\nu)} & \sigma^{[\mu\nu][\rho\sigma]}+\sigma^{[\rho\sigma][\mu\nu]}\end{pmatrix}\geq 0\ , \tag{2.27}$$

is a positive semi-definite matrix. This requirement imposes certain inequality constraints on the transport properties of the hydrodynamic theories that we will study. A summary of the allowed tensor structures in various phases of one-form hydrodynamics is presented in table 4.

A priori, the hydrodynamic fields u^μ, T, and μ_μ are arbitrary degrees of freedom chosen to describe the hydrodynamic fluctuations. In equilibrium, these are unambiguously identified with the timelike isometry $\mathcal{K}$, but in a generic out-of-equilibrium state, they can admit arbitrary non-hydrostatic field redefinitions. We can use this freedom to our advantage and simplify the non-hydrostatic constitutive relations by making a choice of "hydrodynamic frame". The most common of such frames is the Landau frame, which fixes the field redefinition in u^μ and T by choosing $T^{\mu\nu}_{\rm nhs}u_\nu=0$. The redefinition freedom in μ_μ can be similarly used to set $J^{\mu\nu}_{\rm nhs}u_\nu=0$. This leads to

$$\eta^{(\mu\nu)(\rho\sigma)}u_\mu=\chi^{(\mu\nu)[\rho\sigma]}u_\mu=\chi'^{[\mu\nu](\rho\sigma)}u_\mu=\sigma^{[\mu\nu][\rho\sigma]}u_\mu=0\ . \tag{2.28}$$

[8] Generically, $N^\mu_{\rm hs}$ can also include a hydrostatic part transverse to u^μ. Known as Class $\mathrm{H_V}$ constitutive relations or transcendental anomalies, these contributions are completely fixed up to a finite number of constants [10]. For the cases considered here, such terms turn out to be independent of the one-form symmetry sector altogether, and hence have been switched off for simplicity.

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To complete the quadratic form Δ in this frame, we need to further eliminate $u^\mu \delta_{\mathcal{B}} g_{\mu\nu}$ and $u^\mu \delta_{\mathcal{B}} b_{\mu\nu}$ from the non-hydrostatic constitutive relations (2.25), which can be generically done using the conservation equations (2.1). Therefore

$$\eta^{(\mu\nu)(\rho\sigma)} u_\rho = \chi^{(\mu\nu)[\rho\sigma]} u_\rho = \chi'^{[\mu\nu](\rho\sigma)} u_\rho = \sigma^{[\mu\nu][\rho\sigma]} u_\rho = 0 \ . \tag{2.29}$$

Hence, all indices in $\eta^{(\mu\nu)(\rho\sigma)}$, $\chi^{(\mu\nu)[\rho\sigma]}$, $\chi'^{(\mu\nu)[\rho\sigma]}$, and $\sigma^{[\mu\nu][\rho\sigma]}$ can be taken to be projected orthogonally to the fluid velocity. We will not restrict ourselves to this frame choice throughout this work. Instead, we will make a judicious choice of basis based on the hydrodynamic system under consideration, defaulting to the Landau frame when no such natural choice is available.

3 Ordinary one-form fluids

The main topic of interest of this work is one-form superfluids. However, before delving into the intricacies of one-form superfluid dynamics, it is instructive to consider ordinary one-form hydrodynamics first. Even though it is comparatively simpler than the examples that will be studied in later sections, this section provides the first formulation of one-form fluids in which the one-form symmetry is unbroken.

At ideal order, this system is trivial because there are no zero-derivative gauge-invariants that can be constructed from the ideal order hydrodynamic fields μ_μ and $b_{\mu\nu}$ identified in section 2. Consequently, at ideal order one-form fluids are characterised by the same constitutive relations as ordinary neutral fluids. Precisely

$$T^{\mu\nu} = (\epsilon + p)\, u^\mu u^\nu + p\, g^{\mu\nu} + \mathcal{O}(\partial) \ , \quad J^{\mu\nu} = \mathcal{O}(\partial) \ , \tag{3.1}$$

along with the thermodynamic relations

$$\mathrm{d}p = s\, \mathrm{d}T \ , \quad \epsilon + p = s\, T \ , \quad \mathrm{d}\epsilon = T\, \mathrm{d}s \ . \tag{3.2}$$

These constitute relations can be derived from their corresponding hydrostatic free energy density $\mathcal{N} = p(T)$, using (2.23), such that the free energy current is given by $N^\mu = p/T\, u^\mu$. The coefficients ϵ, p, and s are identified as the energy density, isotropic pressure, and entropy density of the fluid respectively. The first-order equations of motion simply imply that

$$u^\mu \nabla_\mu \epsilon + (\epsilon + p) \nabla_\mu u^\mu = 0 \ , \quad \frac{1}{T} P^{\mu\nu} \partial_\nu T + u^\nu \nabla_\nu u^\mu = 0 \ , \tag{3.3}$$

which can be collectively used to eliminate $u^\mu \delta_{\mathcal{B}} g_{\mu\nu}$ from the first-order non-hydrostatic constitutive relations.

At one-derivative order, signatures of one-form symmetry begin to appear. In the hydrostatic sector there is only one gauge-invariant contribution to the hydrostatic free energy density $\mathcal{N}$ at first order, which is given by

$$\mathcal{N} = p(T) - \frac{\alpha(T)}{6} \epsilon^{\mu\nu\rho\sigma} u_\mu H_{\nu\rho\sigma} \ . \tag{3.4}$$

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The transport coefficient α is unconstrained by the adiabaticity equation (second law). The variation of this corrected free energy density, according to eq. (2.23), leads to the hydrostatic constitutive relations

$$\begin{aligned}
T^{\mu\nu}_{\rm hs} &= (\epsilon+p)\,u^\mu u^\nu + p\,g^{\mu\nu} - \frac{1}{6}\epsilon^{\alpha\beta\rho\sigma}u_\alpha H_{\beta\rho\sigma}\frac{\partial(T\alpha)}{\partial T}u^\mu u^\nu - \frac{\alpha}{3}u^{(\mu}\epsilon^{\nu)\lambda\rho\sigma}H_{\lambda\rho\sigma} + \mathcal{O}(\partial^2)\ , \\
J^{\mu\nu}_{\rm hs} &= \nabla_\sigma\left(\alpha\epsilon^{\mu\nu\rho\sigma}u_\rho\right) + \mathcal{O}(\partial^2)\ , \\
N^\mu_{\rm hs} &= \frac{\alpha}{6T}\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma} - \alpha\epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho\left(\frac{\mu_\sigma}{T}\right) + \mathcal{O}(\partial^2)\ .
\end{aligned} \tag{3.5}$$

Note that all the dependence on μ_μ comes via the antisymmetric derivative $\partial_{[\mu}(\mu_{\nu]}/T)$, which is gauge-invariant. The most general non-hydrostatic corrections, in turn, can be decomposed along and transverse to u^μ according to

$$\begin{aligned}
T^{\mu\nu}_{\rm nhs} &= \delta\epsilon\,u^\mu u^\nu + \delta f\,P^{\mu\nu} + 2u^{(\mu}k^{\nu)} + t^{\mu\nu}\ , \\
J^{\mu\nu}_{\rm nhs} &= 2n^{[\mu}u^{\nu]} + s^{\mu\nu}\ .
\end{aligned} \tag{3.6}$$

Here all the tensor structures are transverse to u^μ, while $t^{\mu\nu}$ is symmetric-traceless and $s^{\mu\nu}$ is anti-symmetric. It is possible to use the hydrodynamic redefinition freedom in u^μ and T to set $\delta\epsilon = k^\mu = 0$. There is also a redefinition freedom in μ_μ but since μ_μ does not appear in the ideal order constitutive relations, this redefinition cannot be used to eliminate any first-order structures. Additionaly, the first order equations of motion can be used to remove $u^\mu\delta_{\mathcal{B}}g_{\mu\nu}$ from set of independent first-order structures. Finally, this leads to the following form for the first-order non-hydrostatic corrections

$$\begin{aligned}
\delta f &= -\frac{\zeta T}{2}P^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu} = -\zeta\nabla_\mu u^\mu\ , \\
t^{\mu\nu} &= -\eta\,T P^{\rho\langle\mu}P^{\nu\rangle\sigma}\delta_{\mathcal{B}}g_{\rho\sigma} = -2\eta\,P^{\mu\rho}P^{\nu\sigma}\left(\nabla_{(\rho}u_{\sigma)} - \frac{1}{3}P_{\rho\sigma}\nabla_\lambda u^\lambda\right) \equiv -\eta\sigma^{\mu\nu}\ , \\
n^\mu &= -T\lambda P^{\mu\rho}u^\sigma\delta_{\mathcal{B}}b_{\rho\sigma} = -2\lambda P^{\mu\rho}u^\sigma T\partial_{[\rho}\frac{\mu_{\sigma]}}{T}\ , \\
s^{\mu\nu} &= -T\sigma P^{\mu\rho}P^{\nu\sigma}\delta_{\mathcal{B}}b_{\rho\sigma} = -\sigma P^{\mu\rho}P^{\nu\sigma}\left(2T\partial_{[\rho}\frac{\mu_{\sigma]}}{T} + u^\lambda H_{\lambda\rho\sigma}\right).
\end{aligned} \tag{3.7}$$

Introducing these into the quadratic form in eq. (2.26), the non-negativity of Δ requires that all the non-hydrostatic transport coefficients are non-negative

$$\eta, \zeta, \lambda, \sigma \geq 0\ . \tag{3.8}$$

Thus, all in all, the most generic constitutive relations of a one-form ordinary fluid up to one-derivative order are given as

$$\begin{aligned}
T^{\mu\nu} &= (\epsilon+p)\,u^\mu u^\nu + p\,g^{\mu\nu} - \zeta\,\nabla_\lambda u^\lambda P^{\mu\nu} - \eta\,\sigma^{\mu\nu} \\
&\quad - \frac{1}{6}\epsilon^{\alpha\beta\rho\sigma}u_\alpha H_{\beta\rho\sigma}\frac{\partial(T\alpha)}{\partial T}u^\mu u^\nu - \frac{\alpha}{3}u^{(\mu}\epsilon^{\nu)\lambda\rho\sigma}H_{\lambda\rho\sigma} + \mathcal{O}(\partial^2)\ , \\
J^{\mu\nu} &= \nabla_\sigma\left(\alpha\epsilon^{\mu\nu\rho\sigma}u_\rho\right) + 2\lambda u^{[\mu}P^{\nu]\rho}u^\sigma T\partial_{[\rho}\frac{\mu_{\sigma]}}{T} \\
&\quad - \sigma P^{\mu\rho}P^{\nu\sigma}\left(2T\partial_{[\rho}\frac{\mu_{\sigma]}}{T} + u^\lambda H_{\lambda\rho\sigma}\right) + \mathcal{O}(\partial^2)\ ,
\end{aligned} \tag{3.9}$$

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and satisfy the adiabaticity equation (2.22) with the free-energy current

$$N^\mu = \frac{p}{T}u^\mu + \frac{\alpha}{6T}\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma} - \alpha\epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho\left(\frac{\mu_\sigma}{T}\right) + \mathcal{O}(\partial^2) \quad . \tag{3.10}$$

Out of the 5 transport coefficients appearing at one derivative order, 4 are dissipative and sign definite, while the remaining one does not cause dissipation and is left to be sign-indefinite.

In a global thermal state, characterised by a timelike isometry $\mathcal{K} = (\mathbf{k}^\mu, \Lambda^{\mathbf{k}}_\mu)$, the dynamical fields arrange in an equilibrium configuration

$$\beta^\mu = \mathbf{k}^\mu \ , \quad \Lambda^\beta_\mu = \Lambda^{\mathbf{k}}_\mu \ , \qquad u^\mu = \frac{\mathbf{k}^\mu}{\mathbf{k}} \ , \quad T = \frac{1}{\mathbf{k}} \ , \quad \mu_\mu = \frac{\Lambda^{\mathbf{k}}_\mu + \mathbf{k}^\nu b_{\nu\mu}}{\mathbf{k}}, \tag{3.11}$$

where $\mathbf{k} = \sqrt{-\mathbf{k}^\mu\mathbf{k}_\mu}$. If we choose a basis (t, x^i) such that $\mathbf{k}^\mu = \delta^\mu_t/T_0$, the hydrostatic effective action generating the respective constitute relations can be read out using eq. (3.10) and eq. (2.15) leading to

$$S^{\rm hs}[g_{\mu\nu}, b_{\mu\nu}] = \frac{1}{T_0}\int \mathrm{d}^3x\sqrt{-g}\left[p(T) - \frac{\alpha(T)}{6}\epsilon^{\mu\nu\rho\sigma}u_\mu H_{\nu\rho\sigma}\right]. \tag{3.12}$$

In the next section, it will be shown how the one-form symmetry can be broken and how this breaking can lead to other fields which can modify the ideal order constitutive relations.

4 One-form superfluids

In the previous section, hydrodynamics in the presence of an unbroken one-form symmetry was studied. In this section, this study is extended to include hydrodynamics with a spontaneously broken one-form symmetry by introducing a gapless vector Goldstone mode φ_μ into the generic analysis of section 2. It is observed that this theory is self-dual in the absence of external two-form sources, which is highly reminiscent of the electromagnetic duality of sourceless Maxwell's equations. In addition to the equation of state at ideal order, it is found that the one-form superfluid is characterised by a total of 166 transport coefficients at one-derivative order and hence is not extremely useful from a phenomenological standpoint. However, the various interesting limits/sectors of the theory are highlighted, for which the spectrum of transport coefficients is considerably more manageable. These limits will be investigated in detail in sections 5 and 6. The hydrodynamic theory developed here finds a direct application in describing various phases of plasma. In a certain limit, which we refer to as string fluids, one-form superfluid dynamics provides a dual and conceptually cleaner formulation of *magnetohydrodynamics* describing plasmas with Debye screened electric fields. In another limit, it describes plasmas without free charges, which we refer to as *bound-charge plasmas*. The details of these applications will be given in section 7.

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4.1 Hydrodynamics with spontaneously broken one-form symmetry

In this section, the Josephson condition for one-form superfluids is derived along with the ideal order constitute relations and first-order corrections. The hydrostatic effective action for one-form superfluids is also given.

4.1.1 Vector Goldstone and the Josephson equation

In the theory of zero-form superfluid dynamics, the spontaneous breaking of the global U(1) symmetry gives rise to a scalar Goldstone mode ϕ. Analogously, the Goldstone mode corresponding to a broken global U(1) one-form symmetry is the one-form gauge field φ_μ [28] that under an infinitesimal symmetry transformation $\mathfrak{X} = (\chi^\mu, \Lambda^\chi_\mu)$ transforms as

$$\delta_\mathfrak{X}\varphi_\mu = \pounds_\chi \varphi_\mu - \Lambda^\chi_\mu \ . \tag{4.1}$$

It is useful to introduce the covariant derivative of φ_μ according to

$$\xi_{\mu\nu} = 2\partial_{[\mu}\varphi_{\nu]} + b_{\mu\nu} \ , \tag{4.2}$$

which is gauge-invariant and transforms covariantly under the action of $\mathfrak{X}$, i.e. $\delta_\mathfrak{X}\xi_{\mu\nu} = \pounds_\mathfrak{X}\xi_{\mu\nu}$. In analogy with zero-form superfluids, for which the superfluid velocity is given by $\xi_\mu = \partial_\mu\phi + A_\mu$, we refer to (4.2) as the two-form "superfluid velocity". This superfluid velocity satisfies the Bianchi identity

$$3\partial_{[\mu}\xi_{\nu\rho]} = H_{\mu\nu\rho} \ . \tag{4.3}$$

The existence of φ_μ allows for the definition of a gauge-invariant one-form chemical potential μ^φ_μ such that

$$\mu^\varphi_\mu = \mu_\mu - T\partial_\mu\left(\beta^\nu\varphi_\nu\right) \ , \tag{4.4}$$

where μ_μ was introduced in eq. (2.10). In this symmetry-broken phase, the covariant information contained in $b_{\mu\nu}$, μ_μ, and φ_μ can be exchanged for $\xi_{\mu\nu}$ and μ^φ_μ.

As mentioned in section 2, the dynamics of the Goldstone mode φ_μ is governed by its own equation of motion which can be represented as

$$K^\mu = 0 \ . \tag{4.5}$$

This, along with the conservation equations (2.6), make the system of dynamical equations closed. Our ignorance of the underlying microscopic theory does not allow for a first principle derivation of eq. (4.5). However, using the offshell adiabaticity equation (2.19) for the case at hand

$$\nabla_\mu N^\mu = \frac{1}{2}T^{\mu\nu}\delta_\mathfrak{B} g_{\mu\nu} + \frac{1}{2}J^{\mu\nu}\delta_\mathfrak{B} b_{\mu\nu} + K^\mu\delta_\mathfrak{B}\varphi_\mu + \Delta \ , \quad \Delta \geq 0 \ , \tag{4.6}$$

where $\delta_\mathfrak{B}\varphi_\mu = \beta^\nu\xi_{\nu\mu} - \mu^\varphi_\mu/T$, it is possible to fix the form of eq. (4.5) as in the case of usual superfluids [33]. In particular, at zero order in derivatives using the available hydrodynamic data, the above adiabaticity equation reduces to $-K^\mu\delta_\mathfrak{B}\varphi_\mu + \mathcal{O}(\partial) = \Delta \geq 0$, where $\mathcal{O}(\partial)$ denotes higher derivative corrections. Therefore, it is possible to infer that

$$K^\mu = -T\alpha^{\mu\nu}\delta_\mathfrak{B}\varphi_\nu + \mathcal{O}(\partial) \ , \quad \Delta = T\left(\delta_\mathfrak{B}\varphi_\mu\right)\alpha^{\mu\nu}\left(\delta_\mathfrak{B}\varphi_\nu\right) + \mathcal{O}(\partial) \ , \tag{4.7}$$

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for some positive semi-definite matrix $\alpha^{\mu\nu}$. Since the Goldstone must satisfy eq. (4.5) onshell, the above implies the relation

$$\delta_{\mathcal{B}}\varphi_\mu = \mathcal{O}(\partial) \quad \Longrightarrow \quad \mu^{\varphi}_{\mu} = u^\nu \xi_{\nu\mu} + \mathcal{O}(\partial) \ , \tag{4.8}$$

which is the one-form equivalent of the *Josephson equation* in superfluids $\mu = u^\mu \xi_\mu + \mathcal{O}(\partial)$. Thus μ^{φ}_{μ} does not account for independent degrees of freedom in one-form hydrodynamics. Additionally, the redefinition freedom associated with μ_μ (or correspondingly Λ^{β}_{μ}) can be used to absorb the potential derivative corrections appearing in eq. (4.8). Hence, by redefining μ_μ, the Josephson equation (4.8) can be turned into an exact all-order onshell statement

$$\delta_{\mathcal{B}}\varphi_\mu = 0 \quad \Longrightarrow \quad \mu^{\varphi}_{\mu} = u^\nu \xi_{\nu\mu} \ , \tag{4.9}$$

and eliminate μ^{φ}_{μ} entirely from the hydrodynamic description. Thus, the energy-momentum conservation equation in (2.6) provides dynamics for u^μ and T, while the one-form conservation governs the dynamics of φ_μ.[9] On the other hand, the adiabaticity equation reduces to its simple form in (2.22) as promised earlier. While the final system appears to be similar to its symmetry-unbroken counterpart, it should be noted that the constitutive relations in this case involve φ_μ instead of μ_μ.

4.1.2 Ideal one-form superfluids

Having identified the independent set of hydrodynamic variables, it is straightforward to derive the most general constitutive relations at ideal order. Since we are working with four spacetime dimensions throughout this work, it is useful to introduce an independent set of vectors

$$\zeta_\mu = \xi_{\mu\nu} u^\nu \ , \quad \bar{\zeta}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu \xi_{\rho\sigma} \ , \tag{4.10}$$

satisfying $u_\mu \zeta^\mu = u_\mu \bar{\zeta}^\mu = 0$, which can be thought as electric and magnetic fields associated with $\xi_{\mu\nu}$. Here we have introduced the completely antisymmetric Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$ with conventions $\epsilon_{0123} = \sqrt{-g}$. In turn, eq. (4.10) can be used to decompose the superfluid velocity from eq. (4.2) as

$$\xi_{\mu\nu} = 2u_{[\mu}\zeta_{\nu]} - \epsilon_{\mu\nu\rho\sigma} u^\rho \bar{\zeta}^\sigma \ , \tag{4.11}$$

and to rewrite the Josephon equation (4.9) as $\mu^{\varphi}_{\mu} = -\zeta_\mu$.

Unlike the ordinary one-form fluids studied in section 3, one-form superfluids exhibit signatures of one-form symmetry at ideal order itself. Using the decomposition in eq. (4.10), the most generic form of the hydrostatic free energy density can be shown to take the form

$$\mathcal{N} = P(T, \zeta^2, \bar{\zeta}^2, \zeta\cdot\bar{\zeta}) + \mathcal{O}(\partial) \ . \tag{4.12}$$

[9] To see this, note that when all the μ_μ dependence has been eliminated from the hydrodynamic description, the entire dependence on $b_{\mu\nu}$ in the hydrodynamic constitutive relations comes via $\xi_{\mu\nu}$. Since this is also the source of all φ_μ dependence, for theories admitting an effective action, $K^\mu = 2\nabla_\nu(\delta S/\delta\xi_{\mu\nu}) = 2\nabla_\nu(\delta S/\delta b_{\mu\nu}) = \nabla_\nu J^{\mu\nu}$. In essence, the Josephson equation, that used to originally be the equation of motion for φ_μ, has now been used to algebraically eliminate μ_μ. Therefore, the one-form charge conservation now serves as an equation of motion for φ_μ.

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Performing a variation of the functional arguments with respect to the hydrodynamic variables $\mathcal{B}$ leads to

$$\begin{aligned}
\delta_{\mathcal{B}} T &= \frac{T}{2} u^\mu u^\nu \delta_{\mathcal{B}} g_{\mu\nu}, \\
\delta_{\mathcal{B}} \zeta^2 &= \left(\zeta^2 u^\mu u^\nu - \zeta^\mu \zeta^\nu\right) \delta_{\mathcal{B}} g_{\mu\nu} + 2\zeta^{[\mu} u^{\nu]} \delta_{\mathcal{B}} b_{\mu\nu} \ , \\
\delta_{\mathcal{B}} \bar{\zeta}^2 &= \left(-\bar{\zeta}^2 P^{\mu\nu} + \bar{\zeta}^\mu \bar{\zeta}^\nu + 2u^{(\mu}\epsilon^{\nu)\rho\sigma\tau} u_\rho \bar{\zeta}_\sigma \zeta_\tau\right) \delta_{\mathcal{B}} g_{\mu\nu} - \epsilon^{\mu\nu\rho\sigma} u_\rho \bar{\zeta}_\sigma \delta_{\mathcal{B}} b_{\mu\nu} \ , \\
\delta_{\mathcal{B}} (\zeta \cdot \bar{\zeta}) &= -\frac{1}{2} (\zeta \cdot \bar{\zeta}) g^{\mu\nu} \delta_{\mathcal{B}} g_{\mu\nu} - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \xi_{\rho\sigma} \delta_{\mathcal{B}} b_{\mu\nu} \ .
\end{aligned} \tag{4.13}$$

Using eq. (2.23), the one-form ideal superfluid constitutive relations, free energy, and entropy currents are obtained as

$$\begin{aligned}
T^{\mu\nu} &= \epsilon\, u^\mu u^\nu + \left(P - \bar{q}\, \bar{\zeta}^2 - q_\times\, (\zeta \cdot \bar{\zeta})\right) P^{\mu\nu} - q\, \zeta^\mu \zeta^\nu + \bar{q}\, \left(\bar{\zeta}^\mu \bar{\zeta}^\nu + 2u^{(\mu}\epsilon^{\nu)\rho\sigma\tau} u_\rho \bar{\zeta}_\sigma \zeta_\tau\right) \ , \\
J^{\mu\nu} &= -2u^{[\mu} \left(q\, \zeta^{\nu]} + q_\times\, \bar{\zeta}^{\nu]}\right) - \epsilon^{\mu\nu\rho\sigma} u_\rho \left(\bar{q}\, \bar{\zeta}_\sigma + q_\times\, \zeta_\sigma\right) \ , \\
N^\mu &= \frac{P}{T} u^\mu \ , \\
S^\mu &= N^\mu - \beta_\nu T^{\mu\nu} + \frac{1}{T} \zeta_\nu J^{\mu\nu} = s u^\mu \ ,
\end{aligned} \tag{4.14}$$

where the thermodynamic relations

$$\mathrm{d}P = s\, \mathrm{d}T + \frac{1}{2} q\, \mathrm{d}\zeta^2 + \frac{1}{2} \bar{q}\, \mathrm{d}\bar{\zeta}^2 + q_\times\, \mathrm{d}(\zeta \cdot \bar{\zeta}) \ , \qquad \epsilon + P = s\, T + q\, \zeta^2 + q_\times\, (\zeta \cdot \bar{\zeta}) \ , \tag{4.15}$$

were derived and used to simplify eq. (4.14). From here we can identify P appearing in the free energy density as the thermodynamic pressure. On the other hand, ϵ and s stand for the energy and entropy densities, in addition to the two superfluid densities q and $\bar{q}$, and a cross-density $q_\times$.[10] eqs. (4.14) and (4.15) imply that one-form superfluids are completely characterised by their equation of state $P = P(T, \zeta^2, \bar{\zeta}^2, \zeta \cdot \bar{\zeta})$.

4.1.3 One derivative corrections

Having derived the constitutive relations for an ideal one-form superfluid, it is possible to tackle the marginally more complicated first-order derivative corrections. This complication originates from the fact that there are 3 ideal order mutually orthogonal spatial vectors in one-form superfluids

$$h_1^\mu = \frac{\zeta^\mu}{|\zeta|} \ , \quad h_2^\mu = \frac{\bar{\zeta}^\mu - (\zeta \cdot \bar{\zeta}) \zeta^\mu / \zeta^2}{\sqrt{\bar{\zeta}^2 - (\zeta \cdot \bar{\zeta})^2 / \zeta^2}} \ , \quad h_3^\mu = \frac{\epsilon^{\mu\nu\rho\sigma} u_\nu \zeta_\rho \bar{\zeta}_\sigma}{\sqrt{\zeta^2 \bar{\zeta}^2 - (\zeta \cdot \bar{\zeta})^2}} \ , \tag{4.18}$$

[10] These thermodynamic relations can take a more appealing form if we define

$$\varpi = |\zeta| \ , \quad \bar{\varpi} = |\bar{\zeta}| \ , \quad \rho = q|\zeta| \ , \quad \bar{\rho} = \bar{q}|\bar{\zeta}| \ , \quad \lambda = \zeta \cdot \bar{\zeta} \ , \quad \rho_\times = q_\times \ , \quad p = P - \bar{q}\, \bar{\zeta}^2 - q_\times\, (\zeta \cdot \bar{\zeta}), \tag{4.16}$$

which leads to

$$\epsilon + p = sT + \rho\varpi - \bar{\rho}\bar{\varpi} \ , \quad \mathrm{d}p = s\, \mathrm{d}T + \rho\, \mathrm{d}\varpi - \bar{\varpi}\, \mathrm{d}\bar{\rho} - \lambda\, \mathrm{d}\rho_\times \ . \tag{4.17}$$

However, in the subsequent sections, limits for which ζ_μ or $\bar{\zeta}^\mu$ is taken to be of higher-order in derivatives will be explored. In those situations, these definitions are ill-defined.

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thereby completely breaking the SO(3) rotational symmetry and providing a decomposition for the metric

$$g^{\mu\nu} = -u^\mu u^\nu + \delta^{ab} h^\mu_a h^\nu_b \ , \quad h^\mu_a h^\nu_b = \delta_{ab} \ . \tag{4.19}$$

In terms of these, the corrections to the hydrostatic free energy density (4.12) can be written as

$$\begin{aligned}\mathcal{N} = P &+ f^a_1\, h^\mu_a \partial_\mu T + f^a_2\, \epsilon^{\mu\nu\rho\sigma} u_\mu h_{a\nu} \partial_\rho u_\sigma + f^a_3\, \epsilon^{\mu\nu\rho\sigma} u_\mu h_{a\nu} \partial_\rho \zeta_\sigma + f^a_4\, \epsilon^{\mu\nu\rho\sigma} u_\mu h_{a\nu} \partial_\rho \bar{\zeta}_\sigma \\ &+ 2 f^{ab}_5\, h^\mu_a h^\nu_b \nabla_{(\mu} \zeta_{\nu)} + 2 f^{ab}_6\, h^\mu_a h^\nu_b \nabla_{(\mu} \bar{\zeta}_{\nu)} + \mathcal{O}(\partial^2) \ .\end{aligned} \tag{4.20}$$

Here f_i are the hydrostatic transport coefficients, with f^{ab}_5 and f^{ab}_6 being symmetric and traceless. The respective trace parts lead to total derivative terms which do not lead to independent constitutive relations upon taking a variation. We have not considered any corrections involving $H_{\mu\nu\rho}$ explicitly, as they can be related to $3\partial_{[\mu}\xi_{\nu\rho]}$ using the Bianchi identity. Thus, in total, there are 22 transport coefficients in the hydrostatic sector.[11] As in the ideal order case, it is possible to use eq. (2.23) in order to read out the respective constitutive relations at first order in derivatives but we do not perform this exercise here.

Using the approach detailed in section 2.3, we can derive the constitutive relations in the non-hydrostatic sector. It is convenient to parametrise the stress tensor and charge current as

$$\begin{aligned} T^{\mu\nu}_{\text{nhs}} &= \delta\epsilon\, u^\mu u^\nu + 2\delta k^a\, u^{(\mu} h^{\nu)}_a + \delta t^{ab}\, h^{(\mu}_a h^{\nu)}_b \ , \\ J^{\mu\nu}_{\text{nhs}} &= 2\delta q^a\, u^{[\mu} h^{\nu]}_a + \delta s^{ab}\, h^{[\mu}_a h^{\nu]}_b \ . \end{aligned} \tag{4.21}$$

The terms involving $\delta\epsilon$ and δk^a can be set to zero using the field redefinition freedom inherent to T and u^μ. The field redefinition freedom inherent to μ_μ has been exhausted when turning the Josephson equation into an exact all-order statement (4.9), thus δq^a is generically non-zero. These considerations lead to the set of non-hydrostatic constitutive relations

$$\begin{pmatrix} \delta q^a \\ \delta t^{ab} \\ \delta s^{ab} \end{pmatrix} = -T \begin{pmatrix} \lambda_1^{ac} & \lambda_2^{a(cd)} & \lambda_2^{a[cd]} \\ \lambda_2'^{a(cd)} & \eta^{(ab)(cd)} & \chi^{(ab)[cd]} \\ \lambda_2'^{a[cd]} & \chi'^{[ab](cd)} & \sigma^{[ab][cd]} \end{pmatrix} \begin{pmatrix} u^\mu h^\nu_c \delta_{\mathcal{B}} b_{\mu\nu} \\ \frac{1}{2} h^\mu_c h^\nu_d \delta_{\mathcal{B}} g_{\mu\nu} \\ \frac{1}{2} h^\mu_c h^\nu_d \delta_{\mathcal{B}} b_{\mu\nu} \end{pmatrix} \ , \tag{4.22}$$

where λ_1, λ_2, λ_2', η, χ, χ', and σ are matrices of transport coefficients. There is a total of $12 \times 12 = 144$ non-hydrostatic transport coefficients. Positive semi-definiteness of Δ requires that the symmetric part of the transport coefficient matrix must have all its eigenvalues non-negative. This gives 12 inequality constraints in the non-hydrostatic sector. Onsager's relations may impose further restrictions on the non-hydrostatic transport which we have not considered in this analysis.

[11] In principle, we can remove some terms from the free energy density using the φ_μ equation of motion. The respective contributions to the constitutive relations can be absorbed by redefining φ_μ. We have not analysed these issues here.

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4.1.4 Hydrostatic effective action

At ideal order, the exact same constitutive relations (4.14) along with the thermodynamic relations (4.15) can be obtained from a hydrostatic effective action. Under the assumption that the background manifold admits a timelike isometry $\mathcal{K} = (\mathbf{k}^\mu, \Lambda^{\mathbf{k}}_\mu)$, we can infer the equilibrium configuration for the hydrodynamic fields using eq. (2.10) as

$$\beta^\mu = \mathbf{k}^\mu \ , \quad \Lambda^\beta_\mu = \Lambda^{\mathbf{k}}_\mu \ , \qquad u^\mu = \frac{\mathbf{k}^\mu}{\mathbf{k}} \ , \quad T = \frac{1}{\mathbf{k}} \ , \quad \mu_\mu = \frac{\Lambda^{\mathbf{k}}_\mu + \mathbf{k}^\nu b_{\nu\mu}}{\mathbf{k}} \ , \tag{4.23}$$

where $\mathbf{k} = \sqrt{-\mathbf{k}^\mu \mathbf{k}_\mu}$ is the modulus of the timelike Killing vector field $\mathbf{k}^\mu$. In turn, the hydrostatic effective action, using eq. (4.12) and eq. (2.15), reads

$$S_{\rm hs}[g_{\mu\nu}, b_{\mu\nu}; \varphi_\mu] = \int d^4x \sqrt{-g}\, P(T, \zeta^2, \bar{\zeta}^2, \zeta\cdot\bar{\zeta}) \ . \tag{4.24}$$

Using (2.13) we can readily obtain the currents (4.14). Additionally, by varying the effective action with respect to φ_μ (see eq. (2.14)) yields the equation of motion for equilibrium configurations of φ_μ, specifically

$$\beta^\nu \nabla_\mu (qT\zeta^\mu) = -\epsilon^{\nu\mu\rho\sigma}\nabla_\mu\left(\bar{q}u_\rho\bar{\zeta}_\sigma\right) - \frac{1}{2}\epsilon^{\nu\mu\rho\sigma}\xi_{\rho\sigma}\partial_\mu q_\times - \frac{1}{6}q_\times\epsilon^{\nu\mu\rho\sigma}H_{\mu\rho\sigma} \ , \tag{4.25}$$

where the reader may be reminded of the defining relation in eq. (4.4) which leads to $\zeta_\mu = T\partial_\mu(\beta^\nu\varphi_\nu) - \mu_\mu$. Similarly at one-derivative order, the hydrostatic effective action obtains corrections due to eq. (4.20)

$$\begin{aligned} S_{\rm hs}[g_{\mu\nu}, b_{\mu\nu}; \varphi_\mu] = \int d^4x\sqrt{-g}\Big[& P + f^a_1\, h^\mu_a \partial_\mu T + f^a_2\, \epsilon^{\mu\nu\rho\sigma} u_\mu h_{a\nu}\partial_\rho u_\sigma + f^a_3\, \epsilon^{\mu\nu\rho\sigma} u_\mu h_{a\nu}\partial_\rho\zeta_\sigma \\ & + f^a_4\, \epsilon^{\mu\nu\rho\sigma} u_\mu h_{a\nu}\partial_\rho\bar{\zeta}_\sigma + 2f^{ab}_5\, h^\mu_a h^\nu_b \nabla_{(\mu}\zeta_{\nu)} + 2f^{ab}_6\, h^\mu_a h^\nu_b \nabla_{(\mu}\bar{\zeta}_{\nu)}\Big] \ , \end{aligned} \tag{4.26}$$

which can be used to derive the hydrostatic constitutive relations and φ_μ profiles.

It should be noted that no assumptions were made on the background metric and gauge fields other than the existence of a timelike isometry. As shall be explained in section 5, upon taking appropriate limits, this action describes all equilibrium configurations in string fluids, which includes those of [21, 22, 35] as special cases.

4.2 Special limits of one-form superfluids

An effective theory with 166 arbitrary transport coefficients (and 12 inequalities) at first order in derivatives is perhaps not the most useful effective theory. However, it is possible to identify limits of this general theory with a tractable number of transport coefficients and interesting applications, which are now described:

- *Electromagnetism.* The simplest example encompassed by this general theory is that of electromagnetic fields living alongside a neutral ideal fluid. By simply turning off the coupling between electromagnetic and fluid degrees of freedom, and setting $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} = \xi_{\mu\nu}$, the gauge field φ_μ is directly identified with the electromagnetic

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photon A_μ. The identification $F_{\mu\nu} = (\star J)_{\mu\nu}$ yields the same theory, with the two being related by electromagnetic duality. This case will be described in more detail in section 4.4.

- *String fluid limit.* An interesting limit of one-form superfluids, which will be studied in section 5, is the limit in which ζ^μ enters the constitutive relations while $\bar{\zeta}^\mu$ is simply removed from the theory. As shall be explained in section 5, this limit can be understood as a partial breaking of the one-form symmetry along β^μ in which only the timelike component of the Goldstone mode $\varphi = \beta^\mu \varphi_\mu$ (in terms of which ζ_μ is defined — see eq. (4.4) and eq. (4.9)) enters the constitutive relations. This limit, which will be shown to be exactly equivalent to magnetohydrodynamics in which the electric fields are Debye screened, is characterised by 23 independent transport coefficients.

- *Electric limit.* The electric limit is attained by considering the hierarchy of scales $\zeta^\mu = \mathcal{O}(1)$ and $\bar{\zeta}^\mu = \mathcal{O}(\partial)$, implying that electric fields ζ^μ can be arbitrary but magnetic fields $\bar{\zeta}^\mu$ are weak. In this context, the one-form symmetry is completely broken. This limit is equivalent to the hydrodynamics of magnetically dominated bound-charge plasmas, i.e. plasmas that do not contain free charge carriers and have electric fields derivative suppressed. We will return to this in detail in section 6.

In general, the formalism of one-form superfluids finds applications in many phases of (hot) electromagnetism. A more detailed description and derivation of these connections is given in section 7.

4.3 Self-duality of one-form superfluids

To summarise, the theory of one-form superfluid dynamics developed in the previous sections is governed by the following set of equations[12]

$$\begin{aligned}
\text{Energy-momentum conservation}: \quad & \nabla_\mu T^{\mu\nu} = \frac{3}{2}\nabla^{[\nu}\xi^{\rho\sigma]}J_{\rho\sigma} - \xi^{\nu\rho}\nabla^\sigma J_{\sigma\rho} \ , \\
\varphi_\mu \text{ equation of motion}: \quad & \nabla_\mu J^{\mu\nu} = 0 \ , \\
\varphi_\mu \text{ Bianchi identity}: \quad & \nabla_\mu \star\xi^{\mu\nu} = \star H^\nu \ , \\
\text{Second law of thermodynamics}: \quad & \nabla_\mu N^\mu = \frac{1}{2}T^{\mu\nu}\delta_{\mathcal{B}} g_{\mu\nu} + \frac{1}{2}J^{\mu\nu}\delta_{\mathcal{B}}\xi_{\mu\nu} + \Delta \ , \quad \Delta \geq 0 \ ,
\end{aligned} \tag{4.27}$$

where due to $\delta_{\mathcal{B}}\varphi_\mu = 0$, the following identity holds $\delta_{\mathcal{B}}\xi_{\mu\nu} = \delta_{\mathcal{B}} b_{\mu\nu}$.[13] When the background field strength $H_{\mu\nu\rho}$ vanishes, it is possible to check that under the mapping

$$\begin{aligned}
J^{\mu\nu} \to J^{\mu\nu}_* = \star\xi^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\xi_{\rho\sigma} \ , \qquad & \xi_{\mu\nu} \to \xi^*_{\mu\nu} = \star J_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\rho\sigma} \ , \\
N^\mu \to N^\mu_* = N^\mu - \frac{1}{2}\beta^\mu J^{\rho\sigma}\xi_{\rho\sigma} \ , &
\end{aligned} \tag{4.28}$$

[12] The Hodge duality operation is defined as $(\star\omega)^{\mu_1\cdots\mu_{d-k}} = \frac{1}{k!}\epsilon^{\nu_1\cdots\nu_k\mu_1\cdots\mu_{d-k}}\omega_{\nu_1\cdots\nu_k}$.

[13] In the first equation in (4.27), the term involving the charge current divergence has been included in order to make the self-duality manifest. Onshell, this equation is identical to eq. (2.6) upon using the Bianchi identity and one-form conservation equation.

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these equations map to themselves.[14] This is the self-duality of one-form superfluid dynamics. The operation (4.28) can be seen as a Legendre transform in the one-form sector, so that $J^{\mu\nu}$ become background sources while $\xi_{\mu\nu}$ are seen as the respective responses. What used to be the φ_μ equation of motion, in the Legendre transformed picture becomes the Bianchi identity for some auxiliary gauge field φ^*_μ such that $\xi^*_{\mu\nu} = 2\partial_{[\mu}\varphi^*_{\nu]}$. The equation of motion for φ^*_μ is given by what previously used to be the Bianchi identity. It is interesting to note that even though the free-energy current is Legendre transformed according to (4.28), the physical entropy current in the two pictures is exactly the same, namely

$$S^\mu = N^\mu - T^{\mu\nu}\beta_\nu - J^{\mu\nu}\beta^\rho\xi_{\rho\nu} = N^\mu_* - T^{\mu\nu}\beta_\nu - J^{\mu\nu}_*\beta^\rho\xi^*_{\rho\nu} \ . \tag{4.29}$$

Thus, irrespective of the formulation being used, entropy production remains the same. Additionally, it also ensures that if the constitutive relations in one formalism are tuned in order to satisfy the second law of thermodynamics, then the coefficients in the Legendre transformed picture automatically respect the second law.

The realisation of the self-duality of one-form superfluids has been phrased in abstract terms by means of (4.28). In practice, however, the exact map between transport coefficients in both pictures can be non-trivial. In order to illustrate this, we apply the map (4.28) to one-form superfluids at ideal order in section 4.1.2. The two-form superfluid velocity in the Legendre transformed picture is given by

$$\xi^*_{\mu\nu} = \star J_{\mu\nu} = 2u_{[\mu}\left(\bar{q}\,\bar{\zeta}_{\nu]} + q_\times\zeta_{\nu]}\right) - \epsilon_{\mu\nu\rho\sigma}u^\rho\left(q\zeta^\sigma + q_\times\bar{\zeta}^\sigma\right) \ . \tag{4.30}$$

Comparison with eq. (4.10), where $(\zeta^\mu, \bar{\zeta}^\mu)$ have been replaced by their corresponding Legendre transform vectors $(\zeta^\mu_*, \bar{\zeta}^\mu_*)$ that ought to be determined, it is possible to infer that

$$\begin{aligned} \zeta^*_\mu &= \bar{q}\,\bar{\zeta}_\mu + q_\times\zeta_\mu \ , & \bar{\zeta}^*_\mu &= q\,\zeta_\mu + q_\times\bar{\zeta}_\mu \ , \\ \zeta_\mu &= \frac{\bar{q}}{q\bar{q}-q_\times^2}\bar{\zeta}^*_\mu - \frac{q_\times}{q\bar{q}-q_\times^2}\zeta^*_\mu \ , & \bar{\zeta}_\mu &= \frac{q}{q\bar{q}-q_\times^2}\zeta^*_\mu - \frac{q_\times}{q\bar{q}-q_\times^2}\bar{\zeta}^*_\mu \ . \end{aligned} \tag{4.31}$$

Using these and comparing with (4.14), it is possible to find the respective two-form current via the relation

$$J^{\mu\nu}_* = \star\xi^{\mu\nu} = -2u^{[\mu}\left(q_*\,\zeta^{\nu]}_* + q_{\times *}\,\bar{\zeta}^{\nu]}_*\right) - \epsilon^{\mu\nu\rho\sigma}u_\rho\left(\bar{q}_*\,\bar{\zeta}^*_\sigma + q_{\times *}\,\zeta^*_\sigma\right) \ , \tag{4.32}$$

where the Legendre transformed transport coefficients were identified according to

$$\begin{gathered} q_* = -\frac{q}{q\bar{q}-q_\times^2} \ , \quad \bar{q}_* = -\frac{\bar{q}}{q\bar{q}-q_\times^2} \ , \quad q_{\times *} = \frac{q_\times}{q\bar{q}-q_\times^2} \ , \quad \epsilon_* = \epsilon, \qquad p_* = p \ , \\ P_* = P - q\zeta^2 - \bar{q}\bar{\zeta}^2 - 2q_\times(\zeta\cdot\bar{\zeta}) \ . \end{gathered} \tag{4.33}$$

This identification brings the Legendre transformed stress tensor and charge current to the same form as in eq. (4.14) but with transport coefficients and $(\zeta^\mu, \bar{\zeta}^\mu)$ replaced by their Legendre transformed counterparts. It is worth noticing that the transformation (4.33) is not defined if $q\bar{q} - q_\times^2 = 0$.

[14] In d spacetime dimensions, a similar Legendre transform is expected to map a q-form superfluid to a $(d-q-2)$-form superfluid.

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4.4 Application to hot electromagnetism

As a simple application of one-form superfluid dynamics, consider a neutral fluid subjected to dynamical electromagnetic fields. This is the simplest example of a hot electromagnetic plasma, which we consider in detail in section 7, where the electromagnetic fields are completely decoupled from the fluid degrees of freedom. The dynamics of this system is governed by the energy-momentum conservation and the familiar Maxwell's equations

$$\nabla_\mu T^{\mu\nu} = 0 \ , \quad \nabla_\mu F^{\mu\nu} = 0 \ , \quad \nabla_{[\rho} F_{\mu\nu]} = 0 \ . \tag{4.34}$$

The third equation (Bianchi identity) in (4.34) is solved by introducing the photon A_μ such that $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. The energy-momentum tensor of this theory receives contributions from both the fluid component as well as the electromagnetic fields

$$\begin{aligned} T^{\mu\nu} &= \epsilon_{\rm m}(T) u^\mu u^\nu + p_{\rm m}(T)(g^{\mu\nu} + u^\mu u^\nu) + F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \\ &= \epsilon_{\rm m}(T) u^\mu u^\nu + p_{\rm m}(T)(g^{\mu\nu} + u^\mu u^\nu) \\ &\quad + \frac{1}{2}(E^2 + B^2) u^\mu u^\nu + \frac{1}{2}(E^2 + B^2) P^{\mu\nu} - E^\mu E^\nu - \left(B^\mu B^\nu + 2 u^{(\mu} \epsilon^{\nu)\rho\sigma\tau} u_\rho B_\sigma E_\tau \right) , \end{aligned} \tag{4.35}$$

where we have defined the electric fields $E^\mu = F^{\mu\nu} u_\nu$ and magnetic fields $B^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma}$. The electromagnetic part trivially satisfies the conservation equations for the photon configurations that satisfy the Maxwell's equations (second equation in (4.34)), while the conservation of the fluid part governs the dynamics of u^μ and T.

This setup can be equivalently described by ideal one-form superfluid dynamics. To this aim, we perform the identification $F_{\mu\nu} = \xi_{\mu\nu}$, which implies

$$E^\mu = \zeta^\mu \ , \quad B^\mu = \bar{\zeta}^\mu \ . \tag{4.36}$$

Comparing the energy-momentum tensor in eq. (4.35) with eq. (4.14) we can read out that

$$P = p_{\rm m}(T) + \frac{1}{2}(\zeta^2 - \bar{\zeta}^2) \ , \quad q = -\bar{q} = 1 \ , \quad q_\times = 0 \ , \quad \epsilon = \epsilon_{\rm m}(T) + \frac{1}{2}\left(\zeta^2 + \bar{\zeta}^2\right) \ . \tag{4.37}$$

It follows that the two-form current $J^{\mu\nu} = -\xi^{\mu\nu} = -F^{\mu\nu}$. Having made the identification, the equations of one-form superfluid dynamics in (4.27) with $H_{\mu\nu\rho} = 0$ map directly to (4.34). The respective energy-momentum tensors and Bianchi identities map to each other, while the equation of motion for φ_μ is equivalent to Maxwell's equations. Therefore, the one-form Goldstone φ_μ can be identified with the photon A_μ. The associated hydrostatic free-energy density for this one-form superfluid is given by

$$\mathcal{N} = P = p_{\rm m}(T) - \frac{1}{4} \xi_{\mu\nu} \xi^{\mu\nu} = p_{\rm m}(T) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ . \tag{4.38}$$

This is precisely the Lagrangian density for electromagnetism minimally coupled to a neutral fluid, where the vacuum permeability has been set to unity.

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Due to the self-duality of one-form superfluids (4.30) at $H_{\mu\nu\rho} = 0$, we can also make the identification $F_{\mu\nu} = \xi^*_{\mu\nu} = \star J_{\mu\nu}$. The electric and magnetic fields now reverse their roles

$$E^\mu = -\bar{\zeta}^\mu \ , \quad B^\mu = \zeta^\mu \ , \tag{4.39}$$

while the mapping for transport coefficients remains the same (see eq. (4.33)). This dual description is essentially the consequence of electromagnetic duality of vacuum Maxwell's equations under $E^\mu \to B^\mu$ and $B^\mu \to -E^\mu$. In this case, the Bianchi identity in (4.27) maps to Maxwell equations in (4.34), while the equation of motion for φ_μ maps to the electromagnetic Bianchi identity. In this picture the vector Goldstone φ_μ can be understood as an auxiliary "magnetic photon". The energy momentum tensor (4.35) and the Lagrangian density (4.38), when defined with respect to the Legendre transformed P_* in eq. (4.33), remain invariant.

The relations between one-form superfluids at finite temperature and hot electromagnetism will be considered in more generality in section 7. In any case, the relations established here should provide confidence to the reader that one-form superfluids can be used to construct effective theories where the electromagnetic degrees of freedom interact with the mechanical and thermal degrees of freedom of relativistic matter.

5 String fluids

In this section a theory of parity-violating string fluids is formulated up to first order in derivatives, extending and completing earlier formulations [3, 20, 21]. This theory can be formulated by partially breaking the one-form symmetry along the fluid velocity β^μ, yielding a scalar Goldstone mode φ, or by a direct limit of one-form superfluids as discussed in section 4.2. Both these directions will be described in this section. String fluids provide a dual formulation of MHD that is cast only in terms of symmetries, eliminating the need of introducing the non-propagating degrees of freedom μ (chemical potential) and E^μ (electric fields) in traditional treatments of MHD [24]. The exact relation between the two formulations will be described in detail in section 7.

5.1 Partial breaking of one-form symmetry

String fluids can be obtained directly from one-form fluids discussed in section 3 where the one-form symmetry is spontaneously broken in the direction of the fluid flow. In practice, it implies that the theory admits a scalar Goldstone φ in the hydrodynamic regime along with the usual hydrodynamic fields u^μ, T, and μ_μ introduced in section 2. Under a symmetry transformation $\mathfrak{X}$, φ transforms as

$$\delta_\mathfrak{X}\varphi = \pounds_\chi\varphi - \beta^\mu\Lambda^\chi_\mu \ . \tag{5.1}$$

This new mode allows for the introduction of a new gauge-invariant vector combination ϖh^μ that captures the covariant derivatives of φ, namely

$$\varpi h_\mu = \mu_\mu - T\partial_\mu\varphi \ . \tag{5.2}$$

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Here $h^\mu h_\mu = 1$ and we have isolated the norm ϖ of the vector. It can be verified that, at this stage, u^μ and h^μ are not necessarily orthogonal, instead their inner product satisfies $u^\mu h_\mu = -T^2/\varpi\delta_{\mathcal{B}}\varphi$. The hydrodynamic systems built using these degrees of freedom are referred to as *string fluids*. In particular, the vector h_μ characterises the direction of the strings while ϖ is interpreted as a string chemical potential.

Following a similar procedure as in section 4.1 we can determine the Josephson equation for string fluids. The Goldstone mode φ is accompanied by its equation of motion $K = 0$, which can be used to write down the offshell adiabaticity equation (2.19) in the form

$$\nabla_\mu N^\mu = \frac{1}{2}T^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu} + \frac{1}{2}J^{\mu\nu}\delta_{\mathcal{B}}b_{\mu\nu} + K\delta_{\mathcal{B}}\varphi + \Delta \ , \quad \Delta \geq 0 \ . \tag{5.3}$$

Using the available hydrodynamic data, at ideal order this equation becomes $-K\delta_{\mathcal{B}}\varphi = \Delta \geq 0$, implying that

$$K = -\alpha\delta_{\mathcal{B}}\varphi + \mathcal{O}(\partial) \ , \quad \Delta = \alpha(\delta_{\mathcal{B}}\varphi)^2 + \mathcal{O}(\partial) \ , \quad \alpha \geq 0 \ , \tag{5.4}$$

where α is some transport coefficient. Imposing the φ equation of motion $K = 0$, it follows that $\delta_{\mathcal{B}}\varphi = \mathcal{O}(\partial)$, which in turn implies the Josephson equation for string fluids $u^\mu h_\mu = \mathcal{O}(\partial)$. Analogous to section 4.1, it is possible to use the redefinition freedom associated with μ_μ to absorb potential derivative corrections and to turn it into the exact statement

$$\delta_{\mathcal{B}}\varphi = 0 \quad \Longrightarrow \quad u^\mu h_\mu = 0 \ . \tag{5.5}$$

Thus, the string direction can generically be chosen to be transverse to the fluid flow. Therefore, the independent dynamical fields in string fluids, just like the previous considerations of [21], are u^μ, T, ϖ, and h_μ with $u^\mu u_\mu = -1$, $h^\mu h_\mu = 1$, and $u^\mu h_\mu = 0$. The dynamics for u^μ and T is governed by the energy-momentum conservation in eq. (2.6), while that for ϖ and h_μ by the components of the one-form conservation transverse to u^μ. The component of the one-form conservation along u^μ, on the other hand, acts as a constraint on the allowed field configurations on an initial Cauchy slice. In our picture, this constraint is seen as determining the configurations of the scalar Goldstone φ.[15] Additionally, once eq. (5.5) is imposed, the adiabaticity equation (2.19) reduces to (2.22).

5.2 Ideal string fluids

At ideal order, string fluids are characterised by the free energy density $\mathcal{N} = p(T, \varpi)$. The $\delta_{\mathcal{B}}$ variations of T and ϖ read

$$\delta_{\mathcal{B}}T = \frac{T}{2}u^\mu u^\nu \delta_{\mathcal{B}}g_{\mu\nu} \ , \quad \delta_{\mathcal{B}}\varpi = \frac{\varpi}{2}\left(u^\mu u^\nu - h^\mu h^\nu\right)\delta_{\mathcal{B}}g_{\mu\nu} + u^{[\mu}h^{\nu]}\delta_{\mathcal{B}}\xi_{\mu\nu} \ , \tag{5.6}$$

[15] To see this, note that there are two sources of $b_{\mu\nu}$ dependence in string fluids: $H_{\mu\nu\rho}$ and ϖh_μ. Therefore, for theories admitting an effective action, we can infer that $J^{\mu\nu} = -3\nabla_\lambda(\delta S/\delta H_{\lambda\mu\nu}) + 2u^{[\mu}(\delta S/\delta(\varpi h_{\nu]}))$. On the other hand, all the φ dependence comes from ϖh_μ leading to $K = \nabla_\mu(T\delta S/\delta(\varpi h_\mu)) = (\delta S/\delta(\varpi h_\mu))T^2 u^\nu \delta_{\mathcal{B}}g_{\mu\nu} + Tu_\nu\nabla_\mu J^{\mu\nu}$. We have used that $u^\mu(\delta S/\delta(\varpi h_\mu)) = 0$. Therefore, after the time component of μ_μ has been algebraically eliminated using the φ equation of motion, the time component of the one-form conservation equation serves as the equation of motion for φ.

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and can be used, together with eq. (2.25), to derive the respective constitutive relations. Specifically, these read

$$\begin{aligned} T^{\mu\nu} &= (\epsilon + p)\, u^\mu u^\nu + p\, g^{\mu\nu} - \varpi\rho\, h^\mu h^\nu + \mathcal{O}(\partial) \ , \\ J^{\mu\nu} &= 2\rho\, u^{[\mu} h^{\nu]} + \mathcal{O}(\partial) \ , \end{aligned} \tag{5.7}$$

where the thermodynamic relations

$$\mathrm{d}p = s\, \mathrm{d}T + \rho\, \mathrm{d}\varpi \ , \quad \epsilon + p = s\, T + \rho\, \varpi \ , \tag{5.8}$$

were defined and led to the identification of p as pressure, ϵ as energy density, ρ as string density and s as entropy density. The associated free energy and entropy currents are given as

$$N^\mu = \frac{p}{T} u^\mu \ , \quad S^\mu = s\, u^\mu \ . \tag{5.9}$$

Since Δ at ideal order vanishes, ideal string fluids are non-dissipative.

It is instructive to work out the ideal order equations of motion governing the dynamics of the string fluid hydrodynamic fields. In particular, the components of the energy-momentum conservation imply

$$\nabla_\mu T^{\mu\nu} = \frac{1}{2} H^{\nu\rho\sigma} J_{\rho\sigma} + 2\zeta^{[\nu} u^{\rho]} \nabla^\mu J_{\mu\rho}$$

$$\implies \delta_{\mathcal{B}} s + \frac{s}{2} P^{\mu\nu} \delta_{\mathcal{B}} g_{\mu\nu} = \mathcal{O}(\partial^2) \ , \tag{5.10a}$$

$$u^\mu h^\nu \delta_{\mathcal{B}} g_{\mu\nu} = \mathcal{O}(\partial^2) \ , \tag{5.10b}$$

$$(\epsilon + p) u^\mu \Delta^{\rho\nu} \delta_{\mathcal{B}} g_{\mu\nu} - \rho h^\mu \Delta^{\rho\nu} \delta_{\mathcal{B}} b_{\mu\nu} = \mathcal{O}(\partial^2) \ , \tag{5.10c}$$

while those of the one-form current conservation reduce to

$$\nabla_\mu J^{\mu\nu} = 0 \implies \delta_{\mathcal{B}} \rho + \frac{\rho}{2} \Delta^{\mu\nu} \delta_{\mathcal{B}} g_{\mu\nu} = \mathcal{O}(\partial^2) \ , \tag{5.11a}$$

$$\Delta^{\rho\mu} u^\nu \delta_{\mathcal{B}} b_{\mu\nu} + \varpi \Delta^{\rho\mu} h^\nu \delta_{\mathcal{B}} g_{\mu\nu} = \mathcal{O}(\partial^2) \ , \tag{5.11b}$$

$$\frac{1}{T} \nabla_\mu \left(T \rho h^\mu \right) - \rho T u^\mu h^\nu \delta_{\mathcal{B}} g_{\mu\nu} = \mathcal{O}(\partial^2) \ . \tag{5.11c}$$

Here $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu - h^\mu h^\nu$ and

$$\delta_{\mathcal{B}} g_{\mu\nu} = 2\nabla_{(\mu} \left(\frac{u_{\nu)}}{T} \right) \ , \quad \delta_{\mathcal{B}} b_{\mu\nu} = 2\partial_{[\mu} \left(\frac{\varpi h_{\nu]}}{T} \right) + \frac{u^\sigma}{T} H_{\sigma\mu\nu} \ , \tag{5.12}$$

were used to simplify the expressions. Eqs. (5.10a) to (5.10c), (5.11a)–(5.11b) can be used to eliminate $u^\mu \delta_{\mathcal{B}} g_{\mu\nu}$ and $u^\mu \delta_{\mathcal{B}} b_{\mu\nu}$ from the set of independent first order non-hydrostatic tensors. On the other hand, eq. (5.11c), upon using eq. (5.10b), gives a constraint equation for φ configurations on an initial Cauchy slice

$$\nabla_\mu \left(T \rho h^\mu \right) = 0 \ , \tag{5.13}$$

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which is the no-monopole constraint of [27]. Additionally, the second equation in (5.11) is the induction equation of [27].

As already explained in [24], the introduction of φ in the formulation of string fluid dynamics allows for a well-defined hydrostatic effective action (2.15), where $N^{\mu}_{\rm hs} = (p/T)u^{\mu}$ and from which (5.13) arises as the variation with respect to φ (see eq. (2.14)).

5.2.1 Strings fluids as a limit of one-form superfluids

As mentioned in section 4.2, string fluids as described above can be obtained as a limit of one-form superfluids introduced in section 4.1. This limit is obtained by removing any dependence on $\bar{\zeta}^{\mu}$ from the one-form superfluid theory, in which case the Bianchi identity (4.3) looses its meaning. Comparing (5.1) with (4.1), it is straightforwardly inferred that the Goldstone scalar φ is the component of the Goldstone vector φ_{μ} along β_{μ}, i.e. $\varphi = \beta^{\mu}\varphi_{\mu}$. The complete equivalence is made by comparing (5.2) with (4.4) under the light of the Josephson equations (4.9) and (5.5) leading to the identification

$$\zeta_{\mu} = -\varpi h_{\mu} \ , \quad q = \frac{\rho}{\varpi} \ , \tag{5.14}$$

while the conditions $\bar{q} = q_{\times} = 0$ arise due to the removal of any dependence on $\bar{\zeta}$ from the constitutive relations of section 4.1.2, thus recovering (5.7) from eq. (4.14). Additionally, eq. (5.13) can be obtained from the equilibrium equation (4.25) for φ_{μ}.

5.3 One derivative corrections to string fluids

Having established the ideal order constitutive relations, it is possible to continue the hydrodynamic expansion to one higher order. The results will be a sector of the transport coefficients given in section 4.1.3, which also includes parity-violating terms, hence providing an extension of earlier literature [3, 20, 21].

5.3.1 Hydrostatic corrections

Hydrostatic corrections to ideal string fluids are composed of the first order scalars that can appear in the hydrostatic free energy $\mathcal{N}$ and are non-vanishing in equilibrium. At first order, it is possible to identify a total of 5 transport coefficients

$$\mathcal{N} = p - \frac{\alpha}{6}\epsilon^{\mu\nu\rho\sigma}u_{\mu}H_{\nu\rho\sigma} - \beta\epsilon^{\mu\nu\rho\sigma}u_{\mu}h_{\nu}\partial_{\rho}u_{\sigma} - \tilde{\beta}_1 h^{\mu}\partial_{\mu}T - \tilde{\beta}_2 h^{\mu}\partial_{\mu}\frac{\varpi}{T} - \tilde{\beta}_3\epsilon^{\mu\nu\rho\sigma}u_{\mu}h_{\nu}\partial_{\rho}h_{\sigma} \ . \tag{5.15}$$

Since boundary transport is not being considered, total derivative scalars such as $\nabla_{\mu}h^{\mu}$ can be removed from the independent set. Additionally, the equilibrium condition (5.13) allows us to set $\tilde{\beta}_2 = 0$ and hence only 4 scalars are independent. However, allowing for a non-zero $\tilde{\beta}_2$ will ease comparison with earlier literature in section 7. The terms coupling to α and β are CP-even while those coupling to $\tilde{\beta}_1$, $\tilde{\beta}_2$, $\tilde{\beta}_3$ are CP-odd.[16] The distinguished

[16]The discrete parity symmetry P acts on various quantities as usual, while the quantities odd under the one-form charge conjugation C are $b_{\mu\nu}$, $H_{\mu\nu\rho}$, $\xi_{\mu\nu}$, ζ_{μ}, $\bar{\zeta}^{\mu}$, h_{μ}, and $J^{\mu\nu}$. We are interested in CP-odd terms in string fluids because when relating string fluids to magnetohydrodynamics in section 7.2, these correspond to P-odd terms in magnetohydrodynamics. We have deferred a more exhaustive discussion of the action of discrete symmetries, including CPT, to appendix C.

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CP	*Transport coefficients*
CP-even	$\zeta_\perp\,,\ \zeta_\times\,,\ \zeta'_\times\,,\ \zeta_\parallel\,,\ r_\parallel\,,\ \eta_\parallel\,,\ \tilde{\eta}_\parallel\,,\ r_\perp\,,\ \tilde{r}_\perp\,,\ \eta_\perp\,,\ \tilde{\eta}_\perp$
CP-odd	$\tilde{\kappa}_1\,,\ \tilde{\kappa}'_1\,,\ \tilde{\kappa}_2\,,\ \tilde{\kappa}'_2\,,\ r_\times\,,\ r'_\times\,,\ \tilde{r}_\times\,,\ \tilde{r}'_\times$

Table 5. Transformation properties under CP for non-hydrostatic transport coefficients in string fluids.

notation α for the first transport coefficient is due to the fact that it will play a crucial role later in the mapping to magnetohydrodynamics in section 7. Performing the $\delta_{\mathcal{B}}$ variation of all the one-derivative terms in eq. (5.15) and using eq. (2.23), the contributions of each term to the constitutive relations and free energy current can be obtained, and are given in appendix A.

5.3.2 Non-hydrostatic corrections

In order to derive non-hydrostatic constitutive relations, it is useful to decompose the currents in this sector of the theory along and transverse to u^μ and h^μ, such that

$$\begin{aligned} T^{\mu\nu}_{\text{nhs}} &= \delta\epsilon\ u^\mu u^\nu + \delta f \Delta^{\mu\nu} + \delta\tau\ h^\mu h^\nu + 2\ell^{(\mu}h^{\nu)} + 2k^{(\mu}u^{\nu)} + t^{\mu\nu}\ , \\ J^{\mu\nu}_{\text{nhs}} &= 2\delta\rho\ u^{[\mu}h^{\nu]} + 2m^{[\mu}h^{\nu]} + 2n^{[\mu}u^{\nu]} + \delta s\,\epsilon^{\mu\nu}\ , \end{aligned} \tag{5.16}$$

where $\epsilon^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}u_\rho h_\sigma$ is a parity-odd contribution. In particular, any antisymmetric tensor transverse to u^μ and h^ν in 4 spacetime dimensions only has one degree of freedom and is always proportional to $\epsilon^{\mu\nu}$. Choosing to work in the Landau frame following the discussion in section 2.3, and eliminating $u^\mu\delta_{\mathcal{B}}g_{\mu\nu}$ and $u^\mu\delta_{\mathcal{B}}b_{\mu\nu}$ using the first order equations of motion, the non-hydrostatic constitutive relations can be represented as

$$\begin{aligned} \begin{pmatrix}\delta f\\ \delta\tau\\ \delta s\end{pmatrix} &= -\frac{T}{2}\begin{pmatrix}\zeta_\perp & \zeta_\times & \tilde{\kappa}_1\\ \zeta'_\times & \zeta_\parallel & \tilde{\kappa}_2\\ \tilde{\kappa}'_1 & \tilde{\kappa}'_2 & r_\parallel\end{pmatrix}\begin{pmatrix}\Delta^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu}\\ h^\mu h^\nu\delta_{\mathcal{B}}g_{\mu\nu}\\ \epsilon^{\mu\nu}\delta_{\mathcal{B}}\xi_{\mu\nu}\end{pmatrix}\ , \\ \begin{pmatrix}\ell^\mu\\ m^\mu\end{pmatrix} &= -T\begin{pmatrix}\eta_\parallel & r_\times & \tilde{\eta}_\parallel & \tilde{r}_\times\\ r'_\times & r_\perp & \tilde{r}'_\times & \tilde{r}_\perp\end{pmatrix}\begin{pmatrix}\Delta^{\mu\sigma}h^\nu\delta_{\mathcal{B}}g_{\sigma\nu}\\ \Delta^{\mu\sigma}h^\nu\delta_{\mathcal{B}}\xi_{\sigma\nu}\\ \epsilon^{\mu\sigma}h^\nu\delta_{\mathcal{B}}g_{\sigma\nu}\\ \epsilon^{\mu\sigma}h^\nu\delta_{\mathcal{B}}\xi_{\sigma\nu}\end{pmatrix}\ , \\ t^{\mu\nu} &= -\eta_\perp T\Delta^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}\delta_{\mathcal{B}}g_{\rho\sigma} + \tilde{\eta}_\perp T\epsilon^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}\delta_{\mathcal{B}}g_{\rho\sigma}\ . \end{aligned} \tag{5.17}$$

The redefinition freedom in u^μ and T has been used to set $\delta\epsilon = k^\mu = 0$, whereas the residual freedom in μ_μ after setting $u^\mu h_\mu = 0$ is used to set $\delta\rho = n^\mu = 0$. Here we have introduced 19 non-hydrostatic transport coefficients, which are functions of T and ϖ. In table 5, the transformation properties of these coefficients under CP transformations is summarised. Thus, the first 11 coefficients already identified in [3] are in the CP-even sector and the remaining new 8 coefficients (in *blue*) are in the CP-odd sector and had not been previously identified in the literature. Of these 8 coefficients, 4 can be understood as new current resistivities and are related to the remaining 4 via Onsager's relations under certain assumptions as will be explained in section 5.4.

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	CP-even	*CP-odd*
Non-dissipative non-hydrostatic	4	4
Dissipative	7	4

Table 6. Classes of non-hydrostatic transport coefficients in string fluids

Using the adiabaticity equation (2.22) and after some non-trivial algebra, it is possible to derive that

$$\begin{aligned}\frac{1}{T}\Delta = &\frac{1}{4}\begin{pmatrix}\Delta^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu}\\ h^{\mu}h^{\nu}\delta_{\mathcal{B}}g_{\mu\nu}\\ \epsilon^{\mu\nu}\delta_{\mathcal{B}}\xi_{\mu\nu}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\zeta_{\perp} & \frac{1}{2}(\zeta_{\times}+\zeta'_{\times}) & \frac{1}{2}(\tilde{\kappa}_1+\tilde{\kappa}'_1)\\ \frac{1}{2}(\zeta_{\times}+\zeta'_{\times}) & \zeta_{\parallel} & \frac{1}{2}(\tilde{\kappa}_2+\tilde{\kappa}'_2)\\ \frac{1}{2}(\tilde{\kappa}_1+\tilde{\kappa}'_1) & \frac{1}{2}(\tilde{\kappa}_2+\tilde{\kappa}'_2) & r_{\perp}\end{pmatrix}\begin{pmatrix}\Delta^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu}\\ h^{\mu}h^{\nu}\delta_{\mathcal{B}}g_{\mu\nu}\\ \epsilon^{\mu\nu}\delta_{\mathcal{B}}\xi_{\mu\nu}\end{pmatrix}\\ &+\frac{1}{2}\begin{pmatrix}\Delta^{\mu\sigma}h^{\nu}\delta_{\mathcal{B}}g_{\sigma\nu}+i\epsilon^{\mu\sigma}h^{\nu}\delta_{\mathcal{B}}\xi_{\sigma\nu}\\ \Delta^{\mu\sigma}h^{\nu}\delta_{\mathcal{B}}\xi_{\sigma\nu}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\eta_{\parallel} & \frac{1}{2}\left(r_{\times}+r'_{\times}\right)\\ \frac{1}{2}\left(r_{\times}+r'_{\times}\right) & r_{\perp}-i^2\eta_{\parallel}\end{pmatrix}\\ &\qquad\times\begin{pmatrix}\Delta^{\mu\sigma}h^{\nu}\delta_{\mathcal{B}}g_{\sigma\nu}+i\epsilon^{\mu\sigma}h^{\nu}\delta_{\mathcal{B}}\xi_{\sigma\nu}\\ \Delta^{\mu\sigma}h^{\nu}\delta_{\mathcal{B}}\xi_{\sigma\nu}\end{pmatrix}\\ &+\frac{1}{2}\eta_{\perp}\delta_{\mathcal{B}}g_{\mu\nu}\Delta^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}\delta_{\mathcal{B}}g_{\rho\sigma}\ ,\end{aligned}\tag{5.18}$$

where $i=\frac{1}{2\eta_{\parallel}}\left(\tilde{r}_{\times}-\tilde{r}'_{\times}\right)$. Out of the 19 non-hydrostatic transport coefficients, the following 8 linear combinations trivially drop out of the quadratic form

$$\zeta_{\times}-\zeta'_{\times}\ ,\quad \tilde{\kappa}_1-\tilde{\kappa}'_1\ ,\quad \tilde{\kappa}_2-\tilde{\kappa}'_2\ ,\quad r_{\times}-r'_{\times}\ ,\quad \tilde{r}_{\times}+\tilde{r}'_{\times}\ ,\quad \tilde{\eta}_{\parallel}\ ,\quad \tilde{r}_{\perp}\ ,\quad \tilde{\eta}_{\perp}\ ,\tag{5.19}$$

and hence are left totally unconstrained. These combinations can be identified as the non-hydrostatic non-dissipative transport coefficients, as they do not contribute to dissipation. Finally, requiring $\Delta\geq 0$ gives 6 inequality constraints among the remaining 11 dissipative transport coefficients. In terms of matrices of transport coefficients they can be expressed as

$$\begin{pmatrix}\zeta_{\perp} & \frac{1}{2}(\zeta_{\times}+\zeta'_{\times}) & \frac{1}{2}(\tilde{\kappa}_1+\tilde{\kappa}'_1)\\ \frac{1}{2}(\zeta_{\times}+\zeta'_{\times}) & \zeta_{\parallel} & \frac{1}{2}(\tilde{\kappa}_2+\tilde{\kappa}'_2)\\ \frac{1}{2}(\tilde{\kappa}_1+\tilde{\kappa}'_1) & \frac{1}{2}(\tilde{\kappa}_2+\tilde{\kappa}'_2) & r_{\perp}\end{pmatrix}\geq 0\ ,\quad \begin{pmatrix}\eta_{\parallel} & \frac{1}{2}\left(r_{\times}+r'_{\times}\right)\\ \frac{1}{2}\left(r_{\times}+r'_{\times}\right) & r_{\perp}-i^2\eta_{\parallel}\end{pmatrix}\geq 0\ ,\ \eta_{\perp}\geq 0\ ,\tag{5.20}$$

whereby positive semi-definiteness of a matrix is understood as the requirement that all its eigenvalues are non-negative. In total, therefore, the number of non-hydrostatic transport coefficients can be summarised as in table 6. Under certain assumptions, not all of these 19 transport coefficients are independent as it will be shown via Kubo formulae and Onsager's relations.

5.4 Kubo formulae

Using the variational background method of [36] it is possible to derive Kubo formulae for string fluids, which are of particular interest for evaluating transport coefficients in holographic setups. In what follows, the hydrostatic corrections of section 5.3.2 have been

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ignored and only the non-hydrostatic have been taken into account.[17] It is convenient to split the background coordinates x^μ into the set (t, x^i, z) and to consider a simple equilibrium configuration in a flat background spacetime with vanishing $b_{\mu\nu}$ and velocity profile $u^\mu = \delta^\mu{}_t$, $h^\mu = \pm\delta^\mu{}_z$. In order to obtain Kubo formulae, the one-point functions are introduced

$$\mathbb{T}^{\mu\nu} = \sqrt{-g}\,\langle T^{\mu\nu}\rangle \ , \quad \mathbb{J}^{\mu\nu} = \sqrt{-g}\,\langle J^{\mu\nu}\rangle \ , \tag{5.21}$$

and a small time-dependent but homogeneous in space perturbation around the equilibrium state is performed such that $u^\mu \to u^\mu + \delta u^\mu$, $h^\mu \to h^\mu + \delta h^\mu$, $g_{\mu\nu} \to \eta_{\mu\nu} + \delta h_{\mu\nu}$, and $b_{\lambda\sigma} \to \delta b_{\lambda\sigma}$.[18] These perturbations should be understood as small deformations that generically take the form $\delta b_{\lambda\sigma} = A_{\lambda\sigma} e^{-\omega t}$ for some amplitude matrix $A_{\lambda\sigma}$. According to linear response theory, small variations of (5.21) can be written in terms of retarded Green's functions of frequency ω such that

$$\delta\mathbb{T}^{\mu\nu} = \frac{1}{2}G^{\mu\nu,\lambda\rho}_{TT}\delta h_{\lambda\rho} + \frac{1}{2}G^{\mu\nu,\lambda\rho}_{TJ}\delta b_{\lambda\rho}\,, \quad \delta\mathbb{J}^{\mu\nu} = \frac{1}{2}G^{\mu\nu,\lambda\rho}_{JT}\delta h_{\lambda\rho} + \frac{1}{2}G^{\mu\nu,\lambda\rho}_{JJ}\delta b_{\lambda\rho} \ . \tag{5.22}$$

Evaluating (5.22) for the specific initial equilibrium configuration and writing it in components, it is found that

$$\begin{aligned}
\zeta'_\times &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{zz,ii}_{TT} \ , & \zeta_{||} &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{zz,zz}_{TT} \ ,\\
\tilde\kappa_2\mathrm{sign}(h) &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{zz,ij}_{TJ} \ , & & \\
\eta_{||} &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{zi,zi}_{TT} \ , & \tilde\eta_{||}\mathrm{sign}(h) &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{zi,zj}_{TT} \ ,\\
r_\times &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{zi,iz}_{TJ} \ , & & \\
\tilde r_\times\mathrm{sign}(h) &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{zi,jz}_{TJ} \ , & \tilde\kappa'_1\mathrm{sign}(h) &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{ij,kk}_{JT} \ ,\\
\tilde\kappa'_2\mathrm{sign}(h) &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{ij,zz}_{JT} \ , & & \\
r_{||} &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{ij,ij}_{JJ} \ , & r'_\times &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{iz,iz}_{JT} \ ,\\
r_\perp &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{iz,iz}_{JJ} \ , & & \\
\tilde r'_\times\mathrm{sign}(h) &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{iz,jz}_{JT}\ (i\neq j) \ , & \tilde r_\perp\mathrm{sign}(h) &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{iz,jz}_{JJ}\ (i\neq j) \ ,\\
\zeta_\perp + \frac{(d-3)}{2(d-2)}\eta_\perp &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{ii,ii}_{TT} \ , & \zeta_\times &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{ii,zz}_{TT} \ ,\\
\tilde\kappa_1\mathrm{sign}(h) &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{ii,jk}_{TJ} \ , & & \\
\eta_\perp &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{ij,ij}_{TT}\,,\ (i\neq j) \ , & \tilde\eta_\perp\mathrm{sign}(h) &= \lim_{\omega\to0}\frac{1}{\omega}\,\mathrm{Im}\,G^{ij,ii}_{TT}\,,\ (i\neq j) \ .
\end{aligned} \tag{5.23}$$

[17] In the particular holographic setup of [37], the 4 independent hydrostatic transport coefficients of section 5.3.2 vanished.

[18] Explicitly we find $\delta u^t = \frac{1}{2}\delta h_{tt}$, $\delta v^t_i = \delta u^i + \delta h_{ti}$, $\delta h^z = -\frac{1}{2}\delta h_{zz}$ and $\delta\left(\nabla_{(i}u_{j)}\right) = \partial_t\delta h_{ij}/2$.

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If the microscopic theory in question has some sort of discrete symmetry Θ including time-reversal, we can use the Onsager's relations to relate some of the transport coefficients. For operators $O_a = \{T^{ij}, J^{ij}\}$, the Onsager's relations state (see e.g. [6])

$$G_{O_a O_b}(\omega, h) = i_a i_b G_{O_b O_a}(\omega, \Theta h) \ , \tag{5.24}$$

where i_a are eigenvalues of O_a under Θ. For $\Theta = \text{CT}$ (which is just time-reversal in the dual hot electromagnetism picture), $i_T = i_J = 1$ and $\Theta h = -h$; see appendix C for more details. This leads to the following relations among transport coefficients

$$\zeta_\times = \zeta'_\times \ , \quad r_\times = r'_\times \ , \quad \tilde{r}_\times = \tilde{r}'_\times \ , \quad \tilde{\kappa}_1 = -\tilde{\kappa}'_1 \ , \quad \tilde{\kappa}_2 = -\tilde{\kappa}'_2 \ . \tag{5.25}$$

On the other hand, for $\Theta = \text{CPT}$, $i_T = i_J = 1$ and $\Theta h = h$. In this case, the constraints are slightly different and we get

$$\zeta_\times = \zeta'_\times \ , \quad r_\times = r'_\times \ , \quad \tilde{r}_\times = -\tilde{r}'_\times \ , \quad \tilde{\kappa}_1 = \tilde{\kappa}'_1 \ , \quad \tilde{\kappa}_2 = \tilde{\kappa}'_2 \ . \tag{5.26}$$

Thus, within either of these contexts, there are 4 independent hydrostatic transport coefficients and 14 non-hydrostatic transport coefficients. Hence, string fluids are characterised by a total of 18 transport coefficients at first order in derivatives.

While other phenomenological realisations of string fluids are possible, here we consider it in the context in which $J^{\mu\nu}$ has a positive eigenvalue under time-reversal symmetry and h^μ has a negative eigenvalue. These considerations are motivated by the mapping of string fluids to MHD as will be discussed in section 7. In this context, Onsager's relations for the operators require that

6 Electric limit of one-form superfluids

This section explores the electric limit of one-form superfluids discussed in section 4.2. This limit is characterised by the derivative hierarchy $\zeta^\mu = \mathcal{O}(1)$ and $\bar{\zeta}^\mu = \mathcal{O}(\partial)$, in which case, contrary to the previous section, the Bianchi identity (4.3) plays a relevant role. A discussion on the Bianchi identity and its consequences allows the determination of the relevant hydrodynamic structures. This is followed by the derivation of the first order corrections in the electric limit, yielding a total of 29 transport coefficients (modulo Onsager's relations). As shall be established in section 7.4, this limit provides a dual formulation of magnetic-dominated bound-charge plasmas, which under particular assumptions are directly related to MHD without free charges.

6.1 Bianchi identity and order mixing

The electric limit of one-form superfluids is defined as the regime where the ζ_μ components of $\xi_{\mu\nu}$ are treated at ideal order, while the components $\bar{\zeta}^\mu$ are treated at one-derivative order. Naively, this may appear to be qualitatively similar to string fluids where ζ_μ was treated at ideal order while $\bar{\zeta}^\mu$ was entirely removed from the hydrodynamic description.

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However, there is an important distinction. In particular, note that the Bianchi identities (4.3) relate certain derivatives of ζ_μ to those of $\bar{\zeta}^\mu$. In components

$$\begin{aligned}
&\epsilon^{\mu\nu\rho\sigma} u_\mu \zeta_\nu \partial_\rho u_\sigma - \frac{1}{6}\epsilon^{\mu\nu\rho\sigma} u_\mu H_{\nu\rho\sigma} \\
&\qquad\qquad = \nabla_\mu \bar{\zeta}^\mu - \bar{\zeta}^\mu u^\nu \nabla_\nu u_\mu \ , \\
&\epsilon^{\mu\nu\rho\sigma} u_\nu \zeta_\rho \left(\frac{1}{T}\partial_\sigma T + u^\lambda \nabla_\lambda u_\sigma\right) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu \left(-2T\partial_\rho \frac{\zeta_\sigma}{T} + u^\lambda H_{\lambda\rho\sigma}\right) \\
&\qquad\qquad = \bar{\zeta}^\mu \nabla_\nu u^\nu + T P^\mu{}_\lambda \left(\beta^\nu \nabla_\nu \bar{\zeta}^\lambda - \bar{\zeta}^\nu \nabla_\nu \beta^\lambda\right) \ .
\end{aligned} \tag{6.1}$$

In string fluids, where $\bar{\zeta}^\mu$ is not a dynamical field, these equations are irrelevant. On the other hand, in the electric limit, these equations become important. Upon setting $\bar{\zeta}^\mu = \mathcal{O}(\partial)$, these read

$$\begin{aligned}
\frac{1}{6}\epsilon^{\mu\nu\rho\sigma} u_\mu H_{\nu\rho\sigma} &= \epsilon^{\mu\nu\rho\sigma} u_\mu \zeta_\nu \partial_\rho u_\sigma + \mathcal{O}(\partial^2), \\
P^{\mu\rho} P^{\nu\sigma} \delta_{\mathcal{B}} b_{\rho\sigma} &= -2\zeta^{[\mu} P^{\nu]\sigma} u^\rho \delta_{\mathcal{B}} g_{\rho\sigma} + \mathcal{O}(\partial^2) \ .
\end{aligned} \tag{6.2}$$

Therefore, the first order terms appearing on the left hand side, which used to be independent in string fluids, are no longer independent in the electric limit. This has an important consequence which is referred here as "order mixing" between consecutive derivative orders in the electric fluid constitutive relations. Noting that $\delta_{\mathcal{B}} b_{\mu\nu} = \delta_{\mathcal{B}} \xi_{\mu\nu}$, it is possible to massage the adiabaticity equation (2.22) into

$$\begin{aligned}
\nabla_\mu N^\mu = &\frac{1}{2}\left(T^{\mu\nu} + 2u^{(\mu} P^{\nu)\rho} \zeta^\sigma J_{\rho\sigma}\right) \delta_{\mathcal{B}} g_{\mu\nu} - J^{\rho\sigma} u_\rho P^{[\nu}{}_\sigma u^{\mu]} \delta_{\mathcal{B}} \xi_{\mu\nu} \\
&+ \frac{1}{2} J^{\tau\lambda} P_{\tau\rho} P_{\lambda\sigma} \left(P^{\rho\mu} P^{\sigma\nu} \delta_{\mathcal{B}} \xi_{\mu\nu} + 2\zeta^{[\rho} P^{\sigma]\nu} u^\mu \delta_{\mathcal{B}} g_{\mu\nu}\right) + \Delta \ .
\end{aligned} \tag{6.3}$$

This equation implies, in general, the appearance of k-derivative order terms in $T^{\mu\nu}$ and $J^{\mu\nu}$ if N^μ was being studied at k-derivative order. Since the term in the parentheses in the second line in (6.3) is two-derivative order, we could also have a $(k-1)$-derivative contribution to $J^{\mu\nu}$. Furthermore, since $\delta_{\mathcal{B}} g_{\mu\nu}$ is one-derivative order, terms in the parentheses in the first line in (6.3) must be k-derivative order, leading to certain $(k-1)$ derivative contributions in $T^{\mu\nu}$ as well. In turn, this could lead to the same transport coefficient appearing across consecutive derivative orders.

In the hydrostatic sector, such order-mixing only comes from the terms in $\mathcal{N}$ dependent on $\bar{\zeta}^\mu$. Generically, if attention is being focused on the kth order terms in $\mathcal{N}$ and define

$$\mathcal{R}^{(k-1)}_\mu = \frac{\delta \mathcal{N}_{(k)}}{\delta \bar{\zeta}^\mu} \ , \tag{6.4}$$

such order-mixing contributions are given by

$$T^{\mu\nu}_{(k-1)} \sim -2u^{(\mu}\epsilon^{\nu)\rho\sigma\tau} u_\rho \zeta_\sigma \mathcal{R}^{(k-1)}_\tau \ , \quad J^{\mu\nu}_{(k-1)} \sim -\epsilon^{\mu\nu\rho\sigma} u_\rho \mathcal{R}^{(k-1)}_\sigma \ . \tag{6.5}$$

In the non-hydrostatic sector, on the other hand, no independent transport coefficient appear across derivative orders. However, whereas the inequality constraints imposed by $\Delta \geq 0$ usually only apply to one-derivative dissipative transport coefficients, in this case they can also involve transport coefficients from two-derivative order. This will be made explicit below.

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6.2 Ideal one-form superfluids in the electric limit

Defining $\zeta_\mu = -\varpi h_\mu$ for later convenience and suppressing $\bar{\zeta}^\mu$ to one-derivative order, the ideal one-form superfluid constitutive relations (4.14) become

$$\begin{aligned} T^{\mu\nu} &= \epsilon\, u^\mu u^\nu + p\, P^{\mu\nu} - \rho\varpi\, h^\mu h^\nu + \mathcal{O}(\partial) \ , \\ J^{\mu\nu} &= 2\rho\, u^{[\mu} h^{\nu]} + \varpi q_\times\, \epsilon^{\mu\nu\rho\sigma} u_\rho h_\sigma + \mathcal{O}(\partial) \ , \\ N^\mu &= \frac{p}{T} u^\mu + \mathcal{O}(\partial) \ , \end{aligned} \tag{6.6}$$

where $\rho = q\varpi$ was defined. All the coefficients appearing here are now seen as functions of T and ϖ. Except for the $q_\times$ term highlighted in blue, the constitutive relations of an ideal one-form superfluid in the electric limit are precisely the same as for string fluids given in eq. (5.7) and satisfy the thermodynamic relations (5.8).

The $q_\times$ term, on the other hand, is a manifestation of the order-mixing that was alluded to above. Comparing its form with eq. (6.5), it is possible to infer that it originates from a one-derivative term $q_\times \zeta_\mu \bar{\zeta}^\mu$ in the free energy density. This is, in fact, the case as it can be verified by expanding the ideal one-form superfluid free-energy density (4.12) up to one derivative order, obtaining

$$P(T, \zeta^2, \bar{\zeta}^2, \zeta\cdot\bar{\zeta}) = p(T,\varpi) + q_\times(T,\varpi)\, \zeta\cdot\bar{\zeta} + \mathcal{O}(\partial^2) \ . \tag{6.7}$$

Additionally, due to the presence of the $q_\times$ term, the first order equations of motion significantly modify compared to string fluids. The components of the energy-momentum conservation stay the same as in eq. (5.10), while those of the one-form conservation receive contributions from the $q_\times$ term. Precisely, it is found

$$\nabla_\mu J^{\mu\nu} = 0$$

$$\implies \delta_{\mathcal{B}}\rho + \frac{\rho}{2}\Delta^{\mu\nu}\delta_{\mathcal{B}} g_{\mu\nu} + \frac{q_\times}{6T}\epsilon^{\mu\nu\rho\sigma} h_\mu H_{\nu\rho\sigma} = \mathcal{O}(\partial^2) \ , \tag{6.8a}$$

$$\Delta^{\mu\rho} u^\sigma \delta_{\mathcal{B}} b_{\rho\sigma} + \varpi \Delta^{\mu\rho} h^\sigma \delta_{\mathcal{B}} g_{\rho\sigma} + \frac{\varpi^2}{T\rho}\epsilon^{\mu\nu}\partial_\nu q_\times - \frac{q_\times \varpi}{6T\rho}\Delta^\mu{}_\lambda \epsilon^{\lambda\nu\rho\sigma} H_{\nu\rho\sigma} = \mathcal{O}(\partial^2) \ , \tag{6.8b}$$

$$\frac{1}{T}\nabla_\mu (T\rho h^\mu) - \rho T u^\mu h^\nu \delta_{\mathcal{B}} g_{\mu\nu} + \frac{q_\times}{6}\epsilon^{\mu\nu\rho\sigma} u_\mu H_{\nu\rho\sigma} = \mathcal{O}(\partial^2) \ . \tag{6.8c}$$

These equations imply that, as in the string fluid case, it is still possible to eliminate $u^\mu \delta_{\mathcal{B}} g_{\mu\nu}$ using the first order energy-momentum conservation equations. However, it is no longer possible to eliminate $u^\mu \delta_{\mathcal{B}} b_{\mu\nu}$ in terms of other non-hydrostatic data. This has important consequences for one-derivative non-hydrostatic corrections.

Since the study of one-derivative corrections is the subject of our attention below, it is instructive to expand the ideal one-form superfluid constitutive relations (4.14) to one-derivative order. This expansion gives rise to

$$\begin{aligned} T^{\mu\nu} = {} & \epsilon\, u^\mu u^\nu + p\, P^{\mu\nu} - \rho\varpi\, h^\mu h^\nu + \zeta\cdot\bar{\zeta}\left(T\frac{\partial q_\times}{\partial T} + \varpi\frac{\partial q_\times}{\partial \varpi}\right) u^\mu u^\nu - \zeta\cdot\bar{\zeta}\,\varpi\frac{\partial q_\times}{\partial\varpi} h^\mu h^\nu \\ & - 2\bar{q}\, u^{(\mu}\epsilon^{\nu)\rho\sigma\tau} u_\rho \zeta_\sigma \bar{\zeta}_\tau + \mathcal{O}(\partial^2) \ , \end{aligned}$$

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$$
\begin{aligned}
J^{\mu\nu} &= 2\rho\, u^{[\mu}h^{\nu]} + \zeta\cdot\bar{\zeta}\,\frac{\partial q_\times}{\partial\varpi}2u^{[\mu}h^{\nu]} - 2q_\times u^{[\mu}\bar{\zeta}^{\nu]} - q_\times\epsilon^{\mu\nu\rho\sigma}u_\rho\zeta_\sigma \\
&\quad - \epsilon^{\mu\nu\rho\sigma}u_\rho\left(\bar{q}\,\bar{\zeta}_\sigma + q'_\times(\zeta\cdot\bar{\zeta})\,\zeta_\sigma\right) + \mathcal{O}(\partial^2)\ , \\
N^\mu &= \frac{p}{T}u^\mu + \frac{q_\times}{T}\zeta\cdot\bar{\zeta}\,u^\mu + \mathcal{O}(\partial^2)\ .
\end{aligned}
\tag{6.9}
$$

The contributions from the one-derivative order term $q_\times$ are now complete, while two new order-mixing contributions, $q'_\times$ and $\bar{q}$, from two-derivative order appear. Their origin can be traced back to the free energy density (4.12) expanded up to two-derivative order as

$$
P(T,\zeta^2,\bar{\zeta}^2,\zeta\cdot\bar{\zeta}) = p(T,\varpi)+q_\times(T,\varpi)\zeta\cdot\bar{\zeta}+\frac{1}{2}q'_\times(T,\varpi)(\zeta\cdot\bar{\zeta})^2+\frac{1}{2}\bar{q}(T,\varpi)\bar{\zeta}^2+\mathcal{O}(\partial^3)\ . \tag{6.10}
$$

It is clear from these considerations that order mixing significantly increases the difficulty of studying these hydrodynamic systems, nevertheless it is possible to keep track of it precisely and to obtain constitutive relations in a hydrodynamic expansion.

6.3 One-derivative corrections

6.3.1 Hydrostatic corrections

Above it was shown that taking electric limit of ideal one-form superfluids generates some one-derivative corrections to the respective constitutive relations. However, the constitutive relations can also receive more generic one-derivative corrections allowed by the adiabaticity equation (2.22). Consider first the order mixing terms, whose general expression was given in eq. (6.5). The most generic two-derivative terms in the hydrostatic free energy density involving $\bar{\zeta}^\mu$ can be represented as

$$
\mathcal{N}_{(2)} = \frac{1}{2}\left(q'_\times\zeta_\mu\zeta_\nu + \bar{q}P_{\mu\nu}\right)\bar{\zeta}^\mu\bar{\zeta}^\nu + R_\mu\bar{\zeta}^\mu + \dots\ . \tag{6.11}
$$

Here the quadratic terms in $\bar{\zeta}^\mu$, i.e. $q'_\times$ and $\bar{q}$, are the same as those obtained in eq. (6.10) in ideal one-form superfluids. The linear terms in $\bar{\zeta}^\mu$ are parametrised by a generic one-derivative vector structure R_μ which involves an explicit derivative. It will contain, for example, terms proportional to $P_{\mu\nu}\nabla^\nu T$ and $\epsilon_{\mu\nu\rho\sigma}u^\nu\nabla^\rho u^\sigma$ among many others. Using eq. (6.5), their contribution to one-derivative constitutive relations is given as

$$
\begin{aligned}
T^{\mu\nu}_{\text{hs,order-mixing}} &= -2u^{(\mu}\epsilon^{\nu)\rho\sigma\tau}u_\rho\zeta_\sigma\left(\bar{q}\bar{\zeta}_\tau + R_\tau\right) + \mathcal{O}(\partial^2)\ , \\
J^{\mu\nu}_{\text{hs,order-mixing}} &= -\epsilon^{\mu\nu\rho\sigma}u_\rho\left(\bar{q}\bar{\zeta}_\sigma + q'_\times(\zeta\cdot\bar{\zeta})\zeta_\sigma + R_\sigma\right) + \mathcal{O}(\partial^2)\ , \\
N^\mu_{\text{hs,order-mixing}} &= \mathcal{O}(\partial^2)\ .
\end{aligned}
\tag{6.12}
$$

Secondly, it is necessary to consider explicitly one-derivative order terms in the hydrostatic free-energy density. It is possible to import all the terms directly from string fluids in eq. (5.15), except the α term which is no longer independent due to the Bianchi identity (6.2). Taking into account the contributions mentioned above, the total hydrostatic free energy density for one-derivative constitutive relations reads

$$
\begin{aligned}
\mathcal{N} &= p + q_\times\zeta_\mu\bar{\zeta}^\mu + \frac{1}{2}\left(q'_\times\zeta_\mu\zeta_\nu + \bar{q}P_{\mu\nu}\right)\bar{\zeta}^\mu\bar{\zeta}^\nu + R_\mu\bar{\zeta}^\mu \\
&\quad - \beta\epsilon^{\mu\nu\rho\sigma}u_\mu h_\nu\partial_\rho u_\sigma - \tilde{\beta}_1 h^\mu\partial_\mu T - \tilde{\beta}_2 h^\mu\partial_\mu\frac{\varpi}{T} - \tilde{\beta}_3\epsilon^{\mu\nu\rho\sigma}u_\mu h_\nu\partial_\rho h_\sigma\ .
\end{aligned}
\tag{6.13}
$$

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The contributions from p, $q_\times$, $q'_\times$, and $\bar{q}$ are given in eq. (6.9), from R_μ in eq. (6.12), while those from β and $\tilde{\beta}_i$ can be directly imported from appendix A.1.1. As in the case of string fluids, the equation of motion (6.8c) together with the Bianchi identity (6.2) allow to set $\tilde{\beta}_2 = 0$, thus leading to 3 independent hydrostatic transport coefficients at first order in derivatives. This completes the analysis of first order hydrostatic corrections.

6.3.2 Non-hydrostatic corrections

For the non-hydrostatic contributions, it is useful to parametric the stress tensor and charge current as

$$\begin{aligned} T^{\mu\nu}_{\text{nhs}} &= \delta\epsilon\ u^\mu u^\nu + \delta f \Delta^{\mu\nu} + \delta\tau\ h^\mu h^\nu + 2\ell^{(\mu}h^{\nu)} + 2k^{(\mu}u^{\nu)} + t^{\mu\nu} \ , \\ J^{\mu\nu}_{\text{nhs}} &= 2\delta\rho\ u^{[\mu}h^{\nu]} + 2m^{[\mu}h^{\nu]} + 2n^{[\mu}u^{\nu]} + \delta s\,\epsilon^{\mu\nu} \ . \end{aligned} \tag{6.14}$$

Introducing these into (6.3), it is possible to massage the adiabaticity equation into

$$\begin{aligned} \nabla_\mu N^\mu = &\left(\delta\epsilon - \frac{\delta\rho}{\partial\rho/\partial\varpi}\left(T\frac{\partial\rho}{\partial T} + \varpi\frac{\partial\rho}{\partial\varpi}\right)\right)\frac{1}{2}u^\mu u^\nu\delta_{\mathcal{B}}g_{\mu\nu} + (k^\mu - \varpi m^\mu)\,u^\nu\delta_{\mathcal{B}}g_{\mu\nu} \\ &+ \delta\rho\left(\delta_{\mathcal{B}}\rho + \frac{1}{2}\rho\Delta^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu}\right) + n^\mu\,(u^\nu\delta_{\mathcal{B}}b_{\mu\nu} + \varpi h^\nu\delta_{\mathcal{B}}g_{\mu\nu}) \\ &+ \delta s\frac{1}{2}\epsilon^{\mu\nu}\delta_{\mathcal{B}}b_{\mu\nu} + m^\mu\,(h^\nu\delta_{\mathcal{B}}b_{\mu\nu} + \varpi u^\nu\delta_{\mathcal{B}}g_{\mu\nu}) \\ &+ \left(\delta f - \frac{\rho\delta\rho}{\partial\rho/\partial\varpi}\right)\frac{1}{2}\Delta^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu} + (\delta\tau + \varpi\delta\rho)\,\frac{1}{2}h^\mu h^\nu\delta_{\mathcal{B}}g_{\mu\nu} + (\ell^\mu - \varpi n^\mu)\,h^\nu\delta_{\mathcal{B}}g_{\mu\nu} \\ &+ \frac{1}{2}t^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu} + \Delta \ . \end{aligned} \tag{6.15}$$

The rationale behind this arrangement is that the terms in the third line in (6.15) drop out using the Bianchi identities (6.2), while those in the second line in (6.15) drop out using the first order equations of motion (6.8) when $q_\times$ is zero. However, these terms are important to complete the quadratic form Δ. It is possible to use the redefinition freedom in u^μ and T to set

$$\delta\epsilon = \frac{\delta\rho}{\partial\rho/\partial\varpi}\left(T\frac{\partial\rho}{\partial T} + \varpi\frac{\partial\rho}{\partial\varpi}\right) \ , \qquad k^\mu = \varpi m^\mu \ , \tag{6.16}$$

and eliminate terms in the first line in (6.15). In string fluids, it was possible to use the residual redefinition freedom in μ_μ to set $\delta\rho = n^\mu = 0$ as well. However, in the current context there is no such freedom as it was already used to make the Josephson equation (4.9) exact. Schematically, the non-hydrostatic corrections can be written as

$$\begin{pmatrix} \delta\rho \\ \delta f - \frac{\rho\delta\rho}{\partial\rho/\partial\varpi} \\ \delta\tau + \varpi\delta\rho \\ \delta s \end{pmatrix} = -\frac{T}{2}\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda'_2 & \zeta_\perp & \zeta_\times & \tilde{\kappa}_1 \\ \lambda'_3 & \zeta'_\times & \zeta_\parallel & \tilde{\kappa}_2 \\ \lambda'_4 & \tilde{\kappa}'_1 & \tilde{\kappa}'_2 & r_\parallel \end{pmatrix}\begin{pmatrix} 2\delta_{\mathcal{B}}\rho + \rho\Delta^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu} \\ \Delta^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu} \\ h^\mu h^\nu\delta_{\mathcal{B}}g_{\mu\nu} \\ \epsilon^{\mu\nu}\delta_{\mathcal{B}}b_{\mu\nu} \end{pmatrix} ,$$

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$$\begin{pmatrix} n^\mu \\ \ell^\mu - \varpi n^\mu \\ m^\mu \end{pmatrix} = -T \begin{pmatrix} \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 & \lambda_{10} \\ \lambda_6' & \eta_\parallel & r_\times & \lambda_9' & \tilde{\eta}_\parallel & \tilde{r}_\times \\ \lambda_7' & r_\times' & r_\perp & \lambda_{10}' & \tilde{r}_\times' & \tilde{r}_\perp \end{pmatrix} \begin{pmatrix} \Delta^{\mu\sigma} u^\nu \delta_{\mathcal{B}} b_{\sigma\nu} + \varpi \Delta^{\mu\sigma} h^\nu \delta_{\mathcal{B}} g_{\sigma\nu} \\ \Delta^{\mu\sigma} h^\nu \delta_{\mathcal{B}} g_{\sigma\nu} \\ \Delta^{\mu\sigma} h^\nu \delta_{\mathcal{B}} b_{\sigma\nu} + \varpi \Delta^{\mu\sigma} u^\nu \delta_{\mathcal{B}} g_{\sigma\nu} \\ \epsilon^{\mu\sigma} u^\nu \delta_{\mathcal{B}} b_{\sigma\nu} + \varpi \epsilon^{\mu\sigma} h^\nu \delta_{\mathcal{B}} g_{\sigma\nu} \\ \epsilon^{\mu\sigma} h^\nu \delta_{\mathcal{B}} g_{\sigma\nu} \\ \epsilon^{\mu\sigma} h^\nu \delta_{\mathcal{B}} b_{\sigma\nu} + \varpi \epsilon^{\mu\sigma} u^\nu \delta_{\mathcal{B}} g_{\sigma\nu} \end{pmatrix} ,$$

$$t^{\mu\nu} = -\eta_\perp T \Delta^{\rho\langle\mu} \Delta^{\nu\rangle\sigma} \delta_{\mathcal{B}} g_{\rho\sigma} + \tilde{\eta}_\perp T \epsilon^{\rho\langle\mu} \Delta^{\nu\rangle\sigma} \delta_{\mathcal{B}} g_{\rho\sigma} \quad . \tag{6.17}$$

Since the tensor structures

$$\epsilon^{\mu\nu} \delta_{\mathcal{B}} b_{\mu\nu} \ , \quad \Delta^{\mu\sigma} h^\nu \delta_{\mathcal{B}} b_{\sigma\nu} + \varpi \Delta^{\mu\sigma} u^\nu \delta_{\mathcal{B}} g_{\sigma\nu} \ , \quad \epsilon^{\mu\sigma} h^\nu \delta_{\mathcal{B}} b_{\sigma\nu} + \varpi \epsilon^{\mu\sigma} u^\nu \delta_{\mathcal{B}} g_{\sigma\nu} \ , \tag{6.18}$$

are second order due to the Bianchi identities (6.2), the transport coefficients highlighted in blue are actually second order, but are required for positive definiteness of Δ. The terms highlighted in purple are first order in general but become second order when using the first order equations of motion (6.8) if $q_\times = 0$. In general, the positive definiteness of Δ gives 9 inequalities among these transport coefficients and at first order there is a total of 26 non-hydrostatic transport coefficients. However, the application of this theory to magnetic dominated bound-charge plasmas that is provided in section 7.4 consists of setting $q_\times = 0$ and leads to, upon appropriate identification, 8 non-hydrostatic transport coefficients, namely $\zeta_\perp$, $\zeta_\times$, $\zeta_\times'$, $\zeta_{||}$, $\eta_{||}$, $\tilde{\zeta}_{||}$, $\eta_\perp$, $\tilde{\eta}_\perp$.

7 Hot electromagnetism

Hot electromagnetism is the theory that results from the interaction of electromagnetic degrees of freedom with mechanical and thermal degrees of freedom of matter. At long wavelength and large timescales compared to the mean free path of the microscopic theories, matter can be approximated by a hot plasma and hydrodynamic theory determines the dynamical evolution of fluctuations around equilibrium. In this section, the term *hot electromagnetism* is used to denote the traditional treatments of hydrodynamic regimes of plasmas where the electromagnetic gauge field A_μ is incorporated as dynamical degrees of freedom. After a brief exposure of the different types of regimes that are considered in this work, namely MHD where electric fields are Debye screened, and bound-charge plasmas where they are not, exact dualities between different limits of one-form superfluids considered in the previous sections and the these two regimes are derived.

7.1 Heating up Maxwell's equations

Consider an electromagnetic plasma heated up to a finite temperature. The near equilibrium physics of such a plasma is governed by charged hydrodynamics coupled to dynamical electromagnetic fields. The dynamics of the electromagnetic fields $F_{\mu\nu}$ is governed by Maxwell equations in matter[19]

$$\nabla_\nu F^{\nu\mu} + J^\mu_{\text{matter}} + J^\mu_{\text{ext}} = 0 \ , \tag{7.1a}$$

[19] eq. (7.1) is a modified version of the second equation in (4.34) that accounts for the presence of matter and couplings to external currents.

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along with the Bianchi identity

$$\nabla_{[\mu}F_{\nu\rho]} = 0 \quad . \tag{7.1b}$$

Here J^μ_{ext} denotes an identically conserved background charge current distribution coupled to the plasma such that $\nabla_\mu J^\mu_{\text{ext}} = 0$. Prime examples of J^μ_{ext} include a lattice of ions or an auxiliary field theory source that facilitates the computation of correlation functions. J^μ_{matter} is the charge current associated with the matter component of the plasma and it is not required to be trivially conserved at finite temperature. In fact, the conservation equation $\nabla_\mu J^\mu_{\text{matter}} = 0$, which can be seen as the divergence of eq. (7.1a), serves as an equation of motion for the hydrodynamic chemical potential μ. As already commented in section 4.4, the Bianchi identity (7.1b) is solved by introducing the dynamical photon field A_μ such that $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. Having done that, eq. (7.1a) provides dynamics for 4 physical degrees of freedom in A_μ and μ. In addition, the plasma is characterised by the usual hydrodynamic fields u^μ and T, whose dynamics is governed by energy-momentum conservation

$$\nabla_\mu T^{\mu\nu} = F^{\nu\rho}J_\rho \ , \tag{7.1c}$$

where the total dynamical charge current of the plasma $J^\mu = \nabla_\nu F^{\nu\mu} + J^\mu_{\text{matter}}$ was introduced.

The constitutive relations of hot electromagnetism are written as expressions for $T^{\mu\nu}$ and J^μ in terms of u^μ, T, μ, and $F_{\mu\nu}$, arranged in a derivative expansion. A priori, these may be expected to be exactly the same as ordinary charged fluids with background electromagnetic fields. However, since the electromagnetic fields are dynamical, they can be relevant at ideal order in the derivative expansion, i.e. $F_{\mu\nu} = \mathcal{O}(1)$. This considerably modifies the actual constitutive relations [3, 38]. Similar to ordinary hydrodynamics, the constitutive relations of a plasma are also required to satisfy the second law of thermodynamics. This requirement is formulated in terms of the zero-form version of the adiabaticity equation (2.22), namely

$$\nabla_\mu N^\mu = \frac{1}{2}T^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu} + J^\mu\delta_{\mathcal{B}}A_\mu + \Delta \ , \quad \Delta \geq 0 \ , \tag{7.2}$$

which has to be satisfied for some free energy current N^μ and quadratic form Δ. Here $\delta_{\mathcal{B}}$ denotes an infinitesimal symmetry transformation along $\mathcal{B} = (\beta^\mu, \Lambda^\beta)$ introduced in eq. (2.5), which when applied to the metric and gauge field read

$$\delta_{\mathcal{B}}g_{\mu\nu} = 2\nabla_{(\mu}\left(\frac{u_{\nu)}}{T}\right) \ , \quad \delta_{\mathcal{B}}A_\mu = \partial_\mu\frac{\mu}{T} - \frac{1}{T}E_\mu \ , \tag{7.3}$$

where the electric E^μ and magnetic fields B^μ are defined as

$$E^\mu = F^{\mu\nu}u_\nu \ , \quad B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma} \ , \qquad F_{\mu\nu} = 2u_{[\mu}E_{\nu]} - \epsilon_{\mu\nu\rho\sigma}u^\rho B^\sigma . \tag{7.4}$$

Provided that eq. (7.2) is satisfied, the entropy current, defined as $S^\mu = N^\mu - T^{\mu\nu}u_\nu/T - J^\mu\mu/T$, has positive semi-definite divergence onshell (i.e. once the equations of motion are satisfied).

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7.1.1 Ideal fluid minimally coupled to electromagnetism

As a working example, and to aid intuition, consider the well-known model in the context of MHD [1] of an ideal fluid minimally coupled to Maxwell's electromagnetism via a conductivity term σ in the constitutive relations

$$\begin{aligned} T^{\mu\nu} &= F^{\mu}{}_{\rho}F^{\nu\rho} - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g^{\mu\nu} + \epsilon(T,\mu)u^{\mu}u^{\nu} + p(T,\mu)\left(g^{\mu\nu}+u^{\mu}u^{\nu}\right) \ , \\ J^{\mu} &= \nabla_{\nu}F^{\nu\mu} + q(T,\mu)\,u^{\mu} - \sigma(T,\mu)P^{\mu\nu}\left(T\partial_{\nu}\frac{\mu}{T} - E_{\nu}\right) , \end{aligned} \tag{7.5}$$

where the fluid part of the currents satisfies the usual thermodynamic relations $\mathrm{d}p = s\mathrm{d}T + q\mathrm{d}\mu$ and $\epsilon + p = sT + q\mu$. The energy-momentum tensor includes the purely electromagnetic contribution given in eq. (7.4) alongside the usual fluid contributions. These relations satisfy eq. (7.2) with

$$\begin{aligned} N^{\mu} &= -\frac{1}{4T}F_{\rho\sigma}F^{\rho\sigma}\,u^{\mu} + \frac{1}{T}F^{\mu\nu}\left(T\partial_{\nu}\frac{\mu}{T} - E_{\nu}\right) + \frac{p(T,\mu)}{T}u^{\mu} \ , \\ \Delta &= \frac{\sigma(\mu,T)}{T}P^{\mu\nu}\left(T\partial_{\mu}\frac{\mu}{T} - E_{\mu}\right)\left(T\partial_{\nu}\frac{\mu}{T} - E_{\nu}\right) \ , \end{aligned} \tag{7.6}$$

provided that the conductivity obeys the positivity constraint $\sigma(\mu,T) > 0$. In general, the constitutive relations of the plasma (7.5) could admit further derivative corrections and exhibit more intricate couplings between the electromagnetic and fluid sectors as it will be described later.

Using eq. (7.1), it is possible to work out the equations of motion for this simple plasma model. For the purposes of the current discussion, it suffices to look at eq. (7.1a) which leads to

$$\begin{aligned} q(T,\mu) &= u_{\mu}\nabla_{\nu}F^{\nu\mu} + u_{\mu}J^{\mu}_{\text{ext}} \ , \\ \sigma(T,\mu)E^{\mu} &= -P^{\mu}{}_{\lambda}\nabla_{\nu}F^{\nu\lambda} - P^{\mu}{}_{\lambda}J^{\lambda}_{\text{ext}} + T\sigma(T,\mu)P^{\mu\nu}\partial_{\nu}\frac{\mu}{T} \ . \end{aligned} \tag{7.7}$$

The first equation expresses the point that, to leading order in derivatives, the charge density of the plasma organises itself according to the charge density of the background. The second equation states that, to leading order, the electric fields in the plasma are induced by external currents. Additionally, these two equations algebraically determine the plasma dynamical fields μ and E^{μ} in terms of the other dynamical and background fields of the theory order by order in the derivative expansion.[20] Therefore, μ and E^{μ} do not in general obtain independent dynamics in the hydrodynamic regime of hot electromagnetism. In fact, this statement continues to hold when the most general coupling and derivative corrections are taken into account (see section 7.2). An interesting exception to this, which will be studied below, is the case of plasmas which have $q(T,\mu) = \sigma(T,\mu) = 0$.

[20] The rationale here is that if a dynamical field f satisfies an equation $f = f_0 + \mathcal{F}(\nabla, f)$, where $\mathcal{F}(\nabla, f)$ is at least one order in derivatives, then we can algebraically determine it recursively within the derivative expansion as $f = f_0 + \mathcal{F}(\nabla, f_0 + \mathcal{F}(\nabla, f_0 + \mathcal{F}(\nabla, f_0 + \mathcal{F}(\nabla, f_0 + \ldots))))$.

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7.1.2 The magnetohydrodynamics regime

Consider the sector of hot electromagnetism for which the background currents are stationary to leading order, i.e. the spatial currents are derivative suppressed $P_{\mu\nu}J^{\nu}_{\text{ext}} = \mathcal{O}(\partial)$.[21] An example of such backgrounds is the case of a fixed lattice of ions. From eq. (7.7), it follows that the electric fields in such plasmas are derivative suppressed, i.e. $E_\mu = \mathcal{O}(\partial)$, while the magnetic fields can be arbitrarily large. This is the hydrodynamic incarnation of *Debye screening*: electric fields are screened over large distances due to the presence of free charges.[22] Such hydrodynamic systems are commonly referred to as *magnetohydrodynamics* (MHD) (see e.g. [3]). Under the MHD limit, the energy-momentum tensor in eq. (7.5) becomes

$$T^{\mu\nu} = (\epsilon(T,\mu) + p(T,\mu))\, u^\mu u^\nu + \left(p(T,\mu) - \frac{1}{2}B^2\right) g^{\mu\nu} + B^2 \mathbb{B}^{\mu\nu} + \mathcal{O}(\partial) \ , \tag{7.8}$$

where $\mathbb{B}^{\mu\nu} = P^{\mu\nu} - \hat{B}^\mu \hat{B}^\nu$, $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ and $\hat{B}^\mu = B^\mu/|B|$, with $|B|$ being the modulus of B^μ. For most applications of MHD, it is useful to consider the scenario where the background charge current is entirely derivative suppressed, i.e. $J^\mu_{\text{ext}} = \mathcal{O}(\partial)$, making the requirement of sub-leading external currents "covariant". Such models are applicable when the background charge currents are either non-existent or negligible, as in the case of solar physics. Thus, in addition to the electric fields being screened, such plasmas are electrically neutral over large length scales, i.e. $q(T,\mu) = \mathcal{O}(\partial)$. In this regime, MHD can be reformulated in terms of a string fluid with a global one-form symmetry [24]. This connection will be developed further in section 7.3.

7.1.3 The bound-charge plasma regime

An often unstated requirement for the MHD regime to dominate the hydrodynamics of plasmas is that the plasmas are conducting, i.e. $\sigma(T,\mu) \neq 0$, otherwise the second equation in eq. (7.7) would not impose any restriction on electric fields, and hence they could be arbitrarily large. Consider a fluid which does not contain any free charges. For instance, a gas of neutral atoms which can nonetheless be polarised. In the absence of any free charge carriers, the conductivity $\sigma(T,\mu)$ is identically zero. Over large distances, the charge density $q(T,\mu)$ also adds up to zero. More rigorously, these are plasmas whose constitutive relations do not depend on μ. That is, in the simple case of eq. (7.5), $p = p(T)$ and $\sigma = 0$ leading to $\epsilon = \epsilon(T)$ and $q = 0$ by means of the thermodynamic relations. Consequently, μ drops out from the set of independent degrees of freedom and the charge conservation $\nabla_\mu J^\mu = 0$, which had the role of providing dynamics to μ, becomes identically satisfied implying that

$$\nabla_\mu J^\mu = 0 \quad \Longrightarrow \quad J^\mu = \nabla_\nu M^{\mu\nu} \ , \quad M^{\mu\nu} = -F^{\mu\nu} + M^{\mu\nu}_{\text{matter}} \ , \tag{7.9}$$

[21] One can show that this requirement is frame-invariant by noting that under $u^\mu \to u^\mu + \delta u^\mu$, where $\delta u^\mu = \mathcal{O}(\partial)$ such that $u_\mu \delta u^\mu = 0$, it remains invariant.

[22] The usual requirement for Debye screening, found in traditional textbooks of MHD, is to take the limit $\sigma \to \infty$. From the second equation in (7.7), it is obvious that this has the same effect as that attained by requiring $P_{\mu\nu}J^\nu_{\text{ext}} = \mathcal{O}(\partial)$. However, this "infinite conductivity limit" breaks the hydrodynamic derivative expansion. For this reason, it appears that the requirement $P_{\mu\nu}J^\nu_{\text{ext}} = \mathcal{O}(\partial)$ is more physically sound.

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where $M^{\mu\nu}_{\text{matter}}$ is the antisymmetric polarisation tensor characteristic of the material that constitutes the plasma. The physical content of the leading order Maxwell's equations (7.7) is then that such a system can only be described by hydrodynamics when the background charge current is weak, i.e. $J^{\mu}_{\text{ext}} = \mathcal{O}(\partial)$. The dynamical equations (7.1) and adiabaticity equation (7.2) for a *bound-charge plasma* can be recast as

$$\begin{aligned} &\nabla_\mu T^{\mu\nu} = -F^{\nu\rho} J^{\text{ext}}_{\rho} \ , \quad \nabla_\mu M^{\mu\nu} = J^{\nu}_{\text{ext}} \ , \quad \epsilon^{\mu\nu\rho\sigma}\nabla_\nu F_{\rho\sigma} = 0 \ , \\ &\nabla_\mu N^{\mu} = \frac{1}{2} T^{\mu\nu}\delta_{\mathcal{B}} g_{\mu\nu} + \frac{1}{2} M^{\mu\nu}\delta_{\mathcal{B}} F_{\mu\nu} + \Delta \ , \qquad \Delta \geq 0 \ , \end{aligned} \tag{7.10}$$

where $N^\mu \to N^\mu - M^{\mu\nu}\delta_{\mathcal{B}} A_\nu$ was redefined. Maxwell's electromagnetism in vacuum is self-dual under electromagnetic duality. There is a version of this duality that is still respected by the bound-charge plasma. It may be verified that under the transformation

$$F_{\mu\nu} \to \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\rho\sigma} \ , \quad M^{\mu\nu} \to \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \ , \quad N^\mu \to N^\mu - \frac{1}{2}\beta^\mu M^{\mu\nu} F_{\mu\nu} \ , \tag{7.11}$$

the equations of motion (7.10) map to themselves when $J^{\mu}_{\text{ext}} = 0$ and with the same energy-momentum tensor $T^{\mu\nu}$. In section 7.4, it will be shown that eqs. (7.10) are essentially the governing equations of one-form superfluid dynamics.

7.2 Magnetohydrodynamics

This section deals with the MHD regime of hot electromagnetic plasmas described in section 7.1.2. The ideal order constitutive relations of these plasmas are essentially the same as the constitutive relations of ordinary charged hydrodynamics, except that magnetic fields can be arbitrary large, i.e. $B^\mu = \mathcal{O}(1)$, and the electric fields are derivative suppressed, i.e. $E^\mu = \mathcal{O}(\partial)$. Though many of the results that will be presented in this section already appeared in [3], the details given here provide a cleaner derivation of these results and extends the traditional treatment of MHD to include parity-violating terms.

7.2.1 Ideal magnetohydrodynamics

At ideal order, MHD is characterised by a hydrostatic free energy density of the form $\mathcal{N} = P(T, \mu, B^2)$. This free energy is the most general at ideal order and makes no assumptions on the strength of the coupling between electromagnetic degrees of freedom and thermal degrees of freedom. Using the $\delta_{\mathcal{B}}$ variations of the free arguments with respect to the fields (2.5)

$$\begin{aligned} \delta_{\mathcal{B}} T &= \frac{T}{2} u^\mu u^\nu \delta_{\mathcal{B}} g_{\mu\nu} \ , \quad \delta_{\mathcal{B}}\mu = \frac{\mu}{2} u^\mu u^\nu \delta_{\mathcal{B}} g_{\mu\nu} + u^\mu \delta_{\mathcal{B}} A_\mu \ , \\ \delta_{\mathcal{B}} B^2 &= \left(B^\mu B^\nu - B^2 P^{\mu\nu} - 2u^{(\mu}\epsilon^{\nu)\lambda\rho\sigma} B_\lambda u_\rho E_\sigma \right) \delta_{\mathcal{B}} g_{\mu\nu} - 2\epsilon^{\mu\nu\rho\sigma} B_\rho u_\sigma \nabla_\nu \delta_{\mathcal{B}} A_\mu \ , \end{aligned} \tag{7.12}$$

together with the zero-form version of eq. (2.23) (i.e. with $b_{\mu\nu} \to A_\mu$), it is possible to infer the respective constitutive relations, free energy, and entropy currents. These take the form

$$\begin{aligned} T^{\mu\nu} &= (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} + \varpi |B| \mathbb{B}^{\mu\nu} + 2\varpi u^{(\mu}\epsilon^{\nu)\lambda\rho\sigma} \hat{B}_\lambda u_\rho E_\sigma \ , \\ J^\mu &= q u^\mu - \nabla_\nu \left(\varpi \epsilon^{\mu\nu\rho\sigma} \hat{B}_\rho u_\sigma \right) \ , \end{aligned}$$

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$$N^\mu = \frac{P}{T}u^\mu + \varpi\epsilon^{\mu\nu\rho\sigma}\hat{B}_\rho u_\sigma\left(\partial_\nu\frac{\mu}{T} - \frac{1}{T}E_\nu\right) \ ,$$
$$S^\mu = su^\mu + \nabla_\nu\left(\frac{\mu\varpi}{T}\epsilon^{\mu\nu\rho\sigma}\hat{B}_\rho u_\sigma\right) \ , \tag{7.13}$$

where the thermodynamics can be expressed as

$$\mathrm{d}P = s\mathrm{d}T + q\mathrm{d}\mu - \frac{\varpi}{2|B|}\mathrm{d}B^2 \ , \quad \epsilon + P = sT + q\mu \ . \tag{7.14}$$

Here ϖ is being defined as $-2|B|\partial P/\partial B^2$ and will later be identified with the string chemical potential in the dual higher-form language. Note that from eq. (2.23), the first order terms appear in the ideal MHD constitutive relations but these can be ignored when focusing on zero derivative order. These constitutive relations reduce to the simple model (7.5) upon using $P(T,\mu,B^2) = -B^2/2 + p(T,\mu)$.

The equations of motion (7.1)–(7.1c) at ideal order take the form

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} = F^{\nu\rho}J_\rho \implies & -u^\mu\nabla_\mu\epsilon - (\epsilon+P)\nabla_\mu u^\mu - \varpi|B|\mathbb{B}^{\mu\nu}\nabla_\mu u_\nu = \mathcal{O}(\partial^2) \ , \\
& (\epsilon+P)P^{\mu\nu}\left(\frac{1}{T}\partial_\nu T + u^\lambda\nabla_\lambda u_\nu\right) \\
& \quad + qP^{\mu\nu}\left(T\partial_\nu\frac{\mu}{T} - E_\nu\right) + \epsilon^{\nu\rho\alpha\beta}u_\alpha B_\beta J^{(1)}_\rho = \mathcal{O}(\partial^2) \ , \\
J^\mu + J^\mu_{\text{ext}} = 0 \implies & q(T,\mu,B^2) = u_\mu J^\mu_{\text{ext}} + \mathcal{O}(\partial) \ , \\
& P_{\mu\nu}J^\nu_{\text{ext}} = \mathcal{O}(\partial) \ .
\end{aligned} \tag{7.15}$$

where a component of the Bianchi identity (7.1b)

$$\nabla_\mu B^\mu = B^\mu u^\nu\nabla_\nu u_\mu - \epsilon^{\mu\nu\rho\sigma}E_\mu u_\nu\partial_\rho u_\sigma \ , \tag{7.16}$$

was used. In particular, note the appearance of one derivative corrections to the charge current $J^\mu_{(1)}$ in the transverse components to u^μ of the energy-momentum conservation. The transverse components of the Maxwell's equations imply that the transverse components of the external current sources J^μ_{ext} must be derivative suppressed, as earlier advertised in section 7.1.2. The component along the velocity, on the other hand, implies that $q(T,\mu,B^2) = u_\mu J^\mu_{\text{ext}}$ onshell at ideal order. This equation can be formally solved for μ and leads to the inference

$$\mu = \mu_0(T, B^2, u_\mu J^\mu_{\text{ext}}) + \mathcal{O}(\partial) \ . \tag{7.17}$$

Therefore, μ is not a true independent degree of freedom of the theory. At first order in derivatives, it will be seen that this statement also holds true for electric fields. Thus the magnetic fields are the only true dynamical degrees of freedom in the U(1) sector of magnetohydrodynamics.

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7.2.2 One derivative corrections

The ideal MHD theory described above can be extended to one derivative order in both the hydrostatic and non-hydrostatic sectors. The most generic hydrostatic free energy density at first order is given by

$$\begin{aligned}\mathcal{N} = P &+ M_1 B^\mu \partial_\mu \frac{B^2}{T^4} + M_2 \epsilon^{\mu\nu\rho\sigma} u_\mu B_\nu \partial_\rho B_\sigma \\ &- \frac{M_3}{T} B^\mu \partial_\mu T - M_4 \epsilon^{\mu\nu\rho\sigma} u_\mu B_\nu \partial_\rho u_\sigma + T M_5 B^\mu \partial_\mu \frac{\mu}{T} + \mathcal{O}(\partial^2) \ , \end{aligned} \tag{7.18}$$

where all the coefficients M_i $(i = 1, \ldots, 5)$ are functions of T, μ, and B^2. It is possible to vary these first order contributions so as to obtain the respective contributions to the constitutive relations, which are detailed in appendix A.1.2. The hydrostatic free energy (7.18) had been considered in [3]. However, it is noted here that due to the Bianchi identity (7.1b), the term involving M_1 is not independent and hence M_1 can be set to zero. This has led to an over-counting of independent hydrostatic coefficients in [3]. Nevertheless, for the purposes of comparison with earlier literature, a non-vanishing M_1 coefficient is considered here.

In turn, the non-hydrostatic corrections can be obtained as in previous sections. As in earlier cases, the equations of motion allow to remove $u^\mu \delta_{\mathcal{B}} g_{\mu\nu}$ and $u^\mu \delta_{\mathcal{B}} A_\mu$ from the independent non-hydrostatic tensor structures. The most generic corrections can then be written as

$$\begin{aligned} T^{\mu\nu}_{\text{nhs}} &= \delta\mathcal{F}\mathbb{B}^{\mu\nu} + \delta\mathcal{T}\hat{B}^\mu\hat{B}^\nu + 2\mathcal{L}^{(\mu}\hat{B}^{\nu)} + \mathcal{T}^{\mu\nu} \ , \\ J^\mu_{\text{nhs}} &= \delta\mathcal{S}\hat{B}^\mu + \mathcal{M}^\mu \ , \end{aligned} \tag{7.19}$$

where the different components of the stress tensor and charge current can be written in terms of matrices of transport coefficients

$$\begin{pmatrix} \delta\mathcal{F} \\ \delta\mathcal{T} \\ \delta\mathcal{S} \end{pmatrix} = -T \begin{pmatrix} \zeta_{11} & \zeta_{12} & \tilde{\chi}_1 \\ \zeta'_{12} & \zeta_{22} & \tilde{\chi}_2 \\ \tilde{\chi}'_1 & \tilde{\chi}'_2 & \sigma_\parallel \end{pmatrix} \begin{pmatrix} \frac{1}{2}\mathbb{B}^{\mu\nu}\delta_{\mathcal{B}} g_{\mu\nu} \\ \frac{1}{2}\hat{B}^\mu\hat{B}^\nu\delta_{\mathcal{B}} g_{\mu\nu} \\ \hat{B}^\mu \delta_{\mathcal{B}} A_\mu \end{pmatrix} \ ,$$

$$\begin{pmatrix} \mathcal{L}^\mu \\ \mathcal{M}^\mu \end{pmatrix} = -T \begin{pmatrix} \eta_{11} & \sigma_\times & \tilde{\eta}_{11} & \tilde{\sigma}_\times \\ \sigma'_\times & \sigma_\perp & \tilde{\sigma}'_\times & \tilde{\sigma}_\perp \end{pmatrix} \begin{pmatrix} \mathbb{B}^{\mu\sigma}\hat{B}^\nu\delta_{\mathcal{B}} g_{\sigma\nu} \\ \mathbb{B}^{\mu\sigma}\delta_{\mathcal{B}} A_\sigma \\ \epsilon^{\mu\alpha\beta\sigma} u_\alpha \hat{B}_\beta \hat{B}^\nu \delta_{\mathcal{B}} g_{\sigma\nu} \\ \epsilon^{\mu\alpha\beta\sigma} u_\alpha \hat{B}_\beta \delta_{\mathcal{B}} A_\sigma \end{pmatrix} \ , \tag{7.20}$$

$$\mathcal{T}^{\mu\nu} = -\eta_{22} T \mathbb{B}^{\rho\langle\mu}\mathbb{B}^{\nu\rangle\sigma}\delta_{\mathcal{B}} g_{\rho\sigma} + \tilde{\eta}_{22} T \epsilon^{\rho\alpha\beta\langle\mu} u_\alpha \hat{B}_\beta \mathbb{B}^{\nu\rangle\sigma}\delta_{\mathcal{B}} g_{\rho\sigma} \ . \tag{7.21}$$

The 8 coefficients in blue are parity-violating terms whose existence had been identified in [3] but were not studied in any detail.

7.2.3 Maxwell's equations

In this section it is shown that μ and E^μ are not dynamical degrees of freedom in MHD. Assembling all the contributions from the previous subsections, the most general charge

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current J^μ up to first order in derivatives can be written in the form

$$\begin{aligned}J^\mu =& qu^\mu - \nabla_\nu\left(\varpi\epsilon^{\mu\nu\rho\sigma}\hat{B}_\rho u_\sigma\right) + \Bigg(B^\lambda\partial_\lambda\frac{B^2}{T^4}\frac{\partial M_1}{\partial\mu} + \epsilon^{\lambda\nu\rho\sigma}u_\lambda B_\nu\partial_\rho B_\sigma\frac{\partial M_2}{\partial\mu}\\ &- \frac{1}{T}B^\lambda\partial_\lambda T\frac{\partial M_3}{\partial\mu} - \epsilon^{\lambda\nu\rho\sigma}u_\lambda B_\nu\partial_\rho u_\sigma\frac{\partial M_4}{\partial\mu} + TB^\lambda\partial_\lambda\frac{\mu}{T}\frac{\partial M_5}{\partial\mu} - \frac{1}{T}\nabla_\lambda\left(TM_5B^\lambda\right)\Bigg)u^\mu\\ &- \left(\tilde{\chi}_1'\hat{B}^\mu\mathbb{B}^{\rho\sigma} + \tilde{\chi}_2'\hat{B}^\mu\hat{B}^\rho\hat{B}^\sigma + 2\sigma_\times'\mathbb{B}^{\mu(\rho}\hat{B}^{\sigma)} + 2\tilde{\sigma}_\times'\epsilon^{\mu\alpha\beta(\rho}u_\alpha\hat{B}_\beta\hat{B}^{\sigma)}\right)\frac{T}{2}\delta_{\mathcal{B}}g_{\rho\sigma}\\ &- \left(\sigma_\parallel\hat{B}^\mu\hat{B}^\nu + \sigma_\perp\mathbb{B}^{\mu\nu} + \tilde{\sigma}_\perp\epsilon^{\mu\nu\alpha\beta}u_\alpha\hat{B}_\beta\right)T\delta_{\mathcal{B}}A_\nu + \mathcal{O}(\partial^2)\ .\end{aligned} \tag{7.22}$$

Inserting this current into Maxwell's equations eq. (7.1a), the different components read

$$\begin{aligned}q(T,\mu,B^2) =& u_\mu J^\mu_{\text{ext}} - B^\lambda\partial_\lambda\frac{B^2}{T^4}\frac{\partial M_1}{\partial\mu} - \epsilon^{\lambda\nu\rho\sigma}u_\lambda B_\nu\partial_\rho B_\sigma\frac{\partial M_2}{\partial\mu}\\ &+ \frac{1}{T}B^\lambda\partial_\lambda T\frac{\partial M_3}{\partial\mu} - \left(\frac{\partial M_4}{\partial\mu} + \frac{\varpi}{|B|}\right)B_\mu u_\nu\partial_\rho u_\sigma\\ &- TB^\lambda\partial_\lambda\frac{\mu}{T}\frac{\partial M_5}{\partial\mu} + \frac{1}{T}\nabla_\lambda\left(TM_5B^\lambda\right) + \mathcal{O}(\partial^2),\\ \sigma_\parallel\hat{B}^\nu\delta_{\mathcal{B}}A_\nu =& \frac{1}{2}\mathbb{E}^{\mu\nu}X_{\mu\nu} - \left(\tilde{\chi}_1'\mathbb{B}^{\rho\sigma} + \tilde{\chi}_2'\hat{B}^\rho\hat{B}^\sigma\right)\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma} + \mathcal{O}(\partial^2)\,,\\ \left(\sigma_\perp\mathbb{B}^{\mu\nu} + \tilde{\sigma}_\perp\mathbb{E}^{\mu\nu}\right)\delta_{\mathcal{B}}A_\nu =& \mathbb{E}^{\mu\sigma}\hat{B}^\nu X_{\sigma\nu} - 2\left(\sigma_\times'\mathbb{B}^{\mu(\rho}\hat{B}^{\sigma)} + \tilde{\sigma}_\times'\mathbb{E}^{\mu(\rho}\hat{B}^{\sigma)} - \varpi\mathbb{E}^{\mu(\rho}u^{\sigma)}\right)\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma}\\ &+ \mathcal{O}(\partial^2)\ ,\end{aligned} \tag{7.23}$$

where $\mathbb{E}^{\mu\nu}$ was defined according to $\mathbb{E}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}u_\rho\hat{B}_\sigma$ along with

$$X_{\mu\nu} = 2\partial_{[\mu}\left(\frac{\varpi\hat{B}_{\nu]}}{T}\right) + \frac{u^\lambda}{T}\epsilon_{\lambda\mu\nu\rho}J^\rho_{\text{ext}}\ . \tag{7.24}$$

Recalling that $T\delta_{\mathcal{B}}A_\mu = T\partial_\mu(\mu/T) - E_\mu$, these equations can be used to algebraically determine μ and E_μ in MHD. Below, it is shown precisely how this can be accomplished.

Introducing $J^\mu_{(1)}$, i.e. one-derivative corrections appearing in the charge current (7.22), into the first order equations of motion (7.15) and eliminating $P^{\mu\nu}\delta_{\mathcal{B}}A_\nu$ using eq. (7.23), it is possible to derive the onshell relation

$$P^{\mu(\rho}u^{\sigma)}\delta_{\mathcal{B}}g_{\rho\sigma} = -\left(\frac{|B|}{\epsilon + P + \varpi|B|}\right)\mathbb{B}^{\mu[\rho}\hat{B}^{\sigma]}X_{\rho\sigma} + \mathcal{O}(\partial^2)\ , \tag{7.25}$$

which will be useful in solving for μ and E^μ. For the remainder of this subsection, it is assumed that $u_\mu J^\mu_{\text{ext}} = \mathcal{O}(\partial)$ for simplicity, leading to all the components of the background currents to be derivative suppressed. Under this assumption, eq. (7.23) can be solved for μ and E_μ within the derivative expansion leading to

$$\begin{aligned}\mu =& \mu_0(T,B^2) + \frac{1}{\partial q/\partial\mu}\Bigg[u_\mu J^\mu_{\text{ext}} - B^\lambda\partial_\lambda\frac{B^2}{T^4}\frac{\partial M_1}{\partial\mu} - \epsilon^{\lambda\nu\rho\sigma}u_\lambda B_\nu\partial_\rho B_\sigma\frac{\partial M_2}{\partial\mu}\\ &+ \frac{1}{T}B^\lambda\partial_\lambda T\frac{\partial M_3}{\partial\mu} - \left(\frac{\partial M_4}{\partial\mu} + \frac{\varpi}{|B|}\right)\epsilon^{\lambda\nu\rho\sigma}B_\lambda u_\nu\partial_\rho u_\sigma - TB^\lambda\partial_\lambda\frac{\mu}{T}\frac{\partial M_5}{\partial\mu}\\ &+ \frac{1}{T}\nabla_\lambda\left(TM_5B^\lambda\right)\Bigg]_{\mu=\mu_0} + \mathcal{O}(\partial^2)\ ,\end{aligned}$$

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$$
\begin{aligned}
E^\mu = & TP^{\mu\nu}\partial_\nu\frac{\mu}{T} - \frac{T}{2\sigma_\parallel}\hat{B}^\mu\mathbb{E}^{\rho\sigma}X_{\rho\sigma} + \frac{T}{\sigma_\parallel}\hat{B}^\mu\left(\tilde{\chi}_1'\mathbb{B}^{\rho\sigma} + \tilde{\chi}_2'\hat{B}^\rho\hat{B}^\sigma\right)\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma} \\
& - T\left(\frac{\epsilon+P}{\epsilon+P+\varpi|B|}\right)\left(\frac{\sigma_\perp}{\sigma_\perp^2+\tilde{\sigma}_\perp^2}\mathbb{E}^{\mu\rho}\hat{B}^\sigma + \frac{\tilde{\sigma}_\perp}{\sigma_\perp^2+\tilde{\sigma}_\perp^2}\mathbb{B}^{\mu\rho}\hat{B}^\sigma\right)X_{\rho\sigma} \\
& - 2T\left(\frac{\tilde{\sigma}_\perp\sigma_\times' - \sigma_\perp\tilde{\sigma}_\times'}{\sigma_\perp^2+\tilde{\sigma}_\perp^2}\mathbb{E}^{\mu(\rho}\hat{B}^{\sigma)} - \frac{\tilde{\sigma}_\perp\tilde{\sigma}_\times' + \sigma_\perp\sigma_\times'}{\sigma_\perp^2+\tilde{\sigma}_\perp^2}\mathbb{B}^{\mu(\rho}\hat{B}^{\sigma)}\right)\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma} + \mathcal{O}(\partial^2) \ , \quad (7.26)
\end{aligned}
$$

where eq. (7.25) was used to derive the second equation above and $\mu_0(T, B^2)$ was defined as the root of the equation

$$
q(T, \mu_0(T, B^2), B^2) = \left.\frac{\partial P(T, \mu, B^2)}{\partial\mu}\right|_{\mu=\mu_0(T,B^2)} = 0 \ . \quad (7.27)
$$

Therefore, within the MHD derivative expansion, Maxwell's equations can be used to explicitly eliminate the chemical potential and the electric fields from the hydrodynamic description. As it will be shown in section 7.3, this elimination is the backbone for recasting MHD as the string fluid of section 5.

7.2.4 Kubo formulae and Onsager's relations

Analogously to section 5.4, Kubo formulae can be obtained by perturbing around an initial equilibrium configuration. In the context of MHD, the relevant operators are $O_a = \{T^{\mu\nu}, F_{\mu\nu}\}$, whose one point functions are defined as

$$
\mathbb{T}^{\mu\nu} = \sqrt{-g}\,\langle T^{\mu\nu}\rangle \ , \quad \mathbb{F}_{\mu\nu} = \sqrt{-g}\,\langle F_{\mu\nu}\rangle \ . \quad (7.28)
$$

In order to obtain Kubo formulas in MHD, perturbations of the background metric $g_{\mu\nu}$ and the external currents J^μ_{ext} are performed. Thus, solving for the electric field as in (7.26) is required, at least at the linearised level. The retarded Green's functions, for small time-dependent and spatially homogeneous perturbations $\delta h_{\lambda\rho}$ and δJ^μ_{ext} are defined as in [3]

$$
\begin{aligned}
\delta\mathbb{T}^{\mu\nu} &= \frac{1}{2}G_{TT}^{\mu\nu,\lambda\rho}(\omega)\delta h_{\lambda\rho} - i\omega G_{TF}{}^{\mu\nu}{}_{0\lambda}\delta J^\lambda_{\text{ext}} \ , \\
\delta\mathbb{F}_{\mu\nu} &= \frac{1}{2}G_{FT\,\mu\nu}{}^{\lambda\rho}\delta h_{\lambda\rho} - i\omega G_{FF\,\mu\nu,0\lambda}\delta J^\lambda_{\text{ext}} \ . \quad (7.29)
\end{aligned}
$$

Considering an equilibrium configuration with $u^\mu = \delta^\mu_t$, $\mu = \mu_0 = 0$, and magnetic field aligned in the z-direction with magnitude $B^z = B_0$, it is straightforward to derive the Kubo formulae

$$
\begin{aligned}
\frac{\tilde{\chi}_1'}{\sigma_{||}}\text{sign}(B_0) &= \lim_{\omega\to0}\frac{1}{\omega}\,\text{Im}\,G_{FT}^{tz,xx} \ , \\
\frac{\tilde{\chi}_2'}{\sigma_{||}}\text{sign}(B_0) &= \lim_{\omega\to0}\frac{1}{\omega}\,\text{Im}\,G_{FT}^{tz,zz} \ , \\
-\left(\frac{\sigma_\times'\sigma_\perp}{\sigma_\perp^2+\tilde{\sigma}^2} + \frac{\tilde{\sigma}_\times'\tilde{\sigma}}{\sigma_\perp^2+\tilde{\sigma}^2}\right)\text{sign}(B_0) &= \lim_{\omega\to0}\frac{1}{\omega}\,\text{Im}\,G_{FT}^{xt,xz} \ , \\
-\left(\frac{\tilde{\sigma}_\times'\sigma_\perp}{\sigma_\perp^2+\tilde{\sigma}^2} - \frac{\sigma_\times'\tilde{\sigma}}{\sigma_\perp^2+\tilde{\sigma}^2}\right) &= \lim_{\omega\to0}\frac{1}{\omega}\,\text{Im}\,G_{FT}^{xt,yz} \ ,
\end{aligned}
$$

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$$
\begin{aligned}
-\frac{\tilde{\chi}_1}{\sigma_{||}}\text{sign}(B_0) &= \lim_{\omega\to 0}\frac{1}{\omega}\,\text{Im}\,G_{TF}^{xx,tz}\ ,\\
-\frac{\tilde{\chi}_2}{\sigma_{||}}\text{sign}(B_0) &= \lim_{\omega\to 0}\frac{1}{\omega}\,\text{Im}\,G_{TF}^{zz,tz}\ ,\\
\left(\frac{\sigma_\times\sigma_\perp}{\sigma_\perp^2+\tilde{\sigma}^2}+\frac{\tilde{\sigma}_\times\tilde{\sigma}}{\sigma_\perp^2+\tilde{\sigma}^2}\right)\text{sign}(B_0) &= \lim_{\omega\to 0}\frac{1}{\omega}\,\text{Im}\,G_{TF}^{xz,xt}\ ,\\
\left(\frac{\tilde{\sigma}_\times\sigma_\perp}{\sigma_\perp^2+\tilde{\sigma}^2}-\frac{\sigma_\times\tilde{\sigma}}{\sigma_\perp^2+\tilde{\sigma}^2}\right) &= \lim_{\omega\to 0}\frac{1}{\omega}\,\text{Im}\,G_{TF}^{yz,xt}\ ,\\
\zeta_{22}-\frac{\tilde{\chi}_2\tilde{\chi}_2'}{\sigma_{||}} &= \lim_{\omega\to 0}\frac{1}{\omega}\,\text{Im}\,G_{TT}^{zz,zz}\ ,\\
\zeta_{12}'-\frac{\tilde{\chi}_2\tilde{\chi}_1'}{\sigma_{||}} &= \lim_{\omega\to 0}\frac{1}{\omega}\,\text{Im}\,G_{TT}^{zz,ii}\ ,\\
\zeta_{12}-\frac{\tilde{\chi}_1\tilde{\chi}_2'}{\sigma_{||}} &= \lim_{\omega\to 0}\frac{1}{\omega}\,\text{Im}\,G_{TT}^{ii,zz}\ ,\\
\zeta_{11}-\frac{\tilde{\chi}_1\tilde{\chi}_1'}{\sigma_{||}} &= \lim_{\omega\to 0}\frac{1}{\omega}\,\text{Im}\,G_{TT}^{ii,ii}\ ,\\
\eta_{11}-\frac{\sigma_\perp\left(\sigma_\times\sigma_\times'-\tilde{\sigma}_\times\tilde{\sigma}_\times'\right)+\tilde{\sigma}\left(\sigma_\times\tilde{\sigma}_\times'+\tilde{\sigma}_\times\sigma_\times'\right)}{\sigma_\perp^2+\tilde{\sigma}^2} &= \lim_{\omega\to 0}\frac{1}{\omega}\,\text{Im}\,G_{TT}^{zi,zi}\ ,\\
\tilde{\eta}_{11}-\frac{\sigma_\perp\left(\tilde{\sigma}_\times\sigma_\times'+\sigma_\times\tilde{\sigma}_\times'\right)+\tilde{\sigma}\left(\tilde{\sigma}_\times\tilde{\sigma}_\times'-\sigma_\times\sigma_\times'\right)}{\sigma_\perp^2+\tilde{\sigma}^2} &= \lim_{\omega\to 0}\frac{1}{\omega}\,\text{Im}\,G_{TT}^{zi,zj}\ (i\neq j)\ ,
\end{aligned}
\tag{7.30}
$$

while the remaining G_{FF} correlators were given in [3]. In evaluating the above, the contributions arising from the hydrostatic coefficients M_i were ignored and the assumption $\epsilon+P\gg\varpi|B|$ was made for the sake of simplicity.

For the case at hand, if the microscopic theory has a discrete symmetry Θ, the Onsager's relations require that

$$
G_{O_aO_b}(\omega,B_0)=i_ai_bG_{O_bO_a}(\omega,\Theta B_0)\ , \tag{7.31}
$$

where i_a is the eigenvalue of O_a under Θ. See details in appendix C. If Θ is simply time-reversal, we find the constraints from Onsager's relations to be

$$
\zeta_{12}=\zeta_{12}'\ ,\quad \sigma_\times=\sigma_\times'\ ,\quad \tilde{\sigma}_\times=\tilde{\sigma}_\times'\ ,\quad \tilde{\chi}_1=\tilde{\chi}_1'\ ,\quad \tilde{\chi}_2=\tilde{\chi}_2'\ , \tag{7.32}
$$

which in turn means that parity-violating MHD is characterised by 4 hydrostatic transport coefficients and 14 non-hydrostatic transport coefficients. This is the exact same number as for string fluids in section 5. In the next section, it will be shown how (7.30) can be used to map to transport coefficients in string fluids.

7.3 Magnetohydrodynamics as string fluids

7.3.1 The algorithm of mapping

We now show that magnetohydrodynamics, as formulated above, can be equivalently formulated as a string fluid discussed in section 5, when the external current J^μ_{ext} is derivative

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suppressed. To begin with, note that after using Maxwell's equations to eliminate μ and E_μ in (7.26), the MHD constitutive relations can be schematically represented as

$$\begin{aligned} &T^{\mu\nu}[u^\mu, T, B^\mu, g_{\mu\nu}, J^\mu_{\text{ext}}] \ , \\ &F_{\mu\nu}[u^\mu, T, B^\mu, g_{\mu\nu}, J^\mu_{\text{ext}}] = 2u_{[\mu}E_{\nu]}[u^\mu, T, B^\mu, g_{\mu\nu}, J^\mu_{\text{ext}}] - \epsilon_{\mu\nu\rho\sigma}u^\rho B^\sigma \ , \\ &\mu[u^\mu, T, B^\mu, g_{\mu\nu}, J^\mu_{\text{ext}}] \ . \end{aligned} \tag{7.33}$$

They satisfy the adiabaticity equation eq. (7.2) with J^μ replaced with $-J^\mu_{\text{ext}}$. The dynamical evolution of u^μ and T is governed by the energy-momentum conservation (7.1c), while the evolution of B^μ is governed by the Bianchi identity (7.1b). Note that the constitutive relations for μ do not enter the dynamical equations and hence are not relevant for the hydrodynamic description.

In order to establish a connection between MHD and string fluids, it is appropriate to follow the insight of [20] and note that MHD admits the following two-form current

$$J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \ , \tag{7.34}$$

which is conserved due to the Bianchi identity (7.1b). Physically, the integration of this current over any codimension-2 surface counts the number of magnetic fields lines crossing a given area element. The absence of magnetic monopoles in Maxwell's electromagnetism implies that these magnetic field lines are conserved. Furthermore, since the external current J^μ_{ext} satisfies $\nabla_\mu J^\mu_{\text{ext}} = 0$, it can be locally re-expressed as

$$J^\mu_{\text{ext}} = \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma} \ , \quad H_{\nu\rho\sigma} = 3\partial_{[\nu}b_{\rho\sigma]} \ . \tag{7.35}$$

In this language, the background charge current J^μ_{ext} is traded for a two-form background gauge field $b_{\mu\nu}$, which admits a one-form gauge transformation $b_{\mu\nu} \to b_{\mu\nu} + 2\partial_{[\mu}\Lambda_{\nu]}$. In this section, we assume that $J^\mu_{\text{ext}} = \mathcal{O}(\partial)$, leading to $b_{\mu\nu} = \mathcal{O}(1)$, which is sufficient for most applications of MHD.[23] Armed with the mappings (7.34) and (7.35), it can be verified that the MHD dynamical equations (7.1b) and (7.1c) arrange themselves into

$$\nabla_\mu T^{\mu\nu} = \frac{1}{2}H^{\nu\rho\sigma}J_{\rho\sigma} \ , \quad \nabla_\mu J^{\mu\nu} = 0 \ , \tag{7.36}$$

while the constitutive relations (7.33) can now be represented as

$$\begin{aligned} &T^{\mu\nu}[u^\mu, T, B^\mu, g_{\mu\nu}, b_{\mu\nu}] \ , \\ &J^{\mu\nu}[u^\mu, T, B^\mu, g_{\mu\nu}, b_{\mu\nu}] = 2u^{[\mu}B^{\nu]} + \epsilon^{\mu\nu\rho\sigma}u_\rho E_\sigma[u^\mu, T, B^\mu, g_{\mu\nu}, b_{\mu\nu}] \ . \end{aligned} \tag{7.37}$$

In eq. (7.37), the constitutive relations for μ have been ignored since, as stressed earlier, they do not contribute to the dynamical equations. Eq. (7.36) and (7.37) are precisely those encountered in the context of string fluids in section 5. Eq. (7.36) constitute the

[23] This assumption does not allow us to describe MHD with non-vanishing charge density q. However, in most applications of MHD, like in solar physics, the plasma is assumed to be electrically neutral at the hydrodynamical length scales [1].

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dynamical equations of one-form hydrodynamics given in eq. (2.6), while eq. (7.37) are the respective constitutive relations upon identifying $B^\mu = \rho(T,\varpi)h^\mu + \mathcal{O}(\partial)$.

In order to establish an exact equivalence between the two formulations, it is necessary to ensure that the constraints that follow from the adiabaticity equation, or the second law of thermodynamics, are the same in both the formulations. Consider the following map between the free energy currents

$$N^\mu_{\rm string} = N^\mu_{\rm MHD} + \frac{\varpi}{2T}\epsilon^{\mu\nu\rho\sigma}h_\nu F_{\rho\sigma} + \frac{\mu}{T}J^\mu_{\rm ext} + \nabla_\nu\left(\frac{\mu\varpi}{T}\epsilon^{\mu\nu\rho\sigma}u_\rho h_\sigma\right) \ , \tag{7.38}$$

where $N^\mu_{\rm string}$ denotes the free energy current in string fluids and $N^\mu_{\rm MHD}$ the free energy current in MHD. The last term in eq. (7.38) has a trivially vanishing divergence and has only been included for convenience. It is easily checked that

$$\begin{aligned}\nabla_\mu N^\mu_{\rm string} &= \nabla_\mu N^\mu_{\rm MHD} + J^{\mu\nu}\partial_\mu\left(\frac{\varpi h_\nu}{T}\right) + J^\mu_{\rm ext}\partial_\mu\frac{\mu}{T} + \Delta \\ &= \nabla_\mu N^\mu_{\rm MHD} + \frac{1}{2}J^{\mu\nu}\delta_{\mathcal{B}}b_{\mu\nu} + J^\mu_{\rm ext}\delta_{\mathcal{B}}A_\mu + \Delta \\ &= \frac{1}{2}T^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu} + \frac{1}{2}J^{\mu\nu}\delta_{\mathcal{B}}b_{\mu\nu} + \Delta \ ,\end{aligned} \tag{7.39}$$

and thus we recover the string fluid adiabaticity equation (2.22). This establishes that MHD with $J^\mu_{\rm ext} = \mathcal{O}(\partial)$ is entirely equivalent to one-form string fluids.

7.3.2 Mapping of transport coefficients up to first order

The above discussion established the map between MHD and string fluids in quite abstract terms. However, the explicit mapping between the transport coefficients at first order in derivatives is highly non-trivial. This is the purpose of this section. To begin with, it is necessary to derive the exact map between the magnetic field B^μ in MHD and the string fluid fields h^μ and ϖ at first order in derivatives. As we have already shown below eq. (7.37), at ideal order this is just $B^\mu = \rho(T,\varpi)h^\mu + \mathcal{O}(\partial)$. The first order derivative corrections to string fluids in section 5.3 together with (7.34) and the definition of magnetic fields in (7.4) allow to determine

$$\begin{aligned}B^\mu =& \rho h^\mu - h^\mu\left[\frac{1}{6}\epsilon^{\alpha\beta\rho\sigma}u_\alpha H_{\beta\rho\sigma}\frac{\partial\alpha}{\partial\varpi} - \alpha\epsilon^{\lambda\nu\rho\sigma}u_\lambda h_\nu\partial_\rho u_\sigma + \epsilon^{\alpha\beta\rho\sigma}u_\alpha h_\beta\partial_\rho u_\sigma\frac{\partial\beta}{\partial\varpi}\right. \\ &\left. + h^\lambda\partial_\lambda T\frac{\partial\tilde{\beta}_1}{\partial\varpi} + h^\lambda\partial_\lambda\frac{\varpi}{T}\frac{\partial\tilde{\beta}_2}{\partial\varpi} - \nabla_\lambda\left(\tilde{\beta}_2 h^\lambda\right)\frac{1}{T} + \epsilon^{\alpha\beta\rho\sigma}u_\alpha h_\beta\partial_\rho h_\sigma\frac{\partial\tilde{\beta}_3}{\partial\varpi}\right] \\ &- \alpha\Delta^\mu{}_\nu\epsilon^{\nu\lambda\rho\sigma}u_\lambda\partial_\rho u_\sigma + \frac{\beta}{\varpi}\Delta^\mu{}_\nu\epsilon^{\nu\lambda\rho\sigma}u_\lambda\partial_\rho u_\sigma - \frac{\tilde{\beta}_1}{\varpi}\Delta^{\mu\nu}\partial_\nu T - \frac{\tilde{\beta}_2}{\varpi}\Delta^{\mu\nu}\partial_\nu\frac{\varpi}{T} \\ &+ \Delta^\mu{}_\nu\epsilon^{\nu\lambda\rho\sigma}\left(\frac{T\tilde{\beta}_3}{\varpi^2}u_\lambda\partial_\rho\frac{\varpi h_\sigma}{T} + \frac{1}{T}\nabla_\rho\left(\frac{T\tilde{\beta}_3}{\varpi}u_\lambda h_\sigma\right)\right) + \mathcal{O}(\partial^2) \ .\end{aligned} \tag{7.40}$$

Due to our choice of frame in the non-hydrostatic sector of string fluids, note that the first order corrections to B^μ arise only due to hydrostatic corrections. It is useful to note that $X_{\mu\nu}$ defined in eq. (7.24) maps to

$$X_{\mu\nu} = \delta_{\mathcal{B}}b_{\mu\nu} + \mathcal{O}(\partial^2) \ . \tag{7.41}$$

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Substituting B^μ in the constitutive relations (7.37) allows us to determine the mapping between transport coefficients. Consider first the hydrostatic sector of the two formulations. It is useful to re-express eq. (7.38) as

$$N^\mu_{\rm string} = N^\mu_{\rm MHD} - \beta^\mu \left(\frac{1}{6}\mu\epsilon^{\lambda\nu\rho\sigma} u_\lambda H_{\nu\rho\sigma} + \frac{1}{2}\epsilon^{\lambda\nu\rho\sigma}\varpi u_\lambda h_\nu F_{\rho\sigma} + \mu\varpi\epsilon^{\lambda\nu\rho\sigma} u_\lambda h_\nu \partial_\rho u_\sigma \right) + \epsilon^{\mu\nu\rho\sigma} u_\nu \left(\varpi h_\rho \delta_{\mathcal{B}} A_\sigma - \frac{1}{2}\mu\delta_{\mathcal{B}} b_{\rho\sigma} + \mu\varpi h_\rho u^\lambda \delta_{\mathcal{B}} g_{\lambda\sigma} \right) \ . \tag{7.42}$$

Given that all the transverse components are purely non-hydrostatic, it is straightforward to derive the mapping for the hydrostatic free-energy density between the two formulations

$$\mathcal{N}_{\rm string} = \mathcal{N}_{\rm MHD} - \frac{1}{6}\mu\epsilon^{\lambda\nu\rho\sigma} u_\lambda H_{\nu\rho\sigma} - \varpi J^{\mu\nu} u_\mu h_\nu - \mu\varpi\epsilon^{\lambda\nu\rho\sigma} u_\lambda h_\nu \partial_\rho u_\sigma \ , \tag{7.43}$$

where $\mathcal{N}_{\rm string}$ is given in eq. (5.15) and $\mathcal{N}_{\rm MHD}$ in (7.18). Introducing eq. (7.40) into $\mathcal{N}_{\rm MHD}$ in (7.43), it is possible to infer at ideal order that

$$p(T,\varpi) = P(T,\mu_0(T,\rho^2),\rho^2) - 2\rho^2 \frac{\partial P(T,\mu_0(T,\rho^2),\rho^2)}{\partial\rho^2} \ , \tag{7.44}$$

where, on the right hand side, we understand that $\rho = \rho(T,\varpi)$. Given that $\rho(T,\varpi) = \partial p(T,\varpi)/\partial\varpi$ we can find that[24]

$$\rho = \frac{\partial}{\partial\rho^2}\left[P(T,\mu_0(T,\rho^2),\rho^2) - 2\rho^2 \frac{\partial P(T,\mu_0(T,\rho^2),\rho^2)}{\partial\rho^2} \right] \frac{1}{\partial\varpi(T,\rho^2)/\partial\rho^2} \ , \tag{7.46}$$

which can be solved by

$$\varpi(T,\rho^2) = -2\rho \frac{\partial P(T,\mu_0(T,\rho^2),\rho^2)}{\partial\rho^2} \ , \tag{7.47}$$

yielding the functional definition of ϖ in terms of the MHD thermodynamic potentials. Extending the free-energy density mapping (7.43) to one derivative order leads to the determination of the map between hydrostatic transport coefficients

$$\begin{aligned}
\alpha &= \mu_0 \ , \\
\beta &= M_4\rho + \mu_0\varpi \ , \\
\tilde{\beta}_1 &= \frac{M_3\rho}{T} - 2\rho^2\left(\frac{M_1}{T^4} + M_5\frac{\partial\mu_0}{\partial\rho^2}\right)\left(\frac{\partial\rho}{\partial T} + \frac{\varpi}{T}\frac{\partial\rho}{\partial\varpi}\right) + \frac{4M_1}{T^5}\rho^3 - TM_5\rho\frac{\partial(\mu_0/T)}{\partial T} \ , \\
\tilde{\beta}_2 &= -2\rho^2 T\left(\frac{M_1}{T^4} + M_5\frac{\partial\mu_0}{\partial\rho^2}\right)\frac{\partial\rho}{\partial\varpi}, \\
\tilde{\beta}_3 &= -M_2\rho^2 \ ,
\end{aligned} \tag{7.48}$$

[24]The partial derivatives of $\varpi(T,\rho^2)$ and $\rho(T,\varpi)$ are related by

$$\frac{\partial\varpi(T,\rho^2)}{\partial T} = -\frac{\partial\rho(T,\varpi)/\partial T}{\partial\rho(T,\varpi)/\partial\varpi} \ , \quad \frac{\partial\varpi(T,\rho^2)}{\partial\rho^2} = \frac{1}{2\rho}\frac{1}{\partial\rho(T,\varpi)/\partial\varpi} \ . \tag{7.45}$$

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where all the functions on the right hand side are evaluated at $\mu = \mu_0$. Interestingly, the string fluid transport coefficient α maps to the ideal order chemical potential solution μ_0. This implies that if string fluids are to describe MHD configurations at non-zero chemical potential, the α term in (5.15) is required. This observation was lacking in all previous studies [3, 20, 21]. Note also that the 5 MHD transport coefficients $M_{1,2,3,4,5}$ map to just 4 string fluid transport coefficients β and $\tilde{\beta}_{1,2,3}$. The reason is that, when working in a regime where $J^\mu_{\rm ext} = \mathcal{O}(\partial)$, substituting $\mu = \mu_0(T, B^2) + \mathcal{O}(\partial)$ in eq. (7.18) makes M_5 linearly dependent on the other terms.

On the other hand, the mapping in the non-hydrostatic sector is slightly more involved. When deriving the mapping (7.48) in the hydrostatic sector, it was inherently assumed that the fluid variables T and u^μ are the same in both the formulations. In the hydrostatic sector, this assumption is well founded, as these hydrodynamical fields are fixed to the requirement of u^μ/T aligning along the timelike isometry of the background defining the equilibrium state. However, in full generality, the fields T and u^μ can admit a relative non-hydrostatic redefinition between the two formulations. Since $T^{\mu\nu}_{\rm nhs}$ in both the formulations is chosen such that $T^{\mu\nu}_{\rm nhs} u_\nu = 0$ (i.e. the constitutive relations were expressed in the Landau frame), to find this relative redefinition it suffices to compare

$$T^{\mu\nu}_{\rm MHD,hs}[u^\mu \to u^\mu + \delta u^\mu, T \to T + \delta T] u_\nu = T^{\mu\nu}_{\rm string,hs} u_\nu + \mathcal{O}(\partial^2) \ . \tag{7.49}$$

After a straight-forward, yet involved, computation it can be inferred that the relative change in the fluid velocity δu^μ reads

$$\begin{aligned}
\delta u^\mu = & -\frac{\alpha}{s} h^\mu \frac{1}{2} \epsilon^{\rho\sigma} \delta_{\mathcal{B}} b_{\rho\sigma} + \frac{\sigma_\perp}{\sigma_\perp^2 + \tilde{\sigma}_\perp^2} \frac{\varpi T^2 s}{(\epsilon + p)^2} \Delta^{\mu\rho} h^\sigma \delta_{\mathcal{B}} b_{\rho\sigma} \\
& - \frac{T\varpi}{(\epsilon + p)} \left(\frac{\tilde{\sigma}_\perp}{\sigma_\perp^2 + \tilde{\sigma}_\perp^2} \left(\frac{Ts}{\epsilon + p} \right) + \frac{\alpha}{\varpi} \left(1 - \frac{2\varpi\rho}{\epsilon + p} \right) \right) \epsilon^{\mu\rho} h^\sigma \delta_{\mathcal{B}} b_{\rho\sigma} \\
& + \frac{2T\varpi}{\epsilon + p} \left(\frac{\tilde{\sigma}_\perp \sigma'_\times - \sigma_\perp \tilde{\sigma}'_\times}{\sigma_\perp^2 + \tilde{\sigma}_\perp^2} \Delta^{\mu(\rho} h^{\sigma)} + \frac{\tilde{\sigma}_\perp \sigma'_\times + \sigma_\perp \tilde{\sigma}'_\times}{\sigma_\perp^2 + \tilde{\sigma}_\perp^2} \epsilon^{\mu(\rho} h^{\sigma)} \right) \frac{1}{2} \delta_{\mathcal{B}} g_{\rho\sigma} \ ,
\end{aligned} \tag{7.50}$$

while the relative change in temperature vanishes, i.e. $\delta T = 0$. In fact, given the informed choice of parametrisation of the hydrostatic sector in the two formulations, it turns out that

$$T^{\mu\nu}_{\rm MHD,hs}[u^\mu \to u^\mu + \delta u^\mu, T \to T + \delta T] = T^{\mu\nu}_{\rm string,hs} + \mathcal{O}(\partial^2) \ , \tag{7.51}$$

holds exactly without further non-hydrostatic corrections. For the benefit of inquisitive readers, these details have been relegated to appendix A.2. The remaining step consists of comparing $T^{\mu\nu}_{\rm nhs}$ in the two formulations, along with E_μ in MHD to $-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} u^\nu J^{\rho\sigma}$, taking into account the potential redefinition in E_μ induced by (7.50). In particular, it is found that the field redefinition of u^μ non-trivially mixes E_μ and B^μ leading to a one derivative shift in E_μ such that

$$E_\mu \to E_\mu - |B| \mathbb{E}_{\mu\nu} \delta u^\nu + \mathcal{O}(\partial^2) \ . \tag{7.52}$$

Consequently, the comparison must be performed according to

$$\begin{aligned}
E_\mu - |B| \mathbb{E}_{\mu\nu} \delta u^\nu &= -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} u^\nu J^{\rho\sigma} + \mathcal{O}(\partial^2) \ , \\
T^{\mu\nu}_{\rm MHD,nhs}[E_\mu \to E_\mu - |B| \mathbb{E}_{\mu\nu} \delta u^\nu] &= T^{\mu\nu}_{\rm string,nhs} \ ,
\end{aligned} \tag{7.53}$$

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which leads to a straightforward derivation of the map for non-hydrostatic transport coefficients

$$
\begin{aligned}
\zeta_\perp &= \zeta_{11} - \frac{\tilde{\chi}_1\tilde{\chi}_1'}{\sigma_\parallel} \ , & \zeta_\times &= \zeta_{12} - \frac{\tilde{\chi}_1\tilde{\chi}_2'}{\sigma_\parallel} \ , \\
\zeta_\times' &= \zeta_{12}' - \frac{\tilde{\chi}_2\tilde{\chi}_1'}{\sigma_\parallel} \ , & \zeta_\parallel &= \zeta_{22} - \frac{\tilde{\chi}_2\tilde{\chi}_2'}{\sigma_\parallel} \ , \\
\tilde{\kappa}_1 &= \frac{\tilde{\chi}_1}{\sigma_\parallel} \ , & \tilde{\kappa}_2 &= \frac{\tilde{\chi}_2}{\sigma_\parallel} \ , \\
\tilde{\kappa}_1' &= -\frac{\tilde{\chi}_1'}{\sigma_\parallel} \ , & \tilde{\kappa}_2' &= -\frac{\tilde{\chi}_2'}{\sigma_\parallel} \ , \quad r_\parallel = \frac{1}{\sigma_\parallel} \ , \\
\eta_\parallel &= \eta_{11} - \frac{\sigma_\perp(\sigma_\times\sigma_\times' - \tilde{\sigma}_\times\tilde{\sigma}_\times') + \tilde{\sigma}_\perp(\sigma_\times\tilde{\sigma}_\times' + \tilde{\sigma}_\times\sigma_\times')}{\sigma_\perp^2 + \tilde{\sigma}_\perp^2} \ , \\
r_\perp &= \left(\frac{sT}{\epsilon+p}\right)^2 \frac{\sigma_\perp}{\sigma_\perp^2 + \tilde{\sigma}_\perp^2} \ , \\
\tilde{\eta}_\parallel &= \tilde{\eta}_{11} - \frac{\sigma_\perp(\sigma_\times\tilde{\sigma}_\times' + \tilde{\sigma}_\times\sigma_\times') - \tilde{\sigma}_\perp(\sigma_\times\sigma_\times' - \tilde{\sigma}_\times\tilde{\sigma}_\times')}{\sigma_\perp^2 + \tilde{\sigma}_\perp^2} \ , \\
\tilde{r}_\perp &= \left(\frac{sT}{\epsilon+p}\right)^2 \left(\frac{-\tilde{\sigma}_\perp}{\sigma_\perp^2 + \tilde{\sigma}_\perp^2} + \frac{2\alpha\rho}{sT}\right) \ , \\
r_\times &= \frac{sT}{\epsilon+p}\frac{-\sigma_\perp\tilde{\sigma}_\times + \tilde{\sigma}_\perp\sigma_\times}{\sigma_\perp^2 + \tilde{\sigma}_\perp^2} \ , & r_\times' &= \frac{sT}{\epsilon+p}\frac{-\sigma_\perp\tilde{\sigma}_\times' + \tilde{\sigma}_\perp\sigma_\times'}{\sigma_\perp^2 + \tilde{\sigma}_\perp^2} \ , \\
\tilde{r}_\times &= \frac{sT}{\epsilon+p}\frac{\sigma_\perp\sigma_\times + \tilde{\sigma}_\perp\tilde{\sigma}_\times}{\sigma_\perp^2 + \tilde{\sigma}_\perp^2} \ , & \tilde{r}_\times' &= \frac{sT}{\epsilon+p}\frac{\sigma_\perp\sigma_\times' + \tilde{\sigma}_\perp\tilde{\sigma}_\times'}{\sigma_\perp^2 + \tilde{\sigma}_\perp^2} \ , \\
\eta_\perp &= \eta_{22}, & \tilde{\eta}_\perp &= \tilde{\eta}_{22} \ .
\end{aligned}
\tag{7.54}
$$

This map expresses the fact that the that non-hydrostatic transport coefficients are quite non-trivially related to each other. In addition, the map also embodies the mapping of Onsager's relations found in (5.25) and (7.32). In particular, given that the Onsager relations (7.32) hold in MHD, the relations (5.25) are deduced from this map. Additionally, under the assumptions of $\alpha = 0$ and $\epsilon + P \gg \varpi|B|$, direct comparison of the Kubo formulae (7.30) in MHD with the Kubo formulae in string fluids (5.23) by means of (7.34) leads to a particular case of the map derived above, as expected. The results in this section conclude that MHD with $J^\mu_{\text{ext}} = \mathcal{O}(\partial)$ is completely equivalent to the hydrodynamic theory of string fluids formulated in section 5.

7.4 Bound-charge plasma and one-form superfluids

In this section, we formulate a new hydrodynamic theory describing bound-charge plasmas (i.e. plasmas with only bound charges and no free charge carriers) in the conventional language. We then argue how this theory can be equivalently formulated in terms of one-form superfluids. Because the full details of one-derivative corrections in one-form superfluid dynamics are quite involved, we focus on the ideal sector. However, as an illustration of the robustness of this formulation, we provide the first-order corrections in

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the electric limit of one-form superfluids discussed in section 6, and show that is maps to a certain magnetically dominated sector of bound-charge plasmas.

7.4.1 Ideal bound-charge plasma

In order to obtain the constitutive relations for a bound-charge plasma, it should be noted that the adiabaticity equation in eq. (7.10) is precisely the same in form as eq. (2.22) with $\delta_{\mathcal{B}} b_{\mu\nu} = \delta_{\mathcal{B}} \xi_{\mu\nu}$ replaced with $\delta_{\mathcal{B}} F_{\mu\nu}$ and $J^{\mu\nu}$ replaced with $M^{\mu\nu}$ (defined below eq. (7.9)). Note, however, that this naive identification is only true at the level of the adiabaticity equation. It does not hold true at the level of equations of motion because $M^{\mu\nu}$ is not conserved. A better, albeit slightly non-trivial, relation to one-form superfluids will be proposed in the next subsection. Regardless, this naive identification can be used to write down the constitutive relations of bound charge plasmas. At ideal order, following section 4.1.2, we find that

$$\begin{aligned}
T^{\mu\nu} &= \epsilon\, u^\mu u^\nu + \left(P - \alpha_{BB} B^2 - \alpha_{EB}(E\cdot B)\right) P^{\mu\nu} - \alpha_{EE} E^\mu E^\nu \\
&\quad + \alpha_{BB}\left(B^\mu B^\nu + 2u^{(\mu}\epsilon^{\nu)\rho\sigma\tau} u_\rho B_\sigma E_\tau\right) \ , \\
M^{\mu\nu} &= -2u^{[\mu}\left(\alpha_{EE} E^{\nu]} + \alpha_{EB} B^{\nu]}\right) - \epsilon^{\mu\nu\rho\sigma} u_\rho \left(\alpha_{BB} B_\sigma + \alpha_{EB} E_\sigma\right) \ , \\
N^\mu &= \frac{P}{T} u^\mu \ , \\
S^\mu &= N^\mu - \beta_\nu T^{\mu\nu} + \frac{1}{T} E_\nu M^{\mu\nu} = s u^\mu \ ,
\end{aligned} \tag{7.55}$$

where $P = P(T, E^2, B^2, E\cdot B)$, while the other thermodynamic functions were defined via

$$\begin{aligned}
\mathrm{d}P &= s\,\mathrm{d}T + \frac{1}{2}\alpha_{EE}\mathrm{d}E^2 + \frac{1}{2}\alpha_{BB}\mathrm{d}B^2 + \alpha_{EB}\mathrm{d}(E\cdot B) \ , \\
\epsilon + P &= s\,T + \alpha_{EE} E^2 + \alpha_{EB}(E\cdot B) \ .
\end{aligned} \tag{7.56}$$

P, ϵ, and s are identified as the thermodynamic pressure, energy, and entropy density of the plasma. On the other hand, the coefficients α_{EE}, α_{EB}, and α_{BB} are known as electromagnetic susceptibilities of the plasma. These thermodynamic relations and constitutive relations have been derived earlier in [38], though in a slightly different way. In the special case of an ideal fluid minimally coupled to electromagnetic fields in eq. (7.5), one chooses $P(T, E^2, B^2, E\cdot B) = (E^2 - B^2)/2 + p(T)$, leading to $\alpha_{EE} = 1$, $\alpha_{BB} = -1$ and $\alpha_{EB} = 0$.

It is also possible to work out the one-derivative corrections but they can be trivially read out from section 4.1.3. In particular, there are 166 first order transport coefficients, hinting towards the fact that one-form superfluids and bound-charge plasmas are exactly equivalent theories.

7.4.2 Reinterpretation as one-form superfluids

In deriving the one-form superfluid constitutive relations above, we used the naive similarity between the adiabaticity equations of bound-charge plasmas under the identification $\xi_{\mu\nu} \to F_{\mu\nu}$ and $J^{\mu\nu} \to M^{\mu\nu}$. However, as noted earlier, this identification does not follow through

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to the dynamics of the system. In order to get the correct dynamics, we propose the mapping with the respective Hodge duals

$$J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \ , \quad \xi_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\rho\sigma} \ . \tag{7.57}$$

This is a non-trivial mapping because, in bound-charge plasmas, $F_{\mu\nu}$ is being treated as a constituent field and the constitutive relations are expressed in terms of $M^{\mu\nu}$, while in one-form superfluids $\xi_{\mu\nu}$ is treated as a constituent field and the constitutive relations are expressed in terms of $J^{\mu\nu}$. Nonetheless, it is possible to show that under this identification, the defining equations of bound-charge plasmas map to those of a one-form superfluid, provided the following map of background fields

$$J^{\mu}_{\text{ext}} = \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma} \ , \tag{7.58}$$

and the map of the free-energy current

$$N^{\mu}_{\text{1SF}} = N^{\mu}_{\text{BCP}} - \frac{1}{2}\beta^{\mu}M^{\rho\sigma}F_{\rho\sigma} \ . \tag{7.59}$$

It is worth noting that this is precisely the self-duality operation of one-form superfluid dynamics discussed in section 4.3, except that $H_{\mu\nu\rho}$ is no longer required to vanish. It instead maps to the background currents in bound-charge plasmas.

Since the algebraic operation of the self-duality is the same as the map proposed above, it is possible to directly read out the map between fields and transport coefficients

$$\zeta_{\mu} = \alpha_{BB}B_{\mu} + \alpha_{EB}E_{\mu} \ , \quad \bar{\zeta}_{\mu} = \alpha_{EE}E_{\mu} + \alpha_{EB}B_{\mu} \ , \tag{7.60}$$

and

$$\begin{aligned} q &= -\frac{\alpha_{EE}}{\alpha_{EE}\alpha_{BB} - \alpha_{EB}^2} \ , \quad \bar{q} = -\frac{\alpha_{BB}}{\alpha_{EE}\alpha_{BB} - \alpha_{EB}^2} \ , \quad q_{\times} = \frac{\alpha_{EB}}{\alpha_{EE}\alpha_{BB} - \alpha_{EB}^2} \ , \\ P_{\text{1SF}} &= P_{\text{BCP}} - \alpha_{EE}E^2 - \alpha_{BB}B^2 - 2\alpha_{EB}(E\cdot B) \ . \end{aligned} \tag{7.61}$$

Note that this map is only well-defined if the determinant of magnetic susceptibilities is non-zero, that is

$$\alpha_{EE}\alpha_{BB} - \alpha_{EB}^2 = \frac{1}{q\bar{q} - q_{\times}^2} \neq 0 \ . \tag{7.62}$$

In particular, as outlined in section 5.2.1, in the string fluid limit of one-form superfluids, the coefficients $\bar{q}$ and $q_{\times}$ are zero, leading to a violation of this condition. Therefore, they do not map to a bound-charge plasma, but are instead dual to magnetohydrodynamics as discussed in section 7.3.

7.4.3 Magnetically dominated bound-charge plasma

As an interesting case, consider the regime of bound-charge plasmas where the electric fields are derivative suppressed. The reason for focusing on this case is because of its qualitative

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similarity to magnetohydrodynamics. Expanding eq. (7.55) to one-derivative order we find that

$$\begin{aligned}
T^{\mu\nu} &= \epsilon\, u^\mu u^\nu + P\, P^{\mu\nu} + E\cdot B\left[\left(T\frac{\partial\alpha_{EB}}{\partial T}\right)u^\mu u^\nu - 2B^2\frac{\partial\alpha_{EB}}{\partial B^2}\mathbb{B}^{\mu\nu}\right] \\
&\quad + \alpha_{BB}\left(-B^2\mathbb{B}^{\mu\nu} + 2u^{(\mu}\epsilon^{\nu)\rho\sigma\tau}u_\rho B_\sigma E_\tau\right) + \mathcal{O}(\partial^2)\ , \\
M^{\mu\nu} &= -\alpha_{BB}\epsilon^{\mu\nu\rho\sigma}u_\rho B_\sigma - 2E\cdot B\,\frac{\partial\alpha_{EB}}{\partial B^2}\epsilon^{\mu\nu\rho\sigma}u_\rho B_\sigma - 2\alpha_{EB}\,u^{[\mu}B^{\nu]} - \alpha_{EB}\epsilon^{\mu\nu\rho\sigma}u_\rho E_\sigma \\
&\quad - 2u^{[\mu}\left(\alpha_{EE}E^{\nu]} + \alpha'_{EB}B^{\nu]}E\cdot B\right) + \mathcal{O}(\partial^2)\ , \\
N^\mu &= \frac{P}{T}u^\mu + \frac{\alpha_{EB}}{T}E\cdot B\,u^\mu + \mathcal{O}(\partial^2)\ .
\end{aligned} \tag{7.63}$$

All the transport coefficients appearing here are functions of T and B^2 and satisfy the thermodynamics

$$\mathrm{d}P = s\mathrm{d}T + \frac{1}{2}\alpha_{BB}\mathrm{d}B^2\ ,\quad \epsilon + P = Ts\ . \tag{7.64}$$

In writing these, the ideal superfluid pressure was expanded according to

$$P(T, E^2, B^2, E\cdot B) = P + \alpha_{EB}E\cdot B + \frac{1}{2}\left(\alpha'_{EB}B_\mu B_\nu + \alpha_{EE}P_{\mu\nu}\right)E^\mu E^\nu + \mathcal{O}(\partial^3)\ . \tag{7.65}$$

Note that there are order-mixing terms coupled to α_{EE} and α'_{EB} in eq. (7.63), highlighted in blue, arising from the second order free-energy density affecting the one-derivative constitutive relations. It is possible to add more such terms by introducing a term like $S^\mu E_\mu$ in $\mathcal{N}$ such that S^μ includes all the possible one-derivative hydrostatic structures barring E_μ. Generically, such order-mixing terms only give contributions to the polarisation tensor

$$M^{\mu\nu}_{\text{hs,order-mixing}} = -2u^{[\mu}\left(\alpha'_{EB}B^{\nu]}E\cdot B + \alpha_{EE}E^{\nu]} + S^{\nu]}\right)\ . \tag{7.66}$$

Including the explicitly one-derivative terms, it is further possible to write down 4 hydrostatic derivative corrections, namely

$$\begin{aligned}
\mathcal{N} &= P + \alpha_{EB}E\cdot B + \frac{1}{2}\left(\alpha'_{EB}B_\mu B_\nu + \alpha_{EE}P_{\mu\nu}\right)E^\mu E^\nu + R^\mu E_\mu \\
&\quad + M_1 B^\mu\partial_\mu\frac{B^2}{T^4} + M_2\epsilon^{\mu\nu\rho\sigma}u_\mu B_\nu\partial_\rho B_\sigma - \frac{M_3}{T}B^\mu\partial_\mu T - M_4\epsilon^{\mu\nu\rho\sigma}u_\mu B_\nu\partial_\rho u_\sigma\ .
\end{aligned} \tag{7.67}$$

The contributions from the M_i terms to the constitutive relations have been recorded in appendix A.1.2. This completes the hydrostatic sector.

For the non-hydrostatic terms, we express the constitutive relations as

$$\begin{aligned}
T^{\mu\nu}_{\text{nhs}} &= \delta\mathcal{E}\ u^\mu u^\nu + \delta\mathcal{F}\mathbb{B}^{\mu\nu} + \delta\mathcal{T}\ \hat{B}^\mu\hat{B}^\nu + 2\mathcal{L}^{(\mu}\hat{B}^{\nu)} + 2\mathcal{K}^{(\mu}u^{\nu)} + \mathcal{T}^{\mu\nu}\ , \\
M^{\mu\nu}_{\text{nhs}} &= 2\delta\mathcal{R}\ u^{[\mu}\hat{B}^{\nu]} + 2\mathcal{M}^{[\mu}\hat{B}^{\nu]} + 2\mathcal{N}^{[\mu}u^{\nu]} + \delta\mathcal{S}\,\mathbb{E}^{\mu\nu}\ .
\end{aligned} \tag{7.68}$$

It is possible to use the redefinition freedom in u^μ and T to set $\delta\mathcal{E}$ and $\mathcal{K}^\mu$ to zero. The residual terms can be expanded according to

$$\begin{pmatrix}\delta\mathcal{R}\\ \delta\mathcal{F}\\ \delta\mathcal{T}\\ \delta\mathcal{S}\end{pmatrix} = -\frac{T}{2}\begin{pmatrix}\tau_1 & \tau_2 & \tau_3 & \tau_4\\ \tau'_2 & \zeta_{11} & \zeta_{12} & \tilde{\chi}_1\\ \tau'_3 & \zeta'_{12} & \zeta_{22} & \tilde{\chi}_2\\ \tau'_4 & \tilde{\chi}'_1 & \tilde{\chi}'_2 & \sigma_\parallel\end{pmatrix}\begin{pmatrix}2u^\mu\hat{B}^\nu\delta_{\mathcal{B}}F_{\mu\nu}\\ \mathbb{B}^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu}\\ \hat{B}^\mu\hat{B}^\nu\delta_{\mathcal{B}}g_{\mu\nu}\\ \mathbb{E}^{\mu\nu}\delta_{\mathcal{B}}F_{\mu\nu}\end{pmatrix}\ ,$$

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$$\begin{pmatrix}\mathcal{N}^\mu\\ \mathcal{L}^\mu\\ \mathcal{M}^\mu\end{pmatrix}=-T\begin{pmatrix}\tau_5 & \tau_6 & \tau_7 & \tau_8 & \tau_9 & \tau_{10}\\ \tau_6' & \eta_{11} & \sigma_\times & \tau_9' & \tilde{\eta}_{11} & \tilde{\sigma}_\times\\ \tau_7' & \sigma_\times' & \sigma_\perp & \tau_{10}' & \tilde{\sigma}_\times' & \tilde{\sigma}_\perp\end{pmatrix}\begin{pmatrix}\mathbb{B}^{\mu\sigma}u^\nu\delta_{\mathcal{B}}F_{\sigma\nu}\\ \mathbb{B}^{\mu\sigma}\hat{B}^\nu\delta_{\mathcal{B}}g_{\sigma\nu}\\ \mathbb{B}^{\mu\sigma}\hat{B}^\nu\delta_{\mathcal{B}}F_{\sigma\nu}\\ \mathbb{E}^{\mu\sigma}u^\nu\delta_{\mathcal{B}}F_{\sigma\nu}\\ \mathbb{E}^{\mu\sigma}h^\nu\delta_{\mathcal{B}}g_{\sigma\nu}\\ \mathbb{E}^{\mu\sigma}h^\nu\delta_{\mathcal{B}}b_{\sigma\nu}\end{pmatrix}\ ,$$

$$\mathcal{T}^{\mu\nu}=-\eta_{22}T\mathbb{B}^{\rho\langle\mu}\mathbb{B}^{\nu\rangle\sigma}\delta_{\mathcal{B}}g_{\rho\sigma}+\tilde{\eta}_{22}T\mathbb{E}^{\rho\langle\mu}\mathbb{B}^{\nu\rangle\sigma}\delta_{\mathcal{B}}g_{\rho\sigma}\ . \tag{7.69}$$

The blue terms have been considered here in order to complete the quadratic form. However, they are actually second order contributions to the constitutive relations. This mixing of derivative orders in positivity of the quadratic form is a manifestation of the order mixing considerations explained in section 6.

Having discussed the one-derivative corrections to a bound-charge plasma in the magnetically dominated limit, we now establish a map between these and one-form superfluids. Identifying $F_{\mu\nu}=-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\rho\sigma}$ at ideal order one can trivially find that

$$\zeta^\mu=\alpha_{BB}B^\mu+\mathcal{O}(\partial)\ ,\quad \bar{\zeta}^\mu=\alpha_{EB}B^\mu+\mathcal{O}(\partial)\ . \tag{7.70}$$

Thus, on the one-form superfluid side, a linear combination of ζ^μ and $\bar{\zeta}^\mu$ is derivative suppressed. One such limit was studied in section 6, namely the electric limit. In order to map to this limit, it is necessary to set $\alpha_{EB}(T,B^2)=0$. Having done that, it is possible to show that the theory is exactly equivalent to the electric limit of one-form superfluids. Suppressing the details, the following map is found in the hydrostatic sector

$$\begin{aligned}&p(T,\varpi)=P(T,\rho^2)+\varpi\rho\ ,\\ &q_\times=0\ ,\qquad \bar{q}=-\frac{1}{\alpha_{EE}}\ ,\qquad q_\times'=-\frac{1}{\varpi^2}\left(\frac{1}{\alpha_{EB}'\rho^2+\alpha_{EE}}-\frac{1}{\alpha_{EE}}\right)\ ,\end{aligned} \tag{7.71a}$$

together with the order mixing vectors

$$R_\mu=\left(\frac{h_\mu h_\nu}{\alpha_{EB}'\rho^2+\alpha_{EE}}+\frac{\Delta_{\mu\nu}}{\alpha_{EE}}\right)S^\nu\ , \tag{7.71b}$$

and the pure first-order coefficients

$$\begin{aligned}&\beta=M_4\rho\ ,\qquad \tilde{\beta}_1=-\left(\frac{2M_1\rho^2}{T^4}\frac{\partial\rho}{\partial T}+\frac{2M_1\rho^2\varpi}{T^5}\frac{\partial\rho}{\partial\varpi}-\frac{4M_1}{T^5}\rho^3-\frac{M_3\rho}{T}\right)\ ,\\ &\tilde{\beta}_2=-\frac{2M_1\rho^2}{T^3}\frac{\partial\rho}{\partial\varpi}\ ,\qquad \tilde{\beta}_3=-M_2\rho^2\ .\end{aligned} \tag{7.71c}$$

Here $\rho=\varpi q=|B|$, $h^\mu=-\zeta^\mu/\varpi=B^\mu/|B|$ and

$$\varpi=-2\rho\frac{P(T,\rho^2)}{\partial\rho^2}\ . \tag{7.72}$$

For the first-order terms in the non-hydrostatic sector, the following trivial map is obtained for the energy-momentum tensor

$$\begin{aligned}&\zeta_\perp=\zeta_{11}\ ,\qquad \zeta_\parallel=\zeta_{22}\ ,\qquad \zeta_\times=\zeta_{12}\ ,\qquad \zeta_\times'=\zeta_{12}'\ ,\\ &\eta_\parallel=\eta_{11}\ ,\qquad \tilde{\eta}_\parallel=\tilde{\eta}_{11}\ ,\qquad \eta_\perp=\eta_{22}\ ,\qquad \tilde{\eta}_\perp=\tilde{\eta}_{22}\ ,\end{aligned} \tag{7.73a}$$

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while for the polarisation tensor we have

$$\tau_2 = (\alpha_{EE} + \alpha'_{EB}B^2)\tilde{\kappa}'_1\,, \quad \tau_3 = (\alpha_{EE} + \alpha'_{EB}B^2)\tilde{\kappa}'_2\,, \quad \tau_6 = \alpha_{EE}\tilde{r}'_\times\,, \quad \tau_9 = -\alpha_{EE}r'_\times\,,$$
$$\tilde{\chi}'_1 = -\frac{\lambda_2}{\delta\rho/\delta\varpi}\,, \qquad \tilde{\chi}'_2 = -\frac{\lambda_3}{\delta\rho/\delta\varpi}\,, \qquad \sigma'_\times = \frac{\varpi}{\rho}\lambda_9\,, \qquad \tilde{\sigma}'_\times = -\frac{\varpi}{\rho}\lambda_6\,. \tag{7.73b}$$

Note that the first order terms in the polarisation tensor appear at second order in the charge current. Hence, if we were interested in only the one-derivative corrections to the charge current, as in MHD, these terms can be ignored. Taking this into account, at first order in derivatives there are a total of 8 transport coefficients in the non-hydrostatic sector, given in (7.73a), out of which the Onsager's relation set $\zeta_{12} = \zeta'_{12}$. This exactly matches the number of transport coefficients found in MHD in section 7.2, provided that the current of free charges is removed by setting $\tilde{\kappa}_1 = \tilde{\kappa}'_1 = \tilde{\kappa}_2 = \tilde{\kappa}'_2 = 0$, $r_{||} = r_\perp = \tilde{r}_\perp = 0$, and $\tilde{r}_\times = \tilde{r}'_\times = r_\times = r'_\times = 0$.

8 Outlook

This paper has dealt with the formulation of new hydrodynamic theories with generalised global symmetries capable of describing different hydrodynamic regimes of hot electromagnetism. The precise correspondence between these two classes of theories also required the formulation and extension of hydrodynamic theories with dynamical gauge fields. This included the extension of MHD to the parity-violating sector in section 7.2 and a new effective theory that describes the hydrodynamic regime of non-conducting plasmas (i.e. plasmas without free charge carriers) in section 7.4. Though four out of five hydrodynamic theories that were formulated in this work can be seen as different limits of one theory, the explicit construction of each of them actually required a case-by-case analysis.

The connections between hydrodynamics with generalised global symmetries and hot electromagnetism were made in the sector of the theory where the U(1) one-form symmetry is partially or entirely spontaneously broken. It was proven that the theory of one-form superfluids in the electric limit in section 6, in which the one-form symmetry is completely broken, is equivalent to a theory of magnetically dominated non-conducting plasmas with bound charges in section 7.4. It was also proven that a theory of one-form superfluids in the string fluid limit as in section 5, in which the U(1) one-form symmetry is only partially broken along u^μ, is exactly equivalent to MHD with sub-leading external currents (see section 7.2). This equivalence has thus shown that the U(1) one-form symmetry is spontaneously broken in these two hydrodynamic regimes of hot plasmas.

These two theories described above were focused on the magnetic dominated phase of hot electromagnetism in which the magnetic fields can be arbitrary and the electric fields are weak. The opposite regime, that of electrohydrodynamics, in which the hydrodynamics of plasmas is electrically dominated, is still unexplored but could have interesting applications. This type of theories will also be described by one-form superfluids of section 4 in a different regime or specific limits of one-form superfluids. In certain cases, the theories describing these regimes will be directly related to the theories developed here

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due to electromagnetic dualities or variations thereof, as discussed in section 4.2. This suggests that the connections depicted in figure 1 between one-form (super)fluids and hot electromagnetism admit many other unexplored regimes and intricate relations between them. It would be interesting to understand this broader diagram more precisely by, for instance, classifying all the different hydrodynamic regimes of hot electromagnetism and to investigate whether fluids with generalised global symmetries can actually provide dual formulations for all these different hydrodynamic regimes.

The results of section 5, together with the map given in 7.3, provide a formulation of MHD entirely in terms of conservation laws, including all possible dissipative effects. This has the potential to aid numerical simulations of MHD, as numerical codes are better suited for working with conservation equations instead of dynamical Maxwell equations [27]. As such, the work we have presented here has the potential of aiding progress in the astrophysical context, not only by allowing for numerical studies of dissipative effects in accretion disk physics but also by providing the necessary and sufficient conditions (see section 5) for having equilibrium solutions (without dissipation), which serve as starting points in numerical simulations. In particular, besides providing a time-like Killing vector field, one must solve the no-monopole constraint (5.13) in order to have an equilibrium solution for a scalar Goldstone φ (magnetic scalar potential). This has been used in [24] in order to obtain a new solution of a slowly rotating magnetised star but many other possibilities, such as new accretion disk solutions, are yet to be explored. We also expect this formulation to be useful in the study of stability properties of accretion disk solutions and in probing mechanisms for energy transport with analytic control. We intend to pursue some of these directions elsewhere.

Related to the exploration of the scope of hydrodynamics with generalised global symmetries and its connections with electromagnetism, it was noted throughout this paper that the traditional treatment of MHD, where the electromagnetic photon is incorporated as a dynamical field in the hydrodynamic description, has so far been formulated in greater generality than its counterpart as the string fluid of section 5. The traditional MHD formulation given in section 7.2, extending that of [3], allows for the description of hot plasmas that are not electrically neutral at hydrodynamic length scales, i.e. it is possible to consider a situation in which $u_\mu J^\mu_{\text{ext}} = \mathcal{O}(1)$. It may be the case that the string fluid formulation of section 5 can be generalised in order to incorporate the description of non-electrically neutral plasmas. For instance, treating some of the components of $H_{\mu\nu\lambda}$ as $\mathcal{O}(1)$ instead of $\mathcal{O}(\partial)$ may provide the required generalisation. However, at the present moment, it is not clear whether or not such a formulation exists and whether it would be useful. Nevertheless, we plan on returning to this issue in the future.

A theory of ordinary one-form fluids has also been developed in section 3. This theory, which is rather different from the theory of string fluids of section 5, has unbroken one-form symmetry and had not been considered previously in the literature. It is suggestive to speculate that this effective description could describe yet another hydrodynamic regime of hot plasmas in which the U(1) one-form symmetry is unbroken. A back of the envelope calculation suggests that one-form fluids in the unbroken phase do not describe MHD with weak magnetic fields, as could have been naively expected. It would be interesting to pursue

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this direction further and understand whether one-form fluids could find applications in other phases of matter.

Fluid/gravity dualities have been used to describe earlier versions of string fluids (without the Goldstone mode φ) both in the context of Anti-de Sitter black branes charged under a two-form gauge field [37] and in the context of asymptotically flat supergravity black branes, obtained by a series of duality transformations [39, 40]. Pursuing this line of research further, it would be extremely interesting to construct gravity duals to both the string fluids of section 5, explicitly understanding what φ relates to in the gravity dual, and to the one-form superfluids of section 4, identifying φ_μ in the gravity theory. The analogous fluid/gravity considerations in the case of zero-form superfluids [4, 41] will be useful. It is likely that gravity duals to string fluids, as formulated in section 5, will involve black branes charged under a two-form gauge field and with scalar hair.

The long wavelength perturbations of black branes in supergravity are governed by effective fluid theories with multiple higher-form currents [42]. Starting with the work of [22], it would be interesting to develop higher-form superfluid theories that could be used to study the stability of these gravitational solutions and to aid in finding new stationary black hole solutions via hydrostatic effective actions for the Goldstone modes.

Finally, it should be mentioned that the tools developed here and the viewpoint expressed has repercussions to other hydrodynamic theories with generalised global symmetries such as theories of viscoelasticity [23] and with weakly broken symmetries [25]. In particular, it is likely that some of these theories require the introduction of the vector Goldstone mode φ_μ in order to define a hydrostatic effective action. We leave this line of inquire to future work.

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A Calculational details

A.1 Hydrostatic corrections

In this appendix, the explicit expressions for the first order hydrostatic corrections to various hydrodynamic systems studied in this work are derived. Along with being of inherent phenomenological relevance, these corrections are important when comparing the constitutive relations between one-form superfluids and hot electromagnetism.

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A.1.1 String fluids and electric limit of one-form superfluids

Using the free energy density for string fluids in (5.15), performing a $\delta_{\mathcal{B}}$ variation of each of the terms and using eq. (2.23), it is possible to work out their effect on the hydrostatic constitutive relations. It is useful to parametrise these corrections as

$$\begin{aligned}
T^{\mu\nu}_{\rm hs} &= (\epsilon+p)\,u^\mu u^\nu + p\,g^{\mu\nu} - \varpi\rho\,h^\mu h^\nu + T^{\mu\nu}_{{\rm hs},\alpha} + T^{\mu\nu}_{{\rm hs},\beta} + T^{\mu\nu}_{{\rm hs},\tilde\beta_1} + T^{\mu\nu}_{{\rm hs},\tilde\beta_2} + T^{\mu\nu}_{{\rm hs},\tilde\beta_3} + \mathcal{O}(\partial^2)\,,\\
J^{\mu\nu}_{\rm hs} &= 2\rho\,u^{[\mu}h^{\nu]} + J^{\mu\nu}_{{\rm hs},\alpha} + J^{\mu\nu}_{{\rm hs},\beta} + J^{\mu\nu}_{{\rm hs},\tilde\beta_1} + J^{\mu\nu}_{{\rm hs},\tilde\beta_2} + J^{\mu\nu}_{{\rm hs},\tilde\beta_3} + \mathcal{O}(\partial^2)\,,\\
N^{\mu}_{\rm hs} &= \frac{p}{T}u^\mu + N^{\mu}_{{\rm hs},\alpha} + N^{\mu}_{{\rm hs},\beta} + N^{\mu}_{{\rm hs},\tilde\beta_1} + N^{\mu}_{{\rm hs},\tilde\beta_2} + N^{\mu}_{{\rm hs},\tilde\beta_3} + \mathcal{O}(\partial^2)\,,
\end{aligned} \tag{A.1}$$

where all the explicit terms are given below, specifically

$$\begin{aligned}
T^{\mu\nu}_{{\rm hs},\alpha} &= -\frac{1}{6}\epsilon^{\alpha\beta\rho\sigma}u_\alpha H_{\beta\rho\sigma}\left[\left(\frac{\partial(T\alpha)}{\partial T}+\varpi\frac{\partial\alpha}{\partial\varpi}\right)u^\mu u^\nu - \varpi\frac{\partial\alpha}{\partial\varpi}h^\mu h^\nu\right] - \frac{\alpha}{3}u^{(\mu}\epsilon^{\nu)\lambda\rho\sigma}H_{\lambda\rho\sigma}\ ,\\
J^{\mu\nu}_{{\rm hs},\alpha} &= -\frac{1}{3}\epsilon^{\mu\nu\rho\sigma}u_\mu H_{\nu\rho\sigma}\frac{\partial\alpha}{\partial\varpi}u^{[\mu}h^{\nu]} + \nabla_\sigma\left(\alpha\epsilon^{\mu\nu\rho\sigma}u_\rho\right)\ ,\\
N^{\mu}_{{\rm hs},\alpha} &= \frac{\alpha}{6T}\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma} - \alpha\epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho\left(\frac{\varpi h_\sigma}{T}\right)\ ,
\end{aligned}$$

$$\begin{aligned}
T^{\mu\nu}_{{\rm hs},\beta} &= -\epsilon^{\alpha\beta\rho\sigma}u_\alpha h_\beta\partial_\rho u_\sigma\left[\left(\frac{1}{T^2}\frac{\partial(T^3\beta)}{\partial T}+\varpi^2\frac{\partial(\beta/\varpi)}{\partial\varpi}\right)u^\mu u^\nu - \varpi^2\frac{\partial(\beta/\varpi)}{\partial\varpi}h^\mu h^\nu\right]\\
&\quad - 2u^{(\mu}\epsilon^{\nu)\lambda\rho\sigma}\left(T\beta h_\lambda\partial_\rho\frac{u_\sigma}{T}+\frac{1}{T}\nabla_\rho\left(T\beta h_\lambda u_\sigma\right)\right)\ ,\\
J^{\mu\nu}_{{\rm hs},\beta} &= -\epsilon^{\alpha\beta\rho\sigma}u_\alpha h_\beta\partial_\rho u_\sigma 2\varpi\frac{\partial(\beta/\varpi)}{\partial\varpi}u^{[\mu}h^{\nu]} + 2\frac{\beta}{\varpi}u^{[\mu}\epsilon^{\nu]\lambda\rho\sigma}u_\lambda\partial_\rho u_\sigma\ ,\\
N^{\mu}_{{\rm hs},\beta} &= \frac{\beta}{T^2}\epsilon^{\mu\nu\rho\sigma}h_\nu\partial_\rho(Tu_\sigma)\ ,
\end{aligned}$$

$$\begin{aligned}
T^{\mu\nu}_{{\rm hs},\tilde\beta_1} &= -h^\lambda\partial_\lambda T\left[\left(\frac{\partial(T\tilde\beta_1)}{\partial T}+\varpi^2\frac{\partial(\tilde\beta_1/\varpi)}{\partial\varpi}\right)u^\mu u^\nu - \varpi^2\frac{\partial(\tilde\beta_1/\varpi)}{\partial\varpi}h^\mu h^\nu\right]\\
&\quad - \tilde\beta_1 h^\lambda\partial_\lambda T g^{\mu\nu} + 2\tilde\beta_1 h^{(\mu}\nabla^{\nu)}T + T\nabla_\lambda\left(\tilde\beta_1 h^\lambda\right)u^\mu u^\nu\ ,\\
J^{\mu\nu}_{{\rm hs},\tilde\beta_1} &= -h^\lambda\partial_\lambda T 2\varpi\frac{\partial(\tilde\beta_1/\varpi)}{\partial\varpi}u^{[\mu}h^{\nu]} - 2\frac{\tilde\beta_1}{\varpi}u^{[\mu}\nabla^{\nu]}T\ ,\\
N^{\mu}_{{\rm hs},\tilde\beta_1} &= -\frac{2\tilde\beta_1}{T}u^{[\mu}h^{\lambda]}\partial_\lambda T\ ,
\end{aligned}$$

$$\begin{aligned}
T^{\mu\nu}_{{\rm hs},\tilde\beta_2} &= -h^\lambda\partial_\lambda\frac{\varpi}{T}\left[\left(\frac{\partial(T\tilde\beta_2)}{\partial T}+\varpi^2\frac{\partial(\tilde\beta_2/\varpi)}{\partial\varpi}\right)u^\mu u^\nu - \varpi^2\frac{\partial(\tilde\beta_2/\varpi)}{\partial\varpi}h^\mu h^\nu\right]\\
&\quad - \tilde\beta_2 h^\lambda\partial_\lambda\frac{\varpi}{T}g^{\mu\nu} + 2\tilde\beta_2 h^{(\mu}\nabla^{\nu)}\frac{\varpi}{T} - \nabla_\lambda\left(\tilde\beta_2 h^\lambda\right)\frac{\varpi}{T}h^\mu h^\nu\ ,\\
J^{\mu\nu}_{{\rm hs},\tilde\beta_2} &= -h^\lambda\partial_\lambda\frac{\varpi}{T}2\varpi\frac{\partial(\tilde\beta_2/\varpi)}{\partial\varpi}u^{[\mu}h^{\nu]} - 2\frac{\tilde\beta_2}{\varpi}u^{[\mu}\nabla^{\nu]}\frac{\varpi}{T} + 2\nabla_\lambda\left(\tilde\beta_2 h^\lambda\right)\frac{1}{T}u^{[\mu}h^{\nu]}\ ,\\
N^{\mu}_{{\rm hs},\tilde\beta_2} &= -\frac{2\tilde\beta_2}{T}u^{[\mu}h^{\lambda]}\partial_\lambda\frac{\varpi}{T}\ ,
\end{aligned}$$

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$$
\begin{aligned}
T^{\mu\nu}_{\text{hs},\tilde{\beta}_3} &= -\epsilon^{\alpha\beta\rho\sigma} u_\alpha h_\beta \partial_\rho h_\sigma \left[\left(\frac{1}{T^2} \frac{\partial(T^3 \tilde{\beta}_3)}{\partial T} + \varpi^3 \frac{\partial(\tilde{\beta}_3/\varpi^2)}{\partial \varpi} \right) u^\mu u^\nu - \varpi^3 \frac{\partial(\tilde{\beta}_3/\varpi^2)}{\partial \varpi} h^\mu h^\nu \right] \\
&\quad - 2\tilde{\beta}_3 u^{(\mu} \epsilon^{\nu)\lambda\rho\sigma} h_\lambda \partial_\rho h_\sigma \,, \\
J^{\mu\nu}_{\text{hs},\tilde{\beta}_3} &= -\epsilon^{\alpha\beta\rho\sigma} u_\alpha h_\beta \partial_\rho h_\sigma 2\varpi^2 \frac{\partial(\tilde{\beta}_3/\varpi^2)}{\partial \varpi} u^{[\mu} h^{\nu]} \\
&\quad + 2u^{[\mu} \epsilon^{\nu]\lambda\rho\sigma} \left(\frac{T\tilde{\beta}_3}{\varpi^2} u_\lambda \partial_\rho \frac{\varpi h_\sigma}{T} + \frac{1}{T} \nabla_\rho \left(\frac{T\tilde{\beta}_3}{\varpi} u_\lambda h_\sigma \right) \right) \,, \\
N^\mu_{\text{hs},\tilde{\beta}_3} &= \frac{\tilde{\beta}_3}{T} \epsilon^{\mu\nu\rho\sigma} u_\lambda h_\nu \partial_\rho h_\sigma \quad .
\end{aligned} \tag{A.2}
$$

The hydrostatic corrections in the electric limit of one-form superfluids are obtained from the respective hydrostatic free energy density given in eq. (6.13). The contributions from all terms except β and $\tilde{\beta}_i$ have already been discussed in section 6.3.1. The contribution from the remaining terms is precisely the same as in eq. (A.2) for string fluids.

A.1.2 Magnetohydrodynamics and magnetically dominated bound-charge plasma

Using the MHD free energy density (7.18), performing the relevant variations and ignoring certain second order contributions to the energy-momentum tensor, the constitutive relations are the sum of the following contributions

$$
\begin{aligned}
T^{\mu\nu}_{\text{hs},M_1} &= \left[\left(T \frac{\partial M_1}{\partial T} + \mu \frac{\partial M_1}{\partial \mu} \right) u^\mu u^\nu - 2B^2 \frac{\partial M_1}{\partial B^2} \mathbb{B}^{\mu\nu} \right] B^\lambda \partial_\lambda \frac{B^2}{T^4} \\
&\quad + 2M_1 u^\lambda \partial_\lambda \frac{B^2}{T^4} u^{(\mu} B^{\nu)} - M_1 B^\lambda \partial_\lambda \frac{B^2}{T^4} u^\mu u^\nu + \frac{2B^2}{T^4} \nabla_\lambda \left(M_1 B^\lambda \right) \left(\mathbb{B}^{\mu\nu} + 2u^\mu u^\nu \right) , \\
M^{\mu\nu}_{\text{hs},M_1} &= -2|B| \mathbb{E}^{\mu\nu} \frac{\partial M_1}{\partial B^2} B^\lambda \partial_\lambda \frac{B^2}{T^4} - M_1 \epsilon^{\mu\nu\rho\sigma} u_\rho \partial_\sigma \frac{B^2}{T^4} + \frac{2|B|}{T^4} \nabla_\lambda \left(M_1 B^\lambda \right) \mathbb{E}^{\mu\nu} \quad , \\
J^\mu_{\text{hs},M_1} &= u^\mu B^\lambda \partial_\lambda \frac{B^2}{T^4} \frac{\partial M_1}{\partial \mu} + \nabla_\nu M^{\mu\nu}_{\text{hs},M_1} \quad , \\
N^\mu_{\text{hs},M_1} &= \frac{2M_1}{T} u^{[\mu} B^{\nu]} \partial_\nu \frac{B^2}{T^4} \quad ,
\end{aligned}
$$

$$
\begin{aligned}
T^{\mu\nu}_{\text{hs},M_2} &= \left[\left(\frac{1}{T^2} \frac{\partial(T^3 M_2)}{\partial T} + \mu \frac{\partial M_2}{\partial \mu} \right) u^\mu u^\nu - 2B^2 \frac{\partial M_2}{\partial B^2} \mathbb{B}^{\mu\nu} \right] \epsilon^{\alpha\beta\rho\sigma} u_\alpha B_\beta \partial_\rho B_\sigma \\
&\quad + 2u^{(\mu} \epsilon^{\nu)\lambda\rho\sigma} M_2 B_\lambda \partial_\rho B_\sigma \\
&\quad - \epsilon^{\alpha\beta\rho\sigma} \left(T M_2 u_\alpha \partial_\rho \frac{B_\beta}{T} + \frac{1}{T} \nabla_\rho \left(T M_2 u_\alpha B_\beta \right) \right) \left(2B^{(\mu} P^{\nu)}{}_\sigma - B_\sigma (P^{\mu\nu} + u^\mu u^\nu) \right) , \\
M^{\mu\nu}_{\text{hs},M_2} &= -2|B| \mathbb{E}^{\mu\nu} \frac{\partial M_2}{\partial B^2} \epsilon^{\lambda\tau\rho\sigma} u_\lambda B_\tau \partial_\rho B_\sigma + \epsilon^{\mu\nu\lambda\tau} u_\lambda \epsilon^{ab\rho}{}_\tau \left(M_2 u_a \partial_\rho B_b + \nabla_\rho \left(M_2 u_a B_b \right) \right) \quad , \\
J^\mu_{\text{hs},M_2} &= u^\mu \epsilon^{\lambda\nu\rho\sigma} u_\lambda B_\nu \partial_\rho B_\sigma \frac{\partial M_2}{\partial \mu} + \nabla_\nu M^{\mu\nu}_{\text{hs},M_2} \quad , \\
N^\mu_{\text{hs},M_2} &= -\frac{M_2}{T} \epsilon^{\mu\nu\rho\sigma} B_\nu \partial_\rho B_\sigma \quad ,
\end{aligned}
$$

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$$
\begin{aligned}
T^{\mu\nu}_{\text{hs},M_3} &= -\left[\left(T\frac{\partial(M_3/T)}{\partial T}+\frac{\mu}{T}\frac{\partial M_3}{\partial\mu}\right)u^\mu u^\nu - 2\frac{B^2}{T}\frac{\partial M_3}{\partial B^2}\mathbb{B}^{\mu\nu}\right]B^\lambda\partial_\lambda T \\
&\quad - 2\frac{M_3}{T}u^\lambda\partial_\lambda T u^{(\mu}B^{\nu)} + \frac{M_3}{T}B^\lambda\partial_\lambda T u^\mu u^\nu + T\nabla_\lambda\left(\frac{M_3}{T}B^\lambda\right)u^\mu u^\nu, \\
M^{\mu\nu}_{\text{hs},M_3} &= 2|B|\mathbb{E}^{\mu\nu}\frac{\partial M_3}{\partial B^2}\frac{1}{T}B^\lambda\partial_\lambda T + \frac{M_3}{T}\epsilon^{\mu\nu\rho\sigma}u_\rho\partial_\sigma T \ , \\
J^\mu_{\text{hs},M_3} &= -\frac{1}{T}u^\mu B^\lambda\partial_\lambda T\frac{\partial M_3}{\partial\mu} + \nabla_\nu M^{\mu\nu}_{\text{hs},M_3} \ , \\
N^\mu_{\text{hs},M_3} &= -\frac{2M_3}{T^2}u^{[\mu}B^{\nu]}\partial_\nu T \ , \\
\\
T^{\mu\nu}_{\text{hs},M_4} &= -\left[\left(\frac{1}{T^2}\frac{\partial(T^3M_4}{\partial T}+\mu\frac{\partial M_4}{\partial\mu}\right)u^\mu u^\nu - 2B^2\frac{\partial M_4}{\partial B^2}\mathbb{B}^{\mu\nu}\right]\epsilon^{\alpha\beta\rho\sigma}u_\alpha B_\beta\partial_\rho u_\sigma \\
&\quad + \epsilon^{\beta\alpha\rho\sigma}M_4u_\alpha\partial_\rho u_\sigma\left(2B^{(\mu}P^{\nu)}{}_\beta - B_\beta(P^{\mu\nu}+u^\mu u^\nu)\right) \\
&\quad - 2u^{(\mu}\epsilon^{\nu)\lambda\rho\sigma}\left(TM_4B_\lambda\partial_\rho\frac{u_\sigma}{T}+\frac{1}{T}\nabla_\rho\left(TM_4u_\sigma B_\lambda\right)\right), \\
M^{\mu\nu}_{\text{hs},M_4} &= 2|B|\mathbb{E}^{\mu\nu}\frac{\partial M_4}{\partial B^2}\epsilon^{\lambda\tau\rho\sigma}u_\lambda B_\tau\partial_\rho u_\sigma + 2M_4P^{\mu\rho}P^{\nu\sigma}\partial_{[\rho}u_{\sigma]} \ , \\
J^\mu_{\text{hs},M_4} &= -u^\mu\epsilon^{\lambda\nu\rho\sigma}u_\lambda B_\nu\partial_\rho u_\sigma\frac{\partial M_4}{\partial\mu} + \nabla_\nu M^{\mu\nu}_{\text{hs},M_4} \ , \\
N^\mu_{\text{hs},M_4} &= \frac{M_4}{T^2}\epsilon^{\mu\nu\rho\sigma}B_\nu\partial_\rho(Tu_\sigma) \ , \\
\\
T^{\mu\nu}_{\text{hs},M_5} &= \left[\left(T\frac{\partial(TM_5)}{\partial T}+T\mu\frac{\partial M_5}{\partial\mu}\right)u^\mu u^\nu - 2TB^2\frac{\partial M_5}{\partial B^2}\mathbb{B}^{\mu\nu}\right]B^\lambda\partial_\lambda\frac{\mu}{T} \\
&\quad + 2TM_5u^\lambda\partial_\lambda\frac{\mu}{T}u^{(\mu}B^{\nu)} - TM_5B^\lambda\partial_\lambda\frac{\mu}{T}u^\mu u^\nu \ , \\
M^{\mu\nu}_{\text{hs},M_5} &= -2|B|\mathbb{E}^{\mu\nu}\frac{\partial M_5}{\partial B^2}TB^\lambda\partial_\lambda\frac{\mu}{T} - TM_5\epsilon^{\mu\nu\rho\sigma}u_\rho\partial_\sigma\frac{\mu}{T} \ , \\
J^\mu_{\text{hs},M_5} &= Tu^\mu B^\lambda\partial_\lambda\frac{\mu}{T}\frac{\partial M_5}{\partial\mu} - \frac{1}{T}u^\mu\nabla_\lambda\left(TM_5B^\lambda\right) + \nabla_\nu M^{\mu\nu}_{\text{hs},M_5} \ , \\
N^\mu_{\text{hs},M_5} &= 2M_5u^{[\mu}B^{\nu]}\partial_\nu\frac{\mu}{T} \ .
\end{aligned}
\tag{A.3}
$$

In the case of magnetically dominated bound-charge plasmas with the hydrostatic free energy density (7.67), the contributions from $M_{1,2,3,4}$ to the respective hydrostatic constitutive relations are just given in terms of the MHD expressions in eq. (A.3), except that the transport coefficients are taken to be independent of μ.

A.2 Mapping MHD to string fluids

In this appendix the details of the mapping between MHD and string fluid constitutive relations at first order in derivatives given in section 7.3.2 are provided.

A.2.1 Eliminating chemical potential and electric field

To begin with, we take the hydrostatic energy-momentum tensor for MHD from appendix A.1.2 and introduce it in the solutions for μ for E_μ given in eq. (7.26) coming from

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Maxwell's equations. These are described in terms of 6 transport coefficients $P(T,\mu,B^2)$ and $M_{1,2,3,4,5}(T,\mu,B^2)$. We expand $P(T,\mu,B^2)$ around $\mu=\mu_0(T,B^2)$ up to second order in derivatives

$$P(T,\mu,B^2)=P_0(T,B^2)+\frac{1}{2}P_2(T,B^2)\left(\mu-\mu_0(T,B^2)\right)^2+\mathcal{O}(\partial^3)\ . \tag{A.4}$$

Here we have used the defining relation of μ_0 from eq. (7.27). Representing the μ solution in eq. (7.26) as $\mu=\mu_0+\delta\mu$, up to the first order in derivatives, we can work out

$$\begin{aligned}
P(T,\mu,B^2)&=P_0(T,B^2)+\mathcal{O}(\partial^2)\ ,\\
\varpi(T,\mu,B^2)&=-2|B|\frac{\partial P_0(T,B^2)}{\partial B^2}+2|B|P_2(T,B^2)\frac{\partial\mu_0(T,B^2)}{\partial B^2}\delta\mu+\mathcal{O}(\partial^2)\ ,\\
\epsilon(T,\mu,B^2)&=T\frac{\partial P_0(T,B^2)}{\partial T}-P_0(T,B^2)-T^2\frac{\partial}{\partial T}\left(\frac{\mu_0(T,B^2)}{T}\right)P_2(T,B^2)\delta\mu+\mathcal{O}(\partial^2)\ .
\end{aligned} \tag{A.5}$$

In an analogous manner, we can expand $M_{1,2,3,4,5}(T,\mu,B^2)$ up to the first order in derivatives

$$M_i(T,\mu,B^2)=M_{i,0}(T,B^2)+M_{i,1}(T,B^2)\left(\mu-\mu_0(T,B^2)\right)+\mathcal{O}(\partial^2)\ , \tag{A.6}$$

which after eliminating μ leads to

$$\begin{aligned}
M_i(T,\mu,B^2)&=M_{i,0}(T,B^2)+\mathcal{O}(\partial)\ ,\\
\frac{\partial M_i(T,\mu,B^2)}{\partial T}&=\frac{\partial M_{i,0}(T,B^2)}{\partial T}-M_{i,1}(T,B^2)\frac{\partial\mu_0(T,B^2)}{\partial T}+\mathcal{O}(\partial)\ ,\\
\frac{\partial M_i(T,\mu,B^2)}{\partial B^2}&=\frac{\partial M_{i,0}(T,B^2)}{\partial B^2}-M_{i,1}(T,B^2)\frac{\partial\mu_0(T,B^2)}{\partial B^2}+\mathcal{O}(\partial)\ ,\\
\frac{\partial M_i(T,\mu,B^2)}{\partial\mu}&=M_{i,1}(T,B^2)+\mathcal{O}(\partial)\ .
\end{aligned} \tag{A.7}$$

Schematically, splitting the first order hydrostatic contributions to the MHD energy-momentum tensor as $T^{\mu\nu}_{\text{hs},M_i}=T^{\mu\nu}_{\text{hs},M_{i,0}}+T^{\mu\nu}_{\text{hs},M_{i,1}}$ and plugging in the μ and E_μ solution from eq. (7.26), after a straight-forward computation, we can show that

$$\begin{aligned}
&T^{\mu\nu}_{\text{hs},P}+\sum_i T^{\mu\nu}_{\text{hs},M_{i,1}}\\
&\quad=\left(T\frac{\partial P_0}{\partial T}\right)u^\mu u^\nu+P_0g^{\mu\nu}-2B^2\frac{\partial P_0}{\partial B^2}\mathbb{B}^{\mu\nu}+4T|B|\frac{\partial P_0}{\partial B^2}u^{(\mu}\mathbb{E}^{\nu)\sigma}\partial_\sigma\frac{\mu_0}{T}\\
&\qquad+\left(2B^2\frac{\partial\mu_0}{\partial B^2}\mathbb{B}^{\mu\nu}-T^2\frac{\partial(\mu_0/T)}{\partial T}u^\mu u^\nu\right)\\
&\qquad\times\left[u_\mu J^\mu_{\text{ext}}+\frac{1}{T}\nabla_\lambda\left(TM_{5,0}B^\lambda\right)+2\frac{\partial P_0}{\partial B^2}\epsilon^{\lambda\tau\rho\sigma}B_\lambda u_\tau\partial_\rho u_\sigma\right]\\
&\qquad-4T|B|\frac{\partial P_0}{\partial B^2}u^{(\mu}\left[\frac{1}{\sigma_\perp^2+\tilde{\sigma}^2}\left(\frac{T\partial P_0/\partial T}{T\partial P_0/\partial T-2B^2\partial P_0/\partial B^2}\right)\right.
\end{aligned}$$

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$$\times \left(-\sigma_\perp \mathbb{B}^{\nu)\rho}\hat{B}^\sigma + \tilde{\sigma}_\perp \mathbb{E}^{\nu)\rho}\hat{B}^\sigma\right) X_{\rho\sigma}$$
$$+\frac{2}{\sigma_\perp^2+\tilde{\sigma}_\perp^2}\left(-\left(\tilde{\sigma}_\perp\sigma'_\times - \sigma_\perp\tilde{\sigma}'_\times\right)\mathbb{B}^{\nu)(\rho}\hat{B}^{\sigma)} - \left(\tilde{\sigma}\tilde{\sigma}'_\times + \sigma_\perp\sigma'_\times\right)\mathbb{E}^{\nu)(\rho}\hat{B}^{\sigma)}\right)\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma}\Bigg] \ . \tag{A.8}$$

Interestingly, the $M_{i,1}$ contributions entirely drop out. On the other hand, for the remaining hydrostatic contributions we have

$$\begin{aligned}
T^{\mu\nu}_{\text{hs},M_{1,0}} &= \left[T\frac{\partial M_{1,0}}{\partial T}u^\mu u^\nu - 2B^2\frac{\partial M_{1,0}}{\partial B^2}\mathbb{B}^{\mu\nu}\right]B^\lambda\partial_\lambda\frac{B^2}{T^4} + 2M_{1,0}u^\lambda\partial_\lambda\frac{B^2}{T^4}u^{(\mu}B^{\nu)} \\
&\quad - M_{1,0}B^\lambda\partial_\lambda\frac{B^2}{T^4}u^\mu u^\nu + \frac{2B^2}{T^4}\nabla_\lambda\left(M_{1,0}B^\lambda\right)\left(\mathbb{B}^{\mu\nu}+2u^\mu u^\nu\right) \ , \\
T^{\mu\nu}_{\text{hs},M_{2,0}} &= \left[\frac{1}{T^2}\frac{\partial(T^3M_{2,0})}{\partial T}u^\mu u^\nu - 2B^2\frac{\partial M_{2,0}}{\partial B^2}\mathbb{B}^{\mu\nu}\right]\epsilon^{\alpha\beta\rho\sigma}u_\alpha B_\beta\partial_\rho B_\sigma \\
&\quad + 2M_{2,0}u^{(\mu}\epsilon^{\nu)\lambda\rho\sigma}B_\lambda\partial_\rho B_\sigma \\
&\quad - \epsilon^{\alpha\beta\rho\sigma}\left(TM_{2,0}u_\alpha\partial_\rho\frac{B_\beta}{T}+\frac{1}{T}\nabla_\rho\left(TM_{2,0}u_\alpha B_\beta\right)\right)\left(2B^{(\mu}P^{\nu)}{}_\sigma - B_\sigma(P^{\mu\nu}+u^\mu u^\nu)\right), \\
T^{\mu\nu}_{\text{hs},M_{3,0}} &= -\left[T\frac{\partial(M_{3,0}/T)}{\partial T}u^\mu u^\nu - 2\frac{B^2}{T}\frac{\partial M_{3,0}}{\partial B^2}\mathbb{B}^{\mu\nu}\right]B^\lambda\partial_\lambda T \\
&\quad - 2\frac{M_{3,0}}{T}u^\lambda\partial_\lambda T u^{(\mu}B^{\nu)} + \frac{M_{3,0}}{T}B^\lambda\partial_\lambda T u^\mu u^\nu + T\nabla_\lambda\left(\frac{M_{3,0}}{T}B^\lambda\right)u^\mu u^\nu, \\
T^{\mu\nu}_{\text{hs},M_{4,0}} &= -\left[\frac{1}{T^2}\frac{\partial(T^3M_{4,0})}{\partial T}u^\mu u^\nu - 2B^2\frac{\partial M_{4,0}}{\partial B^2}\mathbb{B}^{\mu\nu}\right]\epsilon^{\alpha\beta\rho\sigma}u_\alpha B_\beta\partial_\rho u_\sigma \\
&\quad + \epsilon^{\beta\alpha\rho\sigma}M_{4,0}u_\alpha\partial_\rho u_\sigma\left(2B^{(\mu}P^{\nu)}{}_\beta - B_\beta(P^{\mu\nu}+u^\mu u^\nu)\right) \\
&\quad - 2u^{(\mu}\epsilon^{\nu)\lambda\rho\sigma}\left(TM_{4,0}B_\lambda\partial_\rho\frac{u_\sigma}{T}+\frac{1}{T}\nabla_\rho\left(TM_{4,0}u_\sigma B_\lambda\right)\right), \\
T^{\mu\nu}_{\text{hs},M_{5,0}} &= \left[T\frac{\partial(TM_{5,0})}{\partial T}u^\mu u^\nu - 2TB^2\frac{\partial M_{5,0}}{\partial B^2}\mathbb{B}^{\mu\nu}\right]B^\lambda\partial_\lambda\frac{\mu_0}{T} \\
&\quad + 2TM_{5,0}u^\lambda\partial_\lambda\frac{\mu_0}{T}u^{(\mu}B^{\nu)} - TM_{5,0}B^\lambda\partial_\lambda\frac{\mu_0}{T}u^\mu u^\nu.
\end{aligned} \tag{A.9}$$

A.2.2 Velocity field redefinition

As we suggested in section 7.3.2, the map between string fluids and MHD can involve a non-trivial non-hydrostatic redefinition of u^μ and T. In the following, we find that it is sufficient to perform a redefinition of u^μ alone. With this hindsight, consider a redefinition of the fluid velocity

$$u^\mu \to u^\mu + \delta u^\mu \ , \tag{A.10}$$

such that $u_\mu\delta u^\mu = 0$, where δu^μ is purely non-hydrostatic. The electromagnetic fields get a contribution from δu^μ via

$$B^\mu \to B^\mu + u^\mu B_\nu\delta u^\nu + \mathcal{O}(\partial^2) \ , \quad E_\mu \to E_\mu - |B|\mathbb{E}_{\mu\nu}\delta u^\nu + \mathcal{O}(\partial^2) \ . \tag{A.11}$$

Note that B^2 is invariant to first order. Interestingly, despite being itself first order, E_μ shifts with a first-order piece. Therefore, the equation determining the E_μ in eq. (7.26)

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modifies to

$$
\begin{aligned}
P^{\mu\nu}\delta_{\mathcal{B}}A_\nu = &\frac{1}{2}\frac{1}{\sigma_\parallel}\hat{B}^\mu\mathbb{E}^{\rho\sigma}\delta_{\mathcal{B}}b_{\rho\sigma} - \frac{1}{\sigma_\parallel}\hat{B}^\mu\left(\tilde{\chi}_1'\mathbb{B}^{\rho\sigma} + \tilde{\chi}_2'\hat{B}^\rho\hat{B}^\sigma\right)\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma} \\
&+ \left(\frac{\epsilon+P}{\epsilon+P+\varpi|B|}\right)\left(\frac{\sigma_\perp}{\sigma_\perp^2+\tilde{\sigma}^2}\mathbb{E}^{\mu\rho}\hat{B}^\sigma + \frac{\tilde{\sigma}_\perp}{\sigma_\perp^2+\tilde{\sigma}^2}\mathbb{B}^{\mu\rho}\hat{B}^\sigma\right)\delta_{\mathcal{B}}b_{\rho\sigma} \\
&+ \left(\frac{\tilde{\sigma}_\perp\sigma_\times' - \sigma_\perp\tilde{\sigma}_\times'}{\sigma_\perp^2+\tilde{\sigma}^2}\mathbb{E}^{\mu(\rho}\hat{B}^{\sigma)} - \frac{\tilde{\sigma}_\perp\tilde{\sigma}_\times' + \sigma_\perp\sigma_\times'}{\sigma_\perp^2+\tilde{\sigma}^2}\mathbb{B}^{\mu(\rho}\hat{B}^{\sigma)}\right)\delta_{\mathcal{B}}g_{\rho\sigma} \\
&- \frac{|B|}{T}\mathbb{E}^{\mu\nu}\delta u_\nu \ , \qquad \text{(A.12)}
\end{aligned}
$$

where we have used the mapping for $X_{\mu\nu}$ given in eq. (7.41). The hydrostatic constitutive relations discussed in the previous subsection get corrected by these redefinition and obtain a δu^μ contribution to the energy-momentum tensor

$$
T^{\mu\nu}_{\delta u} = 2u^{(\mu}\left[T\frac{\partial P_0}{\partial T}\hat{B}^{\nu)}\hat{B}_\lambda + \left(T\frac{\partial P_0}{\partial T} - 2B^2\frac{\partial P_0}{\partial B^2}\right)\mathbb{B}^{\nu)}{}_\lambda\right]\delta u^\lambda \ . \qquad \text{(A.13)}
$$

To find what the relative field redefinition for u^μ between MHD and string fluid is, we need to compare the energy-momentum tensor in the two formulations. Substituting B^μ in eqs. (A.8) and A.9 using eq. (7.40), and invoking the mapping between hydrostatic transport coefficients given in eq. (7.48), we can show that

$$
T^{\mu\nu}_{\text{MHD,hs}} + T^{\mu\nu}_{\delta u} = T^{\mu\nu}_{\text{string,hs}} \ , \qquad \text{(A.14)}
$$

for δu^μ given in eq. (7.50). Here $T^{\mu\nu}_{\text{string,hs}}$ are the hydrostatic corrections to string fluid constitutive relations worked out in appendix A.1.1.

A.2.3 Mapping of non-hydrostatic transport coefficients

We have already mapped the hydrostatic transport coefficients between the two formulations in eq. (7.48) using the hydrostatic free-energy density. To find the mapping between non-hydrostatic transport coefficients, let us substitute δu^μ from eq. (7.50) into eq. (A.12) and obtain

$$
\begin{aligned}
P^{\mu\nu}\delta_{\mathcal{B}}A_\nu = &\frac{\alpha\rho}{\epsilon+p}\Delta^{\mu\sigma}h^\nu\delta_{\mathcal{B}}b_{\sigma\nu} + \frac{1}{2}\frac{1}{\sigma_\parallel}h^\mu\epsilon^{\rho\sigma}\delta_{\mathcal{B}}b_{\rho\sigma} - \frac{1}{\sigma_\parallel}h^\mu\left(\tilde{\chi}_1'\Delta^{\rho\sigma} + \tilde{\chi}_2'h^\rho h^\sigma\right)\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma} \\
&+ \frac{sT}{(\epsilon+p)}\left(\frac{\tilde{\sigma}_\perp\sigma_\times' - \sigma_\perp\tilde{\sigma}_\times'}{\sigma_\perp^2+\tilde{\sigma}^2}\epsilon^{\mu(\rho}h^{\sigma)} - \frac{\tilde{\sigma}_\perp\tilde{\sigma}_\times' + \sigma_\perp\sigma_\times'}{\sigma_\perp^2+\tilde{\sigma}^2}\Delta^{\mu(\rho}h^{\sigma)}\right)\delta_{\mathcal{B}}g_{\rho\sigma} \\
&- \left(\frac{sT}{\epsilon+p}\right)^2\left(\frac{-\tilde{\sigma}_\perp}{\sigma_\perp^2+\tilde{\sigma}^2} + \frac{2\alpha\rho}{sT}\right)\Delta^{\mu\rho}h^\sigma\delta_{\mathcal{B}}b_{\rho\sigma} \\
&+ \left(\frac{sT}{\epsilon+p}\right)^2\frac{\sigma_\perp}{\sigma_\perp^2+\tilde{\sigma}^2}\epsilon^{\mu\rho}h^\sigma\delta_{\mathcal{B}}b_{\rho\sigma} \ . \qquad \text{(A.15)}
\end{aligned}
$$

Comparing it to the version obtained via the identification with string fluids

$$
\begin{aligned}
P^{\mu\nu}\delta_{\mathcal{B}}A_\nu &= P^{\mu\nu}\partial_\nu\frac{\mu}{T} - \frac{1}{T}E^\mu \\
&= P^{\mu\nu}\partial_\nu\frac{\alpha}{T} + \frac{1}{2T}\epsilon^{\mu\nu\rho\sigma}u_\nu J_{\rho\sigma} + \mathcal{O}(\partial^2)
\end{aligned}
$$

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$$
\begin{aligned}
&= \frac{\alpha\rho}{\epsilon+p}\Delta^{\mu\sigma}h^{\nu}\delta_{\mathcal{B}}b_{\sigma\nu} + \left(r'_{\times}\epsilon^{\mu\alpha}h^{\beta} - \tilde{r}'_{\times}\Delta^{\mu\alpha}h^{\beta}\right)\delta_{\mathcal{B}}g_{\alpha\beta} \\
&\quad + \left(r_{\perp}\epsilon^{\mu\alpha}h^{\beta} - \tilde{r}_{\perp}\Delta^{\mu\alpha}h^{\beta}\right)\delta_{\mathcal{B}}b_{\alpha\beta} \\
&\quad + \frac{1}{2}h^{\mu}\left(\tilde{\kappa}'_1\Delta^{\alpha\beta} + \tilde{\kappa}'_2 h^{\mu}h^{\nu}\right)\delta_{\mathcal{B}}g_{\mu\nu} + \frac{1}{2}h^{\mu}r_{\parallel}\epsilon^{\alpha\beta}\delta_{\mathcal{B}}b_{\alpha\beta} \ , \qquad \text{(A.16)}
\end{aligned}
$$

we can read out part of the non-hydrostatic map in eq. (7.54). For the remaining part, we need to compare the non-hydrostatic energy-momentum tensors in the two pictures. This is done trivially by taking the MHD non-hydrostatic energy-momentum tensor, before the field redefinition, from eq. (7.19), substitute for the electric fields using eq. (7.26), and comparing it with the string fluid expressions in eq. (5.16). This finishes the mapping of all the first-order transport coefficients presented in section 7.3.2.

A.3 Mapping magnetically dominated BCP to one-form superfluids

The mapping from magnetically dominated bound-charge plasma to one-form superfluids is considerably less involved because we do not have to eliminate the chemical potential. Furthermore, as it turns out, we do not need to perform a hydrodynamic field redefinition to map the two formulations. Firstly, we note that the magnetic and electric fields in a magnetically dominated plasma are given in terms of the electric limit of one-form superfluids discussed in section 6 according to

$$
\begin{aligned}
B^{\mu} &= J^{\mu\nu}u_{\nu} \\
&= -q\,\zeta^{\mu} + \delta\rho\,h^{\mu} - n^{\mu} \\
&\quad - h^{\mu}\Bigg[\epsilon^{\alpha\beta\rho\sigma}u_{\alpha}h_{\beta}\partial_{\rho}u_{\sigma}\frac{\partial\beta}{\partial\varpi} + h^{\lambda}\partial_{\lambda}T\frac{\partial\tilde{\beta}_1}{\partial\varpi} + h^{\lambda}\partial_{\lambda}\frac{\varpi}{T}\frac{\partial\tilde{\beta}_2}{\partial\varpi} \\
&\qquad\qquad - \nabla_{\lambda}\left(\tilde{\beta}_2 h^{\lambda}\right)\frac{1}{T} + \epsilon^{\alpha\beta\rho\sigma}u_{\alpha}h_{\beta}\partial_{\rho}h_{\sigma}\frac{\partial\tilde{\beta}_3}{\partial\varpi}\Bigg] \\
&\quad + \frac{\beta}{\varpi}\Delta^{\mu}{}_{\nu}\epsilon^{\nu\lambda\rho\sigma}u_{\lambda}\partial_{\rho}u_{\sigma} - \frac{\tilde{\beta}_1}{\varpi}\Delta^{\mu\nu}\partial_{\nu}T - \frac{\tilde{\beta}_2}{\varpi}\Delta^{\mu\nu}\partial_{\nu}\frac{\varpi}{T} \\
&\quad + \Delta^{\mu}{}_{\nu}\epsilon^{\nu\lambda\rho\sigma}\left(\frac{T\tilde{\beta}_3}{\varpi^2}u_{\lambda}\partial_{\rho}\frac{\varpi h_{\sigma}}{T} + \frac{1}{T}\nabla_{\rho}\left(\frac{T\tilde{\beta}_3}{\varpi}u_{\lambda}h_{\sigma}\right)\right) + \mathcal{O}(\partial^2) \ , \\
E^{\mu} &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}u^{\nu}J^{\rho\sigma} \\
&= -q_{\times}\zeta^{\mu} - \left(q'_{\times}\zeta^{\mu}\zeta^{\nu} + \bar{q}P^{\mu\nu}\right)\bar{\zeta}_{\nu} - R^{\mu} + h^{\mu}\delta s + \epsilon^{\mu\nu}m_{\nu} + \mathcal{O}(\partial^2) \ . \qquad \text{(A.17)}
\end{aligned}
$$

Unlike the MHD mapping in eq. (7.40), the magnetic fields do get a non-hydrostatic contribution in a magnetically dominated plasma. Note that for E^{μ} to be $\mathcal{O}(\partial)$, we need to set $q_{\times} = 0$. Using the map between free energy currents in the two formulations given in eq. (7.59), we can find a mapping for hydrostatic free-energy densities according to

$$
\mathcal{N}_{\text{BCP}} = \mathcal{N}_{\text{1SF}} + B^{\mu}\zeta_{\mu} + E_{\mu}\bar{\zeta}^{\mu} \ . \qquad \text{(A.18)}
$$

Plugging in the expressions for B^{μ} and E^{μ} from above, this trivially leads to the hydrostatic sector mapping given in eq. (7.71).

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To map the respective non-hydrostatic sector transport coefficients, we need to explicitly compare the constitutive relations for the energy-momentum tensor in the two formulations, along with the map $M^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\xi_{\rho\sigma}$. After an involved algebra, we find

$$\begin{aligned}
T^{\mu\nu}_{\text{BCP}} &= T^{\mu\nu}_{\text{1SF}} \\
&\quad - \left(\delta f - \frac{\rho\delta\rho}{\partial\rho/\partial\varpi} - \delta\mathcal{F}\right)\Delta^{\mu\nu} - (\delta\tau + \varpi\delta\rho - \delta\mathcal{T})\, h^\mu h^\nu \\
&\quad - 2(\ell^{(\mu} - \varpi n^{(\mu} - \mathcal{L}^{(\mu})h^{\nu)} - (t^{\mu\nu} - \mathcal{T}^{\mu\nu}) + \mathcal{O}(\partial^2) \ , \\
M^{\mu\nu} &= -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\xi_{\rho\sigma} \\
&\quad + 2\left(\delta\mathcal{R} - (\alpha'_{EB}B^2 + \alpha_{EE})\delta s\right) u^{[\mu}h^{\nu]} + \left(\delta\mathcal{S} + \frac{\delta\rho}{\partial\rho/\partial\varpi}\right)\epsilon^{\mu\nu} \\
&\quad - 2h^{[\mu}\left(\mathcal{M}^{\nu]} + \frac{\varpi}{\rho}\epsilon^{\nu]\sigma}n_\sigma\right) - 2u^{[\mu}\left(\mathcal{N}^{\nu]} + \alpha_{EE}\epsilon^{\nu]\lambda}m_\lambda\right) + \mathcal{O}(\partial^2) \ .
\end{aligned} \tag{A.19}$$

Various non-hydrostatic corrections appearing here are defined in eqs. (6.17) and (7.69). Since for the map to work the last two lines in both the expressions above must vanish, this trivially leads to the mapping in the non-hydrostatic sector given in eq. (7.73).

B Comparison with the effective action approach

In this appendix we perform a comparison between the work of [30] and the equilibrium partition function construction that we provided in [24]. Additionally, we use the construction of [30] in order to generalise their results so as to obtain an ideal order effective action for the one-form hydrodynamic theories of section 3 (unbroken phase) and section 4 (fully broken phase).

Following [30, 43], we introduce a "fluid spacetime" with coordinates σ^a. A point on this spacetime represents a "fluid element" parametrised by $\sigma^{i=1,2,3}$ at some choice of internal time σ^0. On this fluid spacetime, we define the coordinate fields $x^\mu(\sigma)$ which represent the physical spacetime coordinates of the fluid element. Under a spacetime diffeomorphism $\chi^\mu(x)$, these fields transform as

$$x^\mu(\sigma) \to x^\mu(\sigma) + \chi^\mu(x(\sigma)) \ . \tag{B.1}$$

When the fluid is charged under a U(1) zero-form symmetry, we also associate with every fluid element a phase field $\phi(\sigma)$. In the case of a one-form symmetry, we instead introduce a one-form phase $\varphi_a(\sigma)$ as in [30]. These phases do not transform under spacetime diffeomorphisms,[25] but shift under the respective gauge transformations $\Lambda^\chi(x)$ and $\Lambda^\chi_\mu(x)$

$$\phi(\sigma) \to \phi(\sigma) - \Lambda^\chi(x(\sigma)), \qquad \varphi_a(\sigma) \to \varphi_a(\sigma) - \frac{\partial x^\mu(\sigma)}{\partial\sigma^a}\Lambda^\chi_\mu(x(\sigma)) \ . \tag{B.2}$$

[25] We can pushforward these phases onto the physical spacetime as $\phi(x) = \phi(\sigma(x))$ and $\varphi_\mu(x) = \frac{\partial\sigma^a(x)}{\partial x^\mu}\varphi_a(\sigma(x))$, which have the expected transformation properties $\delta_\chi\phi(x) = \pounds_\chi\phi(x) - \Lambda^\chi(x)$ and $\delta_\chi\varphi_\mu(x) = \pounds_\chi\varphi_\mu(x) - \Lambda^\chi_\mu(x)$. In this case, the field φ_μ, already introduced in [24], coincides with that defined in (4.1).

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The fields $x^\mu(\sigma)$ together with $\phi(\sigma)$, or $\varphi_a(\sigma)$ for the one-form case, form the effective dynamical fields of hydrodynamics. Given the background fields $g_{\mu\nu}(x)$, $A_\mu(x)$, and $b_{\mu\nu}(x)$ on the physical spacetime, we can define their pullbacks onto the fluid spacetime as

$$\begin{aligned} h_{ab}(\sigma) &= \frac{\partial x^\mu(\sigma)}{\partial \sigma^a}\frac{\partial x^\nu(\sigma)}{\partial \sigma^b} g_{\mu\nu}(x(\sigma)) \ , \\ B_a(\sigma) &= \frac{\partial x^\mu(\sigma)}{\partial \sigma^a} A_\mu(x(\sigma)) + \frac{\partial \phi(\sigma)}{\partial \sigma^a} \ , \\ B_{ab}(\sigma) &= \frac{\partial x^\mu(\sigma)}{\partial \sigma^a}\frac{\partial x^\nu(\sigma)}{\partial \sigma^b} b_{\mu\nu}(x(\sigma)) + \frac{\partial \varphi_b(\sigma)}{\partial \sigma^a} - \frac{\partial \varphi_a(\sigma)}{\partial \sigma^b} \ . \end{aligned} \tag{B.3}$$

These fields have been defined such that they are invariant under the symmetry transformations of the physical spacetime. In fact, they constitute the most general invariants made out of dynamical and background fields.

Given these elements, we wish to construct a Wilsonian effective action for hydrodynamics involving the fields in eq. (B.3), with certain symmetries imposed on the fluid spacetime, so that we can recover the hydrodynamic dynamical equations via a variational principle [43]. The physical picture to keep in mind is that every distinct fluid element, parametrised by σ^i, is evolving along the internal time σ^0. We expect the hydrodynamic description to be invariant under an arbitrary relabelling of the fluid elements and the choice of internal time for each fluid element, leading to the symmetries

$$\sigma^a \to \sigma^a + f^a(\vec{\sigma}) \ . \tag{B.4}$$

Note that we are not allowing for a time-dependent redefinition of σ^a, since we require each fluid element and its choice of internal time to stay the same as it moves through time. The transformations eq. (B.4) are the most general fluid spacetime diffeomorphisms which leave the internal time vector $\partial/\partial\sigma^0$ invariant.

In addition, we allow each fluid element to independently choose the associated U(1) phase, leading to the shift symmetry

$$\phi(\sigma) \to \phi(\sigma) + \lambda(\vec{\sigma}) \ , \quad \varphi_a(\sigma) \to \varphi_a(\sigma) + \lambda_a(\vec{\sigma}) \ . \tag{B.5}$$

Note that we are also requiring the choice of phase to remain the same as the fluid element moves through time. We expect the symmetries (B.5) to hold when the underlying U(1) symmetry is not spontaneously broken. To motivate this, let us consider the zero-form case first. At each point $p = (\sigma^a_p)$ in the fluid spacetime, we can define a charged operator

$$V_p = \exp(i\phi(\sigma_p)) \ . \tag{B.6}$$

Under the shift (B.5), these operators admit a phase rotation

$$V_p \to \exp(i\lambda(\vec{\sigma}_p))\, V_p \ , \tag{B.7}$$

which is independent for every fluid element, but remains fixed as the charged operator moves through time. When the symmetry is spontaneously broken, the system picks a random preferred phase in the ground state and the respective shift symmetry in eq. (B.5)

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should be dropped. In this case, the phase pushforward onto the physical spacetime $\phi(x) = \phi(\sigma(x))$ acts as the Goldstone mode of the broken symmetry, and we are led to the physics of zero-form superfluid dynamics.

In the one-form case, on the other hand, the charged operators are defined over non-local "strings" of fluid elements. Let us consider a space-like curve C in the fluid spacetime defined in terms of an internal length parameter ℓ as $\sigma^a = \sigma^a_C(\ell)$. We can then define the operator

$$V_C = \exp\left(i\int_C \varphi_a(\sigma)\,\mathrm{d}\sigma^a\right) = \exp\left(i\int \varphi_a(\sigma_C(\ell))\frac{\mathrm{d}\sigma^a_C(\ell)}{\mathrm{d}\ell}\,\mathrm{d}\ell\right) \quad . \tag{B.8}$$

Under the shift (B.5), this charged operator acquires a phase rotation given by operator

$$V_C \to \exp\left(i\int \lambda_a(\vec{\sigma}_C(\ell))\frac{\mathrm{d}\sigma^a_C(\ell)}{\mathrm{d}\ell}\,\mathrm{d}\ell\right) V_C \ , \tag{B.9}$$

which is independent for every string of fluid elements, but remains fixed if a string moves uniformly in time: $\sigma^0_C(\ell) \to \sigma^0_C(\ell) + \tau$, where τ is independent of ℓ. Using the analogy with the zero-form case, we understand that when the shift symmetry (B.5) is dropped, the system picks up a preferred one-form phase in its ground state spontaneously breaking the symmetry. The pushforward of the one-form phase $\varphi_\mu(x) = \frac{\partial\sigma^a(x)}{\partial x^\mu}\varphi_a(\sigma(x))$ can be identified with the Goldstone mode of this broken symmetry. Interestingly, in this case there is another choice available to us. We can require the choice of phase to be fixed under a non-uniform movement of the string in time: $\sigma^0_C(\ell) \to \sigma^0_C(\ell) + \tau(\ell)$, which implies dropping the time component of the one-form shift in eq. (B.5) setting $\lambda_0(\sigma) = 0$. Since the $\varphi_0(\sigma)$ component of the phase does not admit any redefinition in this case, we can interpret its pushforward onto the physical spacetime $\varphi(x) = \varphi_0(\sigma(x))$ as a scalar Goldstone. This is the partial symmetry breaking of one-form hydrodynamics eluded to in section 5.[26]

Having identified the dynamical degrees of freedom and symmetries, one can construct the most generic hydrodynamic effective action arranged in a derivative expansion leading to a particular subsector of non-dissipative constitutive relations. We do not repeat this exercise here and we encourage interested readers to consult the relevant papers such as [30, 43]. However, to make contact with the hydrodynamic formulation used in the bulk of this paper, it is instructive to map the dynamical degrees of freedom in the two pictures. Starting with the symmetry unbroken phase, we can identify the hydrodynamic fields $\mathcal{B} = (\beta^\mu, \Lambda^\beta)$ or $\mathcal{B} = (\beta^\mu, \Lambda^\beta_\mu)$ introduced around eq. (2.5) and eq. (2.10) respectively as

$$\beta^\mu(x) = \left.\frac{\partial x^\mu(\sigma)}{\partial\sigma^0}\right|_{\sigma=\sigma(x)}, \qquad \Lambda^\beta(x) = \left.\frac{\partial\phi(\sigma)}{\partial\sigma^0}\right|_{\sigma=\sigma(x)}, \qquad \Lambda^\beta_\mu(x) = \left.\frac{\partial\sigma^a(x)}{\partial x^\mu}\frac{\partial\varphi_a(\sigma)}{\partial\sigma^0}\right|_{\sigma=\sigma(x)}. \tag{B.10}$$

[26] The effective action framework of [30] for MHD/string fluids deals with this partially broken picture of one-form hydrodynamics where $\lambda_0(\sigma) = 0$. The authors rightly note that the pullback of the full one-form phase $\varphi_\mu(x)$ is not a Goldstone in this picture, as we see that the one-form symmetry is only partially broken. However, the authors do not identify the pullback of the time component $\varphi(x)$ as a Goldstone mode either. Note that v1 of [30] on arXiv has a typo in equation (2.18) as we confirmed with the authors: the shift symmetry is only imposed in the spatial directions.

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These are invariant under the fluid spacetime symmetries in eqs. (B.4) and (B.5). In terms of the conventional fields, we equivalently have

$$u^\mu(x) = \frac{1}{\sqrt{-h_{00}(\sigma(x))}} \frac{\partial x^\mu(\sigma)}{\partial \sigma^0}\bigg|_{\sigma=\sigma(x)} , \qquad T(x) = \frac{1}{\sqrt{-h_{00}(\sigma(x))}} ,$$
$$\mu(x) = \frac{B_0(\sigma(x))}{\sqrt{-h_{00}(\sigma(x))}} , \qquad \mu_\mu(x) = \frac{\frac{\partial \sigma^a(x)}{\partial x^\mu} B_{0a}(\sigma(x)) + \frac{\partial}{\partial x^\mu}\varphi_0(\sigma(x))}{\sqrt{-h_{00}(\sigma(x))}} . \tag{B.11}$$

As noted in section 2, the one-form chemical potential $\mu_\mu(x)$ is not gauge-invariant.

When the symmetry is spontaneously broken and eq. (B.5) is relaxed, we can identify the respective Goldstone modes and superfluid velocity as additional fluid spacetime invariants

$$\phi(x) = \phi(\sigma(x)) , \qquad \xi_\mu(x) = \frac{\partial \sigma^a(x)}{\partial x^\mu} B_a(\sigma(x)) ,$$
$$\varphi_\mu(x) = \frac{\partial \sigma^a(x)}{\partial x^\mu}\varphi_a(\sigma(x)) , \qquad \xi_{\mu\nu}(x) = \frac{\partial \sigma^a(x)}{\partial x^\mu}\frac{\partial \sigma^b(x)}{\partial x^\nu} B_{ab}(\sigma(x)) . \tag{B.12}$$

Interestingly, the respective Josephson equations $u^\mu \xi_\mu = \mu$ and $u^\nu \xi_{\nu\mu} = \mu_\mu - T\partial_\mu(\beta^\nu \varphi_\nu)$ given in section 4.1.1 are automatically satisfied. Finally, in the case when the one-form symmetry is only partially broken, the respective scalar Goldstone and string fluid variables can be read out as

$$\varphi(x) = \varphi_0(\sigma(x)) , \quad \varpi h_\mu = \frac{\partial \sigma^a(x)}{\partial x^\mu}\frac{B_{0a}(\sigma(x))}{\sqrt{-h_{00}(\sigma(x))}} . \tag{B.13}$$

Order parameter. The question of whether a global symmetry is spontaneously broken or unbroken can be articulated in terms of an order parameter charged under the symmetry. In the zero-form case, such an order parameter is provided by the expectation value of the vertex operator eq. (B.6), i.e.

$$\langle \exp(i\phi(\sigma_p)) \rangle . \tag{B.14}$$

If this happens to be non-zero when computed within the effective action framework of hydrodynamics, we understand that the symmetry is spontaneously broken and we are in the superfluid phase, otherwise the symmetry is unbroken and we are in the ordinary fluid phase. A similar construction can be extended to one-form symmetries using eq. (B.8) to obtain an order parameter

$$\left\langle \exp\left(i \int_C \varphi_a(\sigma)\, \mathrm{d}\sigma^a \right) \right\rangle . \tag{B.15}$$

If, for large spacelike loops, the expectation value scales as the perimeter of the loop we are in the symmetry broken phase, otherwise we are in the symmetry unbroken or partially broken phase. This order parameter will not distinguish between the partially broken and unbroken phases of one-form symmetry. If we were at equilibrium, we could obtain a plausible operator that will make such distinction by integrating over the Euclidean time circle

$$\left\langle \exp\left(- \int_{S^1} \varphi_a(\vec{\sigma})\, \mathrm{d}\sigma^a_E \right) \right\rangle = \left\langle \exp\left(\frac{i}{T_0}\varphi_0(\vec{\sigma}) \right) \right\rangle . \tag{B.16}$$

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	C	P	T	CPT
T^{tt}, g_{tt}	+	+	+	+
T^{ti}, g_{ti}	+	−	−	+
T^{ij}, g_{ij}	+	+	+	+
J^t, A_t	−	+	+	−
J^i, A_i	−	−	−	−
J^{ti}, b_{ti}	−	−	+	+
J^{ij}, b_{ij}	−	+	−	+

	C	P	T	CPT
u^t	+	+	+	+
u^i	+	−	−	+
T	+	+	+	+
μ	−	+	+	−
ϕ	−	+	−	+
μ_t	−	+	−	+
μ_i	−	−	+	+
ϕ_t, φ	−	+	+	−
ϕ_i	−	−	−	−

	C	P	T	CPT
ξ_t	−	+	+	−
ξ_i	−	−	−	−
F_{ti}, E_i, B^t	−	−	+	+
F_{ij}, B^i, E^t	−	+	−	+
ϖ	+	+	+	+
h_i, ξ_{ti}, ζ_i, $\bar{\zeta}^t$	−	−	+	+
ξ_{ij}, $\bar{\zeta}^i$, h_t, ζ_t	−	+	−	+
H_{tij}	−	+	+	−
H_{ijk}	−	−	−	−

Table 7. Transformation properties of various quantities under the discrete symmetries C, P, and T. The first table summarises properties of conserved currents and the associated sources, the second table of dynamical fields, while the third table of various derived quantities.

Generically, there is no notion of preferred time outside thermal equilibrium to define such an order parameter but within the regime of hydrodynamics, we can use the fluid velocity to define this operator[27]

$$\left\langle \exp\left(\frac{i}{T}u^\mu\varphi_\mu(\sigma)\right)\right\rangle = \langle \exp\left(i\varphi(\sigma)\right)\rangle \quad . \tag{B.17}$$

Whether or not this operator is the required order parameter can be settled by computing it within the effective field theory outlined in this appendix. We leave it here as a speculative note and plan to come back to this question in the future.

C Discrete symmetries

If the physical system in question is invariant under certain discrete symmetries, like chirality (parity) or CPT, on top of the continuous Poincaré and zero/one-form symmetries, they can be used to further constraint the number of allowed transport coefficients. The formulations of one-form hydrodynamics and hot electromagnetism involves distinct sets of conserved quantities and are mapped to each other via a Hodge duality operation, therefore discrete symmetries in the respective pictures do not map to each other trivially. Already in section 5.3, we discussed the action of CP symmetry in string fluids, later noting in section 7.3 that CP-preserving string fluids map to parity or P_{EM}-preserving sector of MHD. Therefore, we devote this appendix to a more careful treatment of discrete symmetries in hot electromagnetism and one-form hydrodynamics.

In order to do so, the first step is to define the action of discrete symmetries on the field content. We introduce three operations: charge conjugation C, parity P, and time-reversal T. Their action on the conserved currents and field content is summarised in table 7. Note that h^μ transforms as a vector under parity, which under the duality operation gets mapped to an axial-vector B^μ. Therefore, on the electromagnetism side,

[27] The effective action construction of [30] contains a gauge symmetry $\varphi_a(\sigma) \to \varphi_a(\sigma) + \partial_a\Lambda(\sigma)$, which doesn't leave the out-of-equilibrium order parameter in eq. (B.17) invariant. Arriving at a correct gauge-invariant order parameter might need some more work which we leave for future considerations. We thank P. Glorioso and D. T. Son for pointing this out to us.

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the parity operation is actually defined in terms of the one-form discrete symmetries as $\mathrm{P}_{\mathrm{EM}} = \mathrm{CP}$. The holds for the time-reversal operator, and we find

$$\mathrm{C}_{\mathrm{EM}} = \mathrm{C} \ , \qquad \mathrm{P}_{\mathrm{EM}} = \mathrm{CP} \ , \qquad \mathrm{T}_{\mathrm{EM}} = \mathrm{CT} \ . \tag{C.1}$$

The charge conjugation operator, of course, is the same in both pictures. Interestingly, the full $(\mathrm{CPT})_{\mathrm{EM}}$ is mapped to CPT in the one-form picture.

With table 7 in place, we can easily work out the nature of various transport coefficients under CP (i.e. P_{EM}) and CPT. Firstly, all the transport coefficients in ordinary one-form hydrodynamics are CP-even. For string fluids, we have already discussed the CP properties of various transport coefficients in section 5.3. Lastly, for generic one-form superfluids, since the transport coefficients can arbitrarily depend on a zero-derivative CP-odd scalar $(\zeta \cdot \bar{\zeta})$, no terms in the constitutive relations have a definite CP behaviour.

As for CPT, it is easy to check that all zero derivative tensor structures in any phase of one-form hydrodynamics are CPT-even. Consequently, all one-derivative transport coefficients are CPT-odd. Before we draw any conclusions from this result, it is worth noting that the shear and bulk viscosity terms in neutral relativistic hydrodynamics are CPT-odd as well (or equivalently PT-odd due to neutrality). This is not surprising due to the dissipative nature of these coefficients. However, this CPT is distinct from the "microscopic" CPT that is implemented, not at the level of the constitutive relations, but more fundamentally at the level of an effective action, hydrostatic partition function, or correlation functions (see for instance [43]). In the hydrostatic sector, microscopic CPT-invariance requires that all the CPT-violating terms in the hydrostatic partition function vanish. Since all the hydrostatic one-derivative scalars are CPT-odd, all the hydrostatic transport coefficients are turned off by requiring microscopic CPT-invariance in all the phases of one-form hydrodynamics. Due to the map between and MHD and string fluids in eqs. (7.48) and 7.54, microscopic CPT-invariance implies, in particular, that the chemical potential μ_0 in MHD must vanish. In the non-hydrostatic sector, on the other hand, microscopic CPT can be implemented using Onsager's relations. For instance, for string fluids, these constraints have been worked out in section 5.4. We leave a more detailed analysis of microscopic CPT in generic one-form superfluids to a future work.

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