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van Bilsen, S.; Linders, D.

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Affordable and Adequate Annuities with Stable Payouts: Fantasy or Reality?*

Servaas van Bilsen[†]
Dept. of Quantitative Economics
University of Amsterdam
and NETSPAR

Daniel H. Linders
Dept. of Mathematics
University of Illinois

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Abstract

This paper introduces a class of unit-linked annuities that extends existing annuities by allowing portfolio shocks to be gradually absorbed into the annuity payouts. Consequently, our new class enables insurers to offer an affordable and adequate annuity with a stable payout stream. We show how to price and adequately hedge the annuity payouts in a general financial environment. In particular, our model accounts for various stylized facts of stock returns such as asymmetry and heavy-tailedness. Furthermore, the generality of our framework makes it possible to explore the impact of a parameter misspecification on the annuity price and the hedging performance.

JEL Classification: G11, G13, G22.

Keywords: Unit-Linked Annuities, Buffering of Portfolio Shocks, General Financial Market, Risk Management Framework, Model Risk.

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[†]Corresponding author. Mailing Address: PO Box 15867, 1001 NJ Amsterdam, The Netherlands. Phone: +31 (0) 20 525 5389.

Over the last few decades, the number of pension plans offering guaranteed lifelong payout streams has declined significantly due to the high costs these plans impose on insurers. On the other hand, the number of pension plans offering variable payout streams has grown rapidly.¹ This paper develops a new unit-linked (i.e., investment-linked) annuity product that is both affordable and adequate, as well as consistent with people's preferences.² More specifically, we propose a new class of unit-linked annuity contracts³ that provides flexibility in tailoring payout streams to individual tastes, while at the same time our proposed annuity contracts cost (significantly) less than a traditional Defined Benefit (DB) pension plan.

The payouts from our proposed annuity contracts depend directly on the stock market performance, as is also the case with existing unit-linked annuity products. However, what is new is that we allow for buffering of portfolio shocks: the impact of a current portfolio shock on a future annuity payout is smaller the sooner an annuity payout occurs. Hence, in the presence of buffering of portfolio shocks, annuity payouts that occur in the near future exhibit a (substantially) lower annualized volatility than annuity payouts that occur in the distant future, which is in line with reference-dependent preferences.⁴ By differentiating between the short-term and the long-term volatility of annuity payouts, insurers are able to offer an adequate annuity product that provides a stable payout stream, while at the same time it costs less than a traditional DB pension plan. Figure 1 shows a typical trajectory of the payout from an annuity contract featuring buffering of portfolio shocks. We observe that the payout stream is smoother than the payout stream of a unit-linked annuity contract that does not differentiate between the short-term and long-term volatility of annuity payouts (also called a standard unit-linked annuity contract).⁵

An annuity contract featuring buffering of portfolio shocks stands thus in sharp contrast to a unit-linked annuity contract in which both short-term and long-term annuity payouts exhibit the same annualized volatility. Consumption and investment models assuming constant relative risk aversion (CRRA) utility provide a justification for this type of restriction on the risk profile of future annuity payouts; see Merton (1969). However, the ability of CRRA utility to describe how

¹See, e.g., Investment Company Institute (2017).

²A unit-linked annuity product is said to be adequate if it provides a sufficiently high expected payout stream.

³This paper does *not* take guarantees into account when valuing the proposed annuity contracts. For the valuation of unit-linked annuities *with* guarantees, see, e.g., Brennan and Schwartz (1976), Milevsky and Posner (2001), Milevsky and Salisbury (2006), Moenig and Bauer (2015), and Fonseca and Ziveyi (2017).

⁴Reference-dependence is one of the most prevailing empirical regularities in consumption (see Baillon et al. (2017) and references therein).

⁵See, e.g., Chai et al. (2011) and Maurer et al. (2013b) for a description of a standard unit-linked annuity contract.

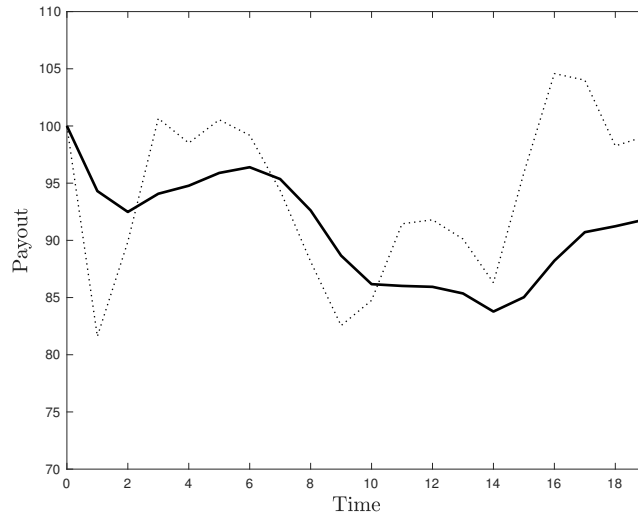


Figure 1. Typical trajectory of payout. The solid line illustrates a typical trajectory of the payout from a unit-linked annuity contract featuring buffering of portfolio shocks. The dotted line shows a trajectory of the payout from a unit-linked annuity contract that does not differentiate between the short-term and long-term volatility of annuity payouts. Both contracts are subject to the same investment shocks. We refer to Figure 5 for the underlying assumptions and parameter values.

individuals actually make decisions under risk is known to be limited. In response, insurers have developed alternative annuity products that aim to be better aligned with people’s preferences. Indeed, some insurers nowadays offer unit-linked annuities whose payouts respond sluggishly, rather than directly, to portfolio shocks; see, e.g., Guillén et al. (2006), Maurer et al. (2013a), and Maurer et al. (2016). These annuities assume that realized investment returns determine the payout dynamics. By contrast, in our framework, the investment portfolio is derived *endogenously* from the desired payout stream.⁶

Our stock return model is a natural generalization of the much celebrated Black and Scholes model. More specifically, we allow standardized stock returns to be distributed according to a distribution for which we can obtain the characteristic function in an analytical form. This (weak) assumption on the stock return distribution makes it possible to compute, in closed-form, arbitrage-free annuity prices and underlying investment strategies that adequately hedge the liabilities of the annuity contracts. Our stock return model is able to capture several stylized facts of stock returns such as asymmetry and heavy-tailedness that the Black and Scholes model fails to explain. Furthermore, the generality of our stock return model enables insurers to explore the impact

⁶Another difference between our proposed annuity contracts and existing annuity contracts is that the computation of the actual payouts from existing annuity contracts featuring smoothing of investment returns is often not fully transparent; see the OECD (2016). Furthermore, insurers may find it difficult to manage the risks inherent to annuity products which exhibit a nontransparent return smoothing mechanism.

of different stock return specifications on the annuity price and the hedging performance.

By buffering portfolio shocks, insurers are able to offer an excessively smooth and excessively sensitive payout stream without charging high annuity prices or reducing expected annuity payouts. Indeed, the low year-on-year volatility of the annuity payout implies an excessively smooth payout stream. Because the current annuity payout responds *less* than one-to-one to a current portfolio shock, future changes in the annuity payout also depend on a current portfolio shock. The payout stream is thus not only excessively smooth but also excessively sensitive. Empirical studies find that aggregate consumption data also exhibit excess smoothness and excess sensitivity; see, e.g., Flavin (1981), Deaton (1987), and Campbell and Deaton (1989). Furthermore, an excessively smooth and excessively sensitive payout stream is consistent with preference models which are based on internal habit formation and loss aversion; see, e.g., Fuhrer (2000), Pagel (2017), Van Bilsen et al. (2017), and Van Bilsen et al. (2018). Indeed, according to these preference models, individuals suffer large welfare losses if consumption tomorrow differs (too) much from consumption today.

We develop a framework to manage the risks inherent to our proposed annuity contracts. In particular, we determine arbitrage-free annuity prices and underlying investment strategies that adequately offset the risks associated with fluctuations in the annuity prices. Using classical pricing techniques, we are able to derive an arbitrage-free discount rate. In the case of buffering of portfolio shocks, the discount rate is larger the further into the future an annuity payout occurs. Indeed, the later an annuity payout occurs, the larger the impact of a current portfolio shock on an annuity payout, and hence the larger the discount rate will be. The discount rate depends not only on the degree of buffering but also on the stock return distribution. In particular, the higher the degree of tail risk in stock returns, the larger the discount rate will be.

We determine the underlying investment strategy by extending the principle of delta hedging – which is familiar from the Black and Scholes model – to our generic stock return model. We find that in the case of buffering of portfolio shocks, the share of the investment portfolio invested in the risky stock decreases over the course of a policyholder’s life. Indeed, portfolio shocks are spread out over a smaller number of years as a policyholder grows older. Hence, to be able to provide a stable payout stream at higher ages, insurers must take less investment risk as a policyholder ages. Because our financial market is incomplete, insurers incur a hedging error. We show that for our choice of the stochastic discount factor, the hedging error is relatively small. Furthermore, a small hedging error is

already achieved when the insurer rebalances the investment portfolio only once a month.⁷ A hedging error introduces the possibility of a shortfall: a situation in which the hedging portfolio is insufficient to cover future annuity payouts. To reduce shortfall risk, insurers typically ask an annuity price such that, with large probability, the charged annuity price is sufficient to meet future annuity payouts. We show that the charged annuity price consists of two components; the first component is equal to the initial value of the hedging portfolio and the second component represents a capital buffer which covers the unhedgeable part of the annuity payouts.

1 A Generic Stock Return Model

1.1 Return Dynamics

We consider a financial market consisting of a risky stock and a risk-free account paying the risk-free interest rate. Insurers adjust annuity payouts at the equidistant time points t_j with time step size $\Delta t > 0$. Hence, $t_j = j\Delta t$ for $j = 0, 1, \dots, J$. Furthermore, we set $t_0 = 0$ and $t_J = T$. We define all random variables and stochastic processes on a common filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_{t_j}\}, \mathbb{P})$ satisfying the usual conditions. Let $\{S_{t_j} \mid j = 0, 1, \dots, J\}$ denote the stock price process. For any $j \in \{1, 2, \dots, J\}$, we model log stock returns as follows:

$$\log \left[\frac{S_{t_j}}{S_{t_{j-1}}} \right] = \mu_{t_j} \Delta t + \sigma_{t_j} \sqrt{\Delta t} A_j, \quad (1)$$

where A_j is a \mathcal{F}_{t_j} -measurable random variable with mean zero and variance one; that is, for any $j \in \{1, 2, \dots, J\}$,

$$\mathbb{E}[A_j] = 0, \quad \text{and} \quad \text{Var}[A_j] = 1. \quad (2)$$

The (annualized) mean rate of return process $\{\mu_{t_j} \mid j = 1, 2, \dots, J\}$ and the (annualized) volatility process $\{\sigma_{t_j} \mid j = 1, 2, \dots, J\}$ are $\mathcal{F}_{t_{j-1}}$ -progressively measurable and satisfy, respectively, $\sum_{j=1}^J |\mu_{t_j}| < \infty$ and $\sum_{j=1}^J \sigma_{t_j}^2 < \infty$. We assume that A_1, A_2, \dots, A_J are identically distributed but not necessarily independent.

Our stock return model (1) generalizes the much celebrated Black and Scholes model in which

⁷In particular, for a realistically calibrated stock return model, we find that the hedging error is smaller than 5% in 99.998% of all scenarios.

each A_j is assumed Gaussian.⁸ Indeed, insurers now have a wide range of stock return specifications at their disposal. In particular, our framework allows to model A_j as a Variance Gamma or Meixner distribution. These distributions model stock returns more accurately than a Gaussian distribution; see, e.g., Merton (1976), Bakshi et al. (1997), Madan et al. (1998), Cont (2001), Schoutens (2003), and Huang and Wu (2004). The generality of our stock return model enables an insurer to incorporate stylized facts of stock returns, such as asymmetry and heavy-tailedness, into his risk management framework and to explore how sensitive the annuity price and the hedging performance are to a misspecification in the underlying stock return distribution; see Sections 3 and 5 for more details.

1.2 Estimating the Stock Return Process

To illustrate our main results, we estimate the stock return process (1). For the sake of convenience, our illustrations assume that the stock return density is stationary. That is, the expected log stock return μ_{t_j} and the stock return volatility σ_{t_j} are time-independent (i.e., $\mu_{t_j} = \mu$ and $\sigma_{t_j} = \sigma$ for all j) and, furthermore, A_1, A_2, \dots, A_J are independent and identically distributed.

We consider a dataset consisting of weekly closing prices of the S&P 500 index from January 3, 2000 until March 16, 2018. Before we calibrate the empirical stock return density, we first standardize the log stock returns. That is, we subtract the (weekly) expected log stock return from the realized log stock returns and then divide by the (weekly) stock return volatility. We find that the annual expected log stock return μ and the annual stock return volatility σ are, respectively, given by

$$\mu = 6.52\%, \quad \text{and} \quad \sigma = 16.38\%. \quad (3)$$

The solid line in Figure 2 shows the empirical probability density function (PDF) of the standardized log stock returns. Denote by m_k the k -th moment of the standardized log stock returns. We find that

$$m_1 = 0, \quad m_2 = 1, \quad m_3 = -0.0263, \quad \text{and} \quad m_4 = 5.36. \quad (4)$$

Because the third empirical moment (i.e., skewness) of the standardized log stock returns is close to zero (which indicates a symmetric stock return distribution) and the fourth empirical moment (i.e., kurtosis) is large (which indicates a stock return distribution with fat tails), we model standardized

⁸The Black and Scholes model also assumes that the expected log stock return μ_{t_j} and the stock return volatility σ_{t_j} do not depend on time. Furthermore, it assumes that A_1, A_2, \dots, A_J are independent.

log stock returns by a symmetric Variance Gamma (VG) distribution:⁹

$$A_j \sim VG(\sigma_{VG}, \nu_{VG}, \theta_{VG}, \mu_{VG}), \text{ with } \theta_{VG} = -\mu_{VG} = 0. \quad (5)$$

We choose the VG parameters such that the stock return distribution is standardized, has zero skewness and its kurtosis matches the empirical kurtosis m_4 . We find the following numerical values:

$$\sigma_{VG} = 1, \nu_{VG} = 0.7853, \theta_{VG} = 0, \text{ and } \mu_{VG} = 0. \quad (6)$$

The dash-dotted line in Figure 2 represents the standard VG density function. For a detailed discussion on how to build standardized Lévy distributions and how to calibrate the VG distribution using historical data or option data, we refer to Seneta (2004), Corcuera et al. (2009), and Linders and Schoutens (2016). The quantile-quantile plots (see Figure 3) show that a standard VG distribution is better capable of fitting the tail behavior of the standardized log stock returns than a standard Gaussian distribution.

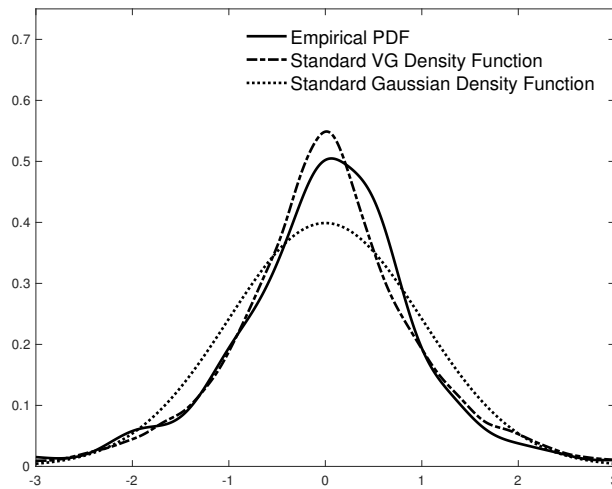
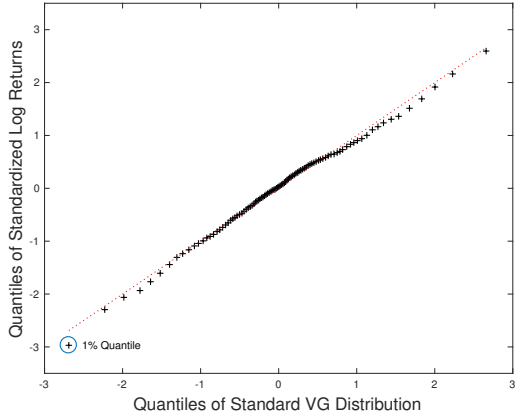


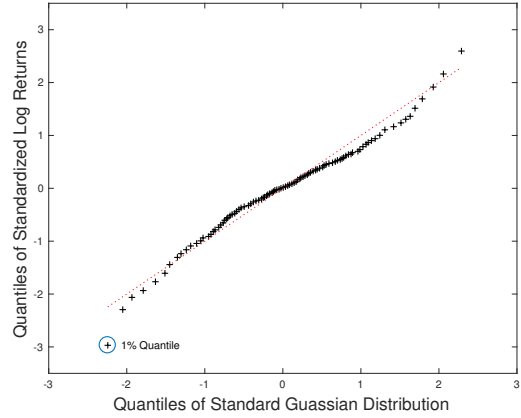
Figure 2. Fitting the empirical stock return density. The solid line shows the empirical probability density function (PDF) of the standardized log stock returns. This empirical PDF is fitted by a standard VG density function (dash-dotted line) and a standard Gaussian density function (dotted line).

It is tempting to use a Gaussian distribution when pricing and hedging unit-linked annuity products. However, Figure 2 and 3 show that such an approach can be dangerous for an insurer. Indeed, the probability of a stock market crash is nearly zero when working with a Gaussian

⁹Appendix A provides the characteristic function of a VG random variable in closed-form. This appendix also states how the empirical moments of the standardized log stock returns are related to the VG parameters.



(a) Standard VG Distribution



(b) Standard Gaussian Distribution

Figure 3. Quantile-quantile plots. This figure illustrates the quantiles of the standardized log stock returns as a function of the quantiles of the standard VG distribution (panel a), and as a function of the standard Gaussian distribution (panel b). The figure shows all quantiles between 1% and 99%, with an increment of 1%. We note that the more similar the theoretical distribution is to the empirical distribution, the better the points will approximate the diagonal line.

distribution, whereas this probability is substantially larger under a heavy-tailed distribution. For example, in the Gaussian case, the probability that the stock price drops by more than 5% in a given day is 0.000221%. Under the VG distribution, this probability is approximately 100 times bigger.¹⁰

In this section, we have calibrated the standard Variance Gamma distribution. However, our framework is not restricted to the standard Variance Gamma distribution. It also allows for other distributions as long as the characteristic function is known in closed-form. Table 3 in Appendix A provides properties of two other distributions that we can use to model the shocks A_1, A_2, \dots, A_J . For instance, if we use a standard Normal Inverse Gaussian (NIG) distribution, then the calibrated parameters will be given by the values as shown in Table 4 (see Appendix A). Appendix A also provides the calibrated NIG density function and a quantile-quantile plot comparing the quantiles of the calibrated NIG distribution with the empirical quantiles.

¹⁰We note that during the period January 1, 2000 to December 31, 2017, the S&P 500 index exhibited 14 days (out of 4526 trading days) with losses exceeding 5% and 12 days with gains exceeding 5%. Based on these numbers, we find that the probability of a decrease in the S&P 500 index by more than 5% equals $14/4526 = 0.31\%$.

2 A Unit-Linked Annuity with Buffering of Portfolio Shocks

2.1 Specification of the Annuity Contract

This section introduces an affordable and adequate unit-linked annuity with a stable payout stream. To understand the payout structure of this annuity, we first recall the payout structure of a *standard unit-linked annuity contract*.¹¹ Such a contract provides a more adequate payout stream (i.e., higher *expected* retirement income) than an equally priced fixed nominal annuity. Indeed, in the case of a standard unit-linked annuity contract, the insurer partly invests the collected annuity premium in the stock market. The flip side of a standard unit-linked annuity contract is, however, that its payout can fluctuate heavily from year to year.

Denote by $\log \widehat{c}_{t_j}$ the log payout from a standard unit-linked annuity contract at time t_j (in what follows, we use the symbol “ $\widehat{}$ ” to represent a standard unit-linked annuity contract). The log payout from a standard unit-linked annuity contract evolves as follows (see, e.g., Maurer et al. (2013b)):

$$\log \widehat{c}_{t_j} = \log \widehat{c}_{t_{j-1}} + \widehat{g}_j \Delta t + \beta_j \sigma_{t_j} \sqrt{\Delta t} A_j, \quad (7)$$

where the parameter \widehat{g}_j is $\mathcal{F}_{t_{j-1}}$ -progressively measurable and models the expected change in the log payout $\log \widehat{c}_{t_{j-1}}$ and the parameter β_j is $\mathcal{F}_{t_{j-1}}$ -progressively measurable and models how the log payout $\log \widehat{c}_{t_{j-1}}$ responds to the (unexpected) stock return shock $\sigma_{t_j} \sqrt{\Delta t} A_j$. If $\beta_j = 0$ for every j , payouts are insensitive to (unexpected) stock return shocks, while if $\beta_j = 1$ for every j , the investment portfolio backing the contract is a pure stock portfolio. We can show that the parameter β_j is equal to the share of the investment portfolio invested in the risky stock (see (33) in Section 3.2.2).¹² In what follows, we call $\beta_j \sigma_{t_j} \sqrt{\Delta t} A_j$ the portfolio shock between time t_{j-1} and time t_j . We can express the log payout $\log \widehat{c}_{t_j}$ in terms of past portfolio shocks as follows:

$$\log \widehat{c}_{t_j} = \log \widehat{c}_0 + (\widehat{g}_1 + \dots + \widehat{g}_j) \Delta t + \beta_1 \sigma_{t_1} \sqrt{\Delta t} A_1 + \dots + \beta_j \sigma_{t_j} \sqrt{\Delta t} A_j. \quad (8)$$

¹¹Our aim is to extend a standard unit-linked annuity contract. However, many annuities are sold with riders. In future work, we intend to investigate how to add guarantees to our contract and how to determine the pricing and hedging of the new contract.

¹²We note that the expected change in the log payout from a standard unit-linked annuity contract, i.e., \widehat{g}_j , equals the difference between the expected investment return and the so-called Assumed Interest Rate (AIR); see, e.g., Maurer et al. (2013b).

It follows from (8) that a current portfolio shock has the same impact on payouts that occur in the near future as on payouts that occur in the distant future. The time distance between the payout date and the date at which the portfolio shock occurs plays no role.

Our aim is to generalize the payout structure (8) such that we can differentiate between the volatility of short-term and long-term annuity payouts. More specifically, we allow a current portfolio shock to have a (much) smaller impact on annuity payouts that occur in the near future than on annuity payouts that occur in the far future. By generalizing (8), insurers are able to offer an adequate unit-linked annuity with a stable payout stream which is *not* more expensive than a standard unit-linked annuity contract. In the following definition, we introduce a class of unit-linked annuities that satisfies this property.

Definition 1. Denote by $\log c_{t_j}$ the log annuity payout at time t_j from an annuity with buffering of portfolio shocks. The log annuity payout $\log c_{t_j}$ is defined as follows:

$$\log c_{t_j} = \log c_0 + (g_1 + \dots + g_j) \Delta t + q_j \beta_1 \sigma_{t_1} \sqrt{\Delta t} A_1 + \dots + q_1 \beta_j \sigma_{t_j} \sqrt{\Delta t} A_j, \quad (9)$$

where the parameter g_j is $\mathcal{F}_{t_{j-1}}$ -progressively measurable and models the expected change in the log annuity payout $\log c_{t_{j-1}}$ and q_j measures the impact of the portfolio shock $\beta_1 \sigma_{t_1} \sqrt{\Delta t} A_1$ on the log annuity payout $\log c_{t_j}$.

We call the function q_j the buffering function which is exogenously specified. An insurance company may specify q_j such that it is consistent with its clients' preferences. The buffering function determines how future log annuity payouts respond to portfolio shocks. More specifically, q_j models the exposure of the log annuity payout $\log c_{t_{k+j}}$ to the portfolio shock $\beta_{k+1} \sigma_{t_{k+1}} \sqrt{\Delta t} A_{k+1}$ ($k \geq 0$).¹³ If the buffering function equals the identity function (i.e., $q_j = 1$ for all j) and the expected change in the log payout is the same as the expected change in the log payout from a standard unit-linked annuity contract (i.e., $g_j = \widehat{g}_j$ for all j), then the payout structure (9) reduces to the payout structure of a standard unit-linked annuity contract (see (8)). In what follows, we assume that the buffering function q_j (weakly) increases with the time distance between the payout date (i.e., t_{k+j}) and the date at which the portfolio shock occurs (i.e., t_{k+1}), so that a current portfolio shock has a (much) smaller impact on annuity payouts that occur in the near future than on annuity payouts that occur in the far

¹³We note that the parameter β_j is no longer equal to the share of the investment portfolio invested in the risky stock; see (33) in Section 3.2.2.

future. We refer to *buffering* of portfolio shocks when the buffering function q_j *strictly* increases with the investment horizon j .

Figure 4(a) illustrates various specifications of the buffering function q_j . The specifications are such that the corresponding (arbitrage-free) annuity prices are all the same, as well as the expected payout streams.¹⁴ The solid line corresponds to the case of exponential buffering. That is,

$$q_j = a_1 - a_2 \exp \{-\eta \cdot j \Delta t\}, \quad (10)$$

where a_1 and a_2 are scaling parameters and η models the degree of buffering. If $\eta \Rightarrow 0$, buffering of portfolio shocks is maximal, while if $\eta \Rightarrow \infty$, buffering of portfolio shocks is absent. Exponential buffering is consistent with preference models based on internal habit formation and loss aversion in which individuals are reluctant to change current consumption following an income shock; see Section 4 for more details.¹⁵ The dash-dotted line shows the case in which $q_j = a_3/N \cdot j \Delta t$ for $j \Delta t < N$ and $q_j = a_3$ for $j \Delta t \geq N$. Here, a_3 is a scaling parameter and N denotes the buffering period (i.e., the number of years it takes to fully translate a portfolio shock into the annuity payout). Hence, the risk exposure of a payout that occurs one period from now is only $1/N$ th of the risk exposure of a payout that occurs N periods from now.¹⁶ The dashed line illustrates the benchmark case in which insurers do not apply buffering of portfolio shocks at all (i.e., $q_j = 1$ for all j). This buffering rule is consistent with standard consumption and investment theory; see the seminal work of Merton (1969).

The main novelty of the payout structure (9) is the buffering function q_j . Our approach allows insurers to actively control the volatility structure of their annuity payouts. Denote by $\widehat{\Sigma}_{t_k}^j$ and $\Sigma_{t_k}^j$ the annualized volatility at time t_k of $\log \widehat{c}_{t_k+j}$ and $\log c_{t_k+j}$, respectively. That is,

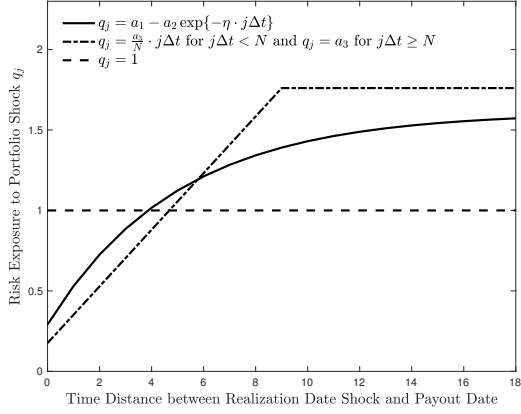
$$\widehat{\Sigma}_{t_k}^j = \sqrt{\frac{\text{Var} \left[\log \widehat{c}_{t_k+j} \mid \mathcal{F}_{t_k} \right]}{j}}, \quad (11)$$

$$\Sigma_{t_k}^j = \sqrt{\frac{\text{Var} \left[\log c_{t_k+j} \mid \mathcal{F}_{t_k} \right]}{j}}. \quad (12)$$

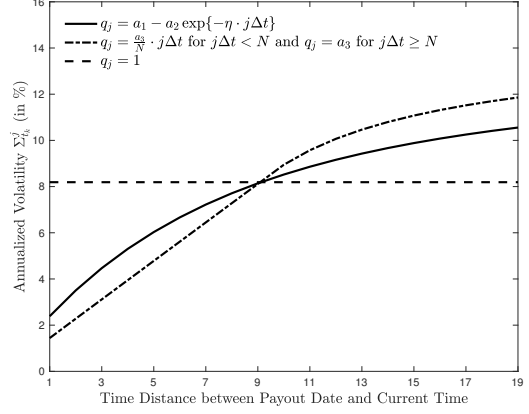
¹⁴To compute the annuity prices and the expected payout streams, we assume the same setting as in Section 2.2.1.

¹⁵See also Fuhrer (2000), Pagel (2017), Van Bilsen et al. (2017), and Van Bilsen et al. (2018).

¹⁶This buffering rule is consistent with the new Dutch legislation on Defined Contribution contracts; see Section 2.3 for further details.



(a) Buffering Function



(b) Volatility Function

Figure 4. Buffering and volatility functions. This figure illustrates various specifications of the buffering function q_j and the volatility function $\Sigma_{t_k}^j$. The solid lines represent the case of exponential buffering; that is, $q_j = a_1 - a_2 \exp\{-\eta \cdot j\Delta t\}$ (with $a_1 = a_2 = 1.6084$ and $\eta = 0.2$). The dash-dotted lines correspond to the case in which $q_j = a_3/N \cdot j\Delta t$ for $j\Delta t < N$ and $q_j = a_3$ for $j\Delta t \geq N$ (with $a_3 = 1.7605$ and $N = 10$). The dashed lines illustrate the standard case in which $q_j = 1$ for all j . The specifications are such that the corresponding annuity prices are all the same, as well as the expected payout streams. To compute the annuity prices and the expected payout streams, we assume the same setting as in Section 2.2.1. The stock return volatility σ_{t_j} is set equal to 16.38% for each j , the parameter β_j to 50% for each j , and the time step Δt to unity.

Insurers can specify the function q_j such that the annualized volatility of $\log c_{t_k+j}$ is relatively low for small horizons (i.e., small j) and relatively high for large horizons (i.e., large j):

$$\Sigma_{t_k}^j < \widehat{\Sigma}_{t_k}^j \text{ for small } j, \text{ and } \Sigma_{t_k}^j \geq \widehat{\Sigma}_{t_k}^j \text{ for large } j. \quad (13)$$

Figure 4(b) shows the annualized volatility $\Sigma_{t_k}^j$ as a function of the time distance between the payout date and the current time for various specifications of the buffering function q_j . The underlying annuity contracts all cost the same, and all provide the same expected payout stream. However, the solid and dash-dotted line correspond to annuity contracts that generate a (much) more stable payout stream than a standard unit-linked annuity contract. Indeed, the short-term annuity payouts of these contracts exhibit a relatively low annualized volatility. We note that individuals can also realize a stable payout stream by buying a fixed nominal annuity. The flip side of such an annuity is, however, that it does not provide an adequate payout stream at an affordable price, especially when interest rates are low.

Guillén et al. (2006) and Maurer et al. (2016) also consider a unit-linked annuity product whose year-on-year volatility is lower than the year-on-year volatility of the underlying investment

portfolio (i.e., the payout stream is excessively smooth).¹⁷ These authors achieve this low year-on-year volatility in current payouts as follows. If the investment portfolio increases in value, then only a fraction of the investment return will be added to the annuity payout. The remainder of the investment return will be retained by the insurer. Conversely, if the investment portfolio decreases in value, then the negative investment return will not be entirely reflected in the annuity payout. The additional payout will be supplemented from the reserve. A major difference between their framework and our framework is that the consumption stream of our individual is excessively sensitive: current realized returns have strong predictive power for future payouts. Also, our annuity contracts determine the investment portfolio *endogenously* from the desired payout stream, while existing annuity contracts assume that the investment portfolio is exogenously given.

2.2 Payout Dynamics

2.2.1 Shock Absorbing Mechanism. This section illustrates the impact of portfolio shocks on current and future payouts. We consider two products: a standard unit-linked annuity contract and a unit-linked annuity featuring buffering of portfolio shocks. Both products have a (fixed) term of 20 years and an initial payout of 100. We choose the (unconditional) expected growth rates \hat{g}_j and g_j such that, given the information available at time 0, expected future payouts are equal to 100. We note that, based on a risk-free interest rate of 1.5%,¹⁸ a fixed nominal annuity only provides a payout of 76.13, which is clearly inadequate.¹⁹ In the case of buffering of portfolio shocks, we assume that the insurer applies exponential buffering (see (10)). We choose the scaling parameter $a_1 = a_2$ such that the prices of the two products are the same; see Section 3.2.1 on how to compute the price of a unit-linked annuity with buffering of portfolio shocks. Log stock returns are distributed according to a VG distribution; see Section 1.2 for the parameter values of the VG distribution. Finally, we set the risk-free interest rate to 1.5%, β_j to 50% for each j , and the time step Δt to unity (i.e., payouts occur yearly).

Figures 5(a) and (b) illustrate the impact of a 40% stock price decline in year one on current and future payouts. In the case of a standard unit-linked annuity contract, insurers fully translate portfolio shocks into current payouts. Hence, after the stock return shock has occurred, the payout

¹⁷See also Jørgensen and Linnemann (2011), Guillén et al. (2013), and Linnemann et al. (2014).

¹⁸The federal funds rate, which is the most important interest rate in the US, currently (i.e., March 2018) equals 1.5%.

¹⁹Here, we assume that an expected annuity payout of 100 would be adequate.

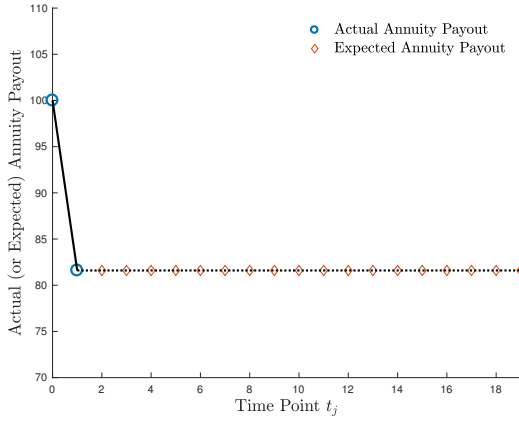
from the standard unit-linked annuity contract drops to $\widehat{c}_{t_1} = \widehat{c}_0 \exp \{\widehat{g}_1 - \beta_1 \cdot 40\%\} = 81.60$. Because the current payout completely absorbs the portfolio shock, the shape of the expected payout stream remains the same; see Figure 5(a). In the case of buffering of portfolio shocks, insurers do not fully translate a portfolio shock into current payouts. As a result, the current payout from the annuity with buffering of portfolio shocks exceeds the current payout from the standard unit-linked annuity contract. In this example, $c_{t_1} = c_0 \exp \{g_1 - q_1 \cdot \beta_1 \cdot 40\%\} = 94.31 > \widehat{c}_{t_1} = 81.60$. The consequence of protecting current payouts is that the shape of the expected payout stream cannot remain the same. More specifically, it becomes a decreasing function of time. Indeed, future payouts bear a large part of a current portfolio shock.

Figures 5(c) and (d) show the impact of a 20% stock price increase in year two (next to the -40% stock return shock in year one) on current and future payouts. As in Figure 5(a), the current payout from the standard unit-linked annuity contract directly absorbs the portfolio shock. That is, $\widehat{c}_{t_2} = \widehat{c}_{t_1} \exp \{\widehat{g}_2 + \beta_2 \cdot 20\%\} = 89.88$; see also Figure 5(c). The impact of the portfolio shock on the current payout from the annuity with buffering of portfolio shocks is much smaller because $q_1 = 29\%$ (i.e., only 29% of the current portfolio shock affects the current annuity payout). Individuals even receive less than last year, because the insurer also translates $q_2 - q_1 = 24\%$ of last year's (negative) portfolio shock into the current annuity payout. Furthermore, as a result of the current stock price increase, the expected payout stream becomes less downward sloping as compared to Figure 5(b).

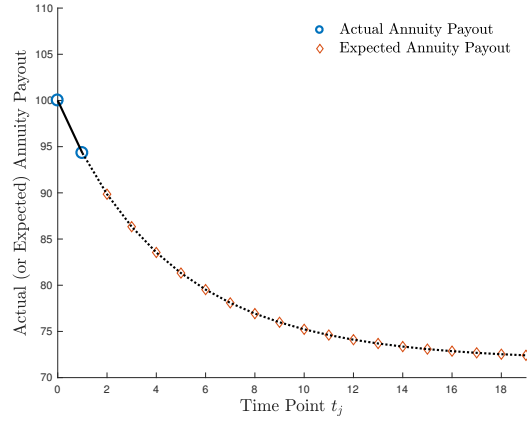
Figures 5(e) and (f) illustrate a sample path of \widehat{c}_{t_j} and c_{t_j} , respectively. In the case of a standard unit-linked annuity contract, the payout stream behaves like a random walk process. In contrast, in the case buffering of portfolio shocks, the payout stream is excessively smooth and excessively sensitive: current annuity payouts under-respond to portfolio shocks and current portfolio shocks have predictive power for future annuity payouts.

2.2.2 Decomposition of Expected Annuity Payouts. In the case of buffering of portfolio shocks, the shape of the expected payout stream cannot remain the same following a stock return shock; see Figure 5. Appendix C shows that the expected value of the annuity payout c_{t_j+h} (conditional on the

Stock Return Shock: -40% in Year 1

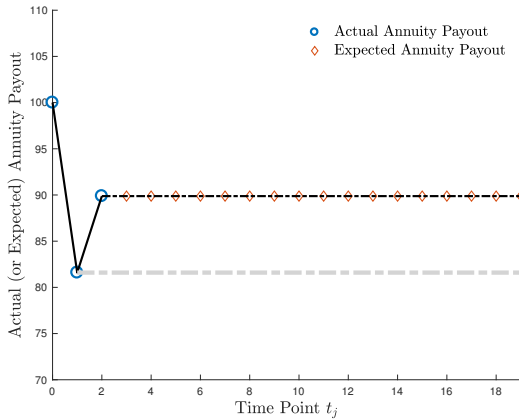


(a) Standard Unit-Linked Annuity

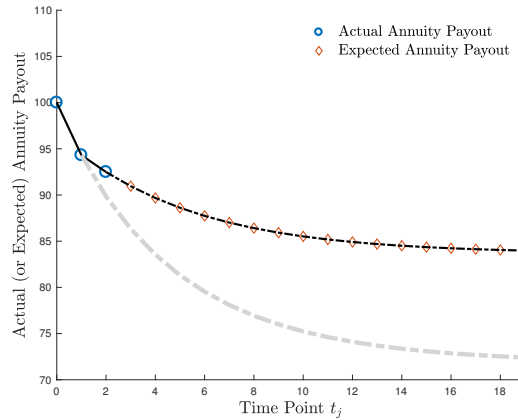


(b) Buffering of Portfolio Shocks

Stock Return Shock: $+20\%$ in Year 2 (next to the -40% Stock Return Shock in Year 1)

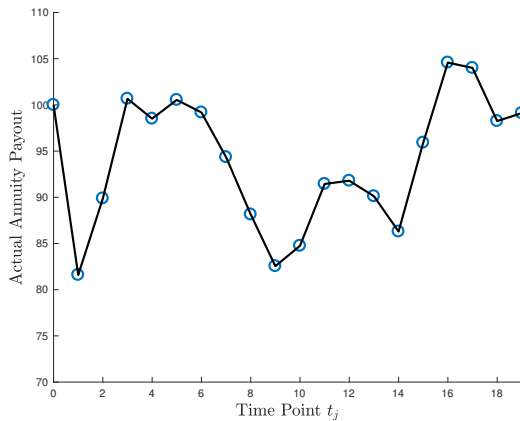


(c) Standard Unit-Linked Annuity

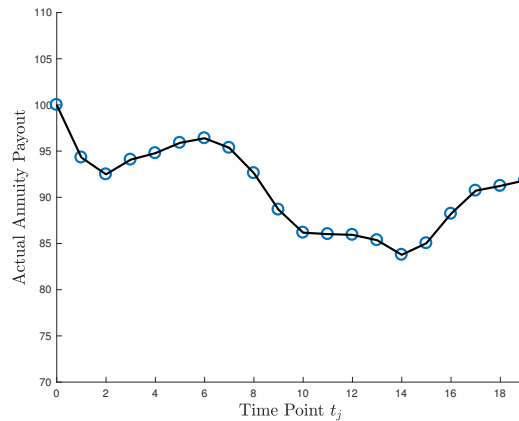


(d) Buffering of Portfolio Shocks

Sample Paths



(e) Standard Unit-Linked Annuity



(f) Buffering of Portfolio Shocks

Figure 5. Shock absorbing mechanisms. The figure shows the impact of stock return shocks on current and future payouts. The left panels assume a standard unit-linked annuity, while the right panels assume buffering of portfolio shocks. We choose the (unconditional) expected growth rates \hat{g}_j and g_j such that, given the information available at time 0, expected future payouts are equal to 100. Log stock returns are distributed according to a VG distribution; see Section 1.2 for the parameter values of the VG distribution. We assume the exponential buffering function (10), with $a_1 = a_2 = 1.6084$ and $\eta = 0.2$. The risk-free interest rate is set equal to 1.5%, the parameter β_j to 50% for each j , and the time step Δt to unity.

information available at time t_j) is given by

$$\begin{aligned} \mathbb{E}_{t_j} [c_{t_{j+h}}] &= c_{t_j} \times F_{t_j}^h \\ &\times \mathbb{E}_{t_j} \left[\exp \left\{ \sum_{k=1}^h g_{j+k} \Delta t + \sum_{k=1}^h q_k \beta_{j+h+1-k} \sigma_{t_{j+h+1-k}} \sqrt{\Delta t} A_{j+h+1-k} \right\} \right], \end{aligned} \quad (14)$$

where $\mathbb{E}_{t_j} [\cdot]$ denotes the expectation conditional on information available at time t_j , and $F_{t_j}^h$ represents the buffering factor which is defined as follows:

$$F_{t_j}^h = \exp \left\{ \sum_{k=1}^j (q_{j+h-k+1} - q_{j-k+1}) \beta_k \sigma_{t_k} \sqrt{\Delta t} A_k \right\}. \quad (15)$$

We note that the random variables A_1, A_2, \dots, A_j in (15) have been realized (i.e., the value of $F_{t_j}^h$ is known at time t_j).

In the case of a standard unit-linked annuity contract, the buffering factor $F_{t_j}^h$ is equal to unity (i.e., past portfolio shocks do not affect the expected payout stream). Indeed, in the absence of buffering of portfolio shocks, past portfolio shocks are already fully reflected into current payouts. The buffering factor $F_{t_j}^h$ is thus the direct consequence of the gradual adjustment of annuity payouts to portfolio shocks. Intuitively, gradual adjustment of annuity payouts gives rise to funding imbalances that must be translated into future annuity payouts. Consequently, future adjustments of annuity payouts become, to some extent, predictable. The buffering factor $F_{t_j}^h$ summarizes the predictable changes in future annuity payouts as a result of past portfolio shocks that have not been reflected into current annuity payouts yet. Indeed, $q_{(j+h-k)+1}$ is the desired risk exposure of the future annuity payout $c_{t_{j+h}}$ to the past portfolio shock $\beta_k \sigma_{t_k} \sqrt{\Delta t} A_k$. Because $q_{(j-k)+1} \beta_k \sigma_{t_k} \sqrt{\Delta t} A_k$ has already been reflected into the current annuity payout c_{t_j} , $q_{(j+h-k)+1} - q_{(j-k)+1}$ represents the *remaining* part of the past portfolio shock $\beta_k \sigma_{t_k} \sqrt{\Delta t} A_k$ that the insurer still needs to translate into the current annuity payout.

To compute the change in the expected annuity payout, insurers need to determine the change in the current annuity payout c_{t_j} as well as the change in the buffering factor $F_{t_j}^h$. The current annuity

payout evolves as follows (see Appendix C):²⁰

$$\Delta \log c_{t_j} = \log c_{t_{j+1}} - \log c_{t_j} = g_{j+1} \Delta t + q_1 \beta_{j+1} \sigma_{t_{j+1}} \sqrt{\Delta t} A_{j+1} + \log F_{t_j}^1. \quad (17)$$

The second term on the right-hand side of the second equality in (17) represents the impact of the current portfolio shock on the current annuity payout. Because q_1 is relatively small in the case of buffering of portfolio shocks, the year-on-year volatility of the annuity payout is low, implying an excessively smooth payout stream. The last term reflects the impact of past portfolio shocks on the current annuity payout. Since not only the current portfolio shock but also past portfolio shocks affect the change in the current annuity payout, the payout process is excessively sensitive.

The log buffering factor $\log F_{t_j}^h$ satisfies (see Appendix C):

$$\Delta \log F_{t_j}^h = \log F_{t_{j+1}}^{h-1} - \log F_{t_j}^h = q_h \beta_{j+1} \sigma_{t_{j+1}} \sqrt{\Delta t} A_{j+1} - \log F_{t_j}^1 - q_1 \beta_{j+1} \sigma_{t_{j+1}} \sqrt{\Delta t} A_{j+1}. \quad (18)$$

The first term on the right-hand side of the second equality in (18) denotes the current portfolio shock that results in a new buffering factor. The last two terms represent portfolio shocks that are being translated into the current annuity payout, so that they are no longer included in the buffering factor.

2.3 Dutch Defined Contribution Contracts

In Dutch Defined Contribution (DC) contracts, a person puts money in an individual investment account and invests the money in the financial market. As of September 2016, individuals are no longer required to convert their total accumulated retirement wealth into a guaranteed lifelong nominal income stream, but may choose to take investment risk during retirement. To prevent large year-on-year fluctuations in pension payouts, the legislation allows stock return shocks to be buffered (with a maximum buffering period of 10 years). Currently, insurance companies in the Netherlands are investigating whether or not they should offer products featuring buffering of

²⁰If g_j , β_j and σ_{t_j} are constant (i.e., $g_j = g$, $\beta_j = \beta$ and $\sigma_{t_j} = \sigma$ for all j) and, furthermore, A_1, A_2, \dots, A_J are independent and identically distributed, then we can write the dynamics of the current annuity payout (17) as an ARMA(1,∞) process. More specifically,

$$\Delta \log c_{t_j} = g \Delta t + \epsilon_{j+1} + \sum_{k=1}^{\infty} \theta_k \epsilon_{j+1-k}, \quad (16)$$

where $\theta_k = \frac{q_{k+1} - q_k}{q_1}$ and $\{\epsilon_j = q_1 \beta \sigma \sqrt{\Delta t} A_j \mid j = 1, 2, \dots, J\}$ is a white noise process ($\epsilon_0 = \epsilon_{-1} = \dots = 0$ by convention).

portfolio shocks. Our framework enables insurers to incorporate buffering of portfolio shocks into unit-linked annuity contracts.

3 Risk Management Framework

Whereas Section 2 took the perspective of the policyholder, this section takes the perspective of the insurer. More specifically, we consider the question: how much money should the insurer ask for an annuity with buffering of portfolio shocks? We show that the price consists of two components. The first component covers the hedgeable part of the contract and follows from arbitrage-free valuation of the future payouts; see Section 3.2. The second component covers the non-hedgeable part of the contract and follows from applying a risk-measure (e.g., Value-at-Risk) to the hedging error (i.e., the difference between the payout and the value of the investment portfolio); see Section 3.3.

Pricing of cash flows typically requires simulation-based techniques. This section, however, provides closed-form expressions for the annuity price and the hedging strategy. To derive the pricing kernel, we assume that stock returns satisfy:

$$\log \left[\frac{S_{t_j}}{S_{t_{j-1}}} \right] = (r_{t_j} + e) \Delta t + \sigma_{t_j} \sqrt{\Delta t} A_j, \quad (19)$$

where $e \geq 0$ denotes the expected excess log stock return²¹ and r_{t_j} represents the risk-free interest rate which is assumed to be $\mathcal{F}_{t_{j-1}}$ -measurable.²² We assume that A_1, A_2, \dots, A_J are independent and identically distributed. Additionally, the shock process $\{A_j \mid j = 1, 2, \dots, J\}$, the interest rate process $\{r_{t_j} \mid j = 1, 2, \dots, J\}$ and the volatility process $\{\sigma_{t_j} \mid j = 1, 2, \dots, J\}$ are assumed to be independent. Furthermore, the characteristic function of the standardized stock return A_j , which is defined as follows:

$$\phi(v) = \mathbb{E} [e^{ivA_j}], \quad (20)$$

is given in an analytical form. Here, $i = \sqrt{-1}$ is the imaginary unit. The cumulative distribution function (CDF) of A_j , however, might be unknown or too cumbersome to work with; see Le Courtois and Walter (2014) on how to numerically back out the CDF from its characteristic function.

²¹Appendix B provides a few properties of the expected excess stock return e .

²²We also assume that r_{t_j} satisfies: $\sum_{j=1}^J |r_{t_j}| < \infty$. Furthermore, we note that without loss of generality, we could also assume that r_{t_j} is \mathcal{F}_{t_j} -measurable. But economically it makes more sense to assume that r_{t_j} is known at time t_{j-1} .

3.1 Pricing of Cash Flows

We assume an arbitrage-free financial market. The fundamental theorem of asset pricing proves the existence of a probability measure \mathbb{Q} , which we call the risk-neutral pricing measure, equivalent to the real-world probability measure \mathbb{P} such that the arbitrage-free price at time t_j of a contingent claim paying off X_T at time T is exactly equal to the discounted expectation of X_T under this probability measure:

$$\text{Time-}t_j \text{ price} = \mathbb{E}_{\mathbb{Q}} \left[e^{-\sum_{k=j+1}^J r_{t_k} \Delta t} X_T \middle| \mathcal{F}_{t_j} \right]. \quad (21)$$

We assume that the contingent claim X_T is \mathcal{F}_T -measurable; that is, the only randomness in X_T comes from the stock price. Since the financial market is incomplete, infinitely many pricing measures \mathbb{Q} exist.²³ Denote by \mathcal{P} the set of probability measures equivalent to \mathbb{P} . Although interest rates and volatilities are described by random processes, we do not assume that interest rate and volatility derivatives are available in our market. Under this assumption, we can define the set \mathcal{Q} of feasible risk-neutral pricing measures as follows:

$$\mathcal{Q} = \left\{ \mathbb{Q} \in \mathcal{P} \mid \mathbb{E}_{\mathbb{Q}} \left[e^{-r_{t_{j+1}} \Delta t} S_{t_{j+1}} \middle| \mathcal{F}_{t_j} \right] = S_{t_j} \text{ for } j = 0, 1, \dots, J-1 \right\}. \quad (22)$$

We search for a pricing measure $\mathbb{Q} \in \mathcal{Q}$ to value cash flows. Brody et al. (2012) also consider the idea of finding a pricing measure for general Lévy processes. The following theorem provides a natural choice for a \mathbb{Q} -measure. This measure has a similar structure as the \mathbb{Q} -measure implied by the Black and Scholes financial market. Furthermore, in Section 3.2.3, we show that our annuity contracts can be hedged almost perfectly with existing assets. Hence, a different choice for the risk-neutral measure will have a small impact on the annuity price.

Theorem 1. *The probability measure \mathbb{Q} with Radon-Nikodym derivative given by*

$$\xi_{t_j} = \exp \left\{ - \sum_{k=1}^j \psi(-\lambda_{t_k}) \Delta t - \sqrt{\Delta t} \sum_{k=1}^j \lambda_{t_k} A_k \right\}, \quad (23)$$

is an arbitrage-free pricing measure; that is, $\mathbb{Q} \in \mathcal{Q}$. For each $k \in \{1, \dots, j\}$, the market price of

²³A perfect hedging strategy does typically not exist when stock prices have jumps.

risk parameter $\lambda_{t_k} > 0$ satisfies the following equation:

$$e = \psi(-\lambda_{t_k}) - \psi(\sigma_{t_k} - \lambda_{t_k}), \quad (24)$$

where

$$\psi(-z) = \frac{1}{\Delta t} \log \phi \left(iz\sqrt{\Delta t} \right). \quad (25)$$

We note that the market price of risk λ_{t_k} has a similar interpretation as the Sharpe ratio in the Black and Scholes model; see also Brody et al. (2012) for a discussion on risk premiums in a Lévy setting. We can now define the stochastic discount factor (SDF) at time t_j as follows:

$$M_{t_j} = e^{-\sum_{k=1}^j r_{t_k} \Delta t} \xi_{t_j}. \quad (26)$$

Using the SDF (26), we can transform a \mathbb{Q} -expectation (see (21)) into a \mathbb{P} -expectation in the following way:

$$\text{Time-}t_j \text{ price} = \mathbb{E}_{\mathbb{Q}} \left[e^{-\sum_{k=j+1}^T r_{t_k} \Delta t} X_T \middle| \mathcal{F}_{t_j} \right] = \mathbb{E}_{\mathbb{P}} \left[\frac{M_T}{M_{t_j}} X_T \middle| \mathcal{F}_{t_j} \right]. \quad (27)$$

Section 3.2 uses (27) to price a unit-linked annuity with buffering of portfolio shocks. In what follows, all expectations are expectations under the real-world probability measure \mathbb{P} . Furthermore, we denote the conditional expectation $\mathbb{E}_{\mathbb{P}} [\cdot | \mathcal{F}_{t_j}]$ by $\mathbb{E}_{t_j} [\cdot]$. Finally, we note that Theorem 6 in Appendix G derives the pricing kernel for the case where interest-rate derivatives are available.

3.2 Valuation of the Hedgeable Part

Insurers can diversify the risks associated with traditional insurance contracts by pooling a large number of independent policies. They can then use premium principles to value such contracts; see, e.g., Goovaerts et al. (2010). Insurers who offer unit-linked annuity products are exposed to a high degree of systematic equity risk; this risk is non-diversifiable and a premium principle cannot cope with the equity risk associated with a unit-linked annuity contract (see, e.g., Feng and Shimizu (2016)). Therefore, hedging must be an integral part of the risk management of unit-linked annuity contracts.²⁴

This section proposes an approximate dynamic hedging strategy consisting of risky stocks and

²⁴We note that there is evidence that some insurers have difficulties managing their unit-linked annuity products; see Kojen and Yogo (2018).

the risk-free account to manage the systemic equity risk associated with a unit-linked annuity with buffering of portfolio shocks. Section 3.2.1 determines the initial value of the hedging portfolio, whereas Section 3.2.2 explicitly states how much wealth the insurer should invest in the risky stock at each moment in time during the lifetime of the contract.²⁵ Finally, Section 3.2.3 evaluates the hedging performance.

We assume from here on that the expected growth rate g_j and the interest rate r_{t_j} are deterministic functions of time.²⁶ In addition, we assume that β_j and σ_{t_j} are constant (i.e., $\beta_j = \beta$ and $\sigma_{t_j} = \sigma$ for every j). In this case, our model is a direct generalization of the Black and Scholes model. In a 2005 report, one of the models recommended by the American Actuarial Society for pricing and hedging equity-linked annuities is the Black and Scholes model, because it is *simple, tractable* and *easy to calibrate*; see Gorski and Brown (2005). We show that our model allows more flexibility to model the stock return distribution while still maintaining the strong points of the Black and Scholes model: simplicity, tractability and easy calibration. Indeed, we can derive intuitive analytical pricing formulas and show how to set up a delta-hedging strategy for managing stock price risk. Note also that similar models have successfully been applied to determine basket option prices (Linders and Schoutens (2016)), CDO prices (Albrecher et al. (2007)), and implied volatility smiles (Corcuera et al. (2009)).

3.2.1 Initial Price. We first determine an arbitrage-free price for a unit-linked annuity with buffering of portfolio shocks. At time 0, this price corresponds to the initial price of a dynamic trading strategy which (approximately) replicates the payouts of the contract. Denote by $V_{t_j}^h$ an arbitrage-free price at time t_j of the future annuity payout c_{t_j+h} . Using formula (27), we express the

²⁵For an introduction to the valuation of contingent claims consisting of actuarial (diversifiable) and financial (non-diversifiable) risks, we refer to Tsanakas et al. (2013), Stadje and Pelsser (2014), Pelsser and Salahnejhad (2016), and Dhaene et al. (2017).

²⁶If g_j and r_{t_j} depend on past stock return shocks, then we are typically still able to derive closed-form expressions for the annuity price and the hedging strategy. Incorporating a stochastic interest rate and a stochastic volatility will require us to determine adequate stochastic processes for these variables as well as a realistic dependence model describing the interactions between the different processes. Moreover, additional derivative instruments have to be added to the financial market in order to be able to hedge the annuity contract.

price $V_{t_j}^h$ as follows:²⁷

$$V_{t_j}^h = \mathbb{E}_{t_j} \left[\frac{M_{t_{j+h}}}{M_{t_j}} c_{t_{j+h}} \right]. \quad (28)$$

Here, M_{t_j} denotes the stochastic discount factor defined in (26). The annuity at time t_j is a collection of payouts due at future dates t_{j+h} ($h = 0, 1, \dots, J - j$). The arbitrage-free annuity price at time t_j , i.e., V_{t_j} , is then defined as follows:

$$V_{t_j} = \sum_{h=0}^{J-j} V_{t_j}^h. \quad (29)$$

The following theorem provides an explicit expression for $V_{t_j}^h$.

Theorem 2. Denote by $F_{t_j}^h$ the buffering factor (see (15)). An arbitrage-free price $V_{t_j}^h$ at time t_j of the future annuity payout $c_{t_{j+h}}$ is now given by

$$V_{t_j}^h = c_{t_j} \times F_{t_j}^h \times A_{t_j}^h, \quad (30)$$

where

$$A_{t_j}^h = \exp \left\{ -\Delta t \sum_{k=1}^h d_{t_j}(k) \right\}, \quad (31)$$

and

$$d_{t_j}(k) = r_{t_{j+k}} - g_{j+k} + \psi(-\lambda) - \psi(q_k \beta \sigma - \lambda). \quad (32)$$

The arbitrage-free annuity price (30) consists of three factors: the current annuity payout c_{t_j} , the buffering factor $F_{t_j}^h$, and the annuity factor $A_{t_j}^h$. Equation (30) shows that $V_{t_j}^h$ is an increasing function of the buffering factor $F_{t_j}^h$. Indeed, a positive *past* portfolio shock increases the expected growth rate of future annuity payouts, thereby increasing the costs of future annuity payouts. The annuity factor $A_{t_j}^h$ captures the future portfolio shocks. This factor depends on the so-called forward discount rate $d_{t_j}(k)$ (see equation (32)) which takes into account the impact of the uncertain portfolio return in the time interval $[t_{j+k-1}, t_{j+k}]$ on the future annuity payout $c_{t_{j+h}}$.

Figure 6 shows the forward discount rate (32) as a function of the time distance between the payout

²⁷Throughout, we abstract away from longevity risk. We note that if survival rates do not depend on stock returns, (28) can be easily extended and becomes:

$$V_{t_j}^h = {}_h p_x \mathbb{E}_{t_j} \left[\frac{M_{t_{j+h}}}{M_{t_j}} c_{t_{j+h}} \right],$$

where ${}_h p_x$ denotes the probability that an individual aged x at time t_j will survive h periods.

date and the current time. The risk-free interest rate is set equal to 1.5%. In the case of a fixed nominal annuity, $d_{t_j}(k)$ reduces to the risk-free interest rate $r_{t_j} = 1.5\%$. As a result, a fixed nominal annuity cannot provide an adequate payout stream at an affordable price in a low interest rate environment (i.e., when r_{t_j} is relatively low). In the case of a unit-linked annuity, future payouts are exposed to (non-diversifiable) portfolio shocks. As a consequence, the discount rate is substantially larger than the risk-free interest rate. In the case of buffering of portfolio shocks, the discount rate increases with the horizon; the further into the future an annuity payout occurs, the larger the exposure of the annuity payout to a current portfolio shock, and hence the higher the discount rate will be. Figure 6 also shows that the discount rate depends on the stock return distribution. In particular, to compensate for the additional tail risk in stock returns, a VG distribution produces a higher discount rate than a Gaussian distribution.

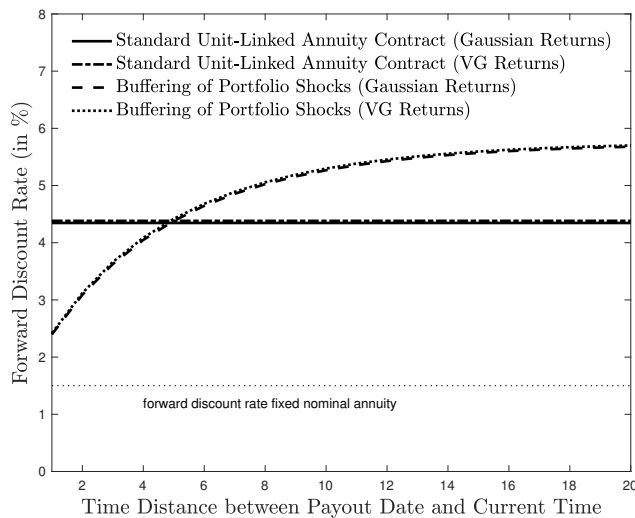


Figure 6. Term structure of forward discount rates. The figure illustrates the forward discount rate as a function of the time distance between the payout date and the current time. The solid line and the dash-dotted line show the case of a standard unit-linked annuity contract, while the dashed line and the dotted line show the case of buffering of portfolio shocks. We set the parameters \hat{g}_j and g_j to zero. Log stock returns are distributed according to either a Gaussian distribution (solid line and dashed line) or a VG distribution (dash-dotted line and dotted line); see Section 1.2 for the parameter values of the VG distribution. We assume the exponential buffering function (10), with $a_1 = a_2 = 1.6084$ and $\eta = 0.2$. The risk-free interest rate r_{t_j} is set equal to 1.5%, the parameter β to 50%, and the time step Δt to unity.

3.2.2 Dynamic Trading Strategy. We consider a dynamic trading strategy consisting of risky stocks and the risk-free account. Theorem 3 below specifies the share of the investment portfolio invested in the risky stock at time t_j .

Theorem 3. Consider a unit-linked annuity with buffering of portfolio shocks. Let α_{t_j} be the share of

the investment portfolio invested in the risky stock at time t_j . Suppose that α_{t_j} is given by

$$\alpha_{t_j} = \beta \sum_{h=1}^{J-j} \frac{V_{t_j}^h}{V_{t_j} - c_{t_j}} q_h. \quad (33)$$

Here, $V_{t_j} = \sum_{h=0}^{J-j} V_{t_j}^h$ and $V_{t_j}^h$ is given by Theorem 2.

Then this trading strategy approximately replicates the annuity payouts of the contract, i.e.,

$$W_{t_j} \approx V_{t_j}, \text{ for any } j = 0, 1, \dots, J, \quad (34)$$

where W_{t_j} denotes the value of the underlying investment portfolio at time t_j . We note that the trading strategy exactly replicates the annuity payouts if A_1, \dots, A_J are normally distributed and $\Delta t \rightarrow 0$.

Figure 7 illustrates the share of the investment portfolio invested in the risky stock as a function of time t_j . We assume the exponential buffering function (10). As shown by this figure, the insurer takes less investment risk as a policyholder becomes older. Intuitively, as time proceeds, portfolio shocks are spread out over a smaller number of years. Hence, to be able to provide a stable payout stream at higher ages, the insurer reduces the share of wealth invested in the risky stock over the course of the contract.

3.2.3 Hedging Performance. Theorem 3 provides a dynamic hedging strategy that (approximately) hedges the payouts of a unit-linked annuity with buffering of portfolio shocks. This section assesses the capability of the investment portfolio to provide an adequate hedge for the future annuity payouts. More specifically, we explore how well the dynamic trading strategy is capable of hedging a single annuity payout c_T with $T = 20$. Consistent with Section 2.2.1, we assume the exponential buffering function (10) (with $a_1 = a_2 = 1.6048$ and $\eta = 0.2$) and set the (unconditional) expected growth rate g_j equal to $-\psi(q_j\beta\sigma)$. Log stock returns are distributed according to a VG distribution; see Section 1.2 for the parameter values of the VG distribution. Finally, we set the risk-free interest rate r_{t_j} to 1.5% for every j , the parameter β to 50%, and the time step Δt to one month.

The value of the investment portfolio W_{t_j} satisfies the following dynamics (with $W_0 = V_0$):

$$\frac{W_{t_{j+1}}}{W_{t_j}} = \alpha_{t_j} \frac{S_{t_{j+1}}}{S_{t_j}} + (1 - \alpha_{t_j}) e^{r_{t_{j+1}} \Delta t}. \quad (35)$$

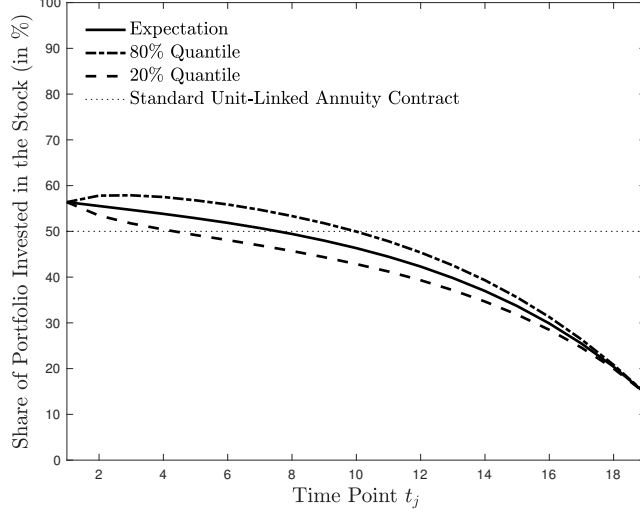


Figure 7. Dynamic trading strategy. The solid line illustrates the expected share of the investment portfolio invested in the risky stock as a function of the time point t_j . The dashed line and dash-dotted line correspond to the 20% and 80% quantile of the portfolio share, respectively. The dotted line shows the portfolio share associated with a standard unit-linked annuity contract. We choose the parameters \hat{g}_j and g_j such that, given the information available at time 0, expected future payouts are equal to 100. Log stock returns are distributed according to a VG distribution; see Section 1.2 for the parameter values of the VG distribution. We assume the exponential buffering function (10), with $a_1 = a_2 = 1.6084$ and $\eta = 0.2$. The risk-free interest rate r_{t_j} is set equal to 1.5% for each j , the parameter β to 50%, and the time step Δt to unity.

Here, α_{t_j} is the share of the investment portfolio invested in the risky stock which, in this particular case, equals $q_{J-j}\beta$; see equation (33). The relative hedging error $\epsilon_T = \epsilon_{20}$ is defined as follows:

$$\epsilon_T = \frac{W_T}{c_T} - 1. \quad (36)$$

We use (36) to measure the performance of the dynamic trading strategy α_{t_j} . First, we simulate the stock price process $\{S_{t_j} \mid j = 0, 1, \dots, J\}$. Then, we determine for each simulated sample path of the stock price, the realizations of the investment portfolio W_{t_j} (see (35)) and the annuity payout c_{t_j} (see (9)). Finally, we determine the relative hedging error ϵ_T (see (36)). Figure 8 shows a histogram of the realizations of the relative hedging error ϵ_T .²⁸ According to this histogram, the probability that the relative hedging error exceeds 5% is 0.0002%. Appendix E evaluates the hedging error associated with a constant-mix strategy. We observe that our strategy α_{t_j} performs very well compared to the constant-mix strategy.

²⁸Appendix D shows the histogram of the realizations of the relative hedging error ϵ_T in case of weekly and yearly rebalancing, and in case of Gaussian distributed stock returns. This appendix also evaluates the hedging performance in case of NIG distributed returns.

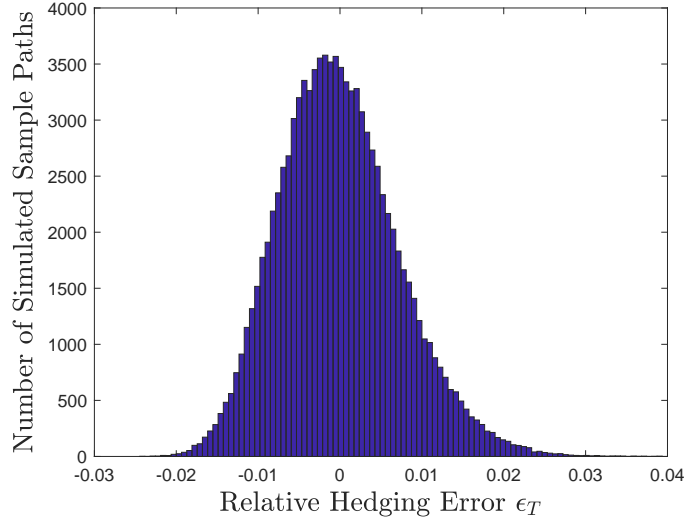


Figure 8. Hedging performance. This figure shows the histogram of the realizations of the relative hedging error ϵ_T based on 100k simulated sample paths for the stock price process $\{S_{t_j} \mid j = 0, 1, \dots, J\}$. The relative hedging error ϵ_T measures the performance of the underlying investment portfolio to hedge the single annuity payout c_T (with $T = 20$). We assume the exponential buffering function (10) (with $a_1 = a_2 = 1.6084$ and $\eta = 0.2$) and set the (unconditional) expected growth rate g_j equal to $-\psi(q_j \beta \sigma)$. Log stock returns are distributed according to a VG distribution; see Section 1.2 for the parameter values of the VG distribution. The risk-free interest rate r_{t_j} is set equal to 1.5% for each j , the parameter β to 50%, and the time step Δt to one month.

3.3 Valuation of the Non-Hedgeable Part

Figure 8 shows that by holding the portfolio W_{t_j} (see Theorem 3), we can adequately hedge the payout c_T . However, c_T is not exactly equal to W_T . We define the residual loss L_T as follows:

$$L_T = c_T - W_T. \quad (37)$$

Figure 8 shows that in some scenarios the hedging error ϵ_T is smaller than zero, representing a loss for the insurer (i.e., $L_T > 0$).

We can determine the charged annuity price P_0 in two steps; first, we determine an adequate hedge (see Section 3.2.1), and second, we value the residual loss using a risk measure. This approach was first proposed in Dhaene et al. (2017) for hybrid claims combining actuarial and financial risks. More specifically, we define the charged annuity price P_0 as follows:

$$P_0 = V_0 + C_0, \quad (38)$$

where $C_0 \geq 0$ denotes a capital buffer which we determine at time 0. The insurer uses the amount

V_0 to buy the hedging portfolio which pays out the amount W_T at maturity T . The remainder $C_0 = P_0 - V_0$ is put in the risk-free account which pays out $C_0 e^{\sum_{k=1}^J r_{t_k} \Delta t}$ at maturity T . We determine the capital buffer C_0 such that the total amount $W_T + C_0 e^{\sum_{k=1}^J r_{t_k} \Delta t}$ exceeds the liability c_T with probability p . More specifically, C_0 solves the following equation:

$$C_0 e^{\sum_{k=1}^J r_{t_k} \Delta t} = \inf \left\{ x_0 \in \mathbb{R} \mid \mathbb{P} \left[W_T + x_0 e^{\sum_{k=1}^J r_{t_k} \Delta t} \geq c_T \right] \geq p \right\}, \quad (39)$$

where p is supposed to be large (values for p usually range from 95% to 99.5%). Our definition of P_0 is equivalent to setting the capital buffer C_0 equal to the Value-at-Risk (VaR) of the residual loss L_T with confidence level p :²⁹

$$C_0 = e^{-\sum_{k=1}^J r_{t_k} \Delta t} \text{VaR}_p[L_T]. \quad (40)$$

We note that there is no unique annuity price P_0 , but a whole set of acceptable annuity prices P_0 . Figure 9 shows the capital buffer C_0 (expressed as a percentage of the initial price V_0) as function of the confidence level p , where we assume the same setting as in Section 3.3.

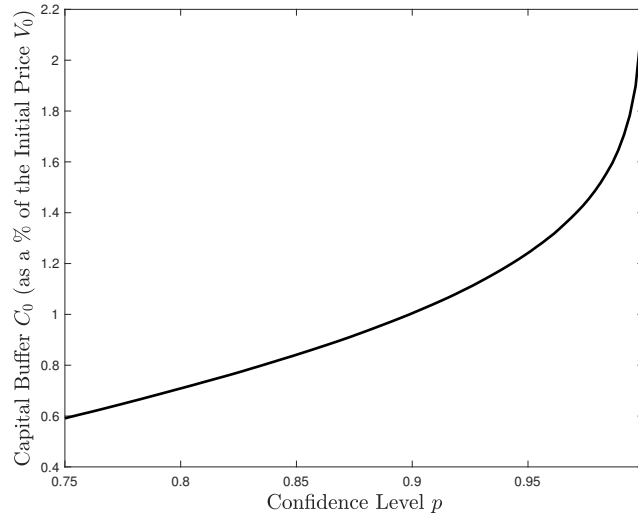


Figure 9. Capital buffer. This figure shows the capital buffer C_0 (expressed as a percentage of the initial price V_0) as a function of the confidence level p . We determine the capital buffer C_0 such that shortfall risk (i.e., $L_T > 0$) is smaller than $1 - p$. The annuity contract considers a single annuity payout c_T (with $T = 20$). We assume the exponential buffering function (10) (with $a_1 = a_2 = 1.6084$ and $\eta = 0.2$) and set the (unconditional) expected growth rate g_j equal to $-\psi(q_j \beta \sigma)$. Log stock returns are distributed according to a VG distribution; see Section 1.2 for the parameter values of the VG distribution. The risk-free interest rate r_{t_j} is set equal to 1.5% for each j , the parameter β to 50%, and the time step Δt to one month.

²⁹We can also determine C_0 using a general risk measure; see, e.g., Artzner et al. (1999) and Dhaene et al. (2003).

3.4 Valuation and Hedging when Interest Rates are Stochastic

Theorems 2 and 3 derive the value and the hedging strategy of a unit-linked annuity with buffering of portfolio shocks under the assumption that the interest rate is a deterministic function of time. This section generalizes our results to a stochastic interest rate environment. Detailed proofs can be found in Appendix G. We assume that the interest rate process $\{r_{t_j} \mid j = 1, 2, \dots, J\}$ is driven by the random variables B_1, \dots, B_J . Moreover, we enlarge our financial market and assume that at any time t_j , one can trade in zero-coupon bonds with maturities t_{j+h} ($h = 1, 2, \dots, J - j$). The time- t_j price of a zero-coupon bond with maturity t_{j+h} is denoted by $P(t_j, t_{j+h})$.

Theorem 7 in Appendix G determines the time- t_j price $V_{t_j}^h$ of the cash flow $c_{t_{j+h}}$ in a stochastic interest rate environment. When interest rates are stochastic, the price $V_{t_j}^h$ can still be expressed as the product of three terms: the current cash flow c_{t_j} , the buffering factor $F_{t_j}^h$, and the annuity factor $A_{t_j}^h$. However, the annuity factor $A_{t_j}^h$ now contains the bond price $P(t_j, t_{j+h})$:

$$A_{t_j}^h = P(t_j, t_{j+h}) \exp \left\{ -\Delta t \sum_{k=1}^h (\psi(-\lambda) - g_{j+k} - \psi(q_k \beta \sigma - \lambda)) \right\}. \quad (41)$$

To replicate the cash flow $c_{t_{j+h}}$, one has to invest in the risk-free asset, the risky stock and a zero-coupon bond with maturity t_{j+h} . Theorem 8 in Appendix G derives that the proportion of wealth invested in the risky stock at time t_j is βq_h . Replication of $c_{t_{j+h}}$ requires investing $\beta q_h V_{t_j}^h$ in the risky stock, $V_{t_j}^h$ in a zero-coupon bond with maturity t_{j+h} , and $-\beta q_h V_{t_j}^h$ in the risk-free asset.

4 Habit Formation and Buffering of Portfolio Shocks

The buffering function can be used to tailor a unit-linked annuity to the risk preferences of a policyholder, since it allows to differentiate between the short-term and the long-term volatility of future annuity payouts; see equation (13). We show in this section that buffering of portfolio shocks is optimal when the individual derives his utility by comparing current consumption with an internal habit level. We also specify the buffering function q_j such that it is consistent with habit formation assuming that the financial market can be described as in Sections 3.1 and 3.2.

We consider an individual with an initial budget W_0 who wants to purchase a unit-linked annuity with payout stream given by $\{c_{t_j} \mid j = 0, 1, \dots, J\}$. The market price to purchase this payout stream

is given by $\sum_{j=0}^J \mathbb{E} [M_{t_j} c_{t_j}]$, where the pricing kernel M_{t_j} is given by (26). We assume that the discounted expected utility U for this individual is given by:³⁰

$$U = \sum_{j=0}^J \mathbb{E} \left[(1 + \delta)^{-j\Delta t} \frac{1}{1 - \gamma} \left(\frac{c_{t_j}}{h_{t_j}} \right)^{1-\gamma} \right], \quad (42)$$

where the habit level h_{t_j} satisfies:

$$\log h_{t_j} - \log h_{t_{j-1}} = \varphi \log c_{t_{j-1}} - \vartheta \log h_{t_{j-1}}. \quad (43)$$

Here, $\delta \geq 0$ denotes the subjective rate of time preference, $\gamma \in (0, \infty)/\{1\}$ represents the coefficient of relative risk aversion, $\vartheta \geq 0$ models the rate at which the habit level depreciates, and $\varphi \leq \vartheta$ measures the relative importance between the initial habit level h_0 and the individual's past consumption choices.

The individual wants to maximize (42) subject to (43) and his dynamic budget constraint. Hence, the individual faces the following maximization problem:

$$\begin{aligned} \max_{c_{t_j}} \quad & \sum_{j=0}^J \mathbb{E} \left[(1 + \delta)^{-j\Delta t} \frac{1}{1 - \gamma} \left(\frac{c_{t_j}}{h_{t_j}} \right)^{1-\gamma} \right] \\ \text{s.t.} \quad & \sum_{j=0}^J \mathbb{E} [M_{t_j} c_{t_j}] \leq W_0, \\ & \log h_{t_j} - \log h_{t_{j-1}} = \varphi \log c_{t_{j-1}} - \vartheta \log h_{t_{j-1}}. \end{aligned} \quad (44)$$

Solving this maximization problem analytically is impossible and one has to use numerical techniques to derive the optimal consumption stream. Here, we follow the ideas proposed in Van Bilsen et al. (2018) and approximate the budget constraint by using a Taylor series expansion around the consumption stream $\left\{ \frac{c_{t_j}}{h_{t_j}} \right\} = 1$. In the following theorem, we show the solution of the approximate maximization problem. A proof of this result is given in Appendix F.

Theorem 4. *Consider the financial market described in Sections 3.1 and 3.2. The optimal*

³⁰We note that the ratio habit model (42) has been employed by many authors, see, e.g., Fuhrer (2000), Abel (1990, 1999), Chan and Kogan (2002), Gómez et al. (2009), Smith and Zhang (2007), Carroll et al. (1997), Carroll et al. (2000), Carroll (2000), and Gomes and Michaelides (2003).

consumption which approximately solves the maximization problem (44) is given by

$$c_{t_j}^* = \exp \left\{ - \sum_{k=0}^{j-1} \frac{\varphi}{\gamma} (1 - (\vartheta - \varphi))^{j-1-k} \widehat{y}_k - \frac{1}{\gamma} \widehat{y}_j + \frac{\lambda}{\gamma} \sqrt{\Delta t} \sum_{k=1}^j A_{j+1-k} q_k \right\} \quad (45)$$

for appropriate choices of \widehat{y}_k . The function q_k is given by

$$q_k = 1 + \sum_{l=0}^{k-2} \varphi (1 - (\vartheta - \varphi))^l. \quad (46)$$

Theorem 4 shows that buffering of portfolio shocks is required to maximize an individual's expected utility in the presence of a habit level.³¹ Indeed, the function q_k defined in equation (46) is a buffering function as defined in Section 2. Furthermore, the optimal consumption $c_{t_j}^*$ turns out to be a unit-linked annuity with buffering of portfolio shocks, with appropriately chosen expected growth rate.³² Note also that Theorem 4 does not make any assumptions regarding the distribution of the stock return shocks.

In the special case where Δt is infinitely small, the individual's optimal consumption choice features the following buffering function q_{s-t} (see Van Bilsen et al. (2018)):

$$q_{s-t} = \begin{cases} 1 + \frac{\varphi}{\vartheta - \varphi} (1 - \exp \{ - (\vartheta - \varphi) (s - t) \}), & \text{if } \vartheta > \varphi, \\ 1 + \varphi \cdot (s - t), & \text{if } \vartheta = \varphi. \end{cases} \quad (47)$$

Here, q_{s-t} denotes the impact of a portfolio shock at time t on a future annuity payout at time $s > t$. It follows from (47) that the parameters φ and $\vartheta - \varphi$, which measure the degree of habit persistence, model the impact of a current portfolio shock on the future growth rates of consumption.

To illustrate the potential benefits of buffering of portfolio shocks, we consider the following situation. Suppose an individual is offered two contracts: a standard unit-linked annuity contract and an annuity with buffering of portfolio shocks. We assume the same setting as in Section 2.2.1. Furthermore, we assume $\varphi = \vartheta$. Table 1 reports for various values of the preference parameters which contract the individual prefers. We observe that for sufficiently high values of $\varphi = \vartheta$ and γ , the individual prefers the contract with buffering of portfolio shocks.

³¹Note that in case the individual derives utility only from absolute levels of consumption (i.e., $h_{t_j} = 1$ for every j), the optimal payout stream is provided by the standard unit-linked annuity contract as specified in equation (7).

³²Here, we implicitly assume that the income provided by the unit-linked annuity is the only source of retirement income.

Parameter $\varphi = \vartheta$	Risk Aversion γ				
	2	3	5	7	10
0	S	S	S	S	S
0.2	S	S	S	S	S
0.4	S	S	B	B	B
0.6	S	B	B	B	B
0.8	S	B	B	B	B

Table 1. Preferred contract. This table reports for various values of the preference parameters which contract the individual prefers. The individual has the choice between two contracts: a standard unit-linked annuity contract (S) and an annuity with buffering of portfolio shocks (B). We assume the same setting as in Section 2.2.1. Furthermore, we assume $\varphi = \vartheta$. The individual’s preferences are described by the ratio internal habit model (see (42)). We set the initial habit level h_0 equal to the initial annuity payout (i.e., 100) and the subjective rate of time preference δ to 1.5%. We note that our findings are quite insensitive to a change in δ .

5 Model Robustness

Model robustness refers to the sensitivity of the (arbitrage-free) annuity price V_0 to small changes in the underlying stock return model. When an insurer determines the annuity price, he must specify the market price of risk parameter λ as well as the standardized stock return distribution; see equation (24) in Section 3.2.1.³³ However, estimation of these parameters using the available data introduces estimation errors. Therefore, we have to explore the impact of a misspecification in the market price of risk parameter and the standardized stock return distribution on the annuity price. Indeed, if an insurer is aware that a misspecification potentially has significant implications for the annuity price, then he might want to charge an additional premium to avoid an (unacceptably) high probability of loss occurrence. A major advantage of our framework is that it encompasses a large variety of stock return models. Hence, our framework allows us to study the impact of assuming different stock return models on the annuity price.³⁴

This section considers the same annuity product as described in Section 3.2.3. Furthermore, it assumes that stock return shocks are distributed according to a VG distribution, with ‘true’ market price of risk parameter $\lambda_{VG} = 0.3874$ and ‘true’ shape parameter $\nu_{VG} = 0.7853$ (see also Section 1.2 for the parameter values of the VG distribution). The insurer has only partial information about the ‘true’ stock return distribution. It correctly employs a VG distribution, but the ‘true’ parameters λ_{VG} and ν_{VG} are unknown. Once these two parameters are estimated, the expected excess rate of return e

³³Here, we implicitly assume that the stock return volatility and the interest rate are exogenously given.

³⁴Our framework also allows us to study the impact of a parameter misspecification on the hedging performance.

follows from (24). The other parameters of the VG distribution (i.e., σ_{VG} , θ_{VG} and μ_{VG}) are assumed to be fixed and thus do not have to be estimated. Denote the estimated parameters by λ_{VG}^* and ν_{VG}^* . We note that the estimated shape parameter ν_{VG}^* unambiguously determines the estimated kurtosis of the standardized stock return distribution. Indeed, the estimated kurtosis κ_{VG}^* equals $3 + 3 \cdot \nu_{VG}^*$. In what follows, we use the estimated kurtosis κ_{VG}^* instead of the estimated shape parameter ν_{VG}^* .

Given the estimated parameters λ_{VG}^* and κ_{VG}^* , we can determine the time-0 (arbitrage-free) annuity price $V_0(\lambda_{VG}^*, \kappa_{VG}^*)$; see Theorem 2.³⁵ Because the estimated parameters λ_{VG}^* and κ_{VG}^* can deviate from their ‘true’ values, the annuity price $V_0(\lambda_{VG}^*, \kappa_{VG}^*)$ may also differ from its ‘true’ value $V_0(\lambda_{VG}, \kappa_{VG})$. We denote this pricing error by $\epsilon_0(\lambda_{VG}^*, \kappa_{VG}^*)$:

$$\epsilon_0(\lambda_{VG}^*, \kappa_{VG}^*) = \frac{V_0(\lambda_{VG}^*, \kappa_{VG}^*)}{V_0(\lambda_{VG}, \kappa_{VG})} - 1. \quad (48)$$

Figure 10 illustrates the pricing error (48) as a function of the estimated market price of risk parameter λ_{VG}^* and the estimated kurtosis κ_{VG}^* . The case $\kappa_{VG}^* = 3$ corresponds to a Gaussian distribution, whereas the case $\kappa_{VG}^* = 5.36$ (and $\lambda_{VG}^* = 0.3874$) corresponds to the ‘true’ VG distribution. As shown by Figure 10, deviations from the ‘true’ parameters may have a significant impact on the annuity price. In the following sections, we illustrate how the annuity price varies with the market price of risk parameter and the kurtosis. For an insurer, it is important to know how sensitive the charged annuity price is to a misspecification in the underlying stock return model. The richness of our framework enables insurers to carry out such a sensitivity analysis.

5.1 Wrong Standardized Stock Return Distribution

This section considers the case where the ‘true’ market price of risk parameter λ_{VG} is used, but a wrong standardized stock return distribution is employed. For example, the ‘true’ market price of risk parameter $\lambda_{VG} = 0.3874$ may be given exogenously by the regulator. However, the insurer must estimate the kurtosis of the standardized stock return distribution. The insurer thus uses an estimated kurtosis which typically differs from the ‘true’ kurtosis when pricing the annuity product.

Figure 10 illustrates that using an estimated kurtosis κ_{VG}^* which is lower than the ‘true’ kurtosis $\kappa_{VG} = 5.36$ leads to a higher annuity price (i.e., positive pricing error). The reason for this is that a reduction in the estimated kurtosis κ_{VG}^* implies a lower estimated mean excess stock return (this

³⁵Alternatively, we can also use the annuity price defined in (38). However, this will not change our conclusions.

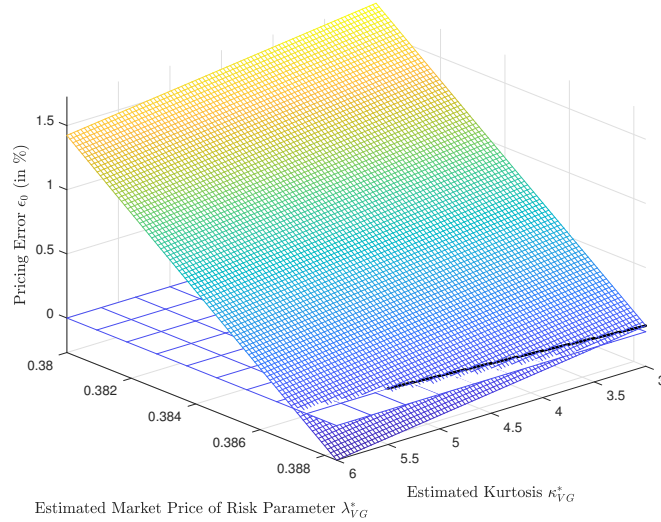


Figure 10. Sensitivity of the annuity price to estimation errors in λ_{VG} and κ_{VG} . This figure shows the pricing error $\epsilon_0(\lambda_{VG}^*, \kappa_{VG}^*)$ as a function of the estimated market price of risk parameter λ_{VG}^* and the estimated kurtosis κ_{VG}^* . The figure considers the same annuity product as described in Section 3.2.3. Furthermore, it assumes that stock return shocks are distributed according to a VG distribution, with ‘true’ market price of risk parameter $\lambda_{VG} = 0.3874$ and ‘true’ kurtosis $\kappa_{VG} = 5.36$. The blue line indicates all combinations of $(\lambda_{VG}^*, \kappa_{VG}^*)$ such that the estimated mean excess stock return equals the ‘true’ mean excess stock return (i.e., the mean stock return implied by the ‘true’ parameter values λ_{VG} and κ_{VG}).

follows from (24), (25) and (49)). This can also be seen from Figure 11 which shows the estimated mean stock return as function of the estimated market price of risk parameter λ_{VG}^* and the estimated kurtosis κ_{VG}^* . Hence, if the insurer employs an estimated kurtosis κ_{VG}^* which is lower than the ‘true’ kurtosis κ_{VG} , then the annuity price is based on a estimated mean stock return that is too low.

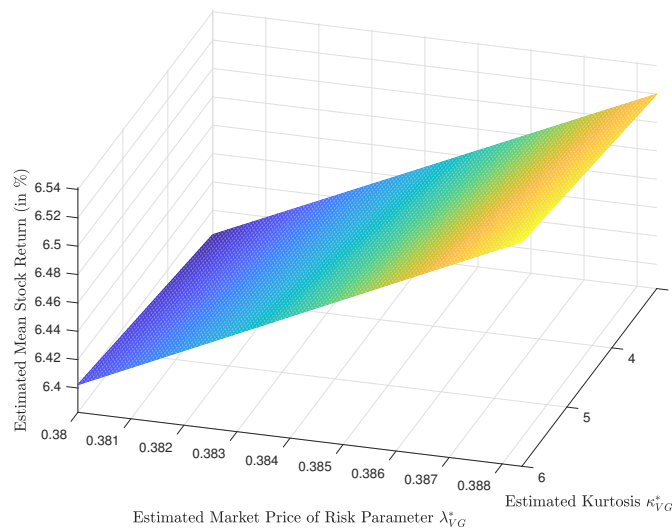


Figure 11. Estimated mean stock return as a function of λ_{VG}^* and κ_{VG}^* . This figure shows the estimated mean stock return as function of the estimated market price of risk parameter λ_{VG}^* and the estimated kurtosis κ_{VG}^* . The figure assumes that stock return shocks are distributed according to a VG distribution; see Section 1.2 for the parameter values of the VG distribution. The risk-free interest rate r_{t_j} equals 1.5%.

5.2 Wrong Market Price of Risk Parameter

This section considers the case where the ‘true’ standardized stock return distribution is employed, but a wrong market price of risk parameter is used when pricing the annuity product. We assume that the ‘true’ kurtosis κ_{VG} is known to the insurer. However, instead of using the ‘true’ market price of risk parameter λ_{VG} , the insurer uses, for example, the market price of risk parameter that follows from the Gaussian distribution (i.e., $\lambda_{VG}^* = 0.3884 > \lambda_{VG} = 0.3874$).

As shown by Figure 10, an increase in the estimated market price of risk parameter λ_{VG}^* leads to a lower annuity price (i.e., negative pricing error). The reason for this is that an estimated market price of risk parameter λ_{VG}^* which is higher than the ‘true’ market price of risk parameter λ_{VG} results in an estimated mean excess stock return that is higher than the ‘true’ mean excess stock return; see also Figure 11.

5.3 Joint Impact of Wrong Market Price of Risk and Wrong Kurtosis

The previous sections have showed that pricing errors can arise when using an estimated market price of risk parameter λ_{VG}^* which differs from the ‘true’ market price of risk parameter λ_{VG} or when using an estimated kurtosis κ_{VG}^* which differs from the ‘true’ kurtosis κ_{VG} . It follows from Figure 11 that changing the underlying assumptions about the market price of risk parameter and the standardized stock return distribution affects the estimated mean excess stock return.

This section assumes that the mean excess stock return is known, but the ‘true’ market price of risk parameter λ_{VG} and the ‘true’ kurtosis κ_{VG} of the stock returns are unknown. We note that the estimated market price of risk parameter λ_{VG}^* unambiguously determines the estimated kurtosis κ_{VG}^* once the mean excess stock return is known. The blue line in Figure 10 indicates the annuity prices $V_0(\lambda_{VG}^*, \kappa_{VG}^*)$ for which the estimated mean excess stock return is equal to the ‘true’ mean excess stock return. We observe that the values on the blue line are all close to the ‘true’ value. The reason for this is that although the insurer makes two different types of errors, these errors have the opposite sign, and combining them results in a relatively small effect on the annuity price.

Table 2 illustrates the impact of assuming a wrong kurtosis and/or a wrong market price of risk parameter on the annuity price. In the case where the estimated parameters are equal to their ‘true’ values, the pricing error is zero. In all other cases, the insurer makes an error by assuming a wrong kurtosis, a wrong market price of risk parameter or both. We note that the joint impact of the two

errors (0.0472%) is (at least) 4 times smaller than the impact of the error due to a wrong kurtosis (0.2339%) or the error due to a wrong market price of risk parameter (-0.1942%).

Estimated Kurtosis	Estimated Market Price of Risk Parameter	
	0.3884	0.3874
5.36	0	-0.1942
3	0.2339	0.0472

Table 2. Impact of wrong market price of risk parameter and wrong kurtosis. This table illustrates the impact of assuming a wrong kurtosis and/or a wrong market price of risk parameter on the annuity price. The numbers denote the pricing error (expressed as a percentage). The table assumes that stock returns are distributed according to a VG distribution, with ‘true’ market price of risk parameter $\lambda_{VG} = 0.3884$ and ‘true’ kurtosis $\kappa_{VG} = 5.36$.

6 Concluding Remarks

The pension landscape is rapidly evolving and this has increased the need for alternative annuity products. This paper has proposed a new class of unit-linked annuities that provides flexibility in tailoring payout streams to individual preferences. More specifically, our proposed annuities allow for buffering of portfolio shocks: the sooner an annuity payout occurs, the smaller its exposure to a current portfolio shock will be. By allowing for buffering of portfolio shocks, insurers are able to offer an affordable and an adequate annuity with a stable payout stream. We have also developed a framework to deal with the risks inherent to our proposed annuities. That is, we have determined annuity prices and underlying investment strategies that adequately hedge the liabilities of the contracts. We have shown that the annuity price consists of two components; the first component is equal to the initial value of the hedging portfolio and the second component corresponds to a capital buffer which covers the unhedgeable part of future annuity payouts. Finally, we have illustrated how insurers can use our framework to explore a parameter misspecification on the annuity price.

We identify various directions for future work. First, we assume that the specification of the buffering function is exogenously given. An alternative approach would be to calibrate the specification of the buffering function using consumption data. Another direction for future work would be to develop a unit-linked annuity contract which not only applies buffering of portfolio shocks but also includes a guarantee (such as a minimum withdrawal guarantee).

A Variance Gamma and other Lévy Distributions

We start this appendix by providing a few properties of the Variance Gamma distribution. Let $\sigma_{VG}, \nu_{VG} > 0$ and $\mu_{VG}, \theta_{VG} \in \mathbb{R}$. The characteristic function of the random variable $X \sim VG(\sigma_{VG}, \nu_{VG}, \theta_{VG}, \mu_{VG})$ is given by

$$\phi(v) = e^{i v \mu_{VG}} (1 - i v \theta_{VG} \nu_{VG} + v^2 \nu_{VG} \sigma_{VG}^2 / 2)^{-1/\nu_{VG}}. \quad (49)$$

The mean and the variance of X are equal to $\mu_{VG} + \theta_{VG}$ and $\sigma_{VG}^2 + \nu_{VG} \theta_{VG}^2$, respectively.

The standardized random variable

$$\bar{X} = \frac{X - \mathbb{E}[X]}{\sqrt{\text{Var}[X]}} \quad (50)$$

is distributed according to:

$$\bar{X} \sim VG(\zeta_{VG} \sigma_{VG}, \nu_{VG}, \zeta_{VG} \theta_{VG}, -\zeta_{VG} \theta_{VG}), \quad (51)$$

where $\zeta_{VG} = 1/\sqrt{\sigma_{VG}^2 + \nu_{VG} \theta_{VG}^2}$.

The empirical moments of \bar{X} are given by

$$m_1 = \mathbb{E}[\bar{X}] = -\zeta_{VG} \theta_{VG} + \zeta_{VG} \theta_{VG} = 0, \quad (52)$$

$$m_2 = \text{Var}[\bar{X}] = (\zeta_{VG} \sigma_{VG})^2 + \nu_{VG} (\zeta_{VG} \theta_{VG})^2 = 1, \quad (53)$$

$$m_3 = 2 (\zeta_{VG} \theta_{VG})^3 \nu_{VG}^2 + 3 (\zeta_{VG} \sigma_{VG})^2 \nu_{VG} \zeta_{VG} \theta_{VG}, \quad (54)$$

$$\begin{aligned} m_4 = & 3 (\zeta_{VG} \sigma_{VG})^4 \nu_{VG} + 12 (\zeta_{VG} \sigma_{VG})^2 \nu_{VG}^2 (\zeta_{VG} \theta_{VG})^2 + 6 (\zeta_{VG} \theta_{VG})^4 \nu_{VG}^3 \\ & + 3 (\zeta_{VG} \sigma_{VG})^4 + 6 (\zeta_{VG} \sigma_{VG})^2 (\zeta_{VG} \theta_{VG})^2 \nu_{VG} + 3 (\zeta_{VG} \theta_{VG})^4 \nu_{VG}^2. \end{aligned} \quad (55)$$

Table 3 provides a few properties of the Normal Inverse Gaussian and the Meixner distribution.

We now calibrate the Normal Inverse Gaussian (NIG) distribution. The calibrated parameter values are given in Table 4.

Finally, Figure 12 shows the calibrated NIG density function (panel a) and a quantile-quantile plot comparing the quantiles of the calibrated NIG distribution with the empirical quantiles (panel b).

	Normal Inverse Gaussian	Meixner
Parameters	$\alpha, \delta > 0, \beta \in (-\alpha, \alpha), \mu \in \mathbb{R}$	$\alpha, \delta > 0, \beta \in (-\pi, \pi), \mu \in \mathbb{R}$
Notation	$NIG(\alpha, \beta, \delta, \mu)$	$MX(\alpha, \beta, \delta, \mu)$
$\phi(u)$	$e^{iu\mu - \delta(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2})}$	$e^{iu\mu} \left(\frac{\cos(\beta/2)}{\cosh((\alpha u - i\beta)/2)} \right)^{2\delta}$
Mean	$\mu + \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}}$	$\mu + \alpha\delta \tan(\beta/2)$
Variance	$\alpha^2\delta(\alpha^2 - \beta^2)^{-3/2}$	$\cos^{-2}(\beta/2)\alpha^2\delta/2$
Standardized version	$NIG\left(\alpha, \beta, (\alpha^2 - \beta^2)^{3/2}, \frac{-(\alpha^2 - \beta^2)\beta}{\alpha^2}\right)$	$MX\left(\alpha, \beta, \frac{2\cos^2(\frac{\beta}{2})}{\alpha^2}, \frac{-\sin(\beta)}{\alpha}\right)$

Table 3. Properties of other distributions. This table provides a few properties of the Normal Inverse Gaussian and the Meixner distribution.

Model Parameters			
α_{NIG}	β_{NIG}	δ_{NIG}	μ_{NIG}
1.1284	0	1.43690	0

Table 4. Calibrated parameter values. This table shows the calibrated parameter values for the Normal Inverse Gaussian (NIG) distribution.

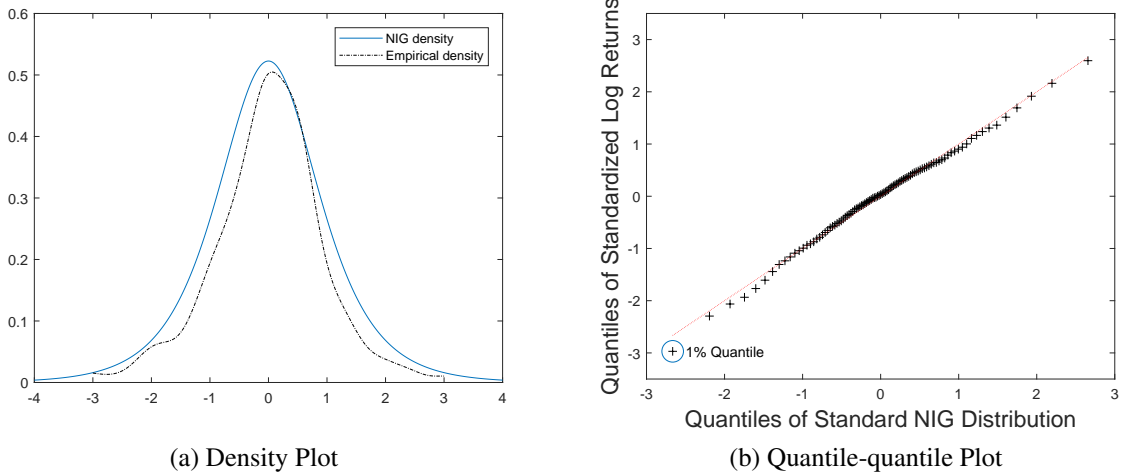


Figure 12. Density function and quantile-quantile plot. This figure illustrates the calibrated NIG density function (panel a) and a quantile-quantile plot comparing the quantiles of the calibrated NIG distribution with the empirical quantiles (panel b). Panel b shows all quantiles between 1% and 99%, with an increment of 1%.

B Properties of the Expected Excess Stock Return

This appendix assumes that the stock return volatility is constant over time (i.e., $\sigma_{t_j} = \sigma$ for every j).

We can write the expected stock price as follows:

$$\begin{aligned}\mathbb{E}[S_{t_j}] &= \mathbb{E}\left[S_0 \exp\left\{\sum_{k=1}^j r_{t_k} \Delta t + e \cdot t_j + \sqrt{\Delta t} \sum_{k=1}^j \sigma A_k\right\}\right] \\ &= S_0 \exp\{e \cdot t_j\} \mathbb{E}\left[\exp\left\{\sum_{k=1}^j r_{t_k} \Delta t\right\}\right] \mathbb{E}\left[\exp\left\{\sqrt{\Delta t} \sum_{k=1}^j \sigma A_k\right\}\right].\end{aligned}\quad (56)$$

Using (20) and the independence of A_1, \dots, A_j , we find

$$\mathbb{E}\left[\exp\left\{\sqrt{\Delta t} \sum_{k=1}^j \sigma A_k\right\}\right] = \prod_{k=1}^j \mathbb{E}\left[\exp\left\{\sigma \sqrt{\Delta t} A_k\right\}\right] = \phi(-i\sigma \sqrt{\Delta t})^j. \quad (57)$$

Using (25), we can write (57) as follows:

$$\mathbb{E}\left[\exp\left\{\sigma \sqrt{\Delta t} \sum_{k=1}^j A_k\right\}\right] = e^{\psi(\sigma)j\Delta t}. \quad (58)$$

Using $j\Delta t = t_j$ and (24), we find

$$\begin{aligned}\mathbb{E}[S_{t_j}] &= S_0 \mathbb{E}\left[\exp\left\{\sum_{k=1}^j r_{t_k} \Delta t\right\}\right] \exp\{(e + \psi(\sigma))t_j\} \\ &= S_0 \mathbb{E}\left[\exp\left\{\sum_{k=1}^j r_{t_k} \Delta t\right\}\right] \exp\{R(\lambda, \sigma)t_j\},\end{aligned}\quad (59)$$

with $R(\lambda, \sigma) = e + \psi(\sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda)$ (use (24)).

We can now prove the following desirable properties of $R(\lambda, \sigma)$.

Theorem 5. *The expected excess stock return $R(\lambda, \sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda)$ is always positive:*

$$R(\lambda, \sigma) > 0 \text{ for any } \lambda, \sigma > 0. \quad (60)$$

Furthermore, $R(\lambda, \sigma)$ is increasing in both its arguments:

$$R(\lambda_2, \sigma) - R(\lambda_1, \sigma) > 0, \text{ and } R(\lambda, \sigma_2) - R(\lambda, \sigma_1) > 0, \quad (61)$$

for $\lambda_1 < \lambda_2$ and $\sigma_1 < \sigma_2$.

We first prove the following property:

$$R(\lambda, \sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda) > 0 \text{ for any } \lambda, \sigma > 0. \quad (62)$$

This property is equivalent to

$$\psi(\sigma) + \psi(-\lambda) > \psi(\sigma - \lambda) \text{ for any } \lambda, \sigma > 0. \quad (63)$$

Using (25) and (20), we can write (63) as follows:

$$\mathbb{E} \left[e^{\sigma\sqrt{\Delta t}A_j} \right] \mathbb{E} \left[e^{-\lambda\sqrt{\Delta t}A_j} \right] > \mathbb{E} \left[e^{(\sigma-\lambda)\sqrt{\Delta t}A_j} \right] \text{ for any } \lambda, \sigma > 0. \quad (64)$$

Define $X = e^{\sigma\sqrt{\Delta t}A_j}$ and $Y = e^{-\lambda\sqrt{\Delta t}A_j}$. By definition,

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] + \text{Cov}(X, Y), \quad (65)$$

where $\text{Cov}(X, Y)$ denotes the covariance between X and Y . Clearly, $\text{Cov}(X, Y) < 0$. Hence,

$$\mathbb{E} \left[e^{(\sigma-\lambda)\sqrt{\Delta t}A_j} \right] = \mathbb{E}[XY] < \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E} \left[e^{\sigma\sqrt{\Delta t}A_j} \right] \mathbb{E} \left[e^{-\lambda\sqrt{\Delta t}A_j} \right] \text{ for any } \lambda, \sigma > 0. \quad (66)$$

We now show that $R(\lambda, \sigma)$ is increasing in both λ and σ . We note that the function $\psi(\sigma)$ is convex.

Indeed,

$$\psi''(\sigma) = \frac{1}{\Delta t} \frac{\mathbb{E} \left[\left(\sqrt{\Delta t}A_j - \frac{\mathbb{E}[\sqrt{\Delta t}A_j e^{\sigma\sqrt{\Delta t}A_j}]}{\mathbb{E}[e^{\sigma\sqrt{\Delta t}A_j}]} \right)^2 e^{\sigma\sqrt{\Delta t}A_j} \right]}{\mathbb{E} \left[e^{\sigma\sqrt{\Delta t}A_j} \right]} > 0. \quad (67)$$

By convexity,

$$\frac{\partial R(\lambda, \sigma)}{\partial \sigma} = \psi'(\sigma) - \psi'(\sigma - \lambda) > 0. \quad (68)$$

In a similar fashion, we find

$$\frac{\partial R(\lambda, \sigma)}{\partial \lambda} = \psi'(\sigma - \lambda) - \psi'(-\lambda) > 0. \quad (69)$$

C Mathematical Proofs

Proof of (14)

Dividing $c_{t_{j+h}}$ by c_{t_j} , we find

$$\frac{c_{t_{j+h}}}{c_{t_j}} = F_{t_j}^h \times \exp \left\{ \sum_{k=1}^h g_{j+k} \Delta t + \sum_{k=1}^h q_k \beta_{j+h+1-k} \sigma_{t_{j+h+1-k}} \sqrt{\Delta t} A_{j+h+1-k} \right\}, \quad (70)$$

where

$$F_{t_j}^h = \exp \left\{ \sum_{k=1}^j (q_{j+h-k+1} - q_{j-k+1}) \beta_k \sigma_{t_k} \sqrt{\Delta t} A_k \right\}. \quad (71)$$

Taking the expectation of (70), we arrive at (14). We note that the parameter g_j can be used to control the expected payout stream.

Proof of (17)

Equation (17) follows from (70) by taking $h = 1$.

Proof of (18)

We have the following expression for $F_{t_j}^h / F_{t_j}^1$:

$$\begin{aligned} \frac{F_{t_j}^h}{F_{t_j}^1} &= \prod_{k=1}^j \exp \left\{ (q_{j+h-(k-1)} - q_{j-k+2}) \beta_k \sigma_{t_k} \sqrt{\Delta t} A_k \right\} \\ &= \prod_{k=1}^j \exp \left\{ (q_{j+1+h-1-(k-1)} - q_{j+1-k+1}) \beta_k \sigma_{t_k} \sqrt{\Delta t} A_k \right\}. \end{aligned} \quad (72)$$

The term $F_{t_{j+1}}^{h-1}$ can be rewritten as follows:

$$\begin{aligned} F_{t_{j+1}}^{h-1} &= \prod_{k=1}^{j+1} \exp \left\{ (q_{j+1+h-1-(k-1)} - q_{j+1-k+1}) \beta_k \sigma_{t_k} \sqrt{\Delta t} A_k \right\} \\ &= \frac{F_{t_j}^h}{F_{t_j}^1} \times \exp \left\{ (q_h - q_1) \beta_{j+1} \sigma_{t_{j+1}} \sqrt{\Delta t} A_{j+1} \right\}. \end{aligned} \quad (73)$$

Taking the logarithm on both sides of (73), we arrive at (18).

Proof of Theorem 1

The pricing measure \mathbb{Q} with Radon-Nikodym derivative ξ_{t_j} is a feasible pricing measure (i.e., $\mathbb{Q} \in \mathcal{Q}$) if the following two conditions are satisfied (for any $j \in \{0, 1, \dots, J\}$):

$$\mathbb{E}_{\mathbb{P}} [\xi_T | \mathcal{F}_{t_j}] = \xi_{t_j}, \quad (74)$$

$$\mathbb{E}_{\mathbb{P}} \left[e^{-\sum_{k=1}^j r_{t_k} \Delta t} \xi_{t_j} S_{t_j} \right] = S_0. \quad (75)$$

We now show that the process $\{\xi_{t_j} | j = 0, 1, \dots, J\}$ given by

$$\xi_{t_j} = \exp \left\{ -\sum_{k=1}^j \psi(-\lambda_{t_k}) \Delta t - \sqrt{\Delta t} \sum_{k=1}^j \lambda_{t_k} A_k \right\} \quad (76)$$

satisfies conditions (74) and (75).

To prove (74), we write ξ_T as follows:

$$\xi_T = \xi_{t_j} \exp \left\{ -\sum_{k=1}^{J-j} \psi(-\lambda_{t_{j+k}}) \Delta t - \sqrt{\Delta t} \sum_{k=1}^{J-j} \lambda_{t_{j+k}} A_{j+k} \right\}. \quad (77)$$

Then we can write:

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} [\xi_T | \mathcal{F}_{t_j}] &= \mathbb{E}_{\mathbb{P}} \left[\xi_{t_j} \exp \left\{ -\sum_{k=1}^{J-j} \psi(-\lambda_{t_{j+k}}) \Delta t - \sqrt{\Delta t} \sum_{k=1}^{J-j} \lambda_{t_{j+k}} A_{j+k} \right\} \middle| \mathcal{F}_{t_j} \right] \\ &= \xi_{t_j} \mathbb{E}_{\mathbb{P}} \left[\mathbb{E}_{\mathbb{P}} \left[\exp \left\{ -\sum_{k=1}^{J-j} \psi(-\lambda_{t_{j+k}}) \Delta t - \sqrt{\Delta t} \sum_{k=1}^{J-j} \lambda_{t_{j+k}} A_{j+k} \right\} \middle| \mathcal{F}_{t_{j-1}} \right] \middle| \mathcal{F}_{t_j} \right] \\ &= \xi_{t_j} \mathbb{E}_{\mathbb{P}} \left[\exp \left\{ -\sum_{k=1}^{J-j-1} \psi(-\lambda_{t_{j+k}}) \Delta t - \sqrt{\Delta t} \sum_{k=1}^{J-j-1} \lambda_{t_{j+k}} A_{j+k} \right\} e^{-\psi(-\lambda_{t_j}) \Delta t} \right. \\ &\quad \left. \mathbb{E}_{\mathbb{P}} \left[\exp \left\{ -\sqrt{\Delta t} \lambda_{t_j} A_j \right\} \middle| \mathcal{F}_{t_{j-1}} \right] \middle| \mathcal{F}_{t_j} \right]. \end{aligned} \quad (78)$$

Using

$$\mathbb{E}_{\mathbb{P}} \left[\exp \left\{ -\lambda_{t_j} \sqrt{\Delta t} A_j \right\} \middle| \mathcal{F}_{t_{j-1}} \right] = e^{\Delta t \psi(-\lambda_{t_j})}, \quad (79)$$

we find

$$\mathbb{E}_{\mathbb{P}} [\xi_T | \mathcal{F}_{t_j}] = \xi_{t_j} \mathbb{E}_{\mathbb{P}} \left[\exp \left\{ -\sum_{k=j}^{J-j-1} \psi(-\lambda_{t_{j+k}}) \Delta t - \sqrt{\Delta t} \sum_{k=1}^{J-j-1} \lambda_{t_{j+k}} A_{j+k} \right\} \middle| \mathcal{F}_{t_j} \right]. \quad (80)$$

Continuing this approach of conditioning on the sets $\mathcal{F}_{t_{j-2}}, \mathcal{F}_{t_{j-3}}, \dots, \mathcal{F}_{t_{j+1}}$, we arrive at

$$\mathbb{E}_{\mathbb{P}} [\xi_T \mid \mathcal{F}_{t_j}] = \xi_{t_j}. \quad (81)$$

To prove (75), we calculate $e^{-\sum_{k=1}^j r_{t_k} \Delta t} \xi_{t_j} S_{t_j}$:

$$\begin{aligned} e^{-\sum_{k=1}^j r_{t_k} \Delta t} \xi_{t_j} S_{t_j} &= S_0 \exp \left\{ - \sum_{k=1}^j (r_{t_k} + \psi(-\lambda_{t_k})) \Delta t - \sqrt{\Delta t} \sum_{k=1}^j \lambda_{t_k} A_k \right\} \\ &\times \exp \left\{ \sum_{k=1}^j r_{t_k} \Delta t + e t_j + \sqrt{\Delta t} \sum_{k=1}^j \sigma_{t_k} A_k \right\}. \end{aligned} \quad (82)$$

Using (24), we find

$$e^{-\sum_{k=1}^j r_{t_k} \Delta t} \xi_{t_j} S_{t_j} = S_0 \exp \left\{ - \sum_{k=1}^j \psi(\sigma_{t_k} - \lambda_{t_k}) \Delta t + \sqrt{\Delta t} \sum_{k=1}^j (\sigma_{t_k} - \lambda_{t_k}) A_k \right\}. \quad (83)$$

We can then write

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[e^{-\sum_{k=1}^j r_{t_k} \Delta t} \xi_{t_j} S_{t_j} \right] &= \mathbb{E}_{\mathbb{P}} \left[S_0 \exp \left\{ - \sum_{k=1}^j \psi(\sigma_{t_k} - \lambda_{t_k}) \Delta t + \sqrt{\Delta t} \sum_{k=1}^j (\sigma_{t_k} - \lambda_{t_k}) A_k \right\} \right] \\ &= \mathbb{E}_{\mathbb{P}} \left[\mathbb{E}_{\mathbb{P}} \left[S_0 \exp \left\{ - \sum_{k=1}^j \psi(\sigma_{t_k} - \lambda_{t_k}) \Delta t + \sqrt{\Delta t} \sum_{k=1}^j (\sigma_{t_k} - \lambda_{t_k}) A_k \right\} \middle| \mathcal{F}_{t_{j-1}} \right] \right] \\ &= \mathbb{E}_{\mathbb{P}} \left[S_0 \exp \left\{ - \sum_{k=1}^{j-1} \psi(\sigma_{t_k} - \lambda_{t_k}) \Delta t + \sqrt{\Delta t} \sum_{k=1}^{j-1} (\sigma_{t_k} - \lambda_{t_k}) A_k \right\} \right. \\ &\quad \left. e^{-\psi(\sigma_{t_j} - \lambda_{t_j}) \Delta t} \mathbb{E}_{\mathbb{P}} \left[e^{\sqrt{\Delta t} (\sigma_{t_j} - \lambda_{t_j}) A_j} \middle| \mathcal{F}_{t_{j-1}} \right] \right] \\ &= \mathbb{E}_{\mathbb{P}} \left[S_0 \exp \left\{ - \sum_{k=1}^{j-1} \psi(\sigma_{t_k} - \lambda_{t_k}) \Delta t + \sqrt{\Delta t} \sum_{k=1}^{j-1} (\sigma_{t_k} - \lambda_{t_k}) A_k \right\} \right]. \end{aligned} \quad (84)$$

Continuing this approach of conditioning on the sets $\mathcal{F}_{t_{j-2}}, \mathcal{F}_{t_{j-3}}, \dots, \mathcal{F}_{t_1}$, we arrive at

$$\mathbb{E}_{\mathbb{P}} \left[e^{-\sum_{k=1}^j r_{t_k} \Delta t} \xi_{t_j} S_{t_j} \right] = S_0. \quad (85)$$

Proof of Theorem 2

We determine $V_{t_j}^h$ using the pricing kernel technique derived in Section 3.1. The arbitrage-free price

$V_{t_j}^h$ at time t_j of the future annuity payout $c_{t_{j+h}}$ is given by

$$V_{t_j}^h = \mathbb{E}_{t_j} \left[\frac{M_{t_{j+h}}}{M_{t_j}} c_{t_{j+h}} \right]. \quad (86)$$

Substituting (26) and (9) into (86), we arrive at

$$\begin{aligned} V_{t_j}^h &= \mathbb{E}_{t_j} \left[\exp \left\{ - \sum_{k=1}^h r_{t_{j+k}} \Delta t - \sum_{k=1}^h \psi(-\lambda) \Delta t - \sqrt{\Delta t} \sum_{k=1}^h \lambda A_{j+k} \right\} \right. \\ &\quad \left. \times c_{t_j} F_{t_j}^h \prod_{k=1}^h \exp \left\{ g_{j+k} \Delta t + q_k \beta \sigma \sqrt{\Delta t} A_{j+h-(k-1)} \right\} \right] \\ &= c_{t_j} F_{t_j}^h \mathbb{E}_{t_j} \left[\exp \left\{ -\Delta t \sum_{k=1}^h (r_{t_{j+k}} + \psi(-\lambda) - g_{j+k}) \right\} \right. \\ &\quad \left. \times \prod_{k=1}^h \exp \left\{ (q_k \beta \sigma - \lambda) \sqrt{\Delta t} A_{j+h-(k-1)} \right\} \right]. \end{aligned} \quad (87)$$

Using the definition of $\psi(q_k \beta \sigma - \lambda)$ (see (25)), we arrive at

$$\mathbb{E}_{t_j} \left[\exp \left\{ (q_k \beta \sigma - \lambda) \sqrt{\Delta t} A_{j+h-(k-1)} \right\} \right] = \exp \{ \Delta t \psi(q_k \beta \sigma - \lambda) \}. \quad (88)$$

Since we assume that the interest rate is a deterministic process, we find the following expression:

$$V_{t_j}^h = c_{t_j} F_{t_j}^h \exp \left\{ -\Delta t \sum_{k=1}^h (r_{t_{j+k}} - g_{j+k} + \psi(-\lambda) - \psi(q_k \beta \sigma - \lambda)) \right\}, \quad (89)$$

which proves the theorem.

Proof of Theorem 3

We start by deriving the dynamics of the arbitrage-free annuity price $V_{t_j}^h$. Using Theorem 2, we find (the second equality follows from (18))

$$\begin{aligned} \frac{V_{t_{j+1}}^{h-1}}{V_{t_j}^h} &= \frac{c_{t_{j+1}} F_{t_{j+1}}^{h-1} A_{t_{j+1}}^{h-1}}{c_{t_j} F_{t_j}^h A_{t_j}^h} \\ &= \frac{c_{t_{j+1}} A_{t_{j+1}}^{h-1}}{c_{t_j} A_{t_j}^h} \frac{e^{(q_h - q_1) \beta \sigma \sqrt{\Delta t} A_{j+1}}}{F_{t_j}^1}. \end{aligned} \quad (90)$$

From (17), we find that

$$c_{t_{j+1}} = c_{t_j} F_{t_j}^1 e^{g_{j+1}\Delta t + q_1\beta\sigma\sqrt{\Delta t}A_{j+1}}. \quad (91)$$

This gives the following expression:

$$\frac{V_{t_{j+1}}^{h-1}}{V_{t_j}^h} = \exp \left\{ g_{j+1}\Delta t + q_h\beta\sigma\sqrt{\Delta t}A_{j+1} \right\} \frac{A_{t_{j+1}}^{h-1}}{A_{t_j}^h}. \quad (92)$$

The annuity factor can be rewritten using (31). We then find the following expression:

$$\frac{A_{t_{j+1}}^{h-1}}{A_{t_j}^h} = \exp \left\{ (r_{t_{j+1}} - g_{j+1} + \psi(-\lambda) - \psi(q_h\beta\sigma - \lambda)) \Delta t \right\}. \quad (93)$$

Substituting (93) in (92), we arrive at

$$\frac{V_{t_{j+1}}^{h-1}}{V_{t_j}^h} = \exp \left\{ (r_{t_{j+1}} + R(\lambda, q_h\beta\sigma) - \psi(q_h\beta\sigma)) \Delta t + q_h\beta\sigma\sqrt{\Delta t}A_{j+1} \right\}. \quad (94)$$

Denote by $W_{t_j}^h$ the value of the investment portfolio at time t_j which replicates the future annuity payout $c_{t_{j+h}}$. We denote the share of $W_{t_j}^h$ invested in the risky stock by $\alpha_{t_j}^h$. This investment portfolio is held for a period Δt and is then re-balanced. We have that $W_{t_j}^h$ satisfies the following dynamic equation:

$$W_{t_{j+1}}^{h-1} = W_{t_j}^h \left(1 - \alpha_{t_j}^h \right) e^{r_{t_{j+1}}\Delta t} + W_{t_j}^h \alpha_{t_j}^h e^{[r_{t_{j+1}} + R(\lambda, \sigma) - \psi(\sigma)]\Delta t + \sigma\sqrt{\Delta t}A_{j+1}}. \quad (95)$$

We can approximate the exponentials as follows:

$$e^{r_{t_{j+1}}\Delta t} \approx 1 + r_{t_{j+1}}\Delta t, \quad (96)$$

$$e^{[r_{t_{j+1}} + R(\lambda, \sigma) - \psi(\sigma)]\Delta t + \sigma\sqrt{\Delta t}A_{j+1}} \approx 1 + [r_{t_{j+1}} + R(\lambda, \sigma) - \psi(\sigma)] \Delta t + \sigma\sqrt{\Delta t}A_{j+1} + \psi(\sigma)\Delta t. \quad (97)$$

The second approximation approximates the geometric rate of return by the arithmetic rate of return.

Substituting (96) and (97) into (95), we arrive at

$$W_{t_{j+1}}^{h-1} \approx W_{t_j}^h (1 - \alpha_{t_j}^h) (1 + r_{t_{j+1}}\Delta t) + W_{t_j}^h \alpha_{t_j}^h \left(1 + [r_{t_{j+1}} + R(\lambda, \sigma)] \Delta t + \sigma\sqrt{\Delta t}A_{j+1} \right). \quad (98)$$

Rearranging terms and using the approximation

$$1 + \left(r_{t_{j+1}} + \alpha_{t_j}^h R(\lambda, \sigma) \right) \Delta t + \alpha_{t_j}^h \sigma \sqrt{\Delta t} A_{j+1} \approx e^{\left[r_{t_{j+1}} + \alpha_{t_j}^h R(\lambda, \sigma) - \psi(\alpha_{t_j}^h \sigma) \right] \Delta t + \alpha_{t_j}^h \sigma \sqrt{\Delta t} A_{j+1}}, \quad (99)$$

we can write

$$W_{t_{j+1}}^{h-1} \approx W_{t_j}^h \exp \left\{ \left[r_{t_{j+1}} + \alpha_{t_j}^h R(\lambda, \sigma) - \psi(\alpha_{t_j}^h \sigma) \right] \Delta t + \alpha_{t_j}^h \sigma \sqrt{\Delta t} A_{j+1} \right\}. \quad (100)$$

We determine the fraction $\alpha_{t_j}^h$ such that the random term in (100) matches the random term in (94):

$$\alpha_{t_j}^h = q_h \beta. \quad (101)$$

We note that $\alpha_{t_j}^h$ is independent of the stock return distribution. Indeed, the characteristic function $\phi(v)$ does not appear in (101). However, the number of stocks the insurer has to hold depends on the initial portfolio value $V_{t_j}^h$. This value does depend on the characteristic function $\phi(v)$. By holding a portfolio with $\alpha_{t_j}^h$ invested in the risky stock and the remaining part invested in the risk-free account, the insurer is able to approximately replicate the future annuity payout $c_{t_{j+h}}$. Indeed, using the approximation

$$\alpha_{t_j}^h R(\lambda, \sigma) \approx R(\lambda, \alpha_{t_j}^h \sigma), \quad (102)$$

we have that $W_{t_j}^h \approx V_{t_j}^h$. At time t_j , the contract consists of a stream of payouts $c_{t_{j+h}}$ ($h = 0, 1, \dots, J - j$). To determine the amount of wealth invested in stocks, we aggregate over $\alpha_{t_j}^h V_{t_j}^h$ ($h = 0, 1, \dots, J - j$):

$$\alpha_{t_j} V_{t_j} = \sum_{h=0}^{J-j} \alpha_{t_j}^h V_{t_j}^h. \quad (103)$$

Substitution of (101) in this expression proves the result.

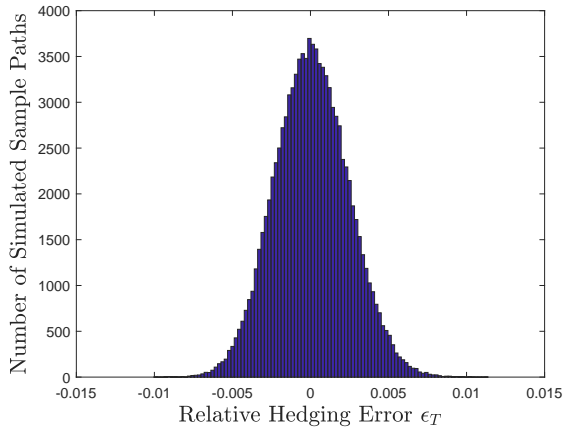
D Hedging Performance

Rebalancing	Distribution	Mean (%)	Volatility	5% Quantile (%)	95% Quantile (%)
Weekly	Gaussian	0.0090	0.0024	-0.37	0.41
	Variance Gamma	-0.0015	0.0035	-0.56	0.59
	Normal Inverse Gaussian	-0.0022	0.0035	-0.55	0.60
Monthly	Gaussian	0.0036	0.0050	-0.75	0.88
	Variance Gamma	-0.0104	0.0073	-1.12	1.27
	Normal Inverse Gaussian	-0.00588	0.0073	-1.10	1.27
Yearly	Gaussian	0.4300	0.0183	-2.17	3.75
	Variance Gamma	-0.0991	0.0259	-3.27	4.72
	Normal Inverse Gaussian	-0.1239	0.0259	-3.21	4.65

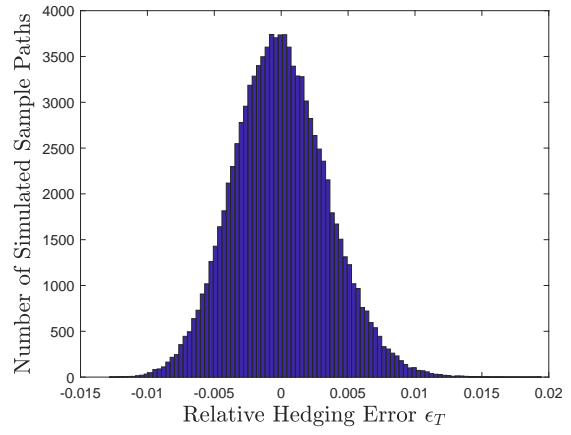
Table 5. Hedging error for different choices of Δt and the underlying stock return distribution.

This table provides summary statistics of the hedging error for different choices of Δt and the underlying stock return distribution. The hedging error measures the performance of the underlying investment portfolio to hedge the single annuity payout c_T (with $T = 20$). We assume the exponential buffering function (10) (with $a_1 = a_2 = 1.6084$ and $\eta = 0.2$) and set the (unconditional) expected growth rate g_j equal to $-\psi(q_j\beta\sigma)$. The risk-free interest rate r_{t_j} is set equal to 1.5% for each j , the parameter β to 50%, and the number of simulations to 100k. For the parameter values of the VG distribution, see Section 1.2. The parameter values of the Normal Inverse Gaussian distribution can be found in Table 4.

Weekly Rebalancing

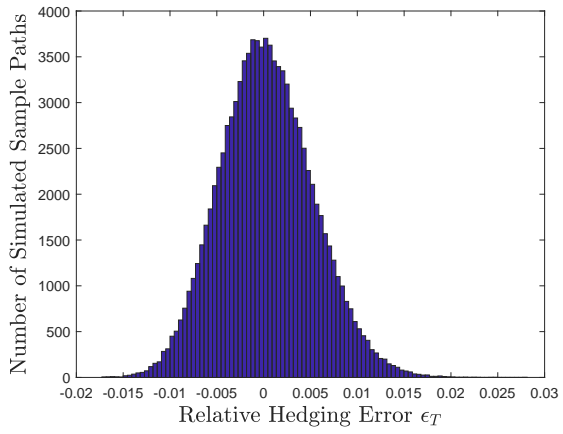


(a) Gaussian Distribution

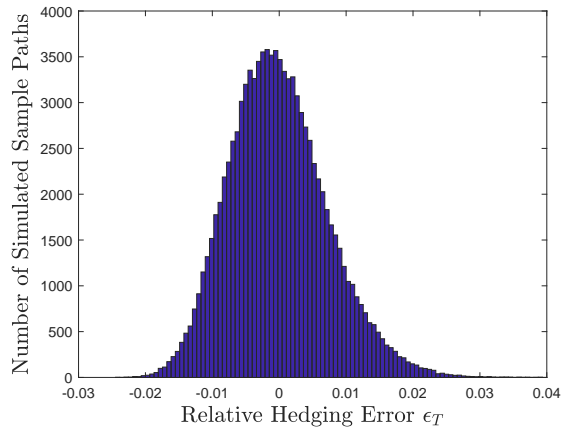


(b) Variance Gamma Distribution

Monthly Rebalancing

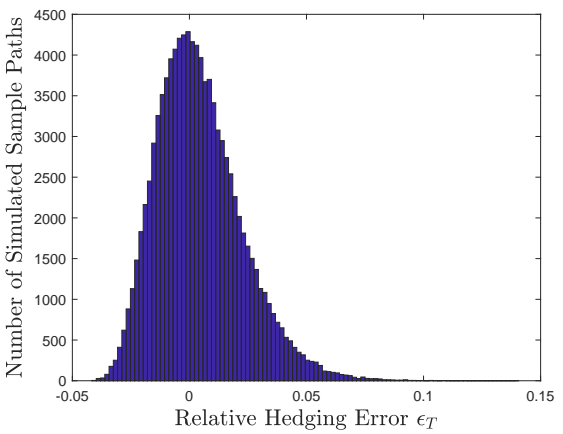


(c) Gaussian Distribution

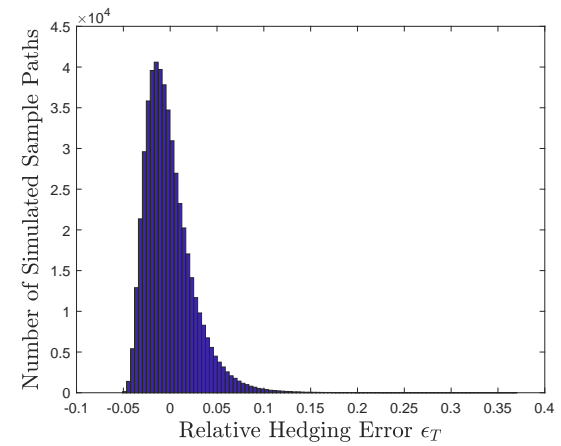


(d) Variance Gamma Distribution

Yearly Rebalancing



(e) Gaussian Distribution



(f) Variance Gamma Distribution

Figure 13. Hedging error: stock return distribution and rebalancing frequency. The figure shows the impact of the stock return distribution and the rebalancing frequency on the hedging performance. Log stock returns are distributed according to a standard Gaussian distribution or a VG distribution; see Section 1.2 for the parameter values of the VG distribution. We assume the exponential buffering function (10), with $a_1 = a_2 = 1.6084$ and $\eta = 0.2$. The risk-free interest rate r_{t_j} is set equal to 1.5% for each j , the parameter β to 50%, and the number of simulations to 100k. The time step Δt equals $1/52$ (upper plots), $1/12$ (middle plots) and 1 (lower plots).

E Constant-Mix Strategy

This appendix computes the hedging error associated with implementing the constant-mix strategy (i.e., a constant share of wealth is invested in the stock market). The following table summarizes the results.

Portfolio Strategy	Mean	Volatility	Quantiles			
			1%	5%	95 %	99%
0%	-0.67	0	-0.67	-0.67	-0.67	-0.67
10%	-0.37	0.26	-0.77	-0.69	0.13	0.48
20%	-0.30	0.24	-0.70	-0.62	0.14	0.43
30%	-0.23	0.21	-0.60	-0.52	0.15	0.38
40%	-0.16	0.18	-0.50	-0.42	0.16	0.35
50%	-0.08	0.16	-0.38	-0.31	0.20	0.34
60%	0.02	0.15	-0.29	-0.21	0.28	0.42
70%	0.12	0.18	-0.25	-0.16	0.44	0.62
80%	0.23	0.26	-0.26	-0.14	0.69	0.94
90%	0.35	0.36	-0.29	-0.15	1.00	1.40
100%	0.48	0.49	-0.34	-0.18	1.39	2.00
Our Strategy	0.00	0.01	-0.01	-0.01	0.01	0.02

Table 6. Hedging error associated with constant-mix strategy. This table provides summary statistics of the hedging error associated with implementing the constant-mix strategy. The hedging error measures the performance of the underlying investment portfolio to hedge the single annuity payout c_T (with $T = 20$). The first column denotes the fixed share of wealth invested in the stock market. The last row represents the summary statistics of the hedging error associated with implementing (33). We assume the exponential buffering function (10) (with $a_1 = a_2 = 1.6084$ and $\eta = 0.2$) and set the (unconditional) expected growth rate g_j equal to $-\psi(q_j\beta\sigma)$. Log stock returns are distributed according to a VG distribution; see Section 1.2 for the parameter values of the VG distribution. The risk-free interest rate r_{t_j} is set equal to 1.5% for each j , the parameter β to 50%, the time step Δt to one month, and the number of simulations to 100k.

F Buffering Function under Habit Formation

This appendix derives the buffering function under the ratio internal habit model (42).³⁶ We assume that A_1, \dots, A_J are independent and identically distributed random variables. In addition, we assume that r_{t_j} is a deterministic function of time and that σ_{t_j} is constant. The pricing kernel is then given by

$$M_{t_j} = \exp \left\{ - \sum_{k=1}^j r_{t_k} \Delta t - \psi(-\lambda)t_j - \lambda \sqrt{\Delta t} \sum_{k=1}^j A_k \right\}. \quad (104)$$

We now consider the following optimization problem:

$$\begin{aligned} \max_{c_{t_j}} \quad & \sum_{j=0}^J \mathbb{E} \left[(1 + \delta)^{-j\Delta t} \frac{1}{1 - \gamma} \left(\frac{c_{t_j}}{h_{t_j}} \right)^{1-\gamma} \right] \\ \text{s.t.} \quad & \sum_{j=0}^J \mathbb{E} [M_{t_j} c_{t_j}] \leq W_0, \\ & \log h_{t_j} - \log h_{t_{j-1}} = \varphi \log c_{t_{j-1}} - \vartheta \log h_{t_{j-1}}, \end{aligned} \quad (105)$$

with $\log h_{t_0} = 0$. Here, W_0 represents the individual's initial wealth.

Denote by \widehat{c}_{t_j} the ratio between c_{t_j} and h_{t_j} , i.e.,

$$\widehat{c}_{t_j} = \frac{c_{t_j}}{h_{t_j}}. \quad (106)$$

Problem (105) is now equivalent to:

$$\begin{aligned} \max_{\widehat{c}_{t_j}} \quad & \sum_{j=0}^J \mathbb{E} \left[(1 + \delta)^{-j\Delta t} \frac{1}{1 - \gamma} (\widehat{c}_{t_j})^{1-\gamma} \right] \\ \text{s.t.} \quad & \sum_{j=0}^J \mathbb{E} [M_{t_j} h_{t_j} \widehat{c}_{t_j}] \leq W_0, \\ & \log h_{t_j} - \log h_{t_{j-1}} = \varphi \log c_{t_{j-1}} - [\vartheta - \varphi] \log h_{t_{j-1}}. \end{aligned} \quad (107)$$

This problem cannot be solved analytically due to the presence of h_{t_j} in the budget constraint. Therefore, we linearize the budget constraint around the trajectory $\{\widehat{c}_{t_j}\} = 1$.

³⁶For the buffering function under the ratio internal habit model in a continuous-time framework, see Van Bilsen et al. (2018).

We need the following partial derivatives:

$$\frac{\partial \sum_{j=0}^J M_{t_j} h_{t_j} \widehat{c}_{t_j}}{\partial \widehat{c}_{t_j}} = M_{t_j} h_{t_j} + \sum_{k=j+1}^J M_{t_k} \frac{\partial h_{t_k} \widehat{c}_{t_k}}{\partial \widehat{c}_{t_j}}, \quad (108)$$

$$\frac{\partial h_{t_k}}{\partial \widehat{c}_{t_j}} = \varphi (1 - (\vartheta - \varphi))^{k-j-1} \frac{h_{t_k}}{\widehat{c}_{t_j}}. \quad (109)$$

Substituting (109) into (108) and evaluating (108) around $\{\widehat{c}_{t_j}\} = 1$, we arrive at

$$\left. \frac{\partial \sum_{j=0}^J M_{t_j} h_{t_j} \widehat{c}_{t_j}}{\partial \widehat{c}_{t_j}} \right|_{\widehat{c}_{t_0}=\dots=\widehat{c}_{t_J}=1} = M_{t_j} + \varphi \sum_{k=j+1}^J M_{t_k} (1 - (\vartheta - \varphi))^{k-j-1}. \quad (110)$$

Define

$$f(\widehat{c}_{t_0}, \dots, \widehat{c}_{t_J}) = \sum_{j=0}^J M_{t_j} h_{t_j} \widehat{c}_{t_j}. \quad (111)$$

By virtue of Taylor series expansion, we have

$$f(\widehat{c}_{t_0}, \dots, \widehat{c}_{t_J}) \approx f(1, \dots, 1) + \sum_{j=0}^J \left. \frac{\partial f(\widehat{c}_{t_0}, \dots, \widehat{c}_{t_J})}{\partial \widehat{c}_{t_j}} \right|_{\widehat{c}_{t_0}=\dots=\widehat{c}_{t_J}=1} (\widehat{c}_{t_j} - 1). \quad (112)$$

Substituting (110) into (112), we find

$$f(\widehat{c}_{t_0}, \dots, \widehat{c}_{t_J}) \approx \sum_{j=0}^J M_{t_j} + \sum_{j=0}^J \left(M_{t_j} + \varphi \sum_{k=j+1}^J M_{t_k} (1 - (\vartheta - \varphi))^{k-j-1} \right) (\widehat{c}_{t_j} - 1). \quad (113)$$

Hence, the budget constraint can be approximated as follows:

$$\begin{aligned} \mathbb{E} \left[\sum_{j=0}^J M_{t_j} h_{t_j} \widehat{c}_{t_j} \right] &= \mathbb{E} [f(\widehat{c}_{t_0}, \dots, \widehat{c}_{t_J})] \\ &\approx \mathbb{E} \left[\sum_{j=0}^J M_{t_j} + \sum_{j=0}^J \left(M_{t_j} + \varphi \sum_{k=j+1}^J M_{t_k} (1 - (\vartheta - \varphi))^{k-j-1} \right) (\widehat{c}_{t_j} - 1) \right] \\ &= \mathbb{E} \left[\sum_{j=0}^J M_{t_j} + \sum_{j=0}^J \mathbb{E}_{t_j} \left\{ \left(M_{t_j} + \varphi \sum_{k=j+1}^J M_{t_k} (1 - (\vartheta - \varphi))^{k-j-1} \right) (\widehat{c}_{t_j} - 1) \right\} \right] \\ &= \mathbb{E} \left[\sum_{j=0}^J M_{t_j} + \sum_{j=0}^J M_{t_j} (1 + \varphi P_{t_j}) (\widehat{c}_{t_j} - 1) \right] \\ &= -\varphi \sum_{j=0}^J \mathbb{E} [M_{t_j} P_{t_j}] + \sum_{j=0}^J \mathbb{E} [M_{t_j} (1 + \varphi P_{t_j}) \widehat{c}_{t_j}]. \end{aligned} \quad (114)$$

Here, we define

$$P_{t_j} = \sum_{k=j+1}^J \mathbb{E}_{t_j} \left[\frac{M_{t_k}}{M_{t_j}} (1 - (\vartheta - \varphi))^{k-j-1} \right]. \quad (115)$$

The approximate optimization problem is now defined as follows:

$$\begin{aligned} \max_{\widehat{c}_{t_j}} \quad & \sum_{j=0}^J \mathbb{E} \left[(1 + \delta)^{-j\Delta t} \frac{1}{1 - \gamma} (\widehat{c}_{t_j})^{1-\gamma} \right] \\ \text{s.t.} \quad & \sum_{j=0}^J \mathbb{E} \left[M_{t_j} (1 + \varphi P_{t_j}) \widehat{c}_{t_j} \right] \leq \widehat{W}_0. \end{aligned} \quad (116)$$

Here, \widehat{W}_0 is defined such that the approximate optimal consumption strategy is budget-feasible.

We can solve (116) analytically. The Lagrangian \mathcal{L} is given by

$$\begin{aligned} \mathcal{L} &= \sum_{j=0}^J \mathbb{E} \left[(1 + \delta)^{-j\Delta t} \frac{1}{1 - \gamma} (\widehat{c}_{t_j})^{1-\gamma} \right] + y \left(\widehat{W}_0 - \sum_{j=0}^J \mathbb{E} \left[M_{t_j} (1 + \varphi P_{t_j}) \widehat{c}_{t_j} \right] \right) \\ &= \sum_{j=0}^J \mathbb{E} \left[(1 + \delta)^{-j\Delta t} \frac{1}{1 - \gamma} (\widehat{c}_{t_j})^{1-\gamma} - y M_{t_j} (1 + \varphi P_{t_j}) \widehat{c}_{t_j} \right] + y \widehat{W}_0. \end{aligned} \quad (117)$$

Here, $y \geq 0$ denotes the Lagrange multiplier associated with the budget constraint. The individual aims to maximize $(1 + \delta)^{-j\Delta t} \frac{1}{1 - \gamma} (\widehat{c}_{t_j})^{1-\gamma} - y M_{t_j} (1 + \varphi P_{t_j}) \widehat{c}_{t_j}$.³⁷ We have that the optimal solution $\widehat{c}_{t_j}^*$ satisfies the following first-order optimality condition:

$$(1 + \delta)^{-j\Delta t} (\widehat{c}_{t_j}^*)^{-\gamma} = y M_{t_j} (1 + \varphi P_{t_j}). \quad (118)$$

After solving the first-order optimality condition, we obtain the following maximum:

$$\widehat{c}_{t_j}^* = \left((1 + \delta)^{j\Delta t} y M_{t_j} (1 + \varphi P_{t_j}) \right)^{-\frac{1}{\gamma}}. \quad (119)$$

The optimal consumption choice is thus given by

$$c_{t_j}^* = h_{t_j}^* \left((1 + \delta)^{j\Delta t} y M_{t_j} (1 + \varphi P_{t_j}) \right)^{-\frac{1}{\gamma}} \quad (120)$$

with $h_{t_j}^*$ the optimal habit level at time t_j implied by the optimal solution.

³⁷Karatzas and Shreve (1998) show that maximizing this expression is equivalent to maximizing (116).

Substituting (104) into (119), we arrive at

$$\widehat{c}_{t_j}^* = \exp \left\{ -\frac{\log \left[(1 + \delta)^{j\Delta t} y \left(1 + \varphi P_{t_j} \right) \right] - \sum_{k=1}^j r_{t_k} \Delta t - \psi(-\lambda)t_j}{\gamma} + \frac{\lambda\sqrt{\Delta t}}{\gamma} \sum_{k=1}^j A_k \right\}. \quad (121)$$

Define $\widehat{y}_j = \log \left[(1 + \delta)^{j\Delta t} y \left(1 + \varphi P_{t_j} \right) \right] - \sum_{k=1}^j r_{t_k} \Delta t - \psi(-\lambda)t_j$. We can now write the habit level $h_{t_j}^*$ as follows:

$$\begin{aligned} h_{t_j}^* &= \exp \left\{ \sum_{k=0}^{j-1} \varphi (1 - (\vartheta - \varphi))^{j-1-k} \log \widehat{c}_k^* \right\} \\ &= \exp \left\{ \sum_{k=0}^{j-1} \varphi (1 - (\vartheta - \varphi))^{j-1-k} \left(-\frac{1}{\gamma} \widehat{y}_k + \frac{\lambda}{\gamma} \sqrt{\Delta t} \sum_{l=1}^k A_l \right) \right\} \\ &= \exp \left\{ -\sum_{k=0}^{j-1} \frac{\varphi}{\gamma} (1 - (\vartheta - \varphi))^{j-1-k} \widehat{y}_k + \frac{\lambda\sqrt{\Delta t}}{\gamma} \sum_{k=1}^{j-1} A_k \sum_{l=k}^{j-1} \varphi (1 - (\vartheta - \varphi))^{j-1-l} \right\}. \end{aligned} \quad (122)$$

Hence,

$$\begin{aligned} c_{t_j}^* &= h_{t_j}^* \exp \left\{ -\frac{1}{\gamma} \widehat{y}_j + \frac{\lambda}{\gamma} \sqrt{\Delta t} \sum_{k=1}^j A_k \right\} \\ &= \exp \left\{ -\sum_{k=0}^{j-1} \frac{\varphi}{\gamma} (1 - (\vartheta - \varphi))^{j-1-k} \widehat{y}_k - \frac{1}{\gamma} \widehat{y}_j + \frac{\lambda}{\gamma} \sqrt{\Delta t} \sum_{k=1}^j A_{j+1-k} q_k \right\} \end{aligned} \quad (123)$$

with

$$q_k = 1 + \sum_{l=0}^{k-2} \varphi (1 - (\vartheta - \varphi))^l. \quad (124)$$

G The Case where Interest Rate Derivatives are Available

This section assumes that the interest rate process $\{r_{t_j} \mid j = 1, 2, \dots, J\}$ is a stochastic process. Consider the random variables B_1, B_2, \dots, B_J . The random variable B_j is assumed to be \mathcal{F}_{t_j} -measurable. Furthermore, we assume for simplicity that the random variables B_1, B_2, \dots, B_J and A_1, A_2, \dots, A_J are all independent and identically distributed. Therefore, the characteristic function of B_j is given by $\phi(v)$.³⁸ We do not pin down the specific dynamics of the interest rate process, but instead write:

$$r_{t_j} = f(B_1, B_2, \dots, B_j). \quad (125)$$

³⁸It is straightforward to generalize our results to the case where the distribution of B_j differs from the one of A_j .

As a result, the realization of the interest rate r_{t_j} is known at time t_j .

Our financial market consists of two risky assets: a stock and a zero-coupon bond with maturity date T_m . The time- t_j price of the zero-coupon bond is denoted by $P(t_j, T_m)$ if $t_j \leq T_m$. We assume that the zero-coupon bond price satisfies the following dynamics:

$$\log \left[\frac{P(t_j, T_m)}{P(t_{j-1}, T_m)} \right] = [r_{t_j} + e_P(t_j, T_m)] \Delta t + \sigma_P(t_j, T_m) \sqrt{\Delta t} B_j \quad (126)$$

with $e_P(t_j, T_m)$ and $\sigma_P(t_j, T_m)$ denoting the expected excess bond return and the diffusion coefficient of the bond return, respectively.

A probability measure \mathbb{Q} belongs to the set of pricing measures \mathcal{Q} if \mathbb{Q} is equivalent to \mathbb{P} and the martingale condition holds for all traded assets:

$$\mathbb{E}_{\mathbb{Q}} [e^{-r_{t_{j+1}} \Delta t} S_{t_{j+1}} | \mathcal{F}_{t_j}] = S_{t_j}, \quad (127)$$

$$\mathbb{E}_{\mathbb{Q}} [e^{-r_{t_{j+1}} \Delta t} P(t_{j+1}, T_m) | \mathcal{F}_{t_j}] = P(t_j, T_m). \quad (128)$$

The following theorem gives the Radon-Nikodym derivative.

Theorem 6. *The probability measure \mathbb{Q} with Radon-Nikodym derivative given by*

$$\zeta_{t_j} = \xi_{t_j} \times \pi_{t_j}, \quad (129)$$

where

$$\xi_{t_j} = e^{-\sum_{k=1}^j \psi(-\lambda_k) \Delta t - \sqrt{\Delta t} \sum_{k=1}^j \lambda_k A_k} \quad (130)$$

and

$$\pi_{t_j} = e^{-\sum_{k=1}^j \psi(-\gamma_k) \Delta t - \sqrt{\Delta t} \sum_{k=1}^j \gamma_k B_k} \quad (131)$$

is an arbitrage-free pricing measure, i.e., $\mathbb{Q} \in \mathcal{Q}$. For each $k \in \{1, \dots, j\}$, the market price of equity risk parameter λ_{t_k} and the market price of interest rate risk parameter γ_{t_k} satisfy the following

equations:

$$e = \psi(-\lambda_{t_k}) - \psi(\sigma_{t_k} - \lambda_{t_k}), \quad (132)$$

$$e_P(t_k, T_m) = \psi(-\gamma_{t_k}) - \psi(\sigma_P(t_k, T_m) - \gamma_{t_k}), \quad (133)$$

with $\psi(-z)$ defined in (25).

Proof. We will prove the following three equalities:

$$\mathbb{E}_{\mathbb{P}} [\xi_{t_{j+1}} \pi_{t_{j+1}} \mid \mathcal{F}_{t_j}] = \xi_{t_j} \pi_{t_j}, \quad (134)$$

$$\mathbb{E}_{\mathbb{P}} \left[e^{-\sum_{k=1}^j r_{t_k} \Delta t} \xi_{t_j} \pi_{t_j} S_{t_j} \right] = S_0, \quad (135)$$

$$\mathbb{E}_{\mathbb{P}} \left[e^{-\sum_{k=1}^j r_{t_k} \Delta t} \xi_{t_j} \pi_{t_j} P(t_j, T_m) \right] = P(0, T_m). \quad (136)$$

We can write

$$\xi_{t_{j+1}} = \xi_{t_j} e^{-\psi(-\lambda_{t_{j+1}}) \Delta t - \sqrt{\Delta t} \lambda_{t_{j+1}} A_{j+1}}, \quad (137)$$

$$\pi_{t_{j+1}} = \pi_{t_j} e^{-\psi(-\gamma_{t_{j+1}}) \Delta t - \sqrt{\Delta t} \gamma_{t_{j+1}} B_{j+1}}. \quad (138)$$

Using the independence between A_{j+1} and B_{j+1} , we arrive at

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} [\xi_{t_{j+1}} \pi_{t_{j+1}} \mid \mathcal{F}_{t_j}] &= \xi_{t_j} \pi_{t_j} \mathbb{E}_{\mathbb{P}} \left[e^{-\psi(-\lambda_{t_{j+1}}) \Delta t - \sqrt{\Delta t} \lambda_{t_{j+1}} A_{j+1}} \right] \\ &\quad \times \mathbb{E}_{\mathbb{P}} \left[e^{-\psi(-\gamma_{t_{j+1}}) \Delta t - \sqrt{\Delta t} \gamma_{t_{j+1}} B_{j+1}} \right]. \end{aligned} \quad (139)$$

Since

$$\mathbb{E}_{\mathbb{P}} \left[e^{-\sqrt{\Delta t} \lambda_{t_{j+1}} A_{j+1}} \right] = e^{\psi(-\lambda_{t_{j+1}}) \Delta t}, \quad (140)$$

$$\mathbb{E}_{\mathbb{P}} \left[e^{-\sqrt{\Delta t} \gamma_{t_{j+1}} B_{j+1}} \right] = e^{\psi(-\gamma_{t_{j+1}}) \Delta t}, \quad (141)$$

we find that

$$\mathbb{E}_{\mathbb{P}} [\xi_{t_{j+1}} \pi_{t_{j+1}} \mid \mathcal{F}_{t_j}] = \xi_{t_j} \pi_{t_j}. \quad (142)$$

To prove (135), we use (132) to write:

$$e^{-\sum_{k=1}^j r_{t_k} \Delta t} \xi_{t_j} \pi_{t_j} S_{t_j} = \pi_{t_j} S_0 e^{-\sum_{k=1}^j \psi(\sigma_{t_k} - \lambda_{t_k}) \Delta t} \times e^{\sum_{k=1}^j (\sigma_{t_k} - \lambda_{t_k}) \sqrt{\Delta t} A_k}. \quad (143)$$

Using the independence between the random variable π_{t_j} and $e^{\sum_{k=1}^j (\sigma_{t_k} - \lambda_{t_k}) \sqrt{\Delta t} A_k}$, we can write

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[e^{-\sum_{k=1}^j r_{t_k} \Delta t} \xi_{t_j} \pi_{t_j} S_{t_j} \right] &= S_0 \mathbb{E}_{\mathbb{P}} \left[\pi_{t_j} \right] \\ &\times \mathbb{E}_{\mathbb{P}} \left[e^{-\sum_{k=1}^j \psi(\sigma_{t_k} - \lambda_{t_k}) \Delta t} e^{\sum_{k=1}^j (\sigma_{t_k} - \lambda_{t_k}) \sqrt{\Delta t} A_k} \right] = S_0. \end{aligned} \quad (144)$$

Finally, we prove (136). We use (133) to write:

$$\begin{aligned} e^{-\sum_{k=1}^j r_{t_k} \Delta t} \xi_{t_j} \pi_{t_j} P(t_j, T_m) &= \xi_{t_j} P(0, T_m) e^{-\sum_{k=1}^j \psi(\sigma_P(t_k, T_m) - \gamma_{t_k}) \Delta t} \\ &\times e^{\sum_{k=1}^j (\sigma_P(t_k, T_m) - \gamma_{t_k}) \sqrt{\Delta t} B_k}. \end{aligned} \quad (145)$$

Using the independence between the random variables ξ_{t_j} and $e^{\sum_{k=1}^j (\sigma_P(t_k, T_m) - \gamma_{t_k}) \sqrt{\Delta t} B_k}$, we find:

$$\mathbb{E}_{\mathbb{P}} \left[e^{-\sum_{k=1}^j r_{t_k} \Delta t} \xi_{t_j} \pi_{t_j} P(t_j, T_m) \right] = P(0, T_m), \quad (146)$$

which completes the proof. ■

It follows from Theorem 6 that in a financial market with stochastic interest rates, the pricing kernel M_{t_j} is given by

$$M_{t_j} = e^{-\sum_{k=1}^j r_{t_k} \Delta t} \times \xi_{t_j} \times \pi_{t_j}. \quad (147)$$

We can determine the price of a \mathcal{F}_T -measurable claim X_T as follows:

$$\text{Time-}t_j \text{ price} = \mathbb{E}_{\mathbb{Q}} \left[e^{-\sum_{k=j+1}^J r_{t_k} \Delta t} X_T \middle| \mathcal{F}_{t_j} \right] = \mathbb{E}_{\mathbb{P}} \left[\frac{M_T}{M_{t_j}} X_T \middle| \mathcal{F}_{t_j} \right]. \quad (148)$$

The following theorem derives the price of a unit-linked annuity with buffering of shocks. We assume that β_j and σ_{t_j} are constant (i.e., $\beta_j = \beta$ and $\sigma_{t_j} = \sigma$ for every j) and that g_j is a deterministic function of time. As a result, λ_{t_j} is constant (i.e., $\lambda_{t_j} = \lambda$ for every j).

Theorem 7. Denote by $F_{t_j}^h$ the buffering factor (see (15)). An arbitrage-free price $V_{t_j}^h$ at time t_j of

the future annuity payout c_{t_j+h} is now given by

$$V_{t_j}^h = c_{t_j} \times F_{t_j}^h \times A_{t_j}^h, \quad (149)$$

where

$$A_{t_j}^h = P(t_j, t_{j+h}) \exp \left\{ -\Delta t \sum_{k=1}^h d_{t_j}(k) \right\}, \quad (150)$$

and

$$d_{t_j}(k) = -g_{j+k} + \psi(-\lambda) - \psi(q_k \beta \sigma - \lambda). \quad (151)$$

Proof. Appendix C gives the proof of Theorem 2. We can easily modify this proof for the case with stochastic interest rates. Using (88), we can write the price $V_{t_j}^h$ as follows

$$\begin{aligned} V_{t_j}^h &= c_{t_j} F_{t_j}^h \mathbb{E}_{t_j} \left[\frac{\pi_{t_{j+h}}}{\pi_{t_j}} \exp \left\{ -\Delta t \sum_{k=1}^h (r_{t_{j+k}} + \psi(-\lambda) - g_{j+k}) \right\} \right. \\ &\quad \left. \times \prod_{k=1}^h \exp \left\{ (q_k \beta \sigma - \lambda) \sqrt{\Delta t} A_{j+h-(k-1)} \right\} \right] \\ &= c_{t_j} F_{t_j}^h \mathbb{E}_{t_j} \left[\exp \left\{ -\Delta t \sum_{k=1}^h r_{t_{j+k}} \right\} \frac{\pi_{t_{j+h}}}{\pi_{t_j}} \right] \\ &\quad \times \exp \left\{ -\Delta t \sum_{k=1}^h (\psi(-\lambda) - g_{j+k} - \psi(q_k \beta \sigma - \lambda)) \right\}. \end{aligned} \quad (152)$$

We find the following expression for the bond price $P(t_j, t_{j+h})$:

$$\begin{aligned} P(t_j, t_{j+h}) &= \mathbb{E}_{t_j} \left[\frac{M_{t_{j+h}}}{M_{t_j}} \cdot 1 \right] \\ &= \mathbb{E}_{t_j} \left[\exp \left\{ -\Delta t \sum_{k=1}^h r_{t_{j+k}} \right\} \frac{\pi_{t_{j+h}}}{\pi_{t_j}} \right] \mathbb{E}_{t_j} \left[\frac{\xi_{t_{j+h}}}{\xi_{t_j}} \right] \\ &= \mathbb{E}_{t_j} \left[\exp \left\{ -\Delta t \sum_{k=1}^h r_{t_{j+k}} \right\} \frac{\pi_{t_{j+h}}}{\pi_{t_j}} \right]. \end{aligned} \quad (153)$$

Note that $\mathbb{E}_{t_j} \left[\frac{\xi_{t_{j+h}}}{\xi_{t_j}} \right] = 1$. We can rewrite (152) as follows:

$$V_{t_j}^h = c_{t_j} F_{t_j}^h P(t_j, t_{j+h}) \exp \left\{ -\Delta t \sum_{k=1}^h (\psi(-\lambda) - g_{j+k} - \psi(q_k \beta \sigma - \lambda)) \right\}, \quad (154)$$

which completes the proof. ■

The following theorem presents the hedging strategy.

Theorem 8. Consider an investment portfolio with value $W_{t_j}^h$ at time t_j . Suppose that $\alpha_{t_j}^h W_{t_j}^h$ is invested in the risky stock, $W_{t_j}^h$ in a zero-coupon bond with maturity t_{j+h} , and $-\alpha_{t_j}^h W_{t_j}^h$ in the risk-free asset. Let the share of wealth invested in the risky stock, i.e., $\alpha_{t_j}^h$, be given by

$$\alpha_{t_j}^h = \beta q_h. \quad (155)$$

Then this trading strategy approximately replicates the annuity payout $c_{t_{j+h}}$ of a unit-linked annuity with buffering of portfolio shocks, i.e.,

$$W_{t_k}^{j+h-k} \approx V_{t_k}^{j+h-k} \text{ for every } k = 0, 1, \dots, j+h. \quad (156)$$

Proof. The dynamics of the arbitrage-free price $V_{t_j}^h$ are given by:

$$\frac{V_{t_{j+1}}^{h-1}}{V_{t_j}^h} = \exp \left\{ g_{j+1} \Delta t + q_h \beta \sigma \sqrt{\Delta t} A_{j+1} \right\} \frac{A_{t_{j+1}}^{h-1}}{A_{t_j}^h}. \quad (157)$$

Using (150), we find:

$$\begin{aligned} \frac{V_{t_{j+1}}^{h-1}}{V_{t_j}^h} &= \frac{P(t_{j+1}, t_{j+h})}{P(t_j, t_{j+h})} \exp \left\{ (R(\lambda, q_h \beta \sigma) - \psi(q_h \beta \sigma)) \Delta t + q_h \beta \sigma \sqrt{\Delta t} A_{j+1} \right\} \\ &= \exp \left\{ (r_{t_{j+1}} + e_P(t_{j+1}, t_{j+h}) + R(\lambda, q_h \beta \sigma) - \psi(q_h \beta \sigma)) \Delta t \right\} \\ &\quad \times \exp \left\{ \sigma_P(t_{j+1}, t_{j+h}) \sqrt{\Delta t} B_{j+1} + q_h \beta \sigma \sqrt{\Delta t} A_{j+1} \right\} \end{aligned} \quad (158)$$

with $R(\lambda, q_h \beta \sigma) = \psi(q_h \beta \sigma) + \psi(-\lambda) - \psi(q_h \beta \sigma - \lambda)$.

We have that $W_{t_j}^h$ satisfies the following dynamic equation:

$$\frac{W_{t_{j+1}}^{h-1}}{W_{t_j}^h} = -\alpha_{t_j}^h e^{r_{t_{j+1}} \Delta t} + \frac{P(t_{j+1}, t_{j+h})}{P(t_j, t_{j+h})} + \alpha_{t_j}^h e^{[r_{t_{j+1}} + R(\lambda, \sigma) - \psi(\sigma)] \Delta t + \sigma \sqrt{\Delta t} A_{j+1}}. \quad (159)$$

We apply the following approximations:

$$e^{r_{t_{j+1}}\Delta t} \approx 1 + r_{t_{j+1}}\Delta t, \quad (160)$$

$$\begin{aligned} \frac{P(t_{j+1}, t_{j+h})}{P(t_j, t_{j+h})} &\approx 1 + r_{t_{j+1}}\Delta t + e_P(t_{j+1}, t_{j+h})\Delta t \\ &+ \sigma_P(t_{j+1}, t_{j+h})\sqrt{\Delta t}B_{j+1} + \psi(\sigma_P(t_{j+1}, t_{j+h}))\Delta t, \end{aligned} \quad (161)$$

$$\begin{aligned} e^{[r_{t_{j+1}} + R(\lambda, \sigma) - \psi(\sigma)]\Delta t + \sigma\sqrt{\Delta t}A_{j+1}} &\approx 1 + [r_{t_{j+1}} + R(\lambda, \sigma) - \psi(\sigma)]\Delta t \\ &+ \sigma\sqrt{\Delta t}A_{j+1} + \psi(\sigma)\Delta t. \end{aligned} \quad (162)$$

Substituting (160), (161) and (162) into (159), we arrive at

$$\begin{aligned} \frac{W_{t_{j+1}}^{h-1}}{W_{t_j}^h} &\approx 1 + r_{t_{j+1}}\Delta t + e_P(t_{j+1}, t_{j+h})\Delta t + \sigma_P(t_{j+1}, t_{j+h})\sqrt{\Delta t}B_{j+1} \\ &+ \psi(\sigma_P(t_{j+1}, t_{j+h}))\Delta t + \alpha_{t_j}^h \left(R(\lambda, \sigma)\Delta t + \sigma\sqrt{\Delta t}A_{j+1} \right). \end{aligned} \quad (163)$$

The left-hand side of (163) can be approximated by

$$e^{[r_{t_{j+1}} + e_P(t_{j+1}, t_{j+h}) + \alpha_{t_j}^h R(\lambda, \sigma) - \psi(\alpha_{t_j}^h \sigma)]\Delta t + \sigma_P(t_{j+1}, t_{j+h})\sqrt{\Delta t}B_{j+1} + \alpha_{t_j}^h \sigma\sqrt{\Delta t}A_{j+1}}. \quad (164)$$

Hence,

$$\begin{aligned} \frac{W_{t_{j+1}}^{h-1}}{W_{t_j}^h} &\approx \exp \left\{ [r_{t_{j+1}} + e_P(t_{j+1}, t_{j+h}) + \alpha_{t_j}^h R(\lambda, \sigma) - \psi(\alpha_{t_j}^h \sigma)]\Delta t \right\} \\ &\times \exp \left\{ \sigma_P(t_{j+1}, t_{j+h})\sqrt{\Delta t}B_{j+1} + \alpha_{t_j}^h \sigma\sqrt{\Delta t}A_{j+1} \right\}. \end{aligned} \quad (165)$$

If we take $\alpha_{t_j}^h = \beta q_h$ and use the approximation (102), we find that the investment portfolio at time t_{j+1} , i.e., $W_{t_{j+1}}^{h-1}$, is approximately equal to $V_{t_{j+1}}^{h-1}$. ■

References

- Abel, A. 1999. Risk Premia and Term Premia in General Equilibrium. *Journal of Monetary Economics* 43:3–33.
- Abel, A. B. 1990. Asset Prices under Habit Formation and Catching up with the Joneses. *The American Economic Review* 80:38–42.
- Albrecher, H., S. A. Ladoucette, and W. Schoutens. 2007. A Generic One-Factor Lévy Model for Pricing Synthetic CDOs. In M. C. F. et al. (ed.), *Advances in Mathematical Finance*, pp. 259–277. Birkhäuser Boston.
- Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath. 1999. Coherent Measures of Risk. *Mathematical Finance* 9:203–228.
- Baillon, A., H. Bleichrodt, and V. Spinu. 2017. Searching for the Reference Point. Working Paper.
- Bakshi, G., C. Cao, and Z. Chen. 1997. Empirical Performance of Alternative Option Pricing Models. *Journal of Finance* 52:2003–2049.
- Brennan, M. J., and E. S. Schwartz. 1976. The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee. *Journal of Financial Economics* 3:195–213.
- Brody, D. C., L. P. Hughston, and E. Mackie. 2012. General Theory of Geometric Lévy Models for Dynamic Asset Pricing. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 468:1778–1798.
- Campbell, J. Y., and A. Deaton. 1989. Why is Consumption so Smooth? *Review of Economic Studies* 56:357–373.
- Carroll, C. D. 2000. Solving Consumption Models with Multiplicative Habits. *Economics Letters* 68:67–77.
- Carroll, C. D., J. Overland, and D. N. Weil. 1997. Comparison Utility in a Growth Model. *Journal of Economic Growth* 2:339–367.
- Carroll, C. D., J. Overland, and D. N. Weil. 2000. Saving and Growth with Habit Formation. *The American Economic Review* 90:341–355.
- Chai, J., W. J. Horneff, R. H. Maurer, and O. S. Mitchell. 2011. Optimal Portfolio Choice over the Life Cycle with Flexible Work, Endogenous Retirement, and Lifetime Payouts. *Review of Finance* 15:875–907.
- Chan, Y. L., and L. Kogan. 2002. Catching Up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices. *Journal of Political Economy* 110:1255–1285.
- Cont, R. 2001. Empirical Properties of Asset Returns: Stylized Facts and Statistical Issues. *Quantitative Finance* 1:223–236.
- Corcuera, J. M., F. Guillaume, P. Leoni, and W. Schoutens. 2009. Implied Lévy Volatility. *Quantitative Finance* 9:383–393.
- Deaton, A. 1987. Life-Cycle Models of Consumption: Is the Evidence Consistent with the Theory? In T. F. Bewley (ed.), *Advances in Econometrics: Fifth World Congress*, vol. 2, pp. 121–148. Cambridge University Press.
- Dhaene, J., B. Stassen, K. Barigou, D. Linders, and Z. Chen. 2017. Fair Valuation of Insurance Liabilities: Merging Actuarial Judgement and Market-Consistency. *Insurance: Mathematics and Economics* 76:14–27.
- Dhaene, J. L. M., M. J. Goovaerts, and R. Kaas. 2003. Economic Capital Allocation Derived from Risk Measures. *North American Actuarial Journal* 7:44–59.
- Feng, R., and Y. Shimizu. 2016. Applications of Central Limit Theorems for Equity-linked Insurance. *Insurance: Mathematics and Economics* 69:138–148.
- Flavin, M. A. 1981. The Adjustment of Consumption to Changing Expectations About Future Income. *Journal of Political Economy* 89:974–1009.
- Fonseca, J. D., and J. Ziveyi. 2017. Valuing Variable Annuity Guarantees on Multiple Assets. *Scandinavian Actuarial Journal* 2017:209–230.
- Fuhrer, J. C. 2000. Habit Formation in Consumption and Its Implications for Monetary-Policy Models. *American Economic Review* 90:367–390.

- Gomes, F., and A. Michaelides. 2003. Portfolio Choice with Internal Habit Formation: A Life-Cycle Model with Uninsurable Labor Income Risk. *Review of Economic Dynamics* 6:729–766.
- Gómez, J.-P., R. Priestley, and F. Zapatero. 2009. Implications of Keeping-Up-with-the-Joneses Behavior for the Equilibrium Cross Section of Stock Returns: International Evidence. *Journal of Finance* 64:2703–2737.
- Goovaerts, M. J., J. Dhaene, and O. Rachedi. 2010. Actuarial Premium Principles. In *Encyclopedia of Quantitative Finance*.
- Gorski, L., and R. Brown. 2005. Recommended approach for setting regulatory risk-based capital requirements for variable annuities and similar products. Tech. rep. Technical report, American Academy of Actuaries Life Capital Adequacy Subcommittee, Boston.
- Guillén, M., P. L. Jørgensen, and J. P. Nielsen. 2006. Return Smoothing Mechanisms in Life and Pension Insurance: Path-Dependent Contingent Claims. *Insurance: Mathematics and Economics* 38:229–252.
- Guillén, M., J. P. Nielsen, A. M. Pérez-Marín, and K. S. Petersen. 2013. Performance Measurement of Pension Strategies: a Case Study of Danish Life-Cycle Products. *Scandinavian Actuarial Journal* 2013:49–68.
- Huang, J.-Z., and L. Wu. 2004. Specification Analysis of Option Pricing Models based on Time-Changed Lévy Processes. *Journal of Finance* 59:1405–1439.
- Investment Company Institute. 2017. Investment Company Fact Book. A Review of Trends and Activities in the U.S. Investment Company Industry.
- Jørgensen, P. L., and P. Linnemann. 2011. A Comparison of Three Different Pension Savings Products with Special Emphasis on the Payout Phase. *Annals of Actuarial Science* 6:137–152.
- Karatzas, I., and S. E. Shreve. 1998. *Methods of Mathematical Finance*, vol. 39. Springer.
- Koijen, R. S. J., and M. Yogo. 2018. The Fragility of Market Risk Insurance. NBER Working Paper.
- Le Courtois, O., and C. Walter. 2014. The Computation of Risk Budgets under the Lévy Process Assumption. *Finance* 35:87–108.
- Linders, D., and W. Schoutens. 2016. Basket Option Pricing and Implied Correlation in a One-Factor Lévy Model. In G. et al. (ed.), *Innovations in Derivatives Markets*, pp. 335–367. Springer International Publishing.
- Linnemann, P., K. Bruhn, and M. Steffensen. 2014. A Comparison of Modern Investment-Linked Pension Savings Products. *Annals of Actuarial Science* 9:72–84.
- Madan, D. B., P. Carr, and E. C. Chang. 1998. The Variance Gamma Process and Option Pricing. *European Finance Review* 2:79–105.
- Maurer, R., O. S. Mitchell, R. Rogalla, and I. Siegelin. 2016. Accounting and Actuarial Smoothing of Retirement Payouts in Participating Life Annuities. *Insurance: Mathematics and Economics* 71:268–283.
- Maurer, R., R. Rogalla, and I. Siegelin. 2013a. Participating Payout Annuities: Lessons from Germany. *ASTIN Bulletin* 43:159–187.
- Maurer, R. H., O. S. Mitchell, R. Rogalla, and V. Kartashov. 2013b. Life Cycle Portfolio Choice with Systematic Longevity Risk and Variable Investment-Linked Deferred Annuities. *Journal of Risk and Insurance* 80:649–676.
- Merton, R. C. 1969. Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. *Review of Economics and Statistics* 51:247–257.
- Merton, R. C. 1976. Option Pricing when Underlying Stock Returns are Discontinuous. *Journal of Financial Economics* 3:125 – 144.
- Milevsky, M. A., and S. E. Posner. 2001. The Titanic Option: Valuation of the Guaranteed Minimum Death Benefit in Variable Annuities and Mutual Funds. *Journal of Risk and Insurance* 68:93–128.
- Milevsky, M. A., and T. S. Salisbury. 2006. Financial Valuation of Guaranteed Minimum Withdrawal Benefits. *Insurance: Mathematics and Economics* 38:21–38.

- Moenig, T., and D. Bauer. 2015. Revisiting the Risk-Neutral Approach to Optimal Policyholder Behavior: A Study of Withdrawal Guarantees in Variable Annuities. *Review of Finance* 20:759–794.
- OECD. 2016. Life Annuity Products and Their Guarantees.
- Pagel, M. 2017. Expectations-Based Reference-Dependent Life-Cycle Consumption. *Review of Economic Studies* 84:885–934.
- Pelsser, A., and A. G. Salahnejhad. 2016. Time-Consistent Actuarial Valuations. *Insurance: Mathematics & Economics* 66:97–112.
- Schoutens, W. 2003. *Lévy processes in Finance: Pricing Financial Derivatives*. Wiley.
- Seneta, E. 2004. Fitting the Variance-Gamma Model to Financial Data. *Journal of Applied Probability* 41:177–187.
- Smith, W. T., and Q. Zhang. 2007. Asset pricing with multiplicative habit and power-expo preferences. *Economics Letters* 94:319–325.
- Stadje, M., and A. Pelsser. 2014. Time-Consistent and Market-Consistent evaluations. *Mathematical Finance* 24:25–65.
- Tsanakas, A., M. Wuthrich, and A. Cerny. 2013. Market Value Margin via Mean-Variance Hedging. *ASTIN Bulletin* 43:301–322.
- Van Bilsen, S., A. L. Bovenberg, and R. J. A. Laeven. 2018. How to Invest and Spend Wealth in Retirement? A Utility-Based Analysis. Working Paper.
- Van Bilsen, S., R. J. A. Laeven, and Th. E. Nijman. 2017. Consumption and Portfolio Choice under Loss Aversion and Endogenous Updating of the Reference Level. Working Paper.