New characterizations of partition functions using connection matrices

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Publication date
2018

Document Version
Final published version

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BART SEVENSTER

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New Characterizations of Partition Functions Using Connection Matrices

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Universiteit van Amsterdam
op gezag van de Rector Magnificus
prof. dr. ir. K.I.J. Maex
ten overstaan van een door het College voor Promoties ingestelde commissie,
in het openbaar te verdedigen in de Agnietenkapel
op maandag 22 oktober 2018, te 14:00 uur

door

Bartholomeus Livius Sevenster

geboren te Bloemendaal

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The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no. 339109.

Voor Val,
voor mijn vader.

Dankwoord

Allereerst wil ik mijn promotor Lex bedanken. In het onderzoek dat ik met hem gedaan heb, heb ik altijd bewondering gehad voor zijn zoektocht naar de meest elegante en inzichtelijke oplossing. Ook zijn commentaar op artikelen die ik met anderen geschreven heb, gaf mij meer inzicht in het onderzoek dat ik zelf gedaan had. Ik ben Lex dan ook erg dankbaar voor het van commentaar voorzien van mijn artikelen en het aandachtig lezen van de talloze eerdere versies van dit proefschrift. Verder wil ik Lex graag bedanken voor de vrijheid die hij mij zowel op academisch als op persoonlijk vlak heeft gegeven.

Zonder mijn copromotor Guus had dit proefschrift niet dezelfde vorm gehad. Guus zijn gedrevenheid en zijn doorzettingsvermogen hebben er vaak toe geleid dat we oplossingen voor problemen hebben gevonden die hopeloos leken. Ik ben Guus dan ook erg dankbaar voor de prettige samenwerking. Mijn twee medepromovendi Bart en Sven ben ik dankbaar voor de wiskundige samenwerking, maar vooral ook voor alles buiten de wiskunde om. De vele rondjes tafeltennis, kopjes koffie en uiteraard de conferenties die we samen bezocht hebben zullen me altijd bijblijven. Verder wil ik ook graag mijn andere directe collega's Ewan, Fabian, Jacob, Lluís, Vaidy en Viresh bedanken voor de prettige samenwerking. Zonder jullie had mijn tijd op de UvA een stuk minder kleur gehad.

Ik wil mijn huisgenoten van de afgelopen vier jaar op de Achillesstraat bedanken voor alle niet-wiskundige afleiding. Natuurlijk wil ik dan ook iedereen die te gast was op de Achillesstraat bedanken voor de BBQ's, Sinterklaasvieringen en alle andere activiteiten die mijn tijd daar tot iets onvergetelijks hebben gemaakt. Om er zeker van te zijn dat ik niemand vergeet, wil ik graag al mijn vrienden bedanken voor alle mooie momenten die we samen hebben gehad en vooral ook voor hun steun in mindere tijden.

Uiteraard wil ik ook mijn familie bedanken voor hun toewijding en steun de afgelopen vier jaar. Allereerst mijn broer Merlijn en mijn zus Dieuwke. De afgelopen vier jaar heb ik veel aan jullie gehad bij academische tegenspoed, maar ik ben jullie vooral dankbaar voor alle mooie herinneringen die ik aan de afgelopen vier jaar met jullie heb. De reizen naar New York en Japan, en natuurlijk alle familiegelegenheden met jullie partners en kinderen. Zonder mijn vader was ik waarschijnlijk nooit aan dit promotietraject begonnen. Niet alleen daar ben ik hem dankbaar voor, maar vooral ook voor zijn niet aflatende interesse in mijn onderzoek, zijn advies en zijn vertrouwen in de goede afloop. Mijn moeder ben ik dankbaar voor alles wat ze in de afgelopen vier jaar voor mij en voor ons gezin betekend heeft.

En, *last but not least*, wil ik Valerie bedanken. In de afgelopen tweeënhalfjaar heb je me niet alleen altijd gesteund - eerst vanuit San Francisco en tegen het einde vanuit Amsterdam - bij de totstandkoming van dit proefschrift, maar ook in alle dingen die ik daarbuiten heb gedaan. Het is lastig om mijn dankbaarheid in woorden uit te drukken, maar ik hoop dat, nu je hier in Nederland bent, ik hetzelfde voor jou kan betekenen.

Bart Sevenster
Amsterdam, September 2018

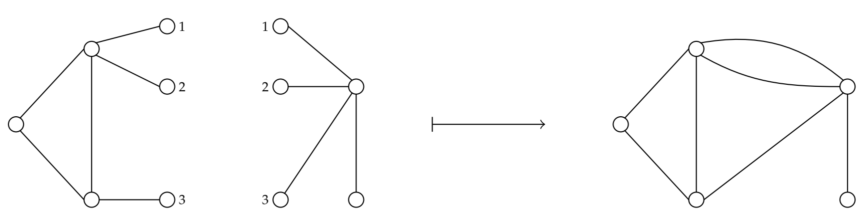
Contents

Chapter 1

Introduction

In mathematics, invariants are functions that help us distinguish between objects. In many areas of mathematics, invariants play an important role: the Vassiliev invariants in knot theory, the Euler characteristic in topology, the chromatic polynomial in graph theory, etc. In this thesis we are interested in invariants of graphs, not so much to help us distinguish between graphs, but to classify classes of invariants of graphs. Our main tool will be the edge connection matrices, which we now define.

For $t \in \mathbb{N}$, a *t-fragment* is a graph with t labeled vertices of degree one labeled $1, 2, \ldots, t$. For two t-fragments F_1, F_2, we define $F_1 * F_2$ to be the graph obtained as follows: we take the disjoint union of F_1 and F_2, and for each pair of equally labeled vertices, we identify the two vertices, remove the new vertex and join its two incident edges into one edge, see the figure below. Note that if F is the 2-fragment on two vertices, labeled 1 and 2, with one edge between those two vertices, then $F * F$ is the *vertexless loop*, which we denote by $\bigcirc$.



The *t-th edge connection matrix* of a graph parameter f is the symmetric matrix $M_{f,t}$ indexed by t-fragments such that the entry at the (F_1, F_2) position is $f(F_1 * F_2)$.

We will be interested in questions of the following type: what is the class of graph parameters f such that for each $t \in \mathbb{N}$ the connection matrix $M_{f,t}$ has a certain property? The two properties we will focus on are being positive semidefinite and having rank that is bounded by some exponential function in t. We will not only restrict ourselves to graphs, but we will also give answers to the corresponding questions for virtual links and 3-graphs, two types of graphs with extra structure at each vertex.

1.1 Background

Partition functions were introduced to the graph theory community by de la Harpe and Jones [15]. Partition functions form a rich class of graph parameters and they appear in different guises throughout mathematics: in quantum information theory they are known as tensor network contractions [24], in statistical physics among others as the partition function of the Ising model [16] and in theoretical computer science they are a subclass of the Holant problems [7].

Weight systems for 3-graphs. Penrose introduced the theory of abstract tensor systems and gave an example showing their relevance in combinatorics by relating them to edge colorings of plane graphs and the four color conjecture [25]. Murphy applied this framework to the structure tensors of metric Lie algebras and showed how these give invariants for cubic graphs embedded in an oriented surface modulo the AS and IHX relations [23]. Through the work of Bar-Natan and Kontsevich the relevance of these invariants in knot theory became apparent [2, 19]. Bar-Natan expanded the work of Penrose and gave a statement about Lie algebras that is equivalent to the four color theorem [1].

Invariants of virtual links. Reidemeister showed that a knot can be described by a knot diagram modulo the three Reidemeister moves [27]. Turaev, expanding on work of Jones [17], gave three conditions on partition functions that correspond to the three Reidemeister moves and showed that a partition function that satisfies these conditions gives a knot invariant [40]. The most famous of these three conditions is the Yang-Baxter equation that has its origin in statistical physics [3, 41]. Kauffman introduced virtual link invariants as the invariants of virtual link diagrams modulo the three Reidemeister moves [18].

Reflection positivity and the orthogonal group. Motivated by a question of Freedman stemming from the area of quantum computing, Freedman, Lovász and Schrijver gave a characterization of partition functions of real vertex coloring models in terms of vertex reflection positivity [12]. They asked if a similar characterization could be given for edge reflection positive graph parameters: those real valued graph parameters f such that $M_{f,t}$ is positive semidefinite for each $t \in \mathbb{N}$. Szegedy solved their question with a novel approach using the invariant theory of the orthogonal group and the Positivstellensatz [39]. Schrijver later gave a strengthening of Szegedy's theorem using a different type of connection matrix [35]. The connection between the orthogonal group and partition functions was further deepened by Draisma, Gijswijt, Lovász, Regts and Schrijver [10]. Regts gave a more detailed exposition of this connection in his PhD thesis [28].

The rank of connection matrices. In the characterization given by Freedman, Lovász and Schrijver [12], the rank of the vertex connection matrices of the graph parameters plays an important role. If f is the partition function of an edge coloring model, then it is not hard to see that $\mathrm{rk}\,(M_{f,t}) \leq f(\bigcirc)^t$ for each $t \in \mathbb{N}$. Schrijver [38] gave a characterization of partition function of edge coloring models in terms of the rank of the edge connection matrices and the value of $\bigcirc$. To prove this, Schrijver extended an algebraic framework that was developed in [12].

1.2 New contributions

We use Schrijver's approach [35] to extend Szegedy's theorem on edge reflection positive graph parameters [39] to invariants for 3-graphs and virtual link invariants. A large part of the proofs of our theorems consists of characterizing which values f can take on $\bigcirc$ if f is edge reflection positive. This requires some representation theory of the symmetric group and a theorem by Hanlon and Wales [14]. Furthermore, the proofs of the theorems use the invariant theory of the orthogonal group and a theorem by Procesi and Schwarz [26]. Outside of the characterizations of which value $\bigcirc$ can take, the proofs of the two theorems follow the same line and are, *mutatis mutandis*, interchangeable. We will also see that the partition functions we find are unique modulo the action of the orthogonal group. This is based on (1.1) and (1.2) mentioned on the following page.

We introduce a new type of graph parameter: skew partition functions. We give a characterization of skew partition functions similar to that of Schrijver [38] for partition functions of edge coloring models. We furthermore give a characterization of skew partition functions that is similar to the characterization of Draisma, Gijswijt, Lovász, Regts, Schrijver [10] for partition functions of edge coloring models. The proof of our characterization makes use of the invariant theory of the symplectic group. This is based on (1.3) mentioned on the following page.

We introduce mixed partition functions, a common generalization of skew partition functions and partition functions of edge coloring models, and we show that for a mixed partition function f there is a constant $r \in \mathbb{R}$ such that $\mathrm{rk}\,(M_{f,t}) \leq r^t$ for each $t \in \mathbb{N}$. We furthermore show that mixed partition functions satisfy certain algebraic identities related to the representation theory of the symmetric group. We will exhibit a connection between the invariant theory of the orthosymplectic supergroup and mixed partition functions. This is based on (1.4) and on unpublished work with G. Regts.

1.2.1 Published papers and contributions

This thesis is based on the following three published papers.

G. Regts, A. Schrijver, B. Sevenster, On partition functions for 3-graphs, *Journal of Combinatorial Theory, Series B* **121** (2016) 421–431. (1.1)

G. Regts, A. Schrijver, B. Sevenster, On the existence of real R-matrices for virtual link invariants, *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* **87** (2017) 435–443. (1.2)

G. Regts, B. Sevenster, Graph parameters from invariants of the symplectic group, *Journal of Combinatorial Theory, Series B* **122** (2017) 844–868. (1.3)

It is furthermore based on the following manuscript written with G. Regts.

G. Regts, B. Sevenster, Mixed partition functions and exponentially bounded edge-connection rank, arXiv preprint, 2018, arXiv:1807.04494. (1.4)

In all four papers the contribution of each of the authors was equivalent.

1.3 Outline of this thesis

Chapter 2. Notation, preliminaries and our results. In this chapter we set up some notation for the rest of the thesis and we recall the definitions of partition functions as given by de la Harpe and Jones [15]. We state the theorem of Szegedy [39] and the theorem of Schrijver [38] and we indicate how we extend these theorems. The formal statement of the theorems follows in Chapter 4, Chapter 7 and Chapter 8.

Chapter 3. Matchings and a theorem by Hanlon and Wales. In this chapter we consider the submatrix of the connection matrix induced by matchings. In the proofs of all the afore mentioned theorems we need a solid understanding of the value of the graph parameters on $\bigcirc$. To this end we discuss a theorem of Hanlon and Wales in [14] and we derive some consequences from this theorem that will be useful later on.

Chapter 4. Skew and mixed partition functions. In this chapter we first give the definition of skew partition functions and we show that they are well-defined. Then we state our theorems on skew partition functions. Next, we use the definition of skew partition functions to define mixed partition functions. We state an algebraic property of mixed partition functions that we will show to hold in Chapter 5. We prove that for a mixed partition function f there exists a contant $r \in \mathbb{R}$ such that $\text{rk}\,(M_{f,t}) \leq r^t$. Finally, we give two more involved examples of mixed partition functions. De la Harpe and Jones [15] asked if evaluations of the characteristic polynomial can be described as the partition function of a spin model. We answer this question negatively, but we do show that evaluations of the characteristic polynomial can be described by mixed partition functions. This chapter is based on (1.3) and (1.4).

Chapter 5. Partition functions and invariant theory. In this chapter we prove the algebraic characterization of skew partition functions given in Chapter 4. The proof uses the invariant theory of the symplectic group and the representation theory of the symmetric group. We furthermore show that mixed partition functions satisfy certain relations that are related to the invariant theory of the symmetric group. The proof of both theorems makes use of a framework developed by Draisma, Gijswijt, Lovász, Regts and Schrijver [10]. This chapter is based on (1.3) and (1.4) and on unpublished work together with G. Regts.

Chapter 6. Partition functions and the algebra of fragments. We give a characterization of skew partition functions similar to that of Schrijver [38] for partition functions of edge coloring models. The proof makes use of the algebra of fragments introduced by Schrijver in [38]. The concept goes back to [12]. This chapter is based on (1.3).

Chapter 7. Reflection positivity for 3-graphs. In this chapter we prove our main theorem on 3-graphs. Most of the work is devoted to analyzing the value that $\bigcirc$ can take. We make use of the theorem of Hanlon and Wales [14] that we discussed in Chapter 3. We furthermore use the invariant theory of the orthogonal group and a theorem of Procesi and Schwarz [26]. This chapter is based on (1.1).

Chapter 8. Reflection positivity for virtual links. In this chapter we prove our main theorem on virtual links. The proof goes along the same lines as the proof of our theorem for 3-graphs, given in Chapter 7. We however need a different combinatorial argument to analyze the value that $\bigcirc$ can take. This chapter is based on (1.2).

Chapter 2

Notation, preliminaries and our results

In this chapter we give the definitions of partition functions of vertex coloring models and partition functions of edge coloring models. Both definitions were given by de la Harpe and Jones [15]. We discuss a theorem of Szegedy [39] that we will extend to partition functions for 3-graphs and to partition functions for virtual links in Chapter 7 and Chapter 8, respectively. We will furthermore indicate how our results are related to work by Schrijver [38] and work by Draisma, Gijswijt, Lovász, Regts and Schrijver [10]. We first set some notation for the rest of this thesis.

2.1 Notation and basic definitions

Sets. We let the natural numbers include zero. So $\mathbb{N} := \{0, 1, 2, \ldots\}$. For $n \in \mathbb{N}$, we define $[n] := \{1, \ldots, n\}$. Note that $[0] = \emptyset$. A *multiset* is a collection of elements where each element can occur more than once. The number of times an element occurs in a multiset is called its *multiplicity*. A *multisubset* of a set S is a multiset consisting of elements of S. We use the same notation for a multiset as for a set. The cardinality of a multiset X is denoted by $|X|$.

Graphs. In this thesis a *graph* is assumed to be finite and can possibly have multiple edges, loops at vertices and free loops (a free loop is an edge of which both ends are glued together). So a graph $G = (V, E)$ consists of a finite set of vertices V together with a finite multiset E of edges, which are either multisubsets of V of cardinality two, or free loops. We think of a free loop in E as an edge $\{u, u\}$ for which $u \notin V$. Let $\overline{\mathcal{G}}$ be the set of all graphs. The graph $(\emptyset, \{\{u, u\}\})$ is called the *vertexless loop* and is denoted by $\bigcirc$. Note that $\{u,u\}$ is a free loop in this graph. If G is a graph, then $V(G)$ denotes its vertex set and $E(G)$ denotes its edge set.

If v is a vertex of a graph G, then $\delta(v)$, the *neighborhood* of v, is the multiset of edges that contain v with multiplicity (so a loop at v occurs twice in $\delta(v)$). We define the *degree* of v to be $d(v) := |\delta(v)|$. The *degree sequence* of a graph G is the non-increasing sequence of the degrees of its vertices. If v is a vertex of degree 2 in a graph $G = (V, E)$ and $\delta(v) = \{\{u, v\}, \{v, w\}\}$, then by *smoothening* v we obtain the graph $(V \setminus \{v\}, (E \setminus \delta(v)) \cup \{u, w\})$, i.e., we remove the vertex v and we remove the edges incident with v and we add an edge between u and w. In particular, if $G = (\{v\}, \{\{v, v\}\})$, i.e., G is a graph with one vertex v and a loop at that vertex, then by smoothening v we obtain the graph $(\emptyset, \{\{v, v\}\}) = \bigcirc$.

Walks. A *walk* in a graph $G = (V, E)$ is a sequence $(v_0, a_1, v_1, \ldots, a_n, v_n)$ such that $v_i \in V$ for each $i \in \{0, \ldots, n\}$ and $a_i \in E$ for each $i \in [n]$, and such that $a_i = \{v_{i-1}, v_i\}$ for each $i \in [n]$. A walk $(v_0, a_1, v_1, \ldots, a_n, v_n)$ is said to *start in* v_0 and *end in* v_n. A *trail* in a graph G is a walk in G in which each edge of G occurs at most once. A *circuit* in a graph G is a trail in G that starts and ends in the same vertex. A *cycle* in a graph G is a circuit in G in which the starting vertex is only seen at the start and at the end, and in which each other vertex of G occurs at most once.

Let $G = (V, E)$ be a graph. For $u, v \in V$, we say that u is *reachable* from v if there is a walk in G that starts in v and ends in u. We say that G is *connected* if $G = \bigcirc$, or if E does not contain any free loops and if for any two vertices $u, v \in V$, the vertex u is reachable from v. If a graph G is the disjoint union of connected graphs $G'_1, \ldots, G'_n$, then $G'_1, \ldots, G'_n$ are referred to as the *connected components* of G.

Directed graphs. In this thesis a *directed graph*, or digraph, is assumed to be finite and can possibly have multiple arcs, loops and free directed loops. So a digraph $D = (V, A)$ consists of a finite set of vertices V together with a finite multiset A of arcs, which are either ordered pairs of elements of V, or free directed loops. We think of a free directed loop in A as an arc (u, u) for which $u \notin V$. We say that an arc (i, j) of a digraph is *outgoing at* i and *incoming at* j and we think of it as being directed from i to j. We say that a digraph is *Eulerian* if at each vertex the number of incoming arcs is equal to the number of outgoing arcs. The *graph underlying a digraph* $D = (V, A)$ is the graph (V, E), where $E = \{\{i, j\} \mid (i, j) \in A\}$.

Graph parameters. Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there is a bijection $\phi : V \to V'$ such that for any two vertices $v_1, v_2 \in V$ the multiplicity of the edge $\{v_1, v_2\}$ in E is the same as the multiplicity of $\{\phi(v_1), \phi(v_2)\}$ in E', and such that the number of free loops in E is equal to the number of free loops in E'. This induces an equivalence relation $\sim$ on $\overline{\mathcal{G}}$. Let $\mathcal{G}$ be the set of graphs, where two elements are considered the same if they are isomorphic, i.e., $\mathcal{G} = \overline{\mathcal{G}}/\sim$.

Let X be a non-empty set. A *graph parameter over* X is a function $f : \mathcal{G} \to X$. If $\mathcal{H} \subseteq \mathcal{G}$, then a function $f : \mathcal{H} \to X$ is also called a graph parameter. A graph parameter f over a commutative ring R is called *multiplicative* if for any two graphs $G, H \in \mathcal{G}$, we have $f(G \cup H) = f(G)f(H)$, where $G \cup H$ is the disjoint union of G and H.

Matrices. Let M be a matrix of which the rows are indexed by the elements of a set $\mathcal{I}$ and the columns are indexed by the elements of a set $\mathcal{J}$. For $i \in \mathcal{I}$ and $j \in \mathcal{J}$ we denote the element in the i-th row and j-th column of the matrix M by $M(i,j)$ or by $M_{i,j}$. We refer to M as an $\mathcal{I} \times \mathcal{J}$ matrix. A (possibly infinite) real symmetric matrix is called *positive semidefinite* if each finite principal submatrix has only non-negative eigenvalues. The rank of a matrix M is denoted by $\mathrm{rk}\,(M)$.

Vector spaces. Let $\mathbb{F}$ be a field. If X is a set, then $\mathbb{F}X$ is the vector space of formal $\mathbb{F}$-linear combinations of elements of X. If $f : X \to \mathbb{F}$ is a function, then we extend f linearly to a function $f : \mathbb{F}X \to \mathbb{F}$.

Let W be a vector space over a field $\mathbb{F}$ of characteristic 0. The *dual space* of W is denoted by W^*. The *tensor algebra* TW of W is defined as

$$TW := \bigoplus_{n=0}^{\infty} W^{\otimes n}. \tag{2.1}$$

Here a tensor $v \in W^{\otimes n}$ for some $n \in \mathbb{N}$ is said to be of *degree* n. The *symmetric algebra* SW is the quotient of TW by the ideal of TW generated by elements of the form $x \otimes y - y \otimes x$, for $x, y \in W$. One can identify SW with the polynomial ring over $\mathbb{F}$ in indeterminates that form a basis of W. An element of SW is called a *symmetric tensor*. The *exterior algebra* $\bigwedge W$ is the quotient of TW by the ideal of TW generated by elements of the form $x \otimes y + y \otimes x$, for $x, y \in W$. An element of $\bigwedge W$ is called an *alternating tensor*. We can write

$$SW = \bigoplus_{n=0}^{\infty} S^n W \quad \text{and} \quad \bigwedge W = \bigoplus_{n=0}^{\infty} \bigwedge\nolimits^n W,$$

by (2.1), as the defining ideals are homogeneous.

Now we introduce two families of vector spaces that we will often encounter in this thesis. First, for $k \in \mathbb{N}$, let $V_k := \mathbb{C}^k$ with standard basis $\{e_1, \ldots, e_k\}$. The image of $e_{i_1} \otimes \cdots \otimes e_{i_n} \in TV_k$ in SV_k under the quotient map is denoted by $\bigodot_{j \in [n]} e_{i_j}$. A basis of SV_k is given by

$$\left\{ \bigodot_{i \in S} e_i \;\middle|\; S \text{ a multisubset of } [k] \right\}.$$

We equip V_k with the standard inner product $(\cdot,\cdot)$ defined, for $v_1, v_2 \in V_k$, by $(v_1, v_2) := v_1^T v_2$. Here v_1^T denotes the transpose of the vector v_1. For $i, j \in [k]$, we have $(e_i, e_j) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta.

Next, for $\ell \in \mathbb{N}$, let $V_{2\ell} := \mathbb{C}^{2\ell}$ with standard basis $\{f_1, \ldots, f_{2\ell}\}$. The image of $f_{i_1} \otimes \cdots \otimes f_{i_n} \in TV_{2\ell}$ in $\bigwedge V_{2\ell}$ under the quotient map is denoted by $f_{i_1} \wedge \cdots \wedge f_{i_n}$. A basis of $\bigwedge V_{2\ell}$ is given by

$$\{f_{i_1} \wedge \cdots \wedge f_{i_n} \mid 1 \leq i_1 < \cdots < i_n \leq 2\ell\}.$$

Let I_ℓ be the $\ell \times \ell$ identity matrix. We equip $V_{2\ell}$ with a skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined, for $v_1, v_2 \in V_{2\ell}$, by

$$\langle v_1, v_2 \rangle := v_1^T \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix} v_2 \,.$$

For $i \in [2\ell]$, define $g_i \in V_{2\ell}$ by

$$g_i := \begin{cases} -f_{i+\ell} & \text{if } i \leq \ell, \\ f_{i-\ell} & \text{if } i > \ell. \end{cases} \tag{2.2}$$

Then $\langle g_i, f_j \rangle = \delta_{i,j}$, for $i, j \in [2\ell]$.

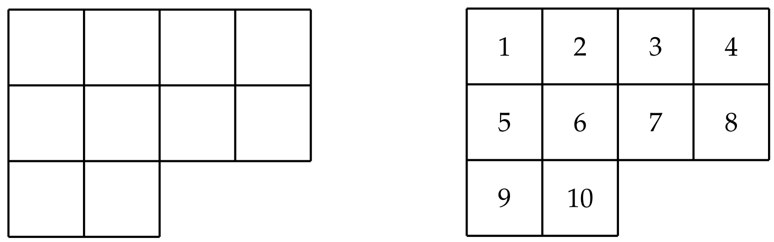
Partitions and Young symmetrizers. The symmetric group on a set X is denoted by S_X. For $n \in \mathbb{N}$, we define $S_n := S_{[n]}$. We briefly recall some concepts from the representation theory of the symmetric group. For more background, see e.g. [34].

A tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $\lambda_1, \ldots, \lambda_r \in \mathbb{N}_{>0}$, $\lambda_1 \geq \cdots \geq \lambda_r$ and $\sum_{i=1}^r \lambda_i = m$ is called a *partition of* m, denoted by $\lambda \vdash m$. The partition λ is called *even* if λ_i is even for each $i \in [r]$. The *Young diagram of shape* λ consists of r left-justified rows of cells such that for $i \in [r]$, row i contains exactly λ_i cells, see Figure 2.1a. The cell of a Young diagram in the i-th row and j-th column is referred to as cell (i, j).

Let $m \in \mathbb{N}$ and let $\lambda \vdash m$. A *Young tableau of shape* λ is a Young diagram of shape λ together with a bijection between the cells of the Young diagram and $[m]$. We refer to such a bijection as a *filling*. If all the values in a Young tableau are ascending when going from left to right in each row and when going from top to bottom in each column in the Young tableau, then it is called a *standard Young tableau*. If $\lambda = (\lambda_1, \ldots, \lambda_r)$, then the filling that assigns $j + \sum_{k=1}^{i-1} \lambda_k$ to the cell (i, j) of the Young diagram of shape λ is called the *standard filling*. The Young tableau of shape λ with the standard filling is referred to as Y_λ. See Figure 2.1b for an example.

Let $R_\lambda \subseteq S_m$ be the subgroup of permutations that preserve each row of Y_λ and let $C_\lambda \subseteq S_m$ be the subgroup of permutations that preserve each column of Y_λ. Then the *Young symmetrizer* $e_\lambda \in \mathbb{C}S_m$ corresponding to λ is given by

$$e_\lambda := \sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma) \sigma \rho, \tag{2.3}$$



(a) The Young diagram for $\lambda = (4,4,2)$. (b) The Young tableau $Y_{(4,4,2)}$.

Figure 2.1: Example of a Young diagram and the standard filling.

where $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ.

Let $k, \ell \in \mathbb{N}$. A partition $\lambda = (\lambda_1, \dots, \lambda_r)$ is called a $(k, 2\ell)$-*hook* if the cell $(k+1, 2\ell+1)$ is not in the Young diagram of shape λ, i.e., if $r \leq k$, or $r > k$ and $\lambda_{k+1} \leq 2\ell$. We define

$$H(k, 2\ell) := \{\lambda \mid \lambda \text{ an even } (k, 2\ell)\text{-hook}\}. \tag{2.4}$$

We say that λ is a $(k, 2\ell)$-*block* if the cell $(k+1, 2\ell+1)$ is in the Young diagram of shape λ, i.e., if $r > k$ and $\lambda_{k+1} > 2\ell$. We define

$$B(k, 2\ell) := \{\lambda \mid \lambda \text{ an even } (k, 2\ell)\text{-block}\}. \tag{2.5}$$

Note that $B(k, 2\ell) \cup H(k, 2\ell)$ is equal to the set consisting of all even partitions.

2.2 Partition functions of vertex coloring models

We first define the partition function of a vertex coloring model. We follow the definition given by Freedman, Lovász and Schrijver [12], which is slightly more general than the definition given by de la Harpe and Jones [15]. Let $\mathcal{G}^* \subset \mathcal{G}$ be the set consisting of graphs G such that G does not have loops at any vertex and such that $\bigcirc$ is not a connected component of G.

Let R be a commutative and unitary ring and let $n \in \mathbb{N}$. A pair (α, B) consisting of a map $\alpha : [n] \to R$ and a symmetric $n \times n$ matrix B over R is called a *vertex coloring model over* R. The *partition function* of the vertex coloring model (α, B) is the graph parameter $p_{(\alpha,B)} : \mathcal{G}^* \to R$ defined, for a graph $G = (V, E) \in \mathcal{G}^*$, by

$$p_{(\alpha,B)}(G) := \sum_{\kappa: V \to [n]} \prod_{v \in V} \alpha(\kappa(v)) \cdot \prod_{\{v_1, v_2\} \in E} B(\kappa(v_1), \kappa(v_2)). \tag{2.6}$$

Note that this is well-defined since B is a symmetric matrix and R is commutative. If $\alpha(i) = 1$ for all $i \in [n]$, then we retrieve the definition given by de la Harpe and Jones. In this special case B is called a *spin model* and $p_B = p_{(\alpha,B)}$ is called the *partition function* of the spin model B.

Labeled graphs and vertex connection matrices. Vertex connection matrices were introduced by Freedman, Lovász and Schrijver [12]. We need some definitions to explain this concept.

For $t \in \mathbb{N}$, a *t-labeled graph* is a graph $G \in \mathcal{G}^*$ with t labeled vertices, labeled $1, 2, \ldots, t$. Let $\mathcal{L}_t$ be the set of t-labeled graphs. Note that $\mathcal{L}_0 = \mathcal{G}^*$. We define the product $L_1 L_2 \in \mathcal{G}^*$ of $L_1, L_2 \in \mathcal{L}_t$ by first taking the disjoint union of L_1 and L_2 and then identifying equally labeled vertices and forgetting the labeling afterwards. This product is called the *vertex product*, see Figure 2.2. In particular, if $L_1, L_2 \in \mathcal{L}_0$, then $L_1 L_2 = L_1 \cup L_2$.

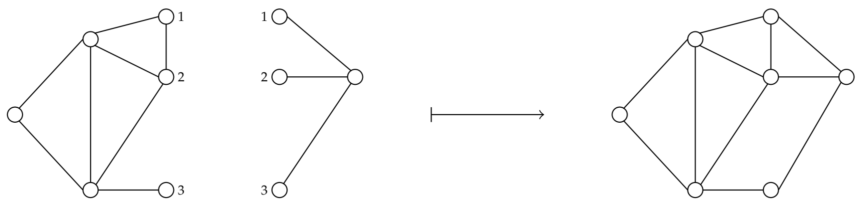


Figure 2.2: The vertex product of two 3-labeled graphs.

For a graph parameter f, we define the matrix $U_{f,t}$ to be the $\mathcal{L}_t \times \mathcal{L}_t$-matrix such that $U_{f,t}(L_1, L_2) = f(L_1 L_2)$ for $L_1, L_2 \in \mathcal{L}_t$. We call this matrix the t-th *vertex connection matrix* of f. A graph parameter $f : \mathcal{G}^* \to \mathbb{R}$ is called *vertex reflection positive* if $U_{f,t}$ is positive semidefinite for each $t \in \mathbb{N}$.

Freedman, Lovász and Schrijver gave the following characterization of vertex reflection positive graph parameters.

Theorem 2.1. *[12, Theorem 2.4] Let $f : \mathcal{G}^* \to \mathbb{R}$ be a graph parameter and $n \in \mathbb{N}$. Then there exist a map $\alpha : [n] \to \mathbb{R}_{\geq 0}$ and a symmetric real-valued $n \times n$ matrix B such that $f = p_{(\alpha,B)}$ if and only if $f(\emptyset) = 1$, f is vertex reflection positive and $\mathrm{rk}\,(U_{f,t}) \leq n^t$ for all $t \in \mathbb{N}$.*

There are several other characterizations of partition functions of vertex coloring models in terms of vertex connection matrices, see for example [36, 37].

2.3 Partition functions of edge coloring models

Partition functions of edge coloring models are defined in a manner similar to that of partition functions of vertex coloring models, with interchanged roles of edges and vertices. Partition functions of edge coloring models were defined in [15] by de la Harpe and Jones. In their terminology they are called vertex models. We stick to the name edge coloring model.

Let $k \in \mathbb{N}$ and let $\mathbb{F}$ be a field of characteristic 0. Let $\{e_1, \ldots, e_k\}$ be the standard basis of the vector space $\mathbb{F}^k$. An element $h \in (S\mathbb{F}^k)^*$ is called a *k-color* *edge coloring model over* $\mathbb{F}$. The *partition function* p_h of the edge coloring model h is the graph parameter over $\mathbb{F}$ defined, for a graph $G = (V, E)$, by

$$p_h(G) := \sum_{\phi: E \to [k]} \prod_{v \in V} h\Big(\bigodot_{a \in \delta(v)} e_{\phi(a)} \Big), \tag{2.7}$$

where we recall that $\delta(v)$ is the multiset consisting of edges incident with v with multiplicities. Note that $p_h(\bigcirc) = k$ and that p_h is multiplicative. If a graph parameter is the partition function of a k-color edge coloring model over $\mathbb{F}$ for some $k \in \mathbb{N}$, then we sometimes refer to it as an *ordinary partition function over* $\mathbb{F}$ to distinguish it from skew partition functions, to be defined later. Note that if $k = 0$, then there is only one element h in $(SV_k)^*$ and p_h is the function that evaluates to 1 on $\emptyset$ and that evaluates to 0 on all other graphs.

Fragments and edge connection matrices. For $t \in \mathbb{N}$, a *t-fragment* is a graph with t labeled vertices of degree one labeled $1, 2, \ldots, t$. We denote the set of all t-fragments by $\mathcal{F}_t$. We now define a gluing operation on $\mathcal{F}_t$. For two t-fragments F_1, F_2 we define $F_1 * F_2 \in \mathcal{G}$ to be the graph obtained as follows. We first take the disjoint union of F_1 and F_2, then we identify equally labeled vertices, and finally we smoothen the labeled vertices and disregard the labeling. See Figure 2.3. It follows from the definitions that $\mathcal{F}_0 = \mathcal{G}$ and that $F_1 * F_2 = F_1 \cup F_2$ if $F_1, F_2 \in \mathcal{F}_0$. Let $t \in \mathbb{N}$. For $i = 1, 2$, let F_i be a $2t$-fragment such that each vertex of F_i is of degree 1 and is labeled. Then each connected component of $F_1 * F_2$ is equal to $\bigcirc$.

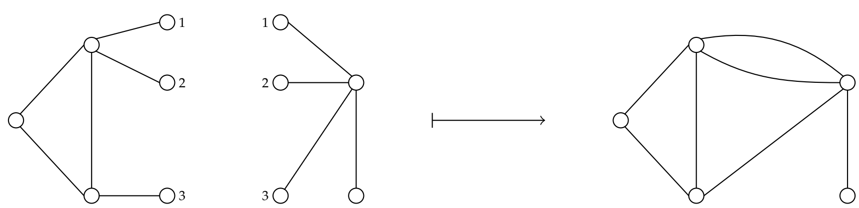


Figure 2.3: The graph obtained by gluing two 3-fragments.

For a graph parameter f, we define its *t-th edge connection matrix* $M_{f,t}$ to be the $\mathcal{F}_t \times \mathcal{F}_t$ matrix such that $M_{f,t}(F_1, F_2) = f(F_1 * F_2)$ for $F_1, F_2 \in \mathcal{F}_t$. We note here that if f is a multiplicative graph parameter, then, for $t_1, t_2 \in \mathbb{N}$, we have

$$\mathrm{rk}\,(M_{f,t_1+t_2}) \geq \mathrm{rk}\,(M_{f,t_1})\mathrm{rk}\,(M_{f,t_2}), \tag{2.8}$$

as $M_{f,t_1} \otimes M_{f,t_2}$ is a submatrix of M_{f,t_1+t_2}.

A graph parameter f over $\mathbb{R}$ is called *edge reflection positive* if $M_{f,t}$ is positive semidefinite for each $t \in \mathbb{N}$. Szegedy gave the following characterization of edge reflection positive graph parameters.

Theorem 2.2. *[39, Theorem 2.2] A graph parameter $f : \mathcal{G} \to \mathbb{R}$ is the partition function of an edge coloring model over $\mathbb{R}$ if and only if $f(\emptyset) = 1$, f is multiplicative and f is edge reflection positive.*

Schrijver [35] gave a strengthening of Szegedy's theorem by introducing a weaker form of reflection positivity. This proof uses a theorem by Procesi and Schwarz [26] instead of the Positivstellensatz. We will use this approach to extend Szegedy's theorem to partition functions for 3-graphs and partition functions for virtual links in Chapter 7 and Chapter 8, respectively.

Schrijver gave the following characterization of partition functions of edge coloring models in terms of the rank of the associated connection matrices.

Theorem 2.3. *[38, Theorem 1] A graph parameter $f : \mathcal{G} \to \mathbb{C}$ is the partition function of an edge coloring model over $\mathbb{C}$ if and only if $f(\emptyset) = 1$, $f(\bigcirc) \in \mathbb{R}$ and*

$$\mathrm{rk}\,(M_{f,t}) \leq f(\bigcirc)^t \tag{2.9}$$

for each $t \in \mathbb{N}$.

Note that the condition $\mathrm{rk}\,(M_{f,t}) \leq f(\bigcirc)^t$ implies that $f(\bigcirc) \geq 0$. A natural question to ask here is what happens if we allow $f(\bigcirc)$ to be negative. In Chapter 4 we will define skew partition functions. A skew partition function can be seen as the partition function of an element $h \in (\bigwedge V_{2\ell})^*$ for some $\ell \in \mathbb{N}$. The definition is a bit more involved than the definition of ordinary partition functions, as signs come into play when working with the exterior algebra. This is why we postpone the definition.

In Theorem 4.3 we will see that skew partition functions are exactly the graph parameters $f : \mathcal{G} \to \mathbb{C}$ such that $f(\emptyset) = 1$, $f(\bigcirc) \leq 0$ and

$$\mathrm{rk}\,(M_{f,2t}) \leq f(\bigcirc)^{2t} \tag{2.10}$$

for each $t \in \mathbb{N}$. We will prove this in Chapter 6.

Algebraic characterizations of partition functions. In [10] Draisma, Gijswijt, Lovász, Regts and Schrijver gave a characterization of ordinary partition functions over an algebraically closed field $\mathbb{F}$ of characteristic 0, using the invariant theory of the orthogonal group. We will give an alternative statement of Theorem 1 in [10] to make the parallels between this theorem and our work more clear.

Let $G = (V, E)$ be a graph. For $n \in \mathbb{N}$ and $u : [2n] \to V$ any map, we define

$$G_u := (V, E \cup \{\{u(2i-1), u(2i)\} \mid i \in [n]\}). \tag{2.11}$$

Recall that for a partition λ we defined C_λ to be the column stabilizer of Y_λ and we defined R_λ to be the row stabilizer of Y_λ, where Y_λ is the Young tableau with shape λ and the standard filling. Let $\lambda \vdash 2n$ be an even partition. Then $n \in \mathbb{N}$ and we define $\mathcal{J}^\lambda \subseteq \mathbb{C}\mathcal{G}$ to be the subspace spanned by

$$\left\{ \sum_{(\sigma,\rho)\in C_\lambda \times R_\lambda} \operatorname{sgn}(\sigma) G_{u\circ\sigma\circ\rho} \;\middle|\; G = (V,E) \in \mathcal{G},\ u : [2n] \to V \right\}. \tag{2.12}$$

Let $k, \ell \in \mathbb{N}$. Recall that $B(k, 2\ell)$ is the set consisting of even partitions λ such that the Young diagram of shape λ contains the cell $(k+1, 2\ell+1)$. We define $\mathcal{J}_{k,2\ell} \subseteq \mathbb{C}\mathcal{G}$ to be

$$\mathcal{J}_{k,2\ell} := \bigoplus_{\lambda \in B(k,2\ell)} \mathcal{J}^\lambda. \tag{2.13}$$

In [10], Draisma, Gijswijt, Lovász, Regts and Schrijver gave a characterization of partition functions of edge coloring models using the First Fundamental Theorem and the Second Fundamental Theorem of invariant theory of the orthogonal group. Over $\mathbb{C}$ their theorem is equivalent to the following statement.

Theorem 2.4. *[10, Theorem 1] Let $k \in \mathbb{N}$. A graph parameter $f : \mathcal{G} \to \mathbb{C}$ is the partition function of an element $h \in (SV_k)^*$ if and only if $f(\emptyset) = 1$, $f(\bigcirc) = k$, f is multiplicative and $f(\mathcal{J}_{k,0}) = 0$.*

For skew partition functions we can give a similar characterization. We will see in Theorem 4.4 that if f is a skew partition function coming from $h \in (\bigwedge V_{2\ell})^*$, then $f(\emptyset) = 1$, f is multiplicative and $f(\mathcal{J}_{0,2\ell}) = 0$. We will also prove a converse to this statement. The proof is inspired by the proof of Draisma, Gijswijt, Lovász, Regts and Schrijver and uses the First Fundamental Theorem of invariant theory of the symplectic group.

In Chapter 4 we will define mixed partition functions, a common generalization of skew partition functions and ordinary partition functions. A mixed partition function can be seen as the partition function of an element $h \in (SV_k \otimes \bigwedge V_{2\ell})^*$ for some $k, \ell \in \mathbb{N}$. In Theorem 4.5 we will see that if f is the partition function of an element $h \in (SV_k \otimes \bigwedge V_{2\ell})^*$ for some $k, \ell \in \mathbb{N}$, then $\operatorname{rk}(M_{f,t}) \leq (k+2\ell)^t$ for each $t \in \mathbb{N}$.

In Theorem 4.6 we will see that if f is the partition function of an element $h \in (SV_k \otimes \bigwedge V_{2\ell})^*$ for some $k, \ell \in \mathbb{N}$, then $f(\mathcal{J}_{k,2\ell}) = 0$. At the end of Chapter 5 we formulate a conjecture saying that the converse of this statement, where we add the assumptions that $f(\emptyset) = 1$, $f(\bigcirc) = k - 2\ell$ and that f is multiplicative, is also true and we will see how one can possibly prove this conjecture using the invariant theory of the orthosymplectic supergroup.

Chapter 3

Matchings and a theorem by Hanlon and Wales

Let $n \in \mathbb{N}$. A *perfect matching on* $[2n]$ is a set M consisting of edges $\{i, j\}$ with $i, j \in [2n]$ such that each vertex in the graph $([2n], M)$ has degree 1. The set of perfect matchings on $[2n]$ is denoted by $\mathcal{M}_{2n}$. For $x \in \mathbb{C}$, let $A_{2n}(x)$ be the symmetric $\mathcal{M}_{2n} \times \mathcal{M}_{2n}$ matrix defined, for $M_1, M_2 \in \mathcal{M}_{2n}$, by

$$A_{2n}(x)(M_1, M_2) := x^{c(M_1 \cup M_2)},$$

where $c(M_1 \cup M_2)$ denotes the number of connected components of the graph $([2n], M_1 \cup M_2)$. For $M \in \mathcal{M}_{2n}$, let $M' = ([2n], M)$ be the $2n$-fragment such that vertex $i \in [2n]$ is labeled i. Then, if f is a multiplicative graph parameter such that $f(\bigcirc) = x$, we see, for $M_1, M_2 \in \mathcal{M}_{2n}$, that

$$x^{c(M_1 \cup M_2)} = f(M_1' * M_2').$$

This shows that $A_{2n}(x)$ can be identified with a submatrix of $M_{f,2n}$ if $f(\bigcirc) = x$ and f is multiplicative.

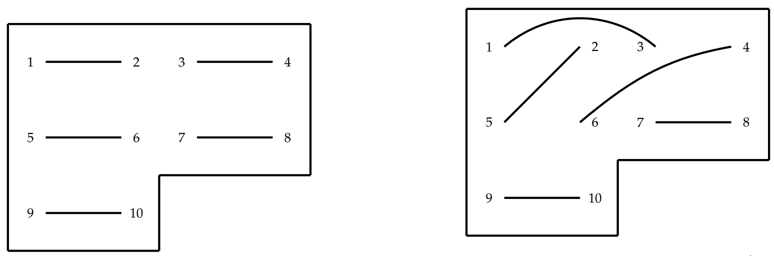
In Proposition 3.1 we will see that $\mathbb{C}\mathcal{M}_{2n}$, the space of formal linear combinations of perfect matchings on $[2n]$, decomposes multiplicity free into a sum of the irreducible representations S^λ, where λ is an even partition of $2n$. It turns out that each S^λ consists of eigenvectors of $A_{2n}(x)$ with the same eigenvalue. We discuss a theorem of Hanlon and Wales [14] that gives a closed expression for these eigenvalues.

3.1 The action of the symmetric group on perfect matchings

Let $n \in \mathbb{N}$. For $\pi \in S_{2n}$ and $M \in \mathcal{M}_{2n}$, let $\pi M = \{\{\pi(i), \pi(j)\} \mid \{i, j\} \in M\}$. This defines an action of S_{2n} on $\mathcal{M}_{2n}$ and we extend this action linearly to an action of $\mathbb{C}S_{2n}$ on $\mathbb{C}\mathcal{M}_{2n}$. We note here that for $\pi \in S_{2n}$ and $M_1, M_2 \in \mathcal{M}_{2n}$, we have

$$c(M_1 \cup M_2) = c(\pi M_1 \cup \pi M_2). \tag{3.1}$$

Let $\lambda \vdash 2n$ be an even partition. Let M be the perfect matching on $[2n]$ with edges $\{2i-1, 2i\}$ for $i \in [n]$. For $\sigma \in C_\lambda$ and $\rho \in R_\lambda$ we can place the matching $\sigma\rho M$ in the Young tableau Y_λ, see Figure 3.1. Sometimes it is convenient to think of $\sigma\rho M$ in this way.



(a) The matching M.

(b) The matching $\sigma\rho M$ for $\rho = (2,3)$ and $\sigma = (2,6)$.

Figure 3.1: Two examples of matchings placed in a Young tableau.

The irreducible representations of S_{2n} are in bijective correspondence with partitions of $2n$. The irreducible representation corresponding to $\lambda \vdash 2n$ is denoted by S^λ. In [34, Ex. 3.12.7], a proof of the following equality is outlined:

$$\sum_{\substack{\lambda \vdash 2n \\ \lambda \text{ even}}} \dim(S^\lambda) = (2n-1)!!, \tag{3.2}$$

where for $m \in \mathbb{N}$, $m!! = m(m-2)\cdots 1$ if m is odd and $m!! = 0$ if m is even. The proof is relatively straightforward, but uses some of the machinery of the Robinson-Schensted-Knuth algorithm, which we do not describe here.

Proposition 3.1. *For $n \in \mathbb{N}$, the S_{2n}-module $\mathbb{C}\mathcal{M}_{2n}$ decomposes multiplicity free as*

$$\mathbb{C}\mathcal{M}_{2n} \cong \bigoplus_{\substack{\lambda \vdash 2n \\ \lambda \text{ even}}} S^\lambda. \tag{3.3}$$

Proof. Let $n \in \mathbb{N}$. According to the representation theory of the symmetric group, $\mathbb{C}\mathcal{M}_{2n}$ decomposes as an S_{2n}-module into

$$\mathbb{C}\mathcal{M}_{2n} \cong \bigoplus_{\lambda \vdash 2n} (S^\lambda)^{\oplus \mu_\lambda}, \tag{3.4}$$

where μ_λ is the multiplicity of the irreducible representation S^λ in $\mathbb{C}\mathcal{M}_{2n}$. We first show that for even $\lambda \vdash 2n$ the multiplicity is non-zero. To this end it suffices to show that M, the perfect matching on $[2n]$ with edges $\{2i-1, 2i\}$ for $i \in [n]$, occurs with non-zero coefficient in $e_\lambda M$.

Let $\sigma \in C_\lambda$ and $\rho \in R_\lambda$ and suppose that $\sigma\rho M = M$. We think of $\sigma\rho M$ as being placed in the Young tableau Y_λ, cf. Figure 3.1. It follows that $\rho M = M$: if this would not be the case then there would be an edge in ρM that is between two non-adjacent columns and hence there would be an edge in $\sigma\rho M$ between two non-adjacent columns, which is not the case for M. So for $\sigma\rho M = M$, we need both σ and ρ to preserve M. The sign of a permutation $\sigma \in C_\lambda$ preserving M is 1. So the coefficient of M in $e_\lambda M$ is positive. Hence $\mu_\lambda \geq 1$ for even $\lambda \vdash 2n$.

By (3.2), each S^λ for even $\lambda \vdash 2n$ can occur at most once and all other irreducible representations do not occur at all, as the dimension of $\mathbb{C}\mathcal{M}_{2n}$ is $(2n-1)!!$. ☐

For $n \in \mathbb{N}$ and $\lambda \vdash 2n$ even, we identify S^λ with the S_{2n}-module of $\mathbb{C}\mathcal{M}_{2n}$ generated by $e_\lambda M$, where $M = \{\{2i-1, 2i\} \mid i \in [n]\}$. We furthermore identify $\mathbb{C}\mathcal{M}_{2n}$ with $\mathbb{C}^{\mathcal{M}_{2n}}$. Under this identification $A_{2n}(x)$ defines a linear transformation of $\mathbb{C}\mathcal{M}_{2n}$.

Lemma 3.2. *Let $x \in \mathbb{C}$, $n \in \mathbb{N}$ and let $\lambda \vdash 2n$ be an even partition. The linear transformation of $\mathbb{C}\mathcal{M}_{2n}$ defined by $A_{2n}(x)$ acts as a scalar on S^λ.*

Proof. Let $n \in \mathbb{N}$ and $x \in \mathbb{C}$. Let $\pi \in S_{2n}$ and let P_π be the permutation matrix that corresponds to the action of π on $\mathbb{C}\mathcal{M}_{2n}$. By (3.1) we know that for all $\pi \in S_{2n}$, we have $A_{2n}(x) = P_\pi^T A_{2n}(x) P_\pi = P_\pi^{-1} A_{2n}(x) P_\pi$.

This shows that $A_{2n}(x)$ gives an S_{2n}-equivariant linear map of $\mathbb{C}\mathcal{M}_{2n}$ to itself. By Proposition 3.1 each irreducible representation in the decomposition occurs with multiplicity one. So by Schur's lemma the linear transformation defined by $A_{2n}(x)$ acts as a scalar on S^λ for even $\lambda \vdash 2n$. ☐

By this lemma, we know that for an even partition $\lambda \vdash 2n$, the space S^λ consists of eigenvectors of $A_{2n}(x)$, all with the same eigenvalue. In the next section we treat a theorem by Hanlon and Wales [14] that gives a closed expression for these eigenvalues.

3.2 The theorem of Hanlon and Wales

For an even partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash 2n$, we define

$$h_\lambda(x) := \prod_{i=1}^{r} \prod_{j=1}^{\frac{1}{2}\lambda_i} (x - i + 2j - 1). \tag{3.5}$$

We can visualize this as follows. Take the grid with numbers placed in it as in Figure 3.2. We place the Young diagram of shape λ in the grid and let S be the multiset of numbers in the grid that are within the Young diagram of shape λ. Now $h_\lambda(x) := \prod_{\alpha \in S}(x + \alpha)$. In Figure 3.2 we have placed the Young

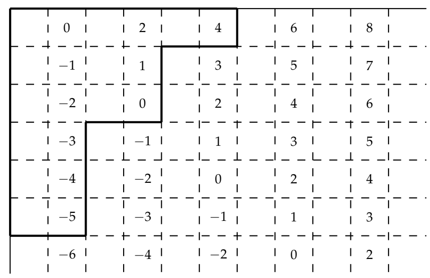


Figure 3.2: The infinite grid with the partition $\lambda = (6, 4, 4, 2, 2, 2)$.

diagram of $\lambda = (6, 4, 4, 2, 2, 2)$ in the grid. We see that $h_\lambda(x) = x(x+2)(x+4)(x-1)(x+1)(x-2)x(x-3)(x-4)(x-5)$ in this case.

We can now state the theorem by Hanlon and Wales [14]. We do not give a proof of the theorem, as it is worked out in full detail in [14].

Theorem 3.3. *[14, Theorem 3.1] Let $x \in \mathbb{C}$, $n \in \mathbb{N}$ and let $\lambda \vdash 2n$ be an even partition. Then S^λ consists of eigenvectors of $A_{2n}(x)$ with eigenvalue $h_\lambda(x)$.*

Let $n \in \mathbb{N}$ and let $\lambda \vdash 2n$ be an even partition. Let $M = \{\{2i-1, 2i\} \mid i \in [n]\}$. Let $R^0_\lambda \subseteq R_\lambda$ be the stabilizer of M in R_λ and let $C^0_\lambda \subseteq C_\lambda$ be the stabilizer of M in C_λ. Then, like we have seen in the proof of Proposition 3.1, the coefficient of M in $e_\lambda M$ is equal to $|R^0_\lambda||C^0_\lambda|$. So the eigenvalue of the vector $e_\lambda M \in S^\lambda$ is equal to the coefficient of M in $A_{2n}(x)(e_\lambda M)$ divided by $|R^0_\lambda||C^0_\lambda|$. This shows that

$$h_\lambda(x) = \frac{1}{|R^0_\lambda||C^0_\lambda|} \sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma) x^{c(\sigma\rho M \cup M)}. \tag{3.6}$$

With Theorem 3.3 we can prove the following lemma that will be useful later on.

Lemma 3.4. *If $x < 0$ and $\operatorname{rk}(A_{2n}(x)) \leq x^{2n}$ for all $n \in \mathbb{N}$, then x is an even integer.*

Proof. Let us first show that x has to be integral. Suppose to the contrary that x is not integral. Note that for any even $\lambda \vdash 2n$ with $n \in \mathbb{N}$, we have that the eigenvalue of S^λ is non-zero, by Theorem 3.3, as all zeroes of $h_\lambda(x)$ are integral. So $A_{2n}(x)$ has full rank and hence $\operatorname{rk}(A_{2n}(x)) = (2n-1)!!$. As there does not exist a constant $c \in \mathbb{R}$ such that $(2n-1)!! \leq c^{2n}$ for all $n \in \mathbb{N}$, this gives a contradiction. So $x \in \mathbb{Z}$.

Now suppose that $x = -2m+1$, for some $m \in \mathbb{N}$. We will show that

$$\sup_{n \in \mathbb{N}_{>0}} (\operatorname{rk}(A_{2n}(-2m+1)))^{1/2n} \geq 2m+1. \tag{3.7}$$

This proves the lemma, as it shows that if $x = -2m+1$, then the rank of $A_{2n}(x)$ is not bounded by x^{2n} for all $n \in \mathbb{N}$. Let $l \in \mathbb{N}_{>0}$ be even and let $\lambda_l = (2m+l, 2m, \ldots, 2m) \vdash 2ml + l$. It follows from Theorem 3.3 that the eigenvalue of S^{λ_l} is non-zero, see Figure 3.3. Indeed, if we imagine the shape of λ_l as a pan with a handle, then all the numbers occurring 'in the pan' are smaller than $2m-1$ and all the number occurring 'in the handle' are larger than $2m-1$. So $h_{\lambda_l}(-2m+1) \neq 0$.

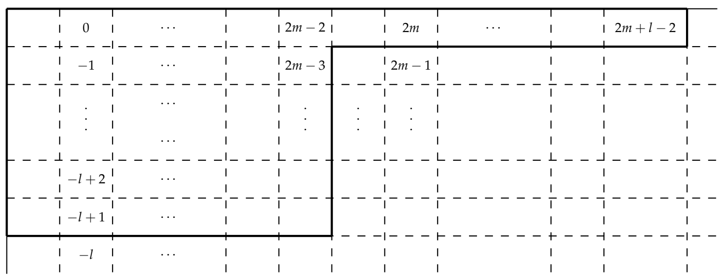


Figure 3.3: The infinite grid with the partition $(2m+l, 2m, \ldots, 2m)$ of $2ml + l$.

Hence $\mathrm{rk}\,(A_{2ml+l}(-2m+1)) \geq \dim(S^{\lambda_l})$. To compute $\dim(S^{\lambda_l})$ we use the hook length formula, see e.g. [34]. We have

$$\dim(S^{\lambda_l}) = \frac{(2ml+l)!}{\prod H_{\lambda_l}(i,j)}, \tag{3.8}$$

where the product in the denominator is over the cells (i,j) in the Young diagram of shape λ_l and $H_{\lambda_l}(i,j)$ is the length of the hook corresponding to cell (i,j). For $k \in [2m]$, the total contribution of the cells in column k to the denominator in (3.8) is

$$(2m+2l-k)\frac{(l+2m-k-1)!}{(2m-k)!} = l!\frac{(2m+2l-k)}{l}\binom{l+2m-k-1}{2m-k}.$$

We define

$$p_k(l) := (2m+2l-k)\binom{l+2m-k-1}{2m-k}.$$

Note that fixing m and k, $p_k(l)$ is a polynomial in l of degree $2m-k+1$. The total contribution to the denominator in (3.8) of the cells in the handle is $l!$. So we find that for $l \in \mathbb{N}$, the denominator in (3.8) is given by

$$\frac{(l!)^{2m+1}}{l^{2m}} \prod_{k=1}^{2m} p_k(l). \tag{3.9}$$

Now define $p(l) := \prod_{k=1}^{2m} p_k(l)$. So (3.8) is equal to

$$\dim(S^{\lambda_l}) = \frac{(2ml+l)!\,l^{2m}}{(l!)^{2m+1}p(l)}.$$

As $p(l)$ is a polynomial of degree $m(2m+1)$ in l, we find that

$$\begin{aligned}\lim_{l\to\infty} \dim(S^{\lambda_l})^{1/(2ml+l)} &= \lim_{l\to\infty} \left(\frac{(2ml+l)!l^{2m}}{(l!)^{2m+1}p(l)}\right)^{1/(2ml+l)} \\ &= \lim_{l\to\infty} \left(\frac{(2ml+l)!}{(l!)^{2m+1}}\right)^{1/(2ml+l)} = 2m+1,\end{aligned}$$

where we use Stirling's approximation in the last equality. This proves (3.7) and hence the lemma follows. $\square$

In Theorem 4.5 we will see that for $m \in \mathbb{N}$, we have $\operatorname{rk}(A_{2n}(-2m)) \leq (2m)^{2n}$ for each $n \in \mathbb{N}$. The following corollary follows immediately from the lemma.

Corollary 3.5. *Let $f : \mathcal{G} \to \mathbb{C}$ be a multiplicative graph parameter. If $f(\bigcirc) < 0$ and $\operatorname{rk}(M_{f,2n}) \leq f(\bigcirc)^{2n}$ for all $n \in \mathbb{N}$, then $f(\bigcirc)$ is an even integer.*

Chapter 4

Skew and mixed partition functions

In this chapter we first define skew partition functions. One can see ordinary partition functions as contractions of a tensor network with respect to a symmetric bilinear form. Skew partition functions can be seen as contractions of a tensor network with respect to a skew-symmetric bilinear form, hence the name. We give an orientation to the edges of the graph to help us contract the tensors in the right order with respect to the skew-symmetric bilinear form. To make sure that this is independent of the orientation of the edges we need to incorporate a sign into the definition. After giving the definition of a skew partition function we will state our main results on skew partition functions.

Using the definition for skew partition functions, we define mixed partition functions. We next state our main results on mixed partition functions. In this chapter we prove one of them, namely that mixed partition functions have exponentially bounded edge connection rank. In Section 4.4 we discuss two more elaborate examples of mixed partition functions. First, we will see that evaluations of the characteristic polynomial of a graph can be described by mixed partition functions. We will furthermore see that the evaluation of the characteristic polynomial of a graph at 0 cannot be described by an ordinary partition function, in doing so we answer a question of de la Harpe and Jones [15]. Next, we will show that integral evaluations of the circuit partition polynomial can be described by mixed partition functions. This chapter is based on [31] and [32].

4.1 Skew partition functions

An *Eulerian graph* is a (not necessarily connected) graph such that each vertex has even degree. Let $G = (V, E)$ be an Eulerian graph. A *local pairing at* *a vertex* v of G is a decomposition κ_v of $\delta(v)$ into ordered pairs, i.e., $\kappa_v = \{(a_1, a_2), \ldots, (a_{d(v)-1}, a_{d(v)})\}$. A *local pairing* κ *of* G is a collection $(\kappa_v)_{v \in V}$, where, for each $v \in V$, κ_v is a local pairing at v. If $(a_i, a_{i+1}) \in \kappa_v$ for some $v \in V$, then we say that the edges a_i and a_{i+1} are *paired* at v.

Recall that a circuit is a closed walk where vertices may occur multiple times, but edges may not. A local pairing κ of G decomposes E into circuits that, after choosing a starting point v_0 and a direction, are of the form $(v_0, a_1, \ldots, a_i, v_i, a_{i+1}, \ldots, v_n)$, where $v_0 = v_n$ and such that a_i and a_{i+1} are paired at v_i for each $i \in [n]$, where we consider the indices modulo n. We refer to this decomposition as the *κ-decomposition* and we refer to a circuit in this decomposition as a *κ-circuit*. We define $c(\kappa)$ to be the total number of free loops in E and κ-circuits in the κ-decomposition.

Let $C = (v_0, a_1, \ldots, a_i, v_i, a_{i+1}, \ldots, v_n)$ be a κ-circuit. Let $i \in [n]$. The edges a_i and a_{i+1} are paired at v_i and if $(a_{i+1}, a_i) \in \kappa_{v_i}$, then we say that (a_{i+1}, a_i) is an *odd pairing* in C. Let ω be an orientation of E. For $i \in [n]$, if a_i is oriented from v_i to v_{i-1} by ω, then we say that a_i is an *odd arc* in C. Let $o(C, \omega, \kappa)$ be the total number of odd arcs and odd pairings in C. One can think of this as walking along the circuit from v_0 to v_n and meanwhile keeping track of the number of arcs that are traversed in the opposite direction and the number of pairings that are traversed in the opposite direction.

Note that the parity of $o(C, \omega, \kappa)$ is independent of the starting vertex and the direction in which we traverse C as the total number of edges and pairings we encounter is even. So the parity of $\sum_C o(C, \omega, \kappa)$, where the sum runs over all κ-circuits C in the κ-decomposition, only depends on κ and ω and we denote it by $o(\omega, \kappa)$.

Recall that for $\ell \in \mathbb{N}$, $V_{2\ell} = \mathbb{C}^{2\ell}$ with standard basis $\{f_1, \ldots, f_{2\ell}\}$ and that $g_i \in V_{2\ell}$, for $i \in [2\ell]$, is defined by

$$g_i := \begin{cases} -f_{i+\ell} & \text{if } i \leq \ell, \\ f_{i-\ell} & \text{if } i > \ell. \end{cases} \tag{4.1}$$

For $\phi : E \to [2\ell]$, $v \in V$, $a \in E$ incident with v and ω an orientation of E, we define

$$b_{\phi,a,\omega,v} := \begin{cases} f_{\phi(a)} & \text{if } a \text{ is incoming at } v \text{ under } \omega, \\ g_{\phi(a)} & \text{if } a \text{ is outgoing at } v \text{ under } \omega. \end{cases} \tag{4.2}$$

Let $G = (V, E)$ be an Eulerian graph with a local pairing κ and an orientation ω of E. For $h = (h^v)_{v \in V}$ with $h^v \in (\bigwedge V_{2\ell})^*$ for each $v \in V$, we define

$$s_h(G, \omega, \kappa) := (-1)^{c(\kappa)+o(\omega,\kappa)} \sum_{\phi: E \to [2\ell]} \prod_{v \in V} h^v\Big(\bigwedge_{(a_1,a_2) \in \kappa_v} b_{\phi,a_1,\omega,v} \wedge b_{\phi,a_2,\omega,v} \Big). \tag{4.3}$$

By skew-symmetry this is independent of the order in which we take the wedge over the elements of κ_v. We see that $s_h(G, \omega, \kappa) = 0$ if G contains a vertex of degree larger than 2ℓ, as $\bigwedge^n V_{2\ell} = 0$ if $n > 2\ell$.

Proposition 4.1. *Let $G = (V, E)$ be an Eulerian graph and let $h = (h^v)_{v \in V}$ with $h^v \in (\bigwedge V_{2\ell})^*$ for each $v \in V$. Then $s_h(G, \omega, \kappa)$ is independent of the choice of an orientation ω of E and a local pairing κ of G.*

Before proving this proposition, we first give the definition of a skew partition function. For $G = (V, E)$ an Eulerian graph and $h = (h^v)_{v \in V}$ with $h^v \in (\bigwedge V_{2\ell})^*$, we define $s_h(G) := s_h(G, \omega, \kappa)$ for some local pairing κ of G and orientation ω of E. This is well-defined by Proposition 4.1.

Definition 4.1. For any element $h \in (\bigwedge V_{2\ell})^*$, the *partition function* $p_h : \mathcal{G} \to \mathbb{C}$ of h is the graph parameter, defined, for a graph G, by

$$p_h(G) := \begin{cases} s_x(G) & \text{if } G \text{ is Eulerian,} \\ 0 & \text{otherwise,} \end{cases} \tag{4.4}$$

where $x = (h^v)_{v \in V}$ with $h^v = h$ for all $v \in V$. If f is the partition function of an element $h \in (\bigwedge V_{2\ell})^*$ for some $\ell \in \mathbb{N}$, then we sometimes refer to f as a *skew partition function*. An element $h \in (\bigwedge V_{2\ell})^*$ is also known as a *skew edge coloring model*.

It follows directly from the definition that skew partition functions are multiplicative. Recall that if p_h is the partition function of $h \in (SV_k)^*$, then $p_h(\bigcirc) = k$. If p_h is the partition function of $h \in (\bigwedge V_{2\ell})^*$, then $p_h(\bigcirc) = -2\ell$. Note that if $\ell = 0$, then there is only one element h in $(\bigwedge V_{2\ell})^*$ and then p_h is the function that evaluates to 1 on $\emptyset$ and that evaluates to 0 on all other graphs.

Let $G = (V, E)$ be an Eulerian graph and let ω be an Eulerian orientation of E. A local pairing κ of G is called *compatible* with ω if for each vertex v and for each $(a_1, a_2) \in \kappa_v$ the arc a_1 is incoming at v under ω and the arc a_2 is outgoing at v under ω. In this case (4.3) reduces to

$$s_h(G, \omega, \kappa) = (-1)^{c(\kappa)} \sum_{\phi : E \to [2\ell]} \prod_{v \in V} h^v\Big(\bigwedge_{(a_1, a_2) \in \kappa_v} f_{\phi(a_1)} \wedge g_{\phi(a_2)} \Big). \tag{4.5}$$

For an Eulerian graph $G = (V, E)$ with a local pairing κ and an orientation ω of E, we define $\epsilon(G, \omega, \kappa) := (-1)^{c(\kappa) + o(\omega, \kappa)}$. We first prove a lemma before proving Proposition 4.1.

Lemma 4.2. *Let $G = (V, E)$ be an Eulerian graph with a local pairing κ and an orientation ω of E. Let $h = (h^v)_{v \in V}$ with $h^v \in (\bigwedge V_{2\ell})^*$ for each $v \in V$. If ω' is obtained from ω by inverting the orientation of an edge, then $s_h(G, \omega', \kappa) = s_h(G, \omega, \kappa)$. Similarly, if κ' is obtained from κ by inverting the order of a pairing at a vertex, then $s_h(G, \omega, \kappa') = s_h(G, \omega, \kappa)$.*

Proof. Let $G = (V, E)$ be an Eulerian graph with a local pairing κ and an orientation ω of E. Define

$$s'_h(G, \omega, \kappa) := \sum_{\phi : E \to [2\ell]} \prod_{v \in V} h^v\Big(\bigwedge_{(a_1, a_2) \in \kappa_v} b_{\phi, a_1, \omega, v} \wedge b_{\phi, a_2, \omega, v} \Big). \tag{4.6}$$

So $s_h(G,\omega,\kappa) = \epsilon(G,\omega,\kappa)s'_h(G,\omega,\kappa)$.

Now to prove the first assertion, suppose $a = \{v_1, v_2\} \in E$ is oriented from v_1 to v_2 under ω and let ω' be obtained from ω by inverting the orientation of a and leaving the orientation of the other edges in E unchanged. The number of odd arcs changes by one in doing so, hence $\epsilon(G,\omega',\kappa) = -\epsilon(G,\omega,\kappa)$.

Let $\phi : E \to [2\ell]$ and let $\phi' : E \to [2\ell]$ be defined by $\phi'(a) = \phi(a) + \ell \mod 2\ell$ and $\phi' = \phi$ for all other edges. For ϕ we see that $b_{\phi,a,\omega,v_1} = g_{\phi(a)}$ and $b_{\phi,a,\omega,v_2} = f_{\phi(a)}$. For ϕ' we see that $b_{\phi',a,\omega',v_1} = f_{\phi'(a)} = f_{\phi(a)+\ell}$ and $b_{\phi',a,\omega',v_2} = g_{\phi'(a)} = g_{\phi(a)+\ell}$. So $s'_h(G,\omega',\kappa) = -s'_h(G,\omega,\kappa)$ by (4.1). This shows that indeed $s_h(G,\omega',\kappa) = s_h(G,\omega,\kappa)$.

To prove the second assertion, suppose that κ' is obtained from κ by changing the order of a pairing at a vertex v. The number of odd pairings changes by one in doing so, hence $\epsilon(G,\omega,\kappa') = -\epsilon(G,\omega,\kappa)$. By skew-symmetry, we see that $s'_h(G,\omega,\kappa') = -s'_h(G,\omega,\kappa)$. This shows that indeed $s_h(G,\omega,\kappa') = s_h(G,\omega,\kappa)$. □

Proof of Proposition 4.1. Let $G = (V, E)$ be an Eulerian graph with a local pairing κ and an orientation ω of E. By the previous lemma, we may assume that ω is an Eulerian orientation and that κ is compatible with ω. So to prove the proposition, it suffices to show that $s_h(G,\omega,\kappa) = s_h(G,\omega',\kappa')$, where κ' is a local pairing of G compatible with some Eulerian orientation ω' of E.

Suppose there exists a vertex v such that (a_1, a_2) and (a_3, a_4) are in κ_v. Let κ' be obtained from κ by replacing (a_1, a_2) and (a_3, a_4) in κ_v by (a_1, a_4) and (a_3, a_2). Note that κ' is still compatible with ω. The parity of the number of κ-circuits is different from the parity of the number of κ'-circuits. So $\epsilon(G,\omega,\kappa) = -\epsilon(G,\omega,\kappa')$. As the evaluation of the tensor at v also changes sign by skew symmetry, this cancels out. We can repeatedly apply these swaps at each v to κ without changing the value of s_h. This shows that we can go from any κ compatible with ω to any other κ' compatible with ω without changing the value of s_h.

Now it remains to show that if ω and ω' are Eulerian orientations of E, then $s_h(G,\omega,\kappa) = s_h(G,\omega',\kappa')$, where κ is any local pairing of G compatible with ω and κ' is any local pairing of G compatible with ω'.

So, to finish the proof, let ω and ω' be Eulerian orientations of G. The symmetric difference of ω and ω', i.e., the set of edges where they do not give the same orientation gives a subgraph of G such that ω restricts to an Eulerian orientation of this subgraph. If $\omega \neq \omega'$, let C be a directed circuit in this graph. By the previous part, the value of $s_h(G,\omega,\kappa)$ is independent of the choice of local pairing κ compatible with ω. So we can choose κ such that C is a κ-circuit. Let ω'' be obtained from ω by inverting the orientation of the edges of C and let κ'' be obtained from κ by flipping the order of each two paired edges of C incident with a vertex v, for all vertices v of C. Then $s_h(G,\omega,\kappa) = s_h(G,\omega'',\kappa'')$. So repeating this until there are no circuits left in the symmetric difference finishes the proof. □

4.1.1 Statement of results on skew partition functions

For skew partition functions we can give a characterization that is close in spirit to Theorem 2.3 by Schrijver.

Theorem 4.3. *A graph parameter $f : \mathcal{G} \to \mathbb{C}$ is a skew partition function if and only if $f(\emptyset) = 1$, $f(\bigcirc) \leq 0$ and*

$$\mathrm{rk}\,(M_{f,2t}) \leq f(\bigcirc)^{2t} \tag{4.7}$$

for each $t \in \mathbb{N}$.

We can also give another characterization of skew partition functions that is close in spirit to the characterization given by Draisma, Gijswijt, Lovász, Regts and Schrijver [10, Theorem 1]. Let $G = (V, E)$ be a graph. Recall that for $n \in \mathbb{N}$ and $u : [2n] \to V$ any map, we defined

$$G_u := (V, E \cup \{\{u(2i-1), u(2i)\} \mid i \in [n]\}).$$

We can now give the characterization.

Theorem 4.4. *Let $\ell \in \mathbb{N}$. A graph parameter $f : \mathcal{G} \to \mathbb{C}$ is the partition function of an element $h \in (\bigwedge V_{2\ell})^*$ if and only if $f(\emptyset) = 1$, $f(\bigcirc) = -2\ell$, f is multiplicative, $f(G) = 0$ if G is not Eulerian and for each graph $G = (V, E)$ and for each map $u : [2\ell + 2] \to V$, we have*

$$\sum_{\rho \in S_{2\ell+2}} f(G_{u \circ \rho}) = 0. \tag{4.8}$$

This formulation is different from our description of this theorem in Chapter 2. In the proof it will become clear how the two formulations are related. To prove Theorem 4.3 we will use Theorem 4.4.

4.2 Mixed partition functions

For a graph $G = (V, E)$ and $F \subseteq E$, the subgraph (V, F) of G is denoted by $G(F)$. If $G(F)$ is Eulerian, then we say that F is Eulerian. For $v \in V$, let $\delta_{E \setminus F}(v)$ be the set of edges (with multiplicities) incident with v that are not in F.

Let $k, \ell \in \mathbb{N}$. An element $h \in (SV_k \otimes \bigwedge V_{2\ell})^*$ is called a *$(k, 2\ell)$-color edge coloring model*. Let $G = (V, E)$ be a graph with an Eulerian subset $F \subseteq E$. Let ω be an orientation of F and let κ be a local pairing of $G(F)$. For $h \in (SV_k \otimes \bigwedge V_{2\ell})^*$, we define $s_h(G, F, \omega, \kappa)$ to be

$$(-1)^{c(\kappa)+o(\omega,\kappa)} \sum_{\substack{\phi: F \to [2\ell] \\ \psi: E \setminus F \to [k]}} \prod_{v \in V} h\Big(\bigodot_{a \in \delta_{E \setminus F}(v)} e_{\psi(a)} \otimes \bigwedge_{(a_1, a_2) \in \kappa_v} b_{\phi, a_1, \omega, v} \wedge b_{\phi, a_2, \omega, v} \Big). \tag{4.9}$$

Fix $\psi : E \setminus F \to [k]$. For $v \in V$, let $h_\psi^v \in (\bigwedge V_{2\ell})^*$ be defined, for $i_1, \ldots, i_n \in [2\ell]$, by

$$h_\psi^v(f_{i_1} \wedge \cdots \wedge f_{i_n}) = h(\bigodot_{a \in \delta_{E\setminus F}(v)} e_{\psi(a)} \otimes f_{i_1} \wedge \cdots \wedge f_{i_n}).$$

Now let $h_\psi = (h_\psi^v)_{v \in V}$. If follows from (4.3) that

$$s_h(G, F, \omega, \kappa) = \sum_{\psi: E\setminus F \to [k]} s_{h_\psi}(G(F), \omega, \kappa).$$

So by Proposition 4.1, $s_h(G, F, \omega, \kappa)$ is independent of the choice of ω and κ. So we can define $s_h(G, F) := s_h(G, F, \omega, \kappa)$ for some choice of orientation ω of F and local pairing κ of $G(F)$. Now the partition function p_h of h is defined, for the graph $G = (V, E)$, by

$$p_h(G) := \sum_{\substack{F \subseteq E \\ F \text{ Eulerian}}} s_h(G, F). \tag{4.10}$$

We sometimes refer to the partition function just defined as a *mixed partition function* to distinguish it from ordinary partition functions and skew partition functions. It follows from the definition that a mixed partition function is multiplicative. Note that if h is a $(k, 2\ell)$-color edge coloring model, then $p_h(\bigcirc) = k - 2\ell$.

Example 4.2. Let h be a $(k, 0)$-color edge coloring model. For a graph $G = (V, E)$ and $F \subseteq E$, we have that if $F \neq \emptyset$, then $s_h(G, F) = 0$. So we find that

$$p_h(G) = s_h(G, \emptyset) = \sum_{\psi: E \to [k]} \prod_{v \in V} h(\bigodot_{a \in \delta(v)} e_{\psi(a)}).$$

So we see that p_h is an ordinary partition function as in (2.7). We similarly see that if h is a $(0, 2\ell)$-color edge coloring model, then p_h is a skew partition function as in (4.4).

Note that for $k, \ell \in \mathbb{N}$, the basis for SV_k and the basis for $\bigwedge V_{2\ell}$ that we defined in Chapter 2 give a basis of $SV_k \otimes \bigwedge V_{2\ell}$, i.e., a basis of $SV_k \otimes \bigwedge V_{2\ell}$ is formed by the

$$\bigodot_{i \in S} e_i \otimes \bigwedge_{i \in T} f_i, \tag{4.11}$$

where S is a multisubset of $[k]$ and $T = \{i_1, \ldots, i_n\}$ with $1 \leq i_1 < \cdots < i_n \leq 2\ell$ (here the wedge over T is taken in ascending order).

Example 4.3. Let h be the $(1, 2)$-color edge coloring model defined on basis elements by $h(f_1 \wedge f_2) = -1$ and $h(e_1 \odot e_1) = 1$ and let h evaluate to zero on all other basis elements of $SV_1 \otimes \bigwedge V_2$. We claim that for any graph G,

$$p_h(G) = \begin{cases} (-1)^{c(G)} & \text{if } G \text{ is two-regular,} \\ 0 & \text{otherwise,} \end{cases} \tag{4.12}$$

where $c(G)$ is the number of connected components of G.

Let $G = (V, E)$ be a graph. Note that as h evaluates to 0 on any tensor that is not of degree 2, we have $p_h(G) = 0$ if G has a vertex that is not of degree 2. So let us assume that each vertex of G has degree 2. Because of the multiplicativity of p_h we may assume that G is connected. We have that $p_h(G) = s_h(G, E) + s_h(G, \emptyset)$ as $\emptyset$ and E are the only Eulerian subsets of E.

Let us first compute $s_h(G, \emptyset)$. There is only one coloring $\psi : E \to [1]$ and for this coloring we see $h(e_1 \otimes e_1) = 1$ at each vertex. So $s_h(G, \emptyset) = 1$.

Let us next compute $s_h(G, E)$. Let ω be an Eulerian orientation of E and let κ be a local pairing of G compatible with ω. The only colorings $\phi : E \to [2]$ that give a non-zero contribution to $s_h(G, F, \omega, \kappa)$ are those that color all edges 1 or that color all edges 2. If $\phi : E \to [2]$ assigns 1 to each edge, then at any vertex we see $h(f_1 \wedge g_1) = -h(f_1 \wedge f_2) = 1$ by (4.1). If $\phi : E \to [2]$ assigns 2 to each edge, then at any vertex we see $h(f_2 \wedge g_2) = h(f_2 \wedge f_1) = 1$, again by (4.1) and skew-symmetry. We see that $\epsilon(G, \omega, \kappa) = -1$ as there are no odd arcs or odd pairings and exactly one κ-circuit. This shows that $s_h(G, E) = -2$.

So we find that $p_h(G) = s_h(G, E) + s_h(G, \emptyset) = -1$. This shows (4.12).

Example 4.4. If $h_0 \in (SV_k)^*$ and $h_1 \in (\bigwedge V_{2\ell})^*$, then let $h = h_0 \otimes h_1 \in (SV_k \otimes \bigwedge V_{2\ell})^*$. For a graph $G = (V, E)$ and $F \subseteq E$ Eulerian, it follows directly from (4.9) that $s_h(G, F) = p_{h_0}(G(E \setminus F)) p_{h_1}(G(F))$. So we find that

$$p_h(G) = \sum_{\substack{F \subseteq E \\ F \text{ Eulerian}}} p_{h_0}(G(E \setminus F)) p_{h_1}(G(F)). \tag{4.13}$$

4.2.1 Statement of results on mixed partition functions

In the next section we will prove that mixed partition functions indeed have exponentially bounded edge connection rank. We state the theorem here.

Theorem 4.5. *Let $k, \ell \in \mathbb{N}$. If $f : \mathcal{G} \to \mathbb{C}$ is the partition function of an element $h \in (SV_k \otimes \bigwedge V_{2\ell})^*$, then*

$$\operatorname{rk}(M_{f,t}) \leq (k + 2\ell)^t$$

for each $t \in \mathbb{N}$.

If f is a skew partition function, then this implies that $\operatorname{rk}(M_{f,2t}) \leq f(\bigcirc)^{2t}$ for each $t \in \mathbb{N}$. In Chapter 5 we will prove the following theorem on mixed partition functions. Recall the definition of $\mathcal{J}_{k,2\ell}$ for $k, \ell \in \mathbb{N}$ given in (2.13).

Theorem 4.6. *Let $k, \ell \in \mathbb{N}$. If $f : \mathcal{G} \to \mathbb{C}$ is the partition function of an element $h \in (SV_k \otimes \bigwedge V_{2\ell})^*$, then $f(\mathcal{J}_{k,2\ell}) = 0$.*

In Section 5.3 we will formulate a conjecture saying that the reverse statement of this theorem also holds (where we add the assumption that $f(\emptyset) = 1$ and that f is multiplicative) and we will see how this is related to the invariant theory of the orthosymplectic supergroup.

4.3 The rank growth of mixed partition functions

In this section we prove Theorem 4.5. We first show a lemma on matchings that will be useful later on. Let $n \in \mathbb{N}$. A *simple arc on* $[2n]$ is an ordered pair (i, j) with $i, j \in [2n]$ and $i \neq j$. A *directed perfect matching on* $[2n]$ is a set M consisting of simple arcs on $[2n]$ such that each vertex in the digraph $([2n], M)$ is incident with exactly one arc. Let $\overrightarrow{\mathcal{M}}_{2n}$ denote the set of directed perfect matchings on $[2n]$. For $\pi \in S_{2n}$ and $M \in \overrightarrow{\mathcal{M}}_{2n}$, we define $\pi M = \{(\pi(i), \pi(j)) \mid (i, j) \in M\}$. This defines an action of S_{2n} on $\overrightarrow{\mathcal{M}}_{2n}$.

For $M, N \in \overrightarrow{\mathcal{M}}_{2n}$, we denote by $o(M \cup N)$ the parity of the number of arcs in $M \cup N$ that need to be flipped to make $([2n], M \cup N)$ into an Eulerian digraph. Since each cycle in $([2n], M \cup N)$ has even length this is well-defined. As before, we define $c(M \cup N)$ to be the number of connected components of the graph underlying $([2n], M \cup N)$. We will refer to these connected components as the connected components of $([2n], M \cup N)$.

Lemma 4.7. *Let $n \in \mathbb{N}$ and let $M, N \in \overrightarrow{\mathcal{M}}_{2n}$. Then the sign of any permutation in S_{2n} that sends M to N is equal to $(-1)^{c(M\cup N)+o(M\cup N)}$.*

Proof. Note that all permutations that send M to N have the same sign, as each permutation in S_{2n} that stabilizes M has trivial sign. We may assume that $([2n], M \cup N)$ consists of a single connected component. Let $\sigma_1, \sigma_2 \in S_{2n}$ be permutations that flip edges of M and N respectively such that

$$([2n], \sigma_1 M \cup \sigma_2 N)$$

is an Eulerian digraph. If the vertices of the cycle are given by $v_1, v_2, \ldots, v_{2n}$ in cyclic order, then the permutation $\tau = (v_1 v_2 \ldots v_{2n})$ has the property that $\tau\sigma_1 M = \sigma_2 N$. So the permutation $\sigma_2^{-1}\tau\sigma_1$ sends M to N. As $2n$ is even, the sign of τ is -1. Per construction we have $\mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2) = (-1)^{o(M\cup N)}$. This proves the lemma. □

Let $k, \ell \in \mathbb{N}$. Recall that V_k is equipped with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ and that $V_{2\ell}$ is equipped with a non-degenerate skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$. We define $V_{k,2\ell} := V_k \oplus V_{2\ell}$. We write an element w of $V_{k,2\ell}$ as $w_0 + w_1$, where $w_0 \in V_k$ and $w_1 \in V_{2\ell}$. We equip $V_{k,2\ell}$ with a non-degenerate bilinear form $[\cdot, \cdot]$ defined, for $x, y \in V_{k,2\ell}$, by

$$[x, y] := (x_0, y_0) + \langle x_1, y_1 \rangle.$$

For each $m \in \mathbb{N}$, the bilinear form $[\cdot, \cdot]$ extends to a bilinear form on $V_{k,2\ell}^{\otimes m}$ and we denote this bilinear form also with $[\cdot, \cdot]$. We note that this form is often called a *super symmetric bilinear* form, cf. [8].

A *directed trail* T in a directed graph $D = (V, A)$ is a sequence

$$(v_0, a_1, \ldots, a_i, v_i, a_{i+1}, \ldots, a_n, v_n)$$

such that $v_i \in V$ for $i \in \{0, \dots, n\}$, $a_i \in A$ for $i \in [n]$, $a_i = (v_{i-1}, v_i)$ for $i \in [n]$ and such that each arc occurs at most once in the sequence. We say that T is a trail *from v_0 to v_n*.

Proof of Theorem 4.5. Our goal is to show that for each $t \in \mathbb{N}$, we can write $M_{f,t}$ as a Gram matrix of vectors in $V_{k,2\ell}^{\otimes t}$ with respect to the bilinear form $[\cdot, \cdot]$. This implies Theorem 4.5.

Let $t \in \mathbb{N}$ and let $F = (V, E)$ be a t-fragment. Recall that a t-fragment is a graph with t vertices of degree 1 labeled $1, \dots, t$. The set of unlabeled vertices of F is denoted by $V'(F)$. A subset $H \subseteq E$ is called *Eulerian* if the degree of each unlabeled vertex in $F(H)$ is even. Let $H \subseteq E$ be Eulerian. Let $S(H)$ be the set of labeled vertices incident with an edge in H. If H is chosen, we refer to $S(H)$ as S. Note that $|S|$ is even because H is Eulerian. We identify the labeled vertices with $[t]$ according to the labeling. Through this identification we view S as a subset of $[t]$.

We now extend some of the definitions we gave for graphs to fragments. An *Eulerian orientation* ω of H is an orientation of the edges of H such that in $F(H)$, at each unlabeled vertex the number of incoming arcs is equal to the number of outgoing arcs. A *local pairing* κ of $F(H)$ is an assignment κ to each $v \in V'(F)$ of a decomposition κ_v of the edges in H incident with v into ordered pairs. The local pairing κ is called *compatible* with a Eulerian orientation ω if for each $v \in V'(F)$ and for each $(a_1, a_2) \in \kappa_v$ the arc a_1 is incoming under ω and the arc a_2 is outgoing under ω.

Now let κ be a local pairing of $F(H)$ compatible with an Eulerian orientation ω of H. Note that κ partitions the edge set of H into circuits and directed trails that begin and end in labeled vertices. We call this decomposition the *κ-decomposition* of H. Let $\hat{c}(\kappa)$ be the number of circuits in the κ-decomposition. Define $M(\omega, \kappa)$ to be the directed perfect matching on $S \subseteq [t]$ such that (i, j) is an arc of $M(\omega, \kappa)$ if there is a directed trail in the κ-decomposition from i to j. Write $S = \{i_1, \dots, i_{|S|}\}$ with $i_1 < \cdots < i_{|S|}$. Let $\operatorname{sgn}(M(\omega, \kappa))$ be the sign of a permutation that sends $M(\omega, \kappa)$ to the directed perfect matching with arcs $(i_1, i_2), \dots, (i_{|S|-1}, i_{|S|})$. This is well-defined by Lemma 4.7.

Let $\chi = (\chi_0, \chi_1)$ with $\chi_0 : [t] \setminus S \to [k]$ and $\chi_1 : S \to [2\ell]$. Such a pair $\chi = (\chi_0, \chi_1)$ is called *consistent* with S. We say that a coloring $\psi : E \setminus H \to [k]$ *extends* χ_0 if, for each $i \in [t] \setminus S$, we have $\chi_0(i) = \psi(a)$, where $a \in E \setminus H$ is the edge incident with i. We denote this by $\psi \sim \chi_0$. Similarly, we say that $\phi : H \to [2\ell]$ extends χ_1 if, for each $i \in S$, we have $\chi_1(i) = \phi(a)$, where $a \in H$ is the edge incident with i. Again, we denote this by $\phi \sim \chi_1$.

For $i \in [t] \setminus S$, let $c_{\chi,\omega,i} = e_{\chi_0(i)}$, and for $i \in S$, let $c_{\chi,\omega,i} = f_{\chi_1(i)}$ if the edge incident with i is incoming at i under ω and let $c_{\chi,\omega,i} = g_{\chi_1(i)}$ if the edge incident with i is outgoing at i under ω. We define the tensor $t'_{h,\chi}(F, H, \omega, \kappa)$

in $V_{k,2\ell}^{\otimes t}$ by

$$t'_{h,\chi}(F,H,\omega,\kappa) :=$$
$$(-1)^{\hat{c}(\kappa)} \sum_{\substack{\psi\sim\chi_0\\ \phi\sim\chi_1}} \prod_{v\in V'(F)} h\Big(\bigodot_{a\in\delta_{E\setminus H}(v)} e_{\psi(a)} \otimes \bigwedge_{(a_1,a_2)\in\kappa_v} f_{\phi(a_1)}\wedge g_{\phi(a_2)}\Big) \bigotimes_{i\in[t]} c_{\chi,\omega,i},$$

where the sum runs over all $\psi : E\setminus H \to [k]$ with $\psi\sim\chi_0$ and all $\phi : H\to[2\ell]$ with $\phi\sim\chi_1$. We define

$$t'_h(F,H,\omega,\kappa) := \sum_{\substack{\chi \text{ consistent}\\ \text{with } S}} t'_{h,\chi}(F,H,\omega,\kappa),$$

and finally we define

$$t_h(F,H,\omega,\kappa) := (-1)^{|S|/4}\mathrm{sgn}(M(\omega,\kappa))t'_h(F,H,\omega,\kappa).$$

We first make an important observation. Let ω' be obtained from ω by inverting the arcs in a directed trail P in the κ-decomposition and let κ' be obtained from κ by inverting all the pairings in the directed trail P (hence κ' is compatible with ω'). Note that $\mathrm{sgn}(M(\omega,\kappa)) = -\mathrm{sgn}(M(\omega',\kappa'))$, as $M(\omega',\kappa')$ is obtained from $M(\omega,\kappa)$ by inverting the direction of one arc. The total number of pairings and arcs in the directed trail P is odd. So similar to what we have seen in the proof of Lemma 4.2, we find that $t'_h(F,H,\omega,\kappa) = -t'_h(F,H,\omega',\kappa')$. This shows that

$$t_h(F,H,\omega,\kappa) = t_h(F,H,\omega',\kappa'). \tag{4.14}$$

Now let $F_1 = (V_1,E_1)$ and $F_2 = (V_2,E_2)$ be two t-fragments with Eulerian subsets $H_1\subseteq E_1$ and $H_2\subseteq E_2$ such that $S(H_1) = S(H_2) = S$. Let $G = (V,E) = F_1 * F_2$. Note that H_1 and H_2 induce an Eulerian subset of E. We denote this set by $H_1 * H_2$. For $i = 1,2$, let ω_i be an Eulerian orientation of H_i with a compatible local pairing κ_i of $F_i(H_i)$. We next show that

$$[t_h(F_1,H_1,\omega_1,\kappa_1), t_h(F_2,H_2,\omega_2,\kappa_2)] = s_h(G,H_1*H_2). \tag{4.15}$$

By (4.14) we may assume that $\omega_1,\kappa_1,\omega_2$ and κ_2 are chosen in such a way that $(S, M(\omega_1,\kappa_1)\cup M(\omega_2,\kappa_2))$ is an Eulerian digraph. By Lemma 4.7 we see that

$$\mathrm{sgn}(M(\omega_1,\kappa_1))\mathrm{sgn}(M(\omega_2,\kappa_2)) = (-1)^{c(M(\omega_1,\kappa_1)\cup M(\omega_2,\kappa_2))},$$

as $o(M(\omega_1,\kappa_1)\cup M(\omega_2,\kappa_2)) = 0$. Furthermore, ω_1 and ω_2 induce an Eulerian orientation ω of H_1*H_2 and the local pairing κ of $G(H_1*H_2)$ induced by κ_1 and κ_2 is compatible with ω. So we find that

$$\mathrm{sgn}(M(\omega_1,\kappa_1))\mathrm{sgn}(M(\omega_2,\kappa_2))(-1)^{\hat{c}(\kappa_1)}(-1)^{\hat{c}(\kappa_2)} = (-1)^{c(\kappa)}. \tag{4.16}$$

Now let $\chi = (\chi_0,\chi_1)$ and $\chi' = (\chi'_0,\chi'_1)$ both be consistent with S. We consider

$$[t'_{h,\chi}(F_1,H_1,\omega_1,\kappa_1), t'_{h,\chi'}(F_2,H_2,\omega_2,\kappa_2)]. \tag{4.17}$$

Note that this is equal to 0 if χ_0 and χ_0' do not agree. Furthermore, as the orientations of ω_1 and ω_2 are opposite at a labeled vertex in S, we see that χ_1 and χ_1' also have to agree for (4.17) to be non-zero. So let us assume that $\chi = \chi'$. Note that as the orientation ω is Eulerian, at half of the vertices in S the arc of H_1 is incoming and the arc of H_2 is outgoing. So at such a vertex i the bilinear form becomes $\langle f_{\chi_1(i)}, g_{\chi_1(i)} \rangle = -1$. At the other half of the vertices in S the arc of H_2 is incoming and the arc of H_2 is outgoing. So at such a vertex i the bilinear form becomes $\langle g_{\chi_1(i)}, f_{\chi_1(i)} \rangle = 1$. These contributions cancel with $(-1)^{|S(H_1)|/4}(-1)^{|S(H_2)|/4}$. Together with (4.16) this shows (4.15).

Now, for $i = 1, 2$, let $H_i \subseteq E_i$ and let ω_i be an Eulerian orientation of H_i with a compatible local pairing κ_i of $F_i(H_i)$. Suppose that $S(H_1) \neq S(H_2)$. Then it follows that

$$[t_h(F_1, H_1, \omega_1, \kappa_1), t_h(F_2, H_2, \omega_2, \kappa_2)] = 0, \tag{4.18}$$

because at i in the symmetric difference of $S(H_1)$ and $S(H_2)$ there occurs an element of V_k at one side of the bilinear form and an element of $V_{2\ell}$ at the other side.

Note that as H_1 and H_2 run over all Eulerian subsets of F_1 and F_2, we have that $H_1 * H_2$ runs over all Eulerian subsets of G. So it follows from (4.15) and (4.18) that

$$\Bigg[\sum_{\substack{H_1 \subseteq E_1 \\ H_1 \text{ Eulerian}}} t_h(F_1, H_1, \omega_1, \kappa_1), \sum_{\substack{H_2 \subseteq E_2 \\ H_2 \text{ Eulerian}}} t_h(F_2, H_2, \omega_2, \kappa_2) \Bigg] = \tag{4.19}$$

$$\sum_{\substack{H \subseteq E \\ H \text{ Eulerian}}} s_h(G, H, \omega, \kappa) = f(G), \tag{4.20}$$

where, for $i = 1, 2$, κ_i is a local pairing of $F_i(H_i)$ compatible with an Eulerian orientation ω_i of H_i. This shows that $M_{f,t}$ indeed is the Gram matrix of a set of vectors in $V_{k,2\ell}^{\otimes t}$ with respect to the bilinear form $[\cdot, \cdot]$. So the rank of $M_{f,t}$ is bounded by $(k + 2\ell)^t$. This proves Theorem 4.5. $\square$

4.4 Examples

In this section we give two more examples of mixed partition functions related to other work. We first show that evaluations of the characteristic polynomial of a graph can be described as partition functions of $(2, 2)$-color edge coloring models and we will see how this is related to a question by de la Harpe and Jones [15]. We also show how integral evaluations of the circuit partition polynomial of a graph can be described by mixed partition functions.

4.4.1 The characteristic polynomial

In this subsection we assume that our graphs do not have $\bigcirc$ as a connected component, as they are irrelevant for the characteristic polynomial. The *adjacency matrix* A of a graph $G = (V, E)$ is the $V \times V$ matrix such that for $i, j \in V$ with $i \neq j$, $A(i, j)$ is the multiplicity of the edge $\{i, j\}$ in E and such that for $i \in V$, $A(i, i)$ is the twice the number of loops at the vertex i. The *characteristic polynomial* $p(G)$ of G is defined as $p(G; t) := \det(tI - A)$. De la Harpe and Jones [15, Problem 1] asked about the existence of a spin model $B(t) \in \mathbb{C}[t]$ such that $p_{B(t)}(G) = p(G; t)$ for each graph G. In the following proposition we shall show that the answer to this question is negative. In fact we show something stronger.

Proposition 4.8. *There does not exist an edge coloring model h such that* $p_h(G) = p(G; 0)$ *for all graphs G.*

This proposition is indeed stronger than we need, since, by a result of Szegedy [39], the partition function of any spin model is equal to the partition function of an ordinary edge coloring model and hence Proposition 4.8 rules out the existence of a spin model of which the partition function equals the characteristic polynomial evaluated at 0. However, we shall show that for each $t \in \mathbb{C}$, there exists a $(2, 2)$-color edge coloring model $h(t)$ such that $p_{h(t)}(G) = p(G; t)$ for all graphs G, cf. Proposition 4.9 below. This may serve as an alternative answer to the question of de la Harpe and Jones.

We now turn to a proof of Proposition 4.8.

Proof of Proposition 4.8. Let us abuse notation and write $\det(G)$ for the determinant of the adjacency matrix of G. Note that for a graph with an even number of vertices we have $p(G; 0) = \det(G)$. We will make use of the characterization of partition functions of edge coloring models as given in [10]. Fix k and consider the graph G consisting of $k + 1$ copies of the 6-cycle C_6. Direct one edge in each cycle and label the endpoints of these arcs 1 up to $k + 1$. For a permutation $\pi \in S_{k+1}$, denote by G_π the graph obtained from G by letting π permute the endpoints of the directed edges. Note that if the permutation π can be written as the product of disjoint cycles $\pi_1, \ldots, \pi_t$, then G_π is the graph consisting of t cycles, of length $6|\pi_1|, \ldots, 6|\pi_t|$ respectively. Here $|\pi_i|$ denotes the length of the cycle π_i; we include cycles of length 1. If $p(G; 0)$ is the partition function of a k-color edge coloring model, then, by [10, Theorem 1], it must satisfy

$$\sum_{\pi \in S_{k+1}} \operatorname{sgn}(\pi) p(G_\pi; 0) = 0. \tag{4.21}$$

It follows from, for example, [6, Section 1.4.3], that $\det(C_k) = 0$ if $k = 0 \mod 4$ and $\det(C_k) < 0$ if $k = 2 \mod 4$. This implies that for $\det(G_\pi)$ to be non-zero none of the cycles $\pi_1, \ldots, \pi_t$ may be of even length. However, if all cycles in the cycle decomposition of π are of odd length, then the parity of the number of these cycles is equal to the parity of $k+1$. So in this case, $\det(G_\pi)$ is strictly positive if this parity is even and strictly negative if this parity is odd for all such permutations π. As all orbits of π are odd we have $\mathrm{sgn}(\pi) = 1$. Since $\det(G_\pi) = p(G_\pi; 0)$ for all π, this shows that $\sum_{\pi \in S_{k+1}} \mathrm{sgn}(\pi) p(G_\pi; 0)$ is either strictly positive or strictly negative. So it is non-zero. So we conclude that (4.21) is violated and hence that $p(\cdot; 0)$ cannot be the partition function of any edge coloring model. $\square$

Proposition 4.9. *For each $t \in \mathbb{C}$, there exists a $(2,2)$-color edge coloring model $h(t)$ such that $p_{h(t)}(G) = p(G; t)$ for all graphs G.*

Proof. Using the Leibniz expansion of the determinant, Sachs [33] gave an expression of the characteristic polynomial of a graph G in terms of certain subgraphs of G. The expression extends to graphs with multiple edges and loops. Let $G = (V, E)$ be a graph. Let $\mathcal{H}$ be the set of $H \subseteq E$ such that each connected component of $G(H)$ is either a vertex, an edge or a cycle. For $H \in \mathcal{H}$, let $e^*(H)$ and $c(H)$ denote the number of connected components of $G(H)$ that are edges and cycles respectively. Let $V[H] \subseteq V$ be the set of vertices of G that are incident with an edge of H. Then Sachs showed that

$$p(G; t) = \sum_{H \in \mathcal{H}} (-1)^{e^*(H)} (-2)^{c(H)} t^{|V| - |V[H]|}. \tag{4.22}$$

We now give a $(2,2)$-color edge coloring model $h = h(t)$ such that $p_h(G) = p(G; t)$ for each $t \in \mathbb{C}$ and graph G. Let h be defined as follows:

$$h(e_1^{\odot i} \otimes f_1 \wedge g_1) = 1 \text{ for } i \in \mathbb{N},$$
$$h(e_1^{\odot i} \odot e_2) = \sqrt{-1} \text{ for } i \in \mathbb{N},$$
$$h(e_1^{\odot i}) = t \text{ for } i \in \mathbb{N},$$

and let h evaluate to 0 on basis elements of $SV_2 \otimes \bigwedge V_2$ that are not in the span of these elements. Now let $F \subseteq E$ be Eulerian. We compute $s_h(G, F)$. If $G(F)$ has a vertex that is not of degree 0 or 2, then $s_h(G, F) = 0$.

So let us assume that each vertex of $G(F)$ has degree 0 or 2. Let ω be an Eulerian orientation of F with a compatible local pairing κ of $G(F)$. Now let $\phi : F \to [2]$ and $\psi : E \setminus F \to [2]$. We first note that for the contribution of ϕ and ψ to $s_h(G, F, \omega, \kappa)$ to be non-zero, we need $\psi^{-1}(2)$ to be a matching in G that is not incident with any edge in F.

Now fix $\psi : E \setminus F \to [2]$ such that $\psi^{-1}(2)$ is a matching in G that is not incident with any edge in F. Let $H = F \cup \psi^{-1}(2) \subseteq E$. Note that at each vertex $v \in V$ that is not incident with H, we see $h(e_1^{\odot d(v)}) = t$. There are $|V| - |V[H]|$ such vertices v. If $v, u \in V$ are two vertices such that $\{u, v\}$ is an isolated edge of (V, H), then at u we see $h(e_1^{\odot d(u)-1} \odot e_2) = \sqrt{-1}$ and at v we see $h(e_1^{\odot d(v)-1} \odot e_2) = \sqrt{-1}$. So two vertices u, v such that $\{u, v\}$ is an isolated edge of (V, H) contribute -1 to the partition function.

Now consider the colorings $\phi : F \to [2]$. Similar to what we have seen in Example 4.12, we have that such a ϕ has non-zero contribution if and only if it is constant on the edges of each 2-regular connected component of (V, H). So there are exactly $2^{c(H)}$ colorings $\phi : F \to [2]$ that have a non-zero contribution. As there are no odd arcs or odd pairings, we find that $(-1)^{c(\kappa)+o(\omega,\kappa)} = (-1)^{c(H)}$. So we see that the total contribution to the partition function of these colorings is exactly

$$(-1)^{e^*(H)}(-2)^{c(H)}t^{|V|-|V[H]|}. \tag{4.23}$$

Now summing over all F and corresponding ϕ and ψ, we find that $p_h(G)$ is indeed equal to $p(G;t)$ by (4.22). □

4.4.2 Evaluations of the circuit partition polynomial

The circuit partition polynomial, introduced, in a slightly different form, by Martin in his thesis [22], is related to Eulerian walks in graphs and to the Tutte polynomial of planar graphs. Several identities for the circuit partition polynomial were established by Bollobás [5] and Ellis-Monaghan [11].

Recall that a circuit is a closed walk where each edge is used at most once. We say that two circuits are equivalent if one can be obtained from the other by possibly changing the starting vertex or the direction of the walk. For a graph $G = (V, E)$, let $X(G)$ be a set of representatives of this equivalence relation. Let $\mathcal{C}(G)$ be the collection of all partitions of E into circuits in $X(G)$. For $C \in \mathcal{C}(G)$, let $|C|$ be the number of circuits in the partition.

The circuit partition polynomial $J(G, x)$ is defined, for a graph G, by

$$J(G, x) := \sum_{C \in \mathcal{C}(G)} x^{|C|}.$$

So if G is not an Eulerian graph, then $J(G, x) = 0$. We clearly have that $J(G \cup H, x) = J(G, x)J(H, x)$ for two graphs G and H and it is natural to define $J(\bigcirc, x) = x$.

For $k \in \mathbb{N}$, it was shown in [5, 11] that $J(G, k)$ can be expressed as

$$J(G, k) = \sum_{A} \prod_{v \in V} \prod_{i=1}^{k} (\deg_{A_i}(v) - 1)!!, \tag{4.24}$$

where A ranges over ordered partitions of E into k subsets $A_1, \ldots, A_k$ such that A_i is Eulerian for all $i \in [k]$.

We express (4.24) as the partition function of $h_0 \in (SV_k)^*$ as follows. For $(\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, we set

$$h_0\Big(\bigodot_{i \in [k]} e_i^{\odot \alpha_i}\Big) := \prod_{i=1}^{k} (\alpha_i - 1)!!. \tag{4.25}$$

Then $p_{h_0}(G) = J(G,k)$ for each graph G.

Bollobás [5] showed that the evaluation of the circuit partition polynomial $J(G,x)$ of a graph at negative even integers -2ℓ can be expressed as

$$J(G,-2\ell) = \sum_{H_1,\ldots,H_\ell} (-2)^{\sum_{i=1}^{\ell} c(H_i)}, \tag{4.26}$$

where this sum runs over all ordered partitions $H_1,\ldots,H_\ell$ of the edge set of G such that for each $i \in [\ell]$ each vertex in (V,H_i) has degree 0 or degree 2 and where $c(H_i)$ is the number of 2-regular connected components of (V,H_i).

We express (4.26) as the partition function of $h_1 \in (\bigwedge V_{2\ell})^*$. To this end we use expression (4.5) for skew partition functions. Let $G = (V,E)$ be an Eulerian graph and let ω be an Eulerian orientation of E with a compatible local pairing κ of G. For $h \in (\bigwedge V_{2\ell})^*$ and $\phi : E \to [\ell]$, we define

$$s_{h,\phi}(G,\omega,\kappa) := (-1)^{c(\kappa)} \sum_{\psi:E\to\{0,\ell\}} \prod_{v\in V} h\Big(\bigwedge_{(a,b)\in\kappa_v} f_{(\phi+\psi)(a)} \wedge g_{(\phi+\psi)(b)}\Big), \tag{4.27}$$

where $(\phi+\psi) : E \to [2\ell]$ is defined as $e \mapsto \phi(e)+\psi(e)$ for $e \in E$. It follows from the proof of Proposition 4.1 that (4.27) is independent of the choice of Eulerian orientation ω and compatible local pairing κ. So we can denote it by $s_{h,\phi}(G)$. Note that

$$s_h(G) = \sum_{\phi:E\to[\ell]} s_{h,\phi}(G). \tag{4.28}$$

For $S \subseteq [\ell]$, we define

$$h_1\Big(\bigwedge_{i\in S} f_i \wedge g_i\Big) = 1 \tag{4.29}$$

and we let h_1 evaluate to 0 on basis elements of $\bigwedge V_{2\ell}$ that are not in the span of the $\bigwedge_{i\in S} f_i \wedge g_i$. Consider a coloring $\phi : E \to [\ell]$ of the edges. We compute $s_{h_1,\phi}(G)$. If, for some $j \in [\ell]$, $G(\phi^{-1}(j))$ has a vertex that is not of degree 0 or 2, then, by (4.29), we see that $s_{h_1,\phi}(G) = 0$. So let us assume that $G(\phi^{-1}(j))$ has only vertices of degree 0 or degree 2 for each $j \in [\ell]$. Let ω be an Eulerian orientation of E such that for each $j \in [\ell]$, each cycle in $G(\phi^{-1}(j))$ is directed. Let κ be a local pairing of G compatible with ω such that for each $j \in [\ell]$ at each vertex v of degree 2 in $G(\phi^{-1}(j))$, the two edges of $G(\phi^{-1}(j))$ at v are paired.

For $j \in [\ell]$, let $c(\phi^{-1}(j))$ be the number of 2-regular connected components of $G(\phi^{-1}(j))$ and let $c(\phi) := \sum_{j=1}^{\ell} c(\phi^{-1}(j))$. We have that $c(\kappa) = c(\phi)$. Note that a coloring $\psi : E \to \{0,\ell\}$ gives a non-zero contribution to (4.27) if and only if it is constant on each connected component of $G(\phi^{-1}(j))$ for each $j \in [\ell]$. Each such coloring contributes $(-1)^{c(\phi)}$ to $s_{h_1,\phi}(G,\omega,\kappa)$ by (4.29). So we find that $s_{h_1,\phi}(G) = (-2)^{c(\phi)}$. And hence by (4.28) we find that $p_{h_1}(G) = J(G,-2\ell)$ according to (4.26).

We next show that mixed partition functions can also express evaluations of the circuit partition polynomial at negative odd integers. In [11], Ellis-Monaghan showed for a graph $G = (V, E)$ that

$$J(G, x + y) = \sum_{A \subseteq E} J(G(A), x) J(G(E \setminus A), y). \tag{4.30}$$

Now, for a negative odd integer $-2\ell + 1$, let $h_0 \in (SV_1)^*$ correspond to $k = 1$ in (4.25) and let $h_1 \in (\bigwedge V_{2\ell})^*$ be as in (4.29). Let $h = h_0 \otimes h_1 \in (SV_1 \otimes \bigwedge V_{2\ell})^*$. Then by (4.13) and (4.30) we find that $p_h(G) = J(G, -2\ell + 1)$.

Chapter 5

Partition functions and invariant theory

This chapter is devoted to proving Theorem 4.4 and Theorem 4.6. We first prove Theorem 4.6 and then use Theorem 4.6 to prove Theorem 4.4. The proof of Theorem 4.4 uses the First Fundamental Theorem of invariant theory (FFT) of the symplectic group and is inspired on the proof of Theorem 1 in [10] by Draisma, Gijwijt, Lovász, Regts and Schrijver. In our proof, however, we use the tensor FFT for the symplectic group, whereas in the proof of Theorem 1 in [10] the authors use the polynomial FFT and the Second Fundamental Theorem of invariant theory for the orthogonal group. The proof of Theorem 4.6 uses a result by Berele and Regev [4] that is related to the invariant theory of the general linear Lie superalgebra. In the last section we will go deeper into this connection and we will see how one could possibly use this connection to prove a converse to Theorem 4.6. This chapter is based on [31] and unpublished work with Guus Regts.

5.1 Proof of Theorem 4.6

Let us restate the theorem. Recall the definition of $\mathcal{J}_{k,2\ell}$ for $k, \ell \in \mathbb{N}$ given in (2.13).

Theorem. *Let $k, \ell \in \mathbb{N}$. If $f : \mathcal{G} \to \mathbb{C}$ is the partition function of an element $h \in (SV_k \otimes \bigwedge V_{2\ell})^*$, then $f(\mathcal{J}_{k,2\ell}) = 0$.*

The main idea of the proof is as follows. In Chapter 3 we defined an S_{2m}-action on $\mathbb{C}\mathcal{M}_{2m}$, the space of formal $\mathbb{C}$-linear combinations of perfect matchings on $[2m]$. We will define an S_{2m}-action on $V_{k,2\ell}^{\otimes 2m}$ and construct an S_{2m}-equivariant map $\tau : \mathbb{C}\mathcal{M}_{2m} \to V_{k,2\ell}^{\otimes 2m}$. The kernel of the map τ will turn out to consist of the modules S^λ such that λ is a $(k, 2\ell)$-block, as defined in

(2.5). We will construct graphs from matchings and using this construction we can prove the theorem. We first develop some framework.

5.1.1 The map p

Let $k, \ell \in \mathbb{N}$. Recall that $V_{k,2\ell} = V_k \oplus V_{2\ell}$, where $V_k = \mathbb{C}^k$ with standard basis $\{e_1, \ldots, e_k\}$ and $V_{2\ell} = \mathbb{C}^{2\ell}$ with standard basis $\{f_1, \ldots, f_{2\ell}\}$. We define

$$\bigwedge\nolimits_0 V_{2\ell} := \bigoplus_{i=0}^{\ell} \bigwedge\nolimits^{2i} V_{2\ell},$$

i.e., $\bigwedge_0 V_{2\ell}$ is the subspace of $\bigwedge V_{2\ell}$ spanned by basis elements of even degree. We define

$$R := S(SV_k \otimes \bigwedge\nolimits_0 V_{2\ell}).$$

We can describe R as the quotient of $T(SV_k \otimes \bigwedge_0 V_{2\ell})$, the tensor algebra of $SV_k \otimes \bigwedge_0 V_{2\ell}$, by the ideal generated by $\{x \otimes y - y \otimes x \mid x, y \in SV_k \otimes \bigwedge_0 V_{2\ell}\}$. Let $n \in \mathbb{N}$ and for $i \in [n]$, let $c_i \in SV_k \otimes \bigwedge_0 V_{2\ell}$. We denote the image of $c_1 \otimes \cdots \otimes c_n \in T(SV_k \otimes \bigwedge_0 V_{2\ell})$ under the quotient map by $\prod_{i=1}^n c_i$.

Through the canonical isomorphisms $V_k \cong (V_k^*)^*$ and $V_{2\ell} \cong (V_{2\ell}^*)^*$, we can view R as the space of regular functions on $(SV_k \otimes \bigwedge_0 V_{2\ell})^*$. We let $p : \mathbb{C}\mathcal{G} \to R$ be the map such that for each $h \in (SV_k \otimes \bigwedge V_{2\ell})^*$ and each $G \in \mathcal{G}$, we have $p(G)(h) = p_h(G)$ (we are abusing notation here, but one should interpret $p(G)(h)$ as the evaluation of h restricted to the subspace $S(SV_k \otimes \bigwedge_0 V_{2\ell})$ of $S(SV_k \otimes \bigwedge V_{2\ell})$). For the sake of completeness, we give the definition of the map p below.

Let $G = (V, E)$ be a graph with $F \subseteq E$ Eulerian. Let ω be an orientation of F and let κ be a local pairing of $G(F)$. We recall from (4.2) that for $\phi : F \to [2\ell]$ and $a \in F$ incident with $v \in V$, we define

$$b_{\phi,a,\omega,v} := \begin{cases} f_{\phi(a)} & \text{if } a \text{ is incoming at } v \text{ under } \omega, \\ g_{\phi(a)} & \text{if } a \text{ is outgoing at } v \text{ under } \omega, \end{cases} \tag{5.1}$$

where g_i for $i \in [2\ell]$ is defined in (4.1). We define $s(G, F, \omega, \kappa) \in R$ as

$$(-1)^{c(\kappa)+o(\omega,\kappa)} \sum_{\substack{\phi: F \to [2\ell] \\ \psi: E \setminus F \to [k]}} \prod_{v \in V} \bigodot_{a \in \delta_{E\setminus F}(v)} e_{\psi(a)} \otimes \bigwedge_{(a_1,a_2) \in \kappa_v} b_{\phi,a_1,\omega,v} \wedge b_{\phi,a_2,\omega,v}. \tag{5.2}$$

This is independent of the choice of κ and ω by a similar argument as in Proposition 4.1. So we define $s(G, F) := s(G, F, \omega, \kappa)$ for some local pairing κ of $G(F)$ and orientation ω of H. Now let $p : \mathbb{C}\mathcal{G} \to R$ be the unique linear map, defined, for a graph $G = (V, E)$, by

$$p(G) := \sum_{\substack{F \subseteq E \\ F \text{ Eulerian}}} s(G, F). \tag{5.3}$$

For $h \in (SV_k \otimes \bigwedge V_{2\ell})^*$ it is immediate that $p(G)(h) = p_h(G)$.

5.1.2 The commutative diagram

Let $D = (d_1, \dots, d_n) \in \mathbb{N}^n$ such that $d_1 \geq \cdots \geq d_n$ and $\sum_{i=1}^n d_i = 2m$ for some $m \in \mathbb{N}$. Let R_D be the subspace of R consisting of the elements

$$\prod_{i=1}^n q_i \text{ with } q_i \in \bigoplus_{j=0}^{\lfloor d_i/2 \rfloor} (S^{d_i - 2j} V_k \otimes \bigwedge^{2j} V_{2\ell}).$$

Let $\mathcal{G}_D$ be the set of graphs with degree sequence D. Then p restricts to a map $p_D : \mathbb{C}\mathcal{G}_D \to R_D$. Recall that $\mathcal{M}_{2m}$ is the set of perfect matchings on $[2m]$. We will next define linear maps μ_D, σ_D and τ so as to make the following diagram commute:

$$\begin{tikzcd}
\mathbb{C}\mathcal{G}_D \arrow[r, "p_D"] & R_D \\
\mathbb{C}\mathcal{M}_{2m} \arrow[u, "\mu_D"] \arrow[r, "\tau"] & V_{k,2\ell}^{\otimes 2m}. \arrow[u, "\sigma_D"']
\end{tikzcd} \tag{5.4}$$

For $j \in [n]$, we define

$$P_j := \left\{ 1 + \sum_{i=1}^{j-1} d_i, 2 + \sum_{i=1}^{j-1} d_i, \dots, d_j + \sum_{i=1}^{j-1} d_i \right\}. \tag{5.5}$$

So $\{P_1, \dots, P_n\}$ is a partition of $[2m]$. Let $\pi_D : [2m] \to \{v_1, \dots, v_n\}$ be the map, defined for $i \in [2m]$, by $\pi_D(i) := v_j$ if $i \in P_j$. Let $\mu_D : \mathbb{C}\mathcal{M}_{2m} \to \mathbb{C}\mathcal{G}_D$ be the unique linear map defined, for a matching $M \in \mathcal{M}_{2m}$, by

$$\mu_D(M) := (\{v_1, \dots, v_n\}, \{\{\pi_D(a), \pi_D(b)\} \mid \{a, b\} \in M\}).$$

Let us now define the map $\tau : \mathbb{C}\mathcal{M}_{2m} \to V_{k,2\ell}^{\otimes 2m}$. Recall that $\overrightarrow{\mathcal{M}}_{2m}$ is the set of directed perfect matchings on $[2m]$. We associate to $M \in \mathcal{M}_{2m}$ a directed perfect matching $\overrightarrow{M} \in \overrightarrow{\mathcal{M}}_{2m}$ by directing each edge from the lower to the higher index, i.e.,

$$\overrightarrow{M} = \{(i, j) \mid \{i, j\} \in M \text{ and } i < j\}.$$

We identify the edges of M with the arcs of $\overrightarrow{M}$, i.e., $\{i, j\} \in M$ with $i < j$ is identified with $(i, j) \in \overrightarrow{M}$.

Let $M \in \mathcal{M}_{2m}$ and $F \subseteq M$. Define $S(F) \subseteq [2m]$ to be $\cup F$, i.e., it is the set consisting of elements i of $[2m]$ such that $i \in a$ for some $a \in F$. We often write S for $S(F)$. Let $\overrightarrow{M}[S]$ be the directed perfect matching on S defined by

$$\overrightarrow{M}[S] := \{(i, j) \mid \{i, j\} \in F \text{ and } i < j\}.$$

Write $S = \{s_1, \ldots, s_{2r}\}$ with $s_1 < \cdots < s_{2r}$. Let N_S be the directed perfect matching on S defined by $N_S := \{(s_{2i-1}, s_{2i}) \mid i \in [r]\}$. We define $\operatorname{sgn}_S(M)$ as

$$\operatorname{sgn}_S(M) := (-1)^{c(N_S \cup \overrightarrow{M}[S]) + o(N_S \cup \overrightarrow{M}[S])}, \tag{5.6}$$

where we recall that for two directed matchings M_1 and M_2 on S we defined $c(M_1 \cup M_2)$ as the number of connected components of $(S, M_1 \cup M_2)$ and where we defined $o(M_1 \cup M_2)$ as the parity of the number of arcs of $M_1 \cup M_2$ that need to be flipped to make $(S, M_1 \cup M_2)$ into an Eulerian digraph. For $\phi : M \to [k+2\ell]$, we define for $i \in [2m]$ and $a \in M$ incident with i

$$b_{\phi,i} := \begin{cases} e_{\phi(a)} & \text{if } \phi(a) \leq k, \\ f_{\phi(a)-k} & \text{if } \phi(a) > k \text{ and a is incoming at } i \text{ in } ([2m], \overrightarrow{M}), \\ g_{\phi(a)-k} & \text{if } \phi(a) > k \text{ and a is outgoing at } i \text{ in } ([2m], \overrightarrow{M}). \end{cases} \tag{5.7}$$

For $M \in \mathcal{M}_{2m}$ and $F \subseteq M$, we define

$$\Phi(F) := \{\phi : M \to [k+2\ell] \mid \phi(a) \in \{k+1, \ldots, k+2\ell\} \text{ iff } a \in F\}$$

and

$$\tau(M, F) := \operatorname{sgn}_{S(F)}(M) \sum_{\phi \in \Phi(F)} \bigotimes_{i \in [2m]} b_{\phi,i}. \tag{5.8}$$

The map $\tau : \mathbb{C}\mathcal{M}_{2m} \to V_{k,2\ell}^{\otimes 2m}$ is the unique linear map defined, for $M \in \mathcal{M}_{2m}$, by

$$\tau(M) := \sum_{F \subseteq M} \tau(M, F). \tag{5.9}$$

To define $\sigma_D : V_{k,2\ell}^{\otimes 2m} \to R_D$, let $c = \bigotimes_{i \in [2m]} c_i \in V_{k,2\ell}^{\otimes 2m}$ with, for each $i \in [2m]$, $c_i \in \{e_1, \ldots, e_k, f_1, \ldots, f_{2\ell}\}$. Recall the definition of P_j in (5.5). For each $j \in [n]$, let

$$P_j^{E(c)} := \{i \in P_j \mid c_i \in \{e_1, \ldots, e_k\}\} \text{ and } P_j^{O(c)} := \{i \in P_j \mid c_i \in \{f_1, \ldots, f_{2\ell}\}\}.$$

We say that c is *balanced* if $|P_j^{O(c)}|$ is even for each $j \in [n]$. Now $\sigma_D : V_{k,2\ell}^{\otimes 2m} \to R_D$ is the unique linear map defined by

$$\sigma_D(c) := \begin{cases} \prod_{j \in [n]} \Big(\bigodot_{i \in P_j^{E(c)}} c_i \otimes \bigwedge_{i \in P_j^{O(c)}} c_i \Big) & \text{if } c \text{ is balanced}, \\ 0 & \text{otherwise}, \end{cases} \tag{5.10}$$

where we take the wedge over the elements in $P_j^{O(c)}$ in ascending order.

Lemma 5.1. *Diagram* (5.4) *commutes, that is, for any* $M \in \mathcal{M}_{2m}$, $\sigma_D(\tau(M)) = p_D(\mu_D(M))$.

Proof. It indeed suffices to show that $\sigma_D(\tau(M)) = p_D(\mu_D(M))$ for each $M \in \mathcal{M}_{2m}$, as the maps involved are linear. So we fix $M \in \mathcal{M}_{2m}$ and set $G := \mu_D(M)$ and write $G = (V, E)$. We say that $F \subseteq M$ is *balanced* if $|S(F) \cap P_j|$ is even for each $j \in [n]$. Note that

$$\sigma_D(\tau(M, F)) = 0 \text{ if } F \text{ is not balanced} \tag{5.11}$$

by the definition of σ_D in (5.10). The sets M and E are in bijection under μ_D and balanced subsets of M correspond one-to-one to Eulerian subsets of E under this bijection. So by (5.11) it suffices to show for a balanced $F \subseteq M$ and $H = \mu_D(F) \subseteq E$ that

$$\sigma_D(\tau(M, F)) = s(G, H), \tag{5.12}$$

as summing over all $F \subseteq M$ then shows that $\sigma_D(\tau(M)) = p_D(\mu_D(M))$.

So let us fix a balanced $F \subseteq M$ and let $S = S(F)$. Let furthermore $H = \mu_D(F)$. We first define a convenient local pairing $\kappa = (\kappa_{v_j})_{j \in [n]}$ of $G(H)$ and a convenient orientation ω of H. For $j \in [n]$, let $P_j \cap S = \{i_1, \ldots, i_{2r_j}\}$ with $i_1 < \cdots < i_{2r_j}$ and for $t \in [2r_j]$, let a_{i_t} be the image under μ_D of the unique edge of M that contains i_t. For $j \in [n]$, we define

$$\kappa_{v_j} := \{(a_{i_1}, a_{i_2}), \ldots, (a_{i_{2r_j-1}}, a_{i_{2r_j}})\}.$$

Let ω be the orientation of H such that an edge $\{v_i, v_j\} \in H$ is oriented from v_i to v_j under ω if $i \leq j$. Note that this orientation corresponds with the orientation of $\overrightarrow{M}[S]$.

Now note that a κ-circuit in $G(H)$ corresponds to a connected component of $(S, N_S \cup \overrightarrow{M}[S])$. Let X be a set of arcs in $N_S \cup \overrightarrow{M}[S]$ such that $(S, N_S \cup \overrightarrow{M}[S])$ becomes an Eulerian digraph after inverting the direction of all arcs in X. Then an arc in $X \cap N_S$ corresponds to an odd pairing in a κ-circuit and an arc in $X \cap \overrightarrow{M}[S]$ corresponds to an odd arc in a κ-circuit. This shows that $\mathrm{sgn}_S(M) = (-1)^{c(\kappa)+o(\omega,\kappa)}$.

Let $\chi \in \Phi(F)$. We define $\phi : H \to [2\ell]$ by $\phi(a) := \chi(\pi_D^{-1}(a)) - k$ for $a \in H$ and we define $\psi : E \setminus H \to [k]$ by $\psi(a) := \chi(\pi_D^{-1}(a))$ for $a \in E \setminus H$. Now we find that

$$\begin{aligned}
\sigma_D(\mathrm{sgn}_S(M) \bigotimes_{i \in [2m]} b_{\chi,i}) &= \mathrm{sgn}_S(M) \prod_{j \in [n]} \bigodot_{i \in P_j \setminus S} b_{\chi,i} \otimes \bigwedge_{i \in P_j \cap S} b_{\chi,i} \\
&= (-1)^{c(\kappa)+o(\omega,\kappa)} \prod_{j=1}^{n} \bigodot_{a \in \delta_{E \setminus H}(v_j)} e_{\psi(a)} \otimes \bigwedge_{(a_1,a_2) \in \kappa_{v_j}} b_{\phi,a_1,\omega,v_j} \wedge b_{\phi,a_2,\omega,v_j},
\end{aligned}$$

where we take the wedge over the elements in $P_j \cap S$ in ascending order. Summing over all $\chi \in \Phi(F)$, we see that $\sigma_D(\tau(M, F)) = s(G, H, \omega, \kappa) = s(G, H)$. So (5.12) holds. This proves the lemma. $\square$

5.1.3 The kernel of τ

To prove Theorem 4.6, we need to understand the kernel of the map τ. To do so, we will use some representation theory. We briefly recall the relevant concepts and results from Chapter 3.

For $\pi \in S_{2m}$ and $M \in \mathcal{M}_{2m}$, we set $\pi M = \{\{\pi(i), \pi(j)\} \mid \{i,j\} \in M\}$. This makes $\mathbb{C}\mathcal{M}_{2m}$ into an S_{2m}-module. The S_{2m}-module $\mathbb{C}\mathcal{M}_{2m}$ decomposes multiplicity free into irreducible representations:

$$\mathbb{C}\mathcal{M}_{2m} = \bigoplus_{\lambda \vdash 2m \text{ even}} S^{\lambda}, \tag{5.13}$$

where S^{λ} is the S_{2m}-module generated by $e_{\lambda}M$ and where $M = \{\{2i-1, 2i\} \mid i \in [m]\}$.

The space $V_{k,2\ell}^{\otimes 2m}$ also has the structure of an S_{2m}-module, that we now define. A basis for $V_{k,2\ell}^{\otimes 2m}$ is given by elements $b = \bigotimes_{j\in[2m]} b_j$ with $b_j \in \{e_1, \dots, e_k, f_1, \dots, f_{2\ell}\}$ for each $j \in [2m]$. To define an action of S_{2m} on $V_{k,2\ell}^{\otimes 2m}$, it suffices to define the action of transpositions of the form $(i, i+1) \in S_{2m}$ with $i \in [2m-1]$ on basis elements. Let $b = \bigotimes_{j\in[2m]} b_j$ be a basis element and let $I = \{j \in [2m] \mid b_j \in \{f_1, \dots, f_{2\ell}\}\}$. We define the action of a transposition $\pi = (i, i+1) \in S_{2m}$ on b by

$$\pi \cdot \bigotimes_{j\in[2m]} b_j = \mathrm{sgn}_I(\pi) \bigotimes_{j\in[2m]} b_{\pi(j)}, \tag{5.14}$$

where $\mathrm{sgn}_I(\pi) = -1$ if $i, i+1 \in I$ and $\mathrm{sgn}_I(\pi) = 1$ otherwise. We extend this linearly to an action on $V_{k,2\ell}^{\otimes 2m}$. For a proof that this really defines an action on $V_{k,2\ell}^{\otimes 2m}$ we refer to [4]. We now show that the map τ preserves the S_{2m}-actions on $\mathbb{C}\mathcal{M}_{2m}$ and $V_{k,2\ell}^{\otimes 2m}$.

Lemma 5.2. *The map τ is S_{2m}-equivariant.*

Proof. Let $M \in \mathcal{M}_{2m}$ and $F \subseteq M$ and let $\pi = (i, i+1) \in S_{2m}$ with $i \in [2m-1]$. Let $\pi F = \{\{\pi(j_1), \pi(j_2)\} \mid \{j_1, j_2\} \in F\} \subseteq \pi M$. By the linearity of τ it suffices to show that $\tau(\pi M, \pi F) = \pi \cdot \tau(M, F)$, as summing over all $F \subseteq M$ then gives that $\tau(\pi M) = \pi \cdot \tau(M)$. Recall that $S = \cup F$.

First suppose that $\{i, i+1\} \in M$. So $\pi M = M$ and $\pi F = F$, and hence we want to show that $\tau(\pi M, \pi F) = \tau(M, F) = \pi \cdot \tau(M, F)$, i.e.,

$$\sum_{\phi\in\Phi(F)} \mathrm{sgn}_S(M) \bigotimes_{j\in[2m]} b_{\phi,j} = \pi \cdot \Big(\sum_{\phi\in\Phi(F)} \mathrm{sgn}_S(M) \bigotimes_{j\in[2m]} b_{\phi,j} \Big). \tag{5.15}$$

If $\{i, i+1\} \notin F$, then π acts with sign 1 on the right hand sign of (5.15) and $b_{\phi,i} = b_{\phi,i+1}$ for every $\phi \in \Phi(F)$. So (5.15) holds in this case. If $\{i, i+1\} \in F$, then π acts with sign -1 on the right hand sign of (5.15). Summing over all

$\phi \in \Phi(F)$ we see that this sign cancels, as the tensor $\sum_{j\in[2\ell]} g_j \otimes f_j$ is skew-symmetric. This shows that (5.15) indeed holds.

Now suppose $\{i, i+1\} \notin M$. Then $\overrightarrow{\pi M} = \pi\overrightarrow{M}$. Let us now show that

$$\mathrm{sgn}_{\pi S}(\pi M) = \mathrm{sgn}_S(\pi)\mathrm{sgn}_S(M). \tag{5.16}$$

Let $\sigma \in S_{2m}$ be a permutation such that $\sigma N_S = \overrightarrow{M}[S]$. So $\mathrm{sgn}_S(M) = \mathrm{sgn}(\sigma)$ by Lemma 4.7.

If $i, i+1 \in S$, then $\mathrm{sgn}_S(\pi) = -1$ and we have that $\pi S = S$. Then

$$\pi\sigma N_{\pi S} = \pi\sigma N_S = \pi(\overrightarrow{M}[S]) = \overrightarrow{\pi M}[\pi S].$$

This shows that

$$\mathrm{sgn}_{\pi S}(\pi M) = \mathrm{sgn}(\pi\sigma) = -\mathrm{sgn}(\sigma) = -\mathrm{sgn}_S(M) = \mathrm{sgn}_S(\pi)\mathrm{sgn}_S(M),$$

by Lemma 4.7. So (5.16) holds in this case.

If at least one of i and $i+1$ is not in S, then $\mathrm{sgn}_S(\pi) = 1$. Then

$$\pi\sigma\pi N_{\pi S} = \pi\sigma\pi\pi N_S = \pi(\overrightarrow{M}[S]) = \overrightarrow{\pi M}[\pi S].$$

This shows that

$$\mathrm{sgn}_{\pi S}(\pi M) = \mathrm{sgn}(\pi\sigma\pi) = \mathrm{sgn}(\sigma) = \mathrm{sgn}_S(M) = \mathrm{sgn}_S(\pi)\mathrm{sgn}_S(M),$$

by Lemma 4.7. So (5.16) holds also in this case.

For $\phi \in \Phi(F)$, let $\phi_\pi \in \Phi(\pi F)$ be defined by $\phi_\pi(\{\pi(i), \pi(j)\}) = \phi(\{i, j\})$ for $\{i, j\} \in M$. Note that $S(\pi F) = \pi S(F) = \pi S$. As $\overrightarrow{\pi M} = \pi\overrightarrow{M}$, we see that for $j \in [2m]$ and $\phi \in \Phi(F)$, we have $b_{\phi_\pi, j} = b_{\phi,\pi(j)}$. Combining all of this, we find

$$\begin{aligned} \mathrm{sgn}_{\pi S}(\pi M) \bigotimes_{j\in[2m]} b_{\phi_\pi,j} &= \mathrm{sgn}_S(\pi)\mathrm{sgn}_S(M) \bigotimes_{j\in[2m]} b_{\phi,\pi(j)} \\ &= \mathrm{sgn}_S(\pi)\mathrm{sgn}_S(M)(\mathrm{sgn}_S(\pi)\pi \cdot \bigotimes_{j\in[2m]} b_{\phi,j}) \\ &= \pi \cdot (\mathrm{sgn}_S(M) \bigotimes_{j\in[2m]} b_{\phi,j}). \end{aligned}$$

Summing over all the $\phi \in \Phi(F)$, we find that $\tau(\pi M, \pi F) = \pi \cdot \tau(M, F)$. This proves the lemma. □

If follows from Lemma 5.2 that the kernel of τ is an S_{2m}-module. Recall that we call a partition λ a $(k, 2\ell)$-hook if the cell $(k+1, 2\ell+1)$ is not in the Young diagram of shape λ. Berele and Regev [4, Theorem 3.20] showed that $V_{k,2\ell}^{\otimes 2m}$ has the following decomposition as an S_{2m}-module:

$$V_{k,2\ell}^{\otimes 2m} \cong \bigoplus_{\substack{\lambda\vdash 2m \\ \lambda \text{ a } (k,2\ell)\text{-hook}}} (S^\lambda)^{\oplus \mu_\lambda}, \tag{5.17}$$

where μ_λ is the multiplicity of the corresponding module S^λ in the decomposition. Note that λ is not required to be even in (5.17). Recall that the bilinear form $[\cdot,\cdot]$ we defined in Section 4.3 extends to a bilinear form on $V_{k,2\ell}^{\otimes 2m}$. We have the following proposition describing the decomposition of the kernel of τ into irreducible representations. Recall that $B(k,2\ell)$ is the set consisting of even partitions λ such that the Young diagram of shape λ contains the cell $(k+1,2\ell+1)$ (so it is not a $(k,2\ell)$-hook).

Proposition 5.3. *The kernel of τ has the following decomposition into irreducible representations:*

$$\ker(\tau) = \bigoplus_{\substack{\lambda\vdash 2m \\ \lambda\in B(k,2\ell)}} S^\lambda, \tag{5.18}$$

where S^λ is the S_{2m}-module of $\mathbb{C}\mathcal{M}_{2m}$ generated by $e_\lambda M$ and where $M = \{\{2i-1,2i\} \mid i\in[m]\}$.

In the proof of this proposition we will work with a different definition of τ. For $M\in\mathcal{M}_{2m}$ and $S\subseteq[2m]$, we say that M is *compatible* with S if for every edge $\{i,j\}\in M$ we have $i,j\in S$ or $i,j\notin S$. If M is compatible with S, then $M[S]$ is the perfect matching on S consisting of the edges $\{i,j\}\in M$ such that $i,j\in S$. We define

$$\tau_S(M) := \begin{cases} \tau(M,M[S]) & \text{if } M \text{ is compatible with S,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.19}$$

It follows from the definition of τ that

$$\tau(M) = \sum_{S\subseteq[2m]} \tau_S(M). \tag{5.20}$$

For $S\subseteq[2m]$, let $\bar{S} = [2m]\setminus S$. Note that

$$V_{k,2\ell}^{\otimes 2m} = \bigoplus_{S\subseteq[2m]} V_k^{\bar{S}}\otimes V_{2\ell}^S,$$

where $V_k^{\bar{S}}\otimes V_{2\ell}^S$ is the subspace of $V_{k,2\ell}^{\otimes 2m}$ consisting of the tensors $\bigotimes_{i\in[2m]} c_i$ with $c_i\in V_k$ if $i\in\bar{S}$ and $c_i\in V_{2\ell}$ if $i\in S$. For $M\in\mathcal{M}_{2m}$, $\tau_S(M)$ is equal to the projection of $\tau(M)$ to $V_k^{\bar{S}}\otimes V_{2\ell}^S$. So we have that

$$\tau(M)\neq 0 \text{ if and only if there exists an } S\subseteq[2m] \text{ such that } \tau_S(M)\neq 0. \tag{5.21}$$

Let $\lambda\vdash 2n$ be an even partition. Let $R_\lambda^0\subseteq R_\lambda$ be the stabilizer of $M = \{\{2i-1,2i\} \mid i\in[n]\}$ and let R'_λ be a set of representatives of the cosets R_λ/R_λ^0. We have that

$$\sum_{\substack{\rho\in R'_\lambda \\ \sigma\in C_\lambda}} \mathrm{sgn}(\sigma)\sigma\rho M = \frac{1}{|R_\lambda^0|}\sum_{\substack{\rho\in R_\lambda \\ \sigma\in C_\lambda}} \mathrm{sgn}(\sigma)\sigma\rho M. \tag{5.22}$$

So, by (3.6), we find that

$$h_\lambda(x) = \frac{1}{|C_\lambda^0|} \sum_{\substack{\rho \in R'_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma) x^{c(\sigma\rho M \cup M)}. \tag{5.23}$$

We now turn to the proof of the proposition.

Proof op Proposition 5.3. The fact that the right-hand side of (5.18) is contained in the kernel of τ follows immediately from (5.13) and (5.17). It therefore suffices to show that, if $\lambda = (\lambda_1, \ldots, \lambda_r)$ is an even partition of $2m$ and λ is a $(k, 2\ell)$-hook, then $\tau(e_\lambda M) \neq 0$, where $M = \{\{2i-1, 2i\} \mid i \in [m]\}$.

Recall that Y_λ is the Young tableau of shape λ with the standard filling. Let $S \subseteq [2m]$ be the set consisting of the numbers contained in columns $1, \ldots, 2\ell$ in Y_λ. We will show that

$$[\tau_S(e_\lambda M), \tau_S(M)] \neq 0. \tag{5.24}$$

By (5.21) this implies that $\tau(e_\lambda M) \neq 0$, as required.

Let $R_\lambda^0 \subseteq R_\lambda$ be the stabilizer of M in R_λ and let R'_λ be a set of coset representatives of R_λ / R_λ^0. To show (5.24), it suffices to show that

$$\sum_{\substack{\rho \in R'_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma) [\tau_S(\sigma\rho M), \tau_S(M)] \neq 0. \tag{5.25}$$

by the linearity of τ and by (5.22). Now let $\rho \in R'_\lambda$ and $\sigma \in C_\lambda$. If the matching $\sigma\rho M$ is not compatible with S, then $\tau_S(\sigma\rho M) = 0$, by the definition of τ_S. As $\sigma\rho M$ is compatible with S if and only if ρM is compatible with S, we can compute (5.25) by only summing over those $\rho \in R'_\lambda$ such that ρM is compatible with S.

Let us now choose R'_λ in a convenient way. Let $S_1 := \bar{S}$ and $S_2 := S$. For $i = 1, 2$, let X_i be the set of cosets ρR_λ^0 such that ρM is compatible with S and such that $(\rho M)[S_i] = M[S_i]$. For $i = 1, 2$, let R_i be a set of representatives ρ' of the cosets in X_i such that ρ' acts trivially on S_i. Now $R_1 R_2 \subseteq S_{2m}$ is a set of representatives of the cosets ρR_λ^0 such that ρM is compatible with S. We extend this set of representatives to a full set of representatives R' of R_λ / R_λ^0.

For $i = 1, 2$, let C_i be the subgroup of C_λ consisting of the permutations σ such that σ acts trivially on S_i. Then $C_1 C_2 = C_\lambda$.

Finally, let $M_1 = M[S]$ and let $M_2 = M[\bar{S}]$ (note that, for $i = 1, 2$, the non-identity elements of R_i and C_i act non-trivially on M_i). We find that

$$\sum_{\substack{\rho \in R'_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma) \big[\tau_S(\sigma\rho M), \tau_S(M)\big] \tag{5.26}$$
$$= (-1)^{|S|/2} \Big(\sum_{\substack{\rho_1 \in R_1 \\ \sigma_1 \in C_1}} \operatorname{sgn}(\sigma_1) (-2\ell)^{c(\sigma_1\rho_1 M_1 \cup M_1)} \Big) \cdot \Big(\sum_{\substack{\rho_2 \in R_2 \\ \sigma_2 \in C_2}} \operatorname{sgn}(\sigma_2) k^{c(\sigma_2\rho_2 M_2 \cup M_2)} \Big).$$

This follows from explicitly computing the form $\langle \cdot, \cdot \rangle$ on positions in S and computing the form $(\cdot, \cdot)$ on the positions in $\bar{S}$, similar to what we did in the proof of Theorem 4.5.

To show that (5.26) is non-zero we use Theorem 3.3 by Hanlon and Wales. Let $\mu = (\mu_1, \ldots, \mu_r)$ with $\mu_i = \min\{2\ell, \lambda_i\}$ for $i \in [r]$. Let $\nu = (\nu_1, \ldots, \nu_k)$ with $\nu_i = \max\{0, \lambda_i - 2\ell\}$ for $i \in [k]$ (if one of the ν_i equals 0 then we disregard it). Then μ is an even partition of $|S|$ and ν is an even partition of $|\bar{S}|$. We find that

$$\sum_{\substack{\rho_1 \in R_1 \\ \sigma_1 \in C_1}} \operatorname{sgn}(\sigma_1)(-2\ell)^{c(\sigma_1\rho_1 M_1 \cup M_1)} = |C_\mu^0| h_\mu(-2\ell) \neq 0,$$

cf. (5.23). Similarly, we find that

$$\sum_{\substack{\rho_2 \in R_2 \\ \sigma_2 \in C_2}} \operatorname{sgn}(\sigma_2) k^{c(\sigma_2\rho_2 M_2 \cup M_2)} = |C_\nu^0| h_\nu(k) \neq 0,$$

cf. (5.23). So we find that (5.26) is non-zero. This proves Proposition 5.3. □

We first derive a corollary from this proposition before proving the theorem.

Corollary 5.4. *Let $n \in \mathbb{N}$. If $n \leq m$ and $\lambda \vdash 2n$ is in $B(k, 2\ell)$, then*

$$\tau\Big(\sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma)\sigma\rho M\Big) = 0,$$

where $M = \{\{2i-1, 2i\} \mid i \in [m]\}$ and where we view S_{2n} as a subgroup of S_{2m} acting on $[2n]$.

Proof. For $w \in \mathbb{N}$, write τ_{2w} for the map $\tau : \mathbb{C}\mathcal{M}_{2w} \to V_{k,2\ell}^{\otimes 2w}$ and write M_{2w} for the perfect matching $\{\{2i-1, 2i\} \mid i \in [w]\}$ on $[2w]$. Let $n, m \in \mathbb{N}$ with $n \leq m$ and let $\lambda \vdash 2n$ be in $B(k, 2\ell)$. We find that

$$\begin{aligned}
\tau\Big(\sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma)\sigma\rho M\Big) &= \sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma)\tau_{2m}(\sigma\rho M_{2m}) \\
&= \sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma)\tau_{2n}(\sigma\rho M_{2n}) \otimes \tau_{2m-2n}(M_{2m-2n}) \\
&= \tau_{2n}\Big(\sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma)\sigma\rho M_{2n}\Big) \otimes \tau_{2m-2n}(M_{2m-2n}) = 0,
\end{aligned}$$

where the last equality follows from Proposition 5.3. This proves the corollary. □

We are now ready to prove Theorem 4.6.

5.1.4 Finishing the proof of Theorem 4.6

Recall that for a graph $G = (V, E)$, $n \in \mathbb{N}$ and $u : [2n] \to V$, we defined

$$G_u := (V, E \cup \{\{u(2i-1), u(2i)\} \mid i \in [n]\}). \tag{5.27}$$

Let $k, \ell \in \mathbb{N}$ and let $h \in (SV_k \otimes \bigwedge V_{2\ell})^*$. Let $n \in \mathbb{N}$ and let $\lambda \vdash 2n$ with $\lambda \in B(k, 2\ell)$. To prove Theorem 4.6, it suffices to show that for a graph $G = (V, E)$ together with a map $u : [2n] \to V$, we have

$$\sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma) p_h(G_{u \circ \sigma \circ \rho}) = 0.$$

So we fix a graph $G = (V, E)$ and a map $u : [2n] \to V$. As $p(G)(h) = p_h(G)$ it suffices to show that

$$\sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma) p(G_{u \circ \sigma \circ \rho}) = 0. \tag{5.28}$$

Let $D = (d_1, \ldots, d_r)$ be the degree sequence of G_u and let $2m = \sum_{i=1}^r d_i$. Let $M = \{\{2i-1, 2i\} \mid i \in [2m]\}$ and let $N \in \mathcal{M}_{2m}$ be such that $\mu_D(N) = G_u$. Let $\pi \in S_{2m}$ such that $\pi N = M$ and such that the edge of N corresponding to $\{u(2i-1), u(2i)\}$ is mapped to $\{2i-1, 2i\}$ for each $i \in [2n]$. Then

$$\mu_D(\pi^{-1}\sigma\rho\pi N) = G_{u \circ \sigma \circ \rho} \text{ for all } (\sigma, \rho) \in C_\lambda \times R_\lambda. \tag{5.29}$$

So we find that

$$\sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma) p(G_{u \circ \sigma \circ \rho}) = \sigma_D(\tau(\sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma) \pi^{-1}\sigma\rho\pi N)) \tag{5.30}$$

$$= \sigma_D(\pi^{-1} \cdot \tau(\sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma)\sigma\rho M)), \tag{5.31}$$

where the first equality follows from (5.29) and Lemma 5.1, and where the last equality follows from the S_{2m}-equivariance of τ. By Corollary 5.4 we have

$$\tau(\sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \operatorname{sgn}(\sigma)\sigma\rho M) = 0,$$

as $\lambda \in B(k, 2\ell)$ and $n \leq m$. So we see that (5.28) indeed holds. This proves Theorem 4.6. □

5.2 Proof of Theorem 4.4

In this section we prove Theorem 4.4. The proof is slightly different from the proof given in [31], but uses the same ideas. The proof uses the invariant theory of the symplectic group and is inspired by the proof of [10, Theorem 1]. We restate Theorem 4.4. Recall the definition of G_u given in (5.27).

Theorem. *Let $\ell \in \mathbb{N}$. A graph parameter $f : \mathcal{G} \to \mathbb{C}$ is the partition function of an element $h \in (\bigwedge V_{2\ell})^*$ if and only if $f(\emptyset) = 1$, $f(\bigcirc) = -2\ell$, f is multiplicative, $f(G) = 0$ if G is not Eulerian and for each graph $G = (V, E)$ and for each map $u : [2\ell + 2] \to V$, we have*

$$\sum_{\rho \in S_{2\ell+2}} f(G_{u\circ\rho}) = 0. \tag{5.32}$$

To prove this theorem, we will use the results we derived in Section 5.1 applied to the case $k = 0$. We briefly recall the relevant results and translate these to our current situation. Let $\ell \in \mathbb{N}$. We have a linear map $p : \mathbb{C}\mathcal{G} \to S(\bigwedge_0 V_{2\ell}) = R$ such that $p(G)(h) = p_h(G)$ for each $h \in (\bigwedge V_{2\ell})^*$. For $m \in \mathbb{N}$, we defined an S_{2m}-equivariant linear map $\tau : \mathbb{C}\mathcal{M}_{2m} \to V_{2\ell}^{\otimes 2m}$. By Proposition 5.3 we have the following description of the kernel of τ:

$$\ker(\tau) = \bigoplus_{\substack{\lambda=(\lambda_1,\dots,\lambda_r)\vdash 2m \text{ even} \\ \lambda_1 \geq 2\ell+2}} S^\lambda. \tag{5.33}$$

According to Corollary 5.4, applied to the partition $\lambda = (2\ell + 2)$ of $2\ell + 2$, we can derive from this that, if $2m \geq 2\ell + 2$, then

$$\tau\Big(\sum_{\rho \in S_{2\ell+2}} \rho M\Big) = 0, \tag{5.34}$$

where $M = \{\{2i - 1, 2i\} \mid i \in [m]\}$ and where we view $S_{2\ell+2}$ as a subgroup of S_{2m} acting on $[2\ell + 2]$.

Let $D = (d_1, \dots, d_n) \in \mathbb{N}^d$ with $d_1 \geq \cdots \geq d_n$ such that $\sum_{i=1}^n d_i = 2m$ for some $m \in \mathbb{N}$. We defined $\mathcal{G}_D$ to be the set of graphs with degree sequence D and R_D to be the subspace of R consisting of elements $\prod_{i=1}^n c_i$ with $c_i \in \bigwedge^{d_i} V_{2\ell}$ for each $i \in [n]$. The map p restricts to a map $p_D : \mathbb{C}\mathcal{G}_D \to R_D$. In Lemma 5.1 we proved that for any $\kappa \in \mathbb{C}\mathcal{M}_{2m}$ we have $p_D(\mu_D(\kappa)) = \sigma_D(\tau(\kappa))$.

The space $\mathbb{C}\mathcal{G}$ has a natural algebra structure, where the multiplication of two graphs is given by their disjoint union. As the map p is multiplicative, it is an algebra homomorphism. Let $\mathcal{J}_{2\ell}$ be the ideal of $\mathbb{C}\mathcal{G}$ spanned by non-Eulerian graphs together with

$$\Big\{ \sum_{\rho \in S_{2\ell+2}} G_{u\circ\rho} \ \Big|\ G = (V, E) \in \mathcal{G},\ u : [2\ell + 2] \to V \Big\}. \tag{5.35}$$

The *symplectic group* $\mathrm{Sp}_{2\ell}$ is the group of $2\ell \times 2\ell$ matrices that preserve the skew-symmetric bilinear form; i.e., for $g \in \mathbb{C}^{2\ell\times 2\ell}$, $g \in \mathrm{Sp}_{2\ell}$ if and only if $\langle gx, gy\rangle = \langle x, y\rangle$ for all $x, y \in V_{2\ell}$. The action of $\mathrm{Sp}_{2\ell}$ on $V_{2\ell}$ extends to an action on $\bigwedge_0 V_{2\ell}$ and hence it also extends to an action on R.

We have the following proposition regarding the image and kernel of p.

Proposition 5.5. *The image of p is equal to $R^{Sp_{2\ell}}$, the space of $Sp_{2\ell}$-invariant elements of R, and the kernel of p is equal to $\mathcal{J}_{2\ell}$.*

Proposition 5.5 actually implies Theorem 4.4. We give the proof of the theorem using the proposition now, and the rest of this section is devoted to proving the proposition.

Proof of Theorem 4.4. Let $h \in (\bigwedge V_{2\ell})^*$. Then p_h is multiplicative, $f(\emptyset) = 1$ and $f(\bigcirc) = -2\ell$. By Proposition 5.5, we know that for any $\gamma = \sum_i \gamma_i G_i \in \mathcal{J}_{2\ell}$ with $\gamma_i \in \mathbb{C}$ and $G_i \in \mathcal{G}$, $p(\gamma) = 0$. Hence $\sum_i \gamma_i p_h(G_i) = \sum_i \gamma_i p(G_i)(h) = p(\gamma)(h) = 0$. This shows that $p_h(G) = 0$ if G is a non-Eulerian graph and

$$\sum_{\rho \in S_{2\ell+2}} p_h(G_{u\circ\rho}) = 0,$$

for each graph $G = (V, E)$ with $u : [2\ell + 2] \to V$.

The proof of the 'if' direction is based on a beautiful, and by now, well-known idea of Szegedy, cf. [39]; see also [10]. We will give the proof.

The idea is to use Hilbert's Nullstellensatz to find a solution $h \in (\bigwedge V_{2\ell})^*$ to the set of equations $f(G) = p(G)(h)$, with $G \in \mathcal{G}$. Since f is multiplicative and maps $\mathcal{J}_{2\ell}$, the kernel of p, to zero, there is a unique algebra homomorphism $\hat{f} : \mathrm{im}\,(p) = R^{\mathrm{Sp}_{2\ell}} \to \mathbb{C}$ such that $f = \hat{f} \circ p$.

If there is no solution $h \in (\bigwedge V_{2\ell})^*$ to the set of equations $f(G) = p(G)(h)$, then, by Hilbert's Nullstellensatz, 1 is contained in the ideal generated by $f(G) - p(G)$. In other words, there exist $G_1, \ldots, G_n$ and $r_1, \ldots, r_n \in R$ such that

$$1 = \sum_{i=1}^{n} r_i(f(G_i) - p(G_i)). \tag{5.36}$$

As the image of p is equal to $R^{\mathrm{Sp}_{2\ell}}$, applying the Reynolds operator of $\mathrm{Sp}_{2\ell}$ to both sides of (5.36), we may assume that each r_i belongs to $R^{\mathrm{Sp}_{2\ell}} = \mathrm{im}\,(p)$ and hence is equal to $p(\eta_i)$ for some linear combination η_i of graphs. Now applying $\hat{f}$ to both sides of (5.36) we obtain

$$1 = \sum_{i=1}^{n} \hat{f}(p(\eta_i))(f(G_i) - \hat{f}(p(G_i))) = \sum_{i=1}^{n} \hat{f}(p(\eta_i))(f(G_i) - f(G_i)) = 0,$$

a contradiction. This finishes the proof. □

The rest of this section is devoted to proving Proposition 5.5. The action of $\mathrm{Sp}_{2\ell}$ respects R_D, for any degree sequence D. So to prove Proposition 5.5, it suffices to show that

$$\mathrm{im}(p_D) = R_D^{\mathrm{Sp}_{2\ell}} \text{ and } \ker(p_D) = \mathcal{J}_{2\ell} \cap \mathbb{C}\mathcal{G}_D \tag{5.37}$$

for each degree sequence $D = (d_1, \ldots, d_n)$. This is trivial if at least one of $d_1, \ldots, d_n$ is odd. So for the rest of the proof we fix $D = (d_1, \ldots, d_n) \in \mathbb{N}^n$ such that $d_1 \geq \cdots \geq d_n$ are all even. Let $2m = \sum_{i=1}^{n} d_i$.

5.2.1 The image of p

We first show that $R_D^{\mathrm{Sp}_{2\ell}} \subseteq \mathrm{im}(p_D)$. To this end, let $q \in R_D^{\mathrm{Sp}_{2\ell}}$. Then there is a $v \in V_{2\ell}^{\otimes 2m}$ with $\sigma_D(v) = q$, by the surjectivity of σ_D. The map σ_D is an equivariant map for the natural action of $\mathrm{Sp}_{2\ell}$ on $V_{2\ell}^{\otimes 2m}$ and the action on R_D. So by applying the Reynolds operator, we can assume that v is invariant under $\mathrm{Sp}_{2\ell}$. It follows from the First Fundamental Theorem of invariant theory for the symplectic group [13, Section 5.4] that

$$\mathrm{im}(\tau) = (V_{2\ell}^{\otimes 2m})^{\mathrm{Sp}_{2\ell}}. \tag{5.38}$$

This shows that v is in the image of τ. So there exists a $\kappa \in \mathbb{C}\mathcal{M}_{2m}$ such that $\sigma_D(\tau(\kappa)) = q$. By Lemma 5.1 we know that $q = \sigma_D(\tau(\kappa)) = p_D(\mu_D(\kappa))$. So $q \in \mathrm{im}(p_D)$. This shows that indeed $R_D^{\mathrm{Sp}_{2\ell}} \subseteq \mathrm{im}(p_D)$.

Next we show that $\mathrm{im}(p_D) \subseteq R_D^{\mathrm{Sp}_{2\ell}}$. To this end, let $r \in \mathrm{im}(p_D)$. Then there is some $\gamma \in \mathbb{C}\mathcal{G}_D$ such that $r = p_D(\gamma)$. By the surjectivity of μ_D there is some $\kappa \in \mathbb{C}\mathcal{M}_{2m}$ such that $\mu_D(\kappa) = \gamma$. Now $\tau(\kappa)$ is invariant under the action of $\mathrm{Sp}_{2\ell}$ by (5.38) and hence $\sigma_D(\tau(\kappa)) \in R_D^{\mathrm{Sp}_{2\ell}}$ as σ_D is an $\mathrm{Sp}_{2\ell}$-equivariant map. We have that $r = p_D(\mu_D(\kappa)) = \sigma_D(\tau(\kappa))$ by Lemma 5.1. So $r \in R_D^{\mathrm{Sp}_{2\ell}}$. This shows that $\mathrm{im}(p_D) \subseteq R_D^{\mathrm{Sp}_{2\ell}}$.

This proves the first part of (5.37) and hence proves the first part of Proposition 5.5.

5.2.2 The kernel of p

We first give an alternative description of $\ker(\tau)$.

Lemma 5.6. *Let $\ell \in \mathbb{N}$ with $m \geq \ell + 1$ and let $M = \{\{2i-1, 2i\} \mid i \in [m]\}$. Then the kernel of τ is equal to the span of*

$$K = \left\{ \pi \sum_{\rho \in S_{2\ell+2}} \rho M \;\middle|\; \pi \in S_{2m} \right\}, \tag{5.39}$$

where $S_{2\ell+2}$ is the subgroup of S_{2m} acting on $[2\ell+2]$.

Proof. We first show that the span of K is contained in the kernel of τ. Let $\pi \in S_{2m}$. We see that

$$\tau(\pi \sum_{\rho \in S_{2\ell+2}} \rho M) = \pi \cdot \tau(\sum_{\rho \in S_{2\ell+2}} \rho M) = 0,$$

where the first equality follows from the S_{2m}-equivariance of τ and where the second equality follows from (5.34). So by the linearity of τ we find that the span of K is indeed contained in the kernel of τ.

We now prove the converse inclusion. By (5.33), it suffices to show, for any element $\kappa \in S^\lambda$ with $\lambda = (\lambda_1, \dots, \lambda_r) \vdash 2m$ such that $\lambda_1 \geq 2\ell + 2$, that κ is in the span of K. Recall that the module S^λ is the S_{2m}-module generated by

$$\sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \mathrm{sgn}(\sigma)\sigma\rho M.$$

Now we write $R_\lambda = \cup_{i \in I} r_i S_{2\ell+2}$, where $\{r_i\}_{i \in I}$ is a set of coset representatives of $R_\lambda / S_{2\ell+2}$. We find that

$$\sum_{\substack{\rho \in R_\lambda \\ \sigma \in C_\lambda}} \mathrm{sgn}(\sigma)\sigma\rho M = \sum_{\substack{i \in I \\ \sigma \in C_\lambda}} \sum_{\rho' \in S_{2\ell+2}} \mathrm{sgn}(\sigma)\sigma r_i \rho' M = \sum_{\substack{i \in I \\ \sigma \in C_\lambda}} \mathrm{sgn}(\sigma)\sigma r_i \sum_{\rho' \in S_{2\ell+2}} \rho' M.$$

For every $\sigma \in C_\lambda$ and for every $i \in I$ we have that $\sigma r_i \sum_{\rho' \in S_{2\ell+2}} \rho' M$ is an element of K. So we see that the kernel of τ is contained in the span of K. This proves the lemma. $\square$

Recall the definition of P_j given in (5.5). Let $Q := S_{P_1} \times \cdots \times S_{P_n} \subseteq S_{2m}$. Let $i, j \in [n]$ such that $|P_i| = |P_j|$. Write $P_i = \{i_1, \dots, i_r\}$ with $i_1 < \cdots < i_r$ and write $P_j = \{j_1, \dots, j_r\}$ with $j_1 < \cdots < j_r$. Let $\nu_{i,j} = (i_1\ j_1) \dots (i_r\ j_r) \in S_{2m}$ and let $T \subseteq S_{2m}$ be the subgroup generated by the elements $\nu_{i,j}$, for $i, j \in [n]$ such that $|P_i| = |P_j|$. Note that $G = TQ$ is the group of permutations in S_{2m} maintaining the partition $\{P_1, \dots, P_n\}$.[1] For a tensor $v \in V_{2\ell}^{\otimes 2m}$, we define

$$v^G := \frac{1}{|T||Q|} \sum_{\pi_1 \in T} \sum_{\pi_2 \in Q} (\pi_1 \pi_2) \cdot v,$$

i.e., we apply the Reynolds operator of the group G to v. Now $(V_{2\ell}^{\otimes 2m})^G$, the subspace of G-invariant elements of $V_{2\ell}^{\otimes 2m}$, is equal to $\{v^G \mid v \in V_{2\ell}^{\otimes 2m}\}$. Note that for any $v \in V_{2\ell}^{\otimes 2m}$ we have that $\sigma_D(v^G) = \sigma_D(v)$. So σ_D restricts to a surjective map from $(V_{2\ell}^{\otimes 2m})^G$ to R_D, because σ_D is surjective. We will now show that σ_D actually restricts to a bijection between $(V_{2\ell}^{\otimes 2m})^G$ and R_D.

Note that $(V_{2\ell}^{\otimes 2m})^Q$, the subspace of Q-invariant elements of $V_{2\ell}^{\otimes 2m}$, is equal to

$$\left\{ \frac{1}{|Q|} \sum_{\pi_2 \in Q} \pi_2 \cdot v \;\middle|\; v \in V_{2\ell}^{\otimes 2m} \right\},$$

which is linearly isomorphic to $\bigotimes_{i=1}^n (\bigwedge^{d_i} V_{2\ell})$. We identify $(V_{2\ell}^{\otimes 2m})^Q$ with $\bigotimes_{i=1}^n (\bigwedge^{d_i} V_{2\ell})$. As d_i is even for each $i \in [n]$, we have that the elements of T act with sign 1 on $V_{2\ell}^{\otimes 2m}$. So we find that $(V_{2\ell}^{\otimes 2m})^G$ is linearly isomorphic to

$$\left\{ \frac{1}{|T|} \sum_{\pi_1 \in T} \pi_1 \cdot v \;\middle|\; v \in \bigotimes_{i=1}^n \left(\bigwedge^{d_i} V_{2\ell}\right) \right\},$$

[1] In [31] we implicitly, and incorrectly, used the larger group S_n instead of the group T. The current proof shows how to modify the proof in [31] to ensure correctness.

which is isomorphic to $\prod_{i=1}^n (\bigwedge^{d_i} V_{2\ell}) = R_D$. This shows that σ_D restricts to a bijection between $(V_{2\ell}^{\otimes 2m})^G$ and R_D.

Using a similar line of reasoning, we find that μ_D restricts to a bijection between $(\mathbb{C}\mathcal{M}_{2m})^G$ and $\mathbb{C}\mathcal{G}_D$, where $(\mathbb{C}\mathcal{M}_{2m})^G$ is the subspace of G-invariant elements of $\mathbb{C}\mathcal{M}_{2m}$.

Let us now show that

$$\ker(p_D) = \mu_D(\ker(\tau)). \tag{5.40}$$

First note that if $\gamma \in \mu_D(\ker(\tau))$, then $\gamma = \mu_D(\kappa)$, for some $\kappa \in \ker(\tau)$, and $p_D(\gamma) = \sigma_D(\tau(\kappa)) = 0$, by Lemma 5.1. This shows that $\gamma \in \ker(p_D)$ and hence that $\mu_D(\ker(\tau)) \subseteq \ker(p_D)$.

For the converse inclusion, let $\gamma \in \ker(p_D)$. Then there is a unique $\kappa \in (\mathbb{C}\mathcal{M}_{2m})^G$ such that $\mu_D(\kappa) = \gamma$, because μ_D restricts to a bijection between $(\mathbb{C}\mathcal{M}_{2m})^G$ and $\mathbb{C}\mathcal{G}_D$. Note that $\tau(\kappa) \in (V_{2\ell}^{\otimes 2m})^G$ by the S_{2m}-equivariance of τ. So $0 = p_D(\gamma) = \sigma_D(\tau(\kappa))$ implies that $\tau(\kappa) = 0$ because σ_D restricts to a bijection between $(V_{2\ell}^{\otimes 2m})^G$ and R_D. This shows that $\gamma \in \mu_D(\ker(\tau))$ and hence that $\ker(p_D) \subseteq \mu_D(\ker(\tau))$. This proves (5.40).

We finally prove that $\ker(p_D) = \mathbb{C}\mathcal{G}_D \cap \mathcal{J}_{2\ell}$, which finishes the proof of Proposition 5.5. By (5.40), this is equivalent to showing that $\mu_D(\ker(\tau)) = \mathbb{C}\mathcal{G}_D \cap \mathcal{J}_{2\ell}$. To show this we will relate the elements of K in (5.39) to graphs $G = (V, E)$ with a map $u : [2\ell + 2] \to V$.

We first show that $\mu_D(\ker(\tau)) \subseteq C\mathcal{G}_D \cap \mathcal{J}_{2\ell}$. Let $\gamma \in \mu_D(\ker(\tau))$. By Lemma 5.6 and linearity, we may assume that $\gamma = \mu_D(\kappa)$ with

$$\kappa = \pi \sum_{\rho \in S_{2\ell+2}} \rho M,$$

for some $\pi \in S_{2m}$, where $M = \{\{2i-1, 2i\} \mid i \in [m]\}$. Let $(V, E') = \mu_D(\pi M)$. We define

$$E := E' \setminus \{\{\mu_D(\pi(2i-1)), \mu_D(\pi(2i))\} \mid i \in [\ell + 1]\}.$$

Now let $G = (V, E)$ and let $u : [2\ell + 2] \to V$ be defined, for $i \in [2\ell + 2]$, by $u(i) := \mu_D(\pi(i))$. For each $\rho \in S_{2\ell+2}$, we see that $G_{u\circ\rho} = \mu_D(\pi\rho M)$. So we find that

$$\mu_D(\kappa) = \sum_{\rho \in S_{2\ell+2}} G_{u\circ\rho}.$$

This shows that $\gamma \in \mathcal{J}_{2\ell} \cap \mathcal{G}_D$ and hence that $\mu_D(\ker(\tau)) \subseteq C\mathcal{G}_D \cap \mathcal{J}_{2\ell}$.

For the converse inclusion, let $\gamma \in \mathcal{J}_{2\ell} \cap \mathcal{G}_D$. By (5.35), we may assume that

$$\gamma = \sum_{\rho \in S_{2\ell+2}} G_{u\circ\rho},$$

where $G = (V, E)$ is a graph and where $u : [2\ell + 2] \to V$ is a map such that G_u has degree sequence D. Choose $N \in \mathcal{M}_{2m}$ such that $\mu_D(N) = G_u$. Let

$\pi \in S_{2m}$ be a permutation such that $\pi^{-1}N = M$, where $M = \{\{2i-1, 2i\} \mid i \in [m]\}$, and such that $u(i) = \mu_D(\pi(i))$ for each $i \in [2\ell+2]$.

For each $\rho \in S_{2\ell+2}$, we find that $\mu_D(\pi\rho\pi^{-1}N) = G_{u\circ\rho}$. This shows that

$$\gamma = \sum_{\rho \in S_{2\ell+2}} G_{u\circ\rho} = \sum_{\rho \in S_{2\ell+2}} \mu_D(\pi\rho\pi^{-1}N) = \mu_D(\pi \sum_{\rho \in S_{2\ell+2}} \rho M).$$

So $\gamma \in \mu_D(\ker(\tau))$ by Lemma 5.6 and hence $\mathbb{C}\mathcal{G}_D \cap \mathcal{J}_{2\ell} \subseteq \mu_D(\ker(\tau))$. This proves the second part of (5.37) and hence proves the second part of Proposition 5.5. $\square$

5.3 Connections with the invariant theory of the orthosymplectic supergroup

In the previous section we have seen how the invariant theory of the symplectic group is related to skew partition functions. In this section we will sketch how the invariant theory of the orthosymplectic supergroup is related to mixed partition functions. Let us first formulate a conjecture.

Conjecture 5.7. *Let $k, \ell \in \mathbb{N}$. Then a graph parameter $f : \mathcal{G} \to \mathbb{C}$ is the partition function of an element $h \in (SV_k \otimes \bigwedge V_{2\ell})^*$ if and only if $f(\bigcirc) = k - 2\ell$, $f(\emptyset) = 1$, f is multiplicative and $f(\mathcal{J}_{k,2\ell}) = 0$.*

Note that Theorem 4.6 gives the forward implication in this conjecture. We hope to prove the converse implication using the invariant theory of the orthosymplectic supergroup. We sketch some of the ideas.

Lehrer and Zhang gave the FFT and the SFT of invariant theory for the orthosymplectic supergroup in [20] and [21]. They prove a more general statement than we need. For a direct reference, see [42]. We also refer to [42] for more background on the orthosymplectic supergroup. We first give the necessary background on super vector spaces.

A super vector space W is a vector space with a $\mathbb{Z}/2\mathbb{Z}$ grading, i.e., $W = W_0 \oplus W_1$. For a homogeneous element $w \in W_i$ we define the parity $|w|$ of w to be i. The subspace W_0 is also referred to as the *even* part of W and the subspace W_1 is also referred to as the *odd* part of W. The space $\mathrm{End}(W)$ naturally inherits the structure of a super vector space: we write $\mathrm{End}(W) = \mathrm{End}(W)_0 \oplus \mathrm{End}(W)_1$, where $\mathrm{End}(W)_0$ consists of those $X \in \mathrm{End}(W)$ such that $|Xw| = |w|$ for all homogeneous $w \in W$ and where $\mathrm{End}(W)_1$ consists of those $X \in \mathrm{End}(W)$ such that $|Xw| = |w| + 1 \mod 2$ for all homogeneous $w \in W$.

The *super symmetric algebra* $\mathcal{S}(W)$ over the super vector space W is defined as the quotient of the tensor algebra TW by the ideal generated by elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x$, for x, y homogeneous elements of W. The algebra $\mathcal{S}(W)$ inherits a super structure: for homogeneous $x_1, \ldots, x_n$, the image of $x_1 \otimes \cdots \otimes x_n$ under the quotient map has parity $\sum_{i=1}^{n} |x_i|$.

We briefly recall some relevant linear algebra. Let $k, \ell \in \mathbb{N}$. Recall that V_k is equipped with a symmetric bilinear form $(\cdot,\cdot)$ and that $V_{2\ell}$ is equipped with a skew-symmetric bilinear form $\langle\cdot,\cdot\rangle$. We extended these bilinear forms to a bilinear form $[\cdot,\cdot]$ on $V_{k,2\ell}$. In the previous section we have seen that $\mathrm{Sp}_{2\ell}$, the symplectic group, acts on $\bigwedge V_{2\ell}$. The *orthogonal group* O_k is the group of $k \times k$ matrices that preserve the symmetric bilinear form; i.e., for $g \in \mathbb{C}^{k\times k}$, $g \in \mathrm{O}_k$ if and only if $(gx, gy) = (x, y)$ for all $x, y \in V_k$. The orthogonal group O_k has a natural action on SV_k. Combining these two actions, we find that the group $\mathrm{O}_k \times \mathrm{Sp}_{2\ell}$ acts on $R = S(SV_k \otimes \bigwedge_0 V_{2\ell})$.

If $\ell = 0$, then the image of p is exactly the space of O_k-invariant elements in $S(SV_k)$ and if $k = 0$, then the image of p is exactly the space of $\mathrm{Sp}_{2\ell}$-invariant elements of $S(\bigwedge_0 V_{2\ell})$. For k, ℓ both positive, we find that the image of p is still contained in the space of $\mathrm{O}_k \times \mathrm{Sp}_{2\ell}$-invariant elements of $S(SV_k \otimes \bigwedge_0 V_{2\ell})$, but unfortunately equality does not hold. This is where the orthosymplectic supergroup comes into play.

We view $V_{k,2\ell} = V_k \oplus V_{2\ell}$ as a super vector space where V_k is the even part of $V_{k,2\ell}$ and $V_{2\ell}$ is the odd part of $V_{k,2\ell}$. The orthosymplectic Lie superalgebra $\mathfrak{osp}(V_{k,2\ell}) \subseteq \mathrm{End}(V_{k,2\ell})$ is the Lie superalgebra preserving the form $[\cdot,\cdot]$, i.e., for each $X \in \mathfrak{osp}(V_{k,2\ell})$, we have $[Xv, w] - (-1)^{|X||v|}[v, Xw] = 0$ for all $v, w \in V_{k,2\ell}$, where we assume all elements involved to be homogenous. We interpret the orthosymplectic supergroup $\mathrm{OSp}(V_{k,2\ell})$ as a pair $(\mathrm{O}_k \times \mathrm{Sp}_{2\ell}, \mathfrak{osp}(V_{k,2\ell}))$. The action of $g \in \mathrm{O}_k \times \mathrm{Sp}_{2\ell}$ on $V_{k,2\ell}^{\otimes 2m}$ is given by the diagonal action. So if $v = v_1 \otimes \cdots \otimes v_{2m} \in V_{k,2\ell}^{\otimes 2m}$, then $g \cdot v = gv_1 \otimes \cdots \otimes gv_{2m}$. The action of $X \in \mathfrak{osp}(V_{k,2\ell})$ on v is given by

$$X \cdot v = \sum_{i=1}^{2m} (-1)^{|X|(\sum_{j=1}^{i-1} |v_j|)} v_1 \otimes \cdots \otimes v_{i-1} \otimes Xv_i \otimes v_{i+1} \otimes \cdots \otimes v_{2m}, \tag{5.41}$$

where we assume all elements involved to be homogeneous. If M is an $\mathrm{OSp}(V_{k,2\ell})$-module, then $M^{\mathrm{OSp}(V_{k,2\ell})}$, the subspace of $\mathrm{OSp}(V_{k,2\ell})$-invariants of M is defined as

$$\{v \in M \mid X \cdot v = 0 \text{ and } g \cdot v = v \text{ for all } X \in \mathfrak{osp}(V_{k,2\ell}) \text{ and all } g \in \mathrm{O}_k \times \mathrm{Sp}_{2\ell}\}.$$

It follows from the work of Lehrer and Zhang [20] that the image of τ in (5.9) is actually equal to $(V_{k,2\ell}^{\otimes 2m})^{\mathrm{OSp}(V_{k,2\ell})}$.

We view $S(SV_k \otimes \bigwedge_0 V_{2\ell})$ as a linear subspace of $\mathcal{S}(\mathcal{S}(V_{k,2\ell}))$. There is a natural action of $\mathrm{O}_k \times \mathrm{Sp}_{2\ell}$ on $\mathcal{S}(\mathcal{S}(V_{k,2\ell}))$. Similar to what we have seen in the previous section, we can describe $\mathcal{S}(\mathcal{S}(V_{k,2\ell}))$ as a direct sum of quotients of $V_{k,2\ell}^{\otimes 2m}$ by subgroups of S_{2m}, where m runs through $\mathbb{N}$. As the action of $\mathfrak{osp}(V_{k,2\ell})$ commutes with the S_{2m}-action on $V_{k,2\ell}^{\otimes 2m}$, there is a natural action of $\mathfrak{osp}(V_{k,2\ell})$ on $\mathcal{S}(\mathcal{S}(V_{k,2\ell}))$.

If we project the space of $(\mathrm{O}_k \times \mathrm{Sp}_{2\ell}, \mathfrak{osp}(V_{k,2\ell}))$-invariants in $\mathcal{S}(\mathcal{S}(V_{k,2\ell}))$ to $S(SV_k \otimes \bigwedge_0 V_{2\ell})$, then we get exactly the image of p. In future work we hope that by using a similar argument as in the proof of Proposition 5.5 we can use the Nullstellensatz to show that the reverse statement in Conjecture 5.7 indeed is true.

Chapter 6

Partition functions and the algebra of fragments

In this chapter we prove Theorem 4.3. One direction of follows directly from Theorem 4.5. For the other direction, we make use of Theorem 4.4 and the algebra of fragments, as defined by Schrijver in [38] (the concept goes back to [12]). This chapter is based on [31].

6.1 The algebra $\mathcal{A}_t$

We now give an algebra structure to the fragments. Let $t \in \mathbb{N}$. Recall that a $2t$-fragment is a graph with $2t$ labeled vertices of degree 1 labeled $1, \ldots, 2t$. The set of $2t$-fragments is denoted by $\mathcal{F}_{2t}$. A labeled vertex of a fragment is referred to as an *open end*.

It is convenient to refer to the open ends labeled $1, \ldots, t$ of elements of $\mathcal{F}_{2t}$ as the *left open ends* and we relabel these as $l_1, \ldots, l_t$; the open ends labeled $t+1, \ldots, 2t$ are referred to as *right open ends* and we relabel these as $r_1, \ldots, r_t$.

Let G be a graph with two vertices v_1 and v_2 of degree 1. The graph obtained by *gluing* v_1 *and* v_2 is the graph obtained from G by identifying v_1 and v_2 and subsequently smoothening the identified vertex. If we glue labeled vertices in a fragment, then we disregard the labeling of the identified vertices after the gluing operation.

For $F_1, F_2 \in \mathcal{F}_{2t}$, let F_1F_2 be the $2t$-fragment obtained from the disjoint union of F_1 and F_2 by gluing the open end labeled r_i of F_1 and the open end labeled l_i of F_2, for $i = 1, \ldots, t$. We extend this bilinearly to obtain an associative multiplication on $\mathbb{C}\mathcal{F}_{2t}$, making $\mathbb{C}\mathcal{F}_{2t}$ into an associative algebra. Notice that this extends the algebra structure we defined on $\mathbb{C}\mathcal{G} = \mathbb{C}\mathcal{F}_0$, where the multiplication of two graphs is given by their disjoint union. The unit, $\mathbf{1}_t$, in $\mathbb{C}\mathcal{F}_{2t}$ is given by t disjoint edges $e_1, \ldots, e_t$ such that the endpoints of e_i are labeled l_i and r_i. Following [12], we sometimes call elements of $\mathbb{C}\mathcal{F}_{2t}$ *quantum fragments*, and *quantum graphs* if $t = 0$. Recall that, for $F_1, F_2 \in \mathcal{F}_{2t}$, we defined $F_1 * F_2$ to be the graph obtained by taking the disjoint union of F_1 and F_2 and gluing equally labeled open ends.

Let $f : \mathcal{G} \to \mathbb{C}$ be a multiplicative graph parameter. We define

$$\mathcal{I}_{2t} := \{\gamma \in \mathbb{C}\mathcal{F}_{2t} \mid f(\gamma * F) = 0 \text{ for all } F \in \mathcal{F}_{2t}\}.$$

Then $\mathcal{I}_{2t}$ is a two-sided ideal in $\mathbb{C}\mathcal{F}_{2t}$. So

$$\mathcal{A}_t := \mathbb{C}\mathcal{F}_{2t}/\mathcal{I}_{2t}$$

is an associative algebra. We have a non-degenerate symmetric bilinear form on $\mathcal{A}_t$ defined by $(x, y) \mapsto f(x * y)$ for $x, y \in \mathcal{A}_t$ (this is well-defined as $f(x * y)$ is independent of the choice of representatives x and y). Assume that there exists an $r \in \mathbb{R}$ such that $\mathrm{rk}(M_{f,2t}) \leq r^{2t}$ and for each t. Note that $\mathcal{I}_{2t}$ can be identified with the kernel of the matrix $M_{f,2t}$ and hence

$$\dim(\mathcal{A}_t) = \mathrm{rk}(M_{f,2t}) \leq r^{2t} \tag{6.1}$$

for each $t \in \mathbb{N}$. Define $\tau : \mathcal{A}_t \to \mathbb{C}$, for $x \in \mathcal{A}_t$, by

$$\tau(x) := f(x * \mathbf{1}_t). \tag{6.2}$$

Using this function τ and the fact that $\dim(\mathcal{A}_t)$ is bounded by r^{2t} for each t, Schrijver [38, Propositions 5 and 6] showed that the algebras $\mathcal{A}_t$ have some useful properties that we will use in the proof of Theorem 4.3. Recall that an *idempotent* in an algebra A is an element $x \in A$ such that $x^2 = x$.

6.2 Proof of Theorem 4.3

We recall the statement of the theorem.

Theorem. *A graph parameter $f : \mathcal{G} \to \mathbb{C}$ is a skew partition function if and only if $f(\emptyset) = 1$, $f(\bigcirc) \leq 0$ and*

$$\mathrm{rk}\,(M_{f,2t}) \leq f(\bigcirc)^{2t} \tag{6.3}$$

for each $t \in \mathbb{N}$.

Proof. We first prove the forward direction. If f is a skew partition function, then there is an $\ell \in \mathbb{N}$ and an element $h \in (\bigwedge V_{2\ell})^*$ such that f is the partition function of h. Then $f(\emptyset) = 1$, $f(\bigcirc) = -2\ell \leq 0$ and (6.3) holds for each $t \in \mathbb{N}$ by Theorem 4.5.

Let us now prove the other direction. Let $f : \mathcal{G} \to \mathbb{C}$ be a graph parameter such that $f(\emptyset) = 1$, $f(\bigcirc) \leq 0$ and such that (6.3) holds for each $t \in \mathbb{N}$. As the rank of $M_{f,0}$ is at most 1 and $f(\emptyset) = 1$, f is multiplicative. If $f(\bigcirc) = 0$, then

$M_{f,2t} = 0$ for all $t > 0$, so $f(G) = 0$ if $G \neq \emptyset$. Hence f is the partition function of the unique $h \in (\bigwedge V_{2\ell})^*$ with $\ell = 0$. So we can assume that $f(\bigcirc) < 0$. It follows from Corollary 3.5 that there exists an $\ell \in \mathbb{N}_{>0}$ such that $f(\bigcirc) = -2\ell$.

In Proposition 5 in [38] Schrijver showed that the algebra $\mathcal{A}_t$ is semisimple for each $t \in \mathbb{N}$. Let us now show that

$$\text{if } x \text{ is a non-zero idempotent in } \mathcal{A}_t \text{, then } \begin{cases} \tau(x) \in \mathbb{N}_{<0} \text{ if } t \text{ is odd,} \\ \tau(x) \in \mathbb{N}_{>0} \text{ if } t \text{ is even.} \end{cases} \tag{6.4}$$

This follows almost directly from Proposition 6 in [38] by Schrijver. It follows from the proof of Proposition 6 in [38] that for an idempotent x of $\mathcal{A}_t$ we have that $\tau(x) \in \mathbb{Z}$ and

$$|\tau(x)| \leq |f(\bigcirc)^t|. \tag{6.5}$$

Note that $\tau(\mathbf{1}_t) = f(\bigcirc)^t$. If x is an idempotent of $\mathcal{A}_t$, then $\mathbf{1}_t - x$ is also an idempotent. For an idempotent x, we find that

$$\tau(\mathbf{1}_t - x) = \tau(\mathbf{1}_t) - \tau(x) = f(\bigcirc)^t - \tau(x). \tag{6.6}$$

If t is odd, then (6.6) and (6.5) imply that $\tau(x) \leq 0$. If t is even, then (6.6) and (6.5) imply that $\tau(x) \geq 0$. Schrijver furthermore shows that for a non-zero idempotent x, we have $\tau(x) \neq 0$. This shows (6.4).

Let $k, m \in \mathbb{N}$. Then, following [38], for $\pi \in S_m$, let $P_{k,\pi}$ be the $2km$-fragment consisting of km disjoint edges $e_{i,j}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, k$, where $e_{i,j}$ connects the vertices labeled $j + (i-1)k$ and $km + j + (\pi(i) - 1)k$. We define $q_{k,m}$ to be

$$q_{k,m} := \sum_{\pi \in S_m} P_{k,\pi}.$$

Let $o(\pi)$ be the number of orbits of the permutation π. If $m > (2\ell)^k$ and k is odd, we have

$$\begin{aligned} \tau(q_{k,m}) &= \sum_{\pi \in S_m} ((-2\ell)^k)^{o(\pi)} = (-1)^m \sum_{\pi \in S_m} (-1)^{m-o(\pi)} (2\ell)^{ko(\pi)} \\ &= (-1)^m \sum_{\pi \in S_m} \operatorname{sgn}(\pi) ((2\ell)^k)^{o(\pi)} = 0, \end{aligned} \tag{6.7}$$

since $\sum_{\pi \in S_m} \operatorname{sgn}(\pi) x^{o(\pi)} = x(x-1)\cdots(x-m+1)$.

We apply Theorem 4.4 to show that f is a skew partition function. Recall the definition of G_u given in (5.27). Using (6.7), we first show that, for each graph $G = (V, E)$ and $u : [2\ell + 2] \to V$, f satisfies

$$\sum_{\rho \in S_{2\ell+2}} f(G_{u \circ \rho}) = 0. \tag{6.8}$$

Let $m = 2\ell + 2$ and consider $q_{1,m}$. Then, by (6.7), we have $\tau(q_{1,m}) = 0$. Since $\frac{1}{m!}\, q_{1,m}$ is an idempotent, (6.4) implies that $q_{1,m} = 0$ in $\mathcal{A}_{\ell+1}$. In other words, $q_{1,m} \in \mathcal{I}_m$.

Now let $G = (V, E)$ be a graph and let $u : [m] \to V$ be such that G_u is Eulerian. Let $F \in \mathcal{F}_m$ be obtained from G as follows: for each $i \in [m]$ add an open end v_i labeled i to the graph and add the edge $\{v_i, u(i)\}$ to the graph. Then for all $\rho \in S_m$, we have that $F * P_{1,\rho} = G_{u\circ\rho}$. Hence, as $q_{1,m} \in \mathcal{I}_m$,

$$0 = f(F * q_{1,m}) = \sum_{\rho \in S_m} f(G_{u\circ\rho}), \tag{6.9}$$

proving (6.8).

Finally, we show that $f(G) = 0$ if G is non-Eulerian. Let $G = (V, E)$ be a graph with $v \in V$ such that $d(v) = k$ is odd. Define fragments $F_0, F_1 \in \mathcal{F}_k$ as follows: F_0 has $k + 1$ vertices, of which k are open ends and of which one has degree k and is a neighbor of all open ends (i.e., it is a star of which all the vertices of degree 1 are labeled); F_1 is obtained from G by removing v from G and for each loop $\{v, v\}$ at v adding an edge between two open ends to G and for each non-loop edge $\{u, v\}$ adding an edge $\{u, w\}$, where w is an open end, to G. Then $F_0 * F_1 = G$. Now take m such that $m > (2\ell)^k$. Then $\frac{1}{m!}q_{k,m}$ is an idempotent and by (6.7), $\tau(q_{k,m}) = 0$, and so, by (6.4), $q_{k,m}$ is actually 0 in $\mathcal{A}_{km}$. Now, take m copies of both F_0 and F_1 and create a fragment $F \in \mathcal{F}_{km}$ from their disjoint union as follows: the end labeled j in F_i gets label $ikm + j + k(n-1)$ in the n-th copy of F_i. Then, as $q_{k,m} \in \mathcal{I}_{mk}$,

$$0 = f(F * q_{k,m}) = m!(f(F_0 * F_1)^m) = m!(f(G)^m).$$

So $f(G) = 0$. Now it follows from Theorem 4.4 that f is indeed a skew partition function. $\square$

Chapter 7

Reflection positivity for 3-graphs

There is a connection between invariants for 3-graphs and knot invariants, but we will not go into detail here. For more information, see the book by Chmutov, Duzhin and Mostovoy [9]. In this chapter we prove a theorem on 3-graphs similar to Theorem 2.2 by Szegedy. We follow an approach by Schrijver that makes use of a theorem of Procesi and Schwarz [35]. This chapter is based on [29].

7.1 Partition functions and k-joins for 3-graphs

We restrict ourselves to $\mathbb{R}$, but the following concepts can be defined over any field. A 3-*graph* is a non-empty connected cubic graph with at each vertex a cyclic order of the edges incident with it (a *cubic graph* is a graph of which each vertex has degree 3). The collection of 3-graphs is denoted by $\mathcal{T}$. The graph $\bigcirc$ is also an element of $\mathcal{T}$. We let $\mathcal{T}'$ be the collection of finite disjoint unions of 3-graphs.

For $n \in \mathbb{N}$, the linear space of tensors in $(\mathbb{R}^n)^{\otimes 3}$ that are invariant under the natural action of the cyclic group C_3 on $(\mathbb{R}^n)^{\otimes 3}$ is denoted by $((\mathbb{R}^n)^{\otimes 3})^{C_3}$. An element $c = (c_{ijk})_{i,j,k=1}^n$ of $((\mathbb{R}^n)^{\otimes 3})^{C_3}$ is called a 3-*graph edge coloring model over* $\mathbb{R}$. For any 3-graph $G = (V, E)$ and 3-graph edge coloring model $c \in ((\mathbb{R}^n)^{\otimes 3})^{C_3}$, define

$$f_c(G) := \sum_{\phi: E \to [n]} \prod_{v \in V} c_{\phi(e_1)\phi(e_2)\phi(e_3)}, \tag{7.1}$$

where, when $v \in V$ is chosen, e_1, e_2, e_3 denote the edges incident with v, in cyclic order. This is well-defined as $c \in ((\mathbb{R}^n)^{\otimes 3})^{C_3}$. Now f_c is the *partition* *function* of the 3-graph edge coloring model $c = (c_{ijk})_{i,j,k=1}^n$. It follows directly that $f_c(\bigcirc) = n$.

Let $\mathbb{R}[\mathcal{T}]$ denote the commutative $\mathbb{R}$-algebra freely generated by the collection of 3-graphs. Any function from $\mathcal{T}$ to any $\mathbb{R}$-algebra can be extended uniquely to an algebra homomorphism on $\mathbb{R}[\mathcal{T}]$. We identify the product $G_1 \cdots G_k$ of 3-graphs in $\mathbb{R}[\mathcal{T}]$ with the disjoint union of $G_1, \ldots, G_k$, which is a cubic graph with a cyclic ordering at each vertex. So the collection $\mathcal{T}'$ of cubic graphs with a cyclic ordering at each vertex corresponds to the set of monomials in $\mathbb{R}[\mathcal{T}]$.

Let $k \in \mathbb{N}$. For G and H in $\mathcal{T}'$, the *k-join* $G \overset{k}{\vee} H$ is the element of $\mathbb{R}[\mathcal{T}]$ obtained as follows. We first take the disjoint union of G and H. Then we choose distinct vertices $u_1, \ldots, u_k$ of G and distinct vertices $v_1, \ldots, v_k$ of H, and, for $i = 1, \ldots, k$ we apply the following transformation, where the orientations at v_i and u_i are clockwise

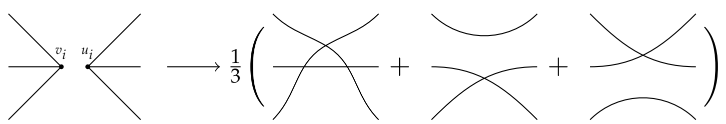


Figure 7.1: The join operation for 3-graphs

Note that we join the two triples of edges in cyclic order. We denote this element of $\mathbb{R}[\mathcal{T}]$ by $G_{u_1,\ldots,u_k} * H_{v_1,\ldots,v_k}$. Finally, $G \overset{k}{\vee} H$ is obtained by adding up these elements of $\mathbb{R}[\mathcal{T}]$ over all choices of distinct $u_1, \ldots, u_k \in V(G)$ and distinct $v_1, \ldots, v_k \in V(H)$:

$$G \overset{k}{\vee} H := \sum_{u_1,\ldots,u_k \in V(G)} \sum_{v_1,\ldots,v_k \in V(H)} G_{u_1,\ldots,u_k} * H_{v_1,\ldots,v_k}. \tag{7.2}$$

A function $f : \mathcal{T} \to \mathbb{R}$ is called *weakly reflection positive* if for each $k \in \mathbb{N}$ the $\mathcal{T}' \times \mathcal{T}'$ matrix $M_{f,k}$ defined by $M_{f,k}(G, H) := f(G \overset{k}{\vee} H)$ is positive semidefinite.

We can extend $G \overset{k}{\vee} H$ bilinearly to a bilinear function $\mathbb{R}[\mathcal{T}] \times \mathbb{R}[\mathcal{T}] \to \mathbb{R}[\mathcal{T}]$. Then weak reflection positivity means that $f(\gamma \overset{k}{\vee} \gamma) \geq 0$ for each $\gamma \in \mathbb{R}[\mathcal{T}]$ and each $k \in \mathbb{N}$. We can now state our main theorem on 3-graphs.

Theorem 7.1. *A function $f : \mathcal{T} \to \mathbb{R}$ is the partition function of some 3-graph edge coloring model over $\mathbb{R}$ if and only if f is weakly reflection positive.*

Let us see how this is related to Theorem 2.2 by Szegedy. Recall that for $t \in \mathbb{N}$ a t-fragment is a graph with t labeled vertices of degree one labeled $1, \ldots, t$. The t-th edge connection matrix of a parameter f is indexed by t-fragments with entry $f(F_1 * F_2)$ at the (F_1, F_2) position. We could give a similar definition of fragment in the 3-graph setting: a connected graph with t vertices of degree one labeled $1, \ldots, t$ such that all unlabeled vertices have degree three and a cyclic ordering of the edges incident with it. The connection matrix is then obtained by gluing equally labeled vertices and then smoothening the vertices of degree two to obtain a 3-graph. Following an argument of Schrijver [35, Corollary 1a], we can write $G_1 \overset{t}{\vee} G_2$ for any two 3-graphs G_1, G_2 as

$$G_1 \overset{t}{\vee} G_2 = (\frac{1}{3^t} \sum F_i) * (\frac{1}{3^t} \sum F_j),$$

where the sums run over certain sets of $3t$-fragments. This shows that the characterization using t-joins is stronger than the one given using t-fragments, as the condition is weaker.

Before proving the theorem, we first derive a corollary for real-valued weight systems. If $f : \mathcal{T} \to \mathbb{R}$ is a function that respects the relation $G = -G'$, where G' is obtained from G by reversing the cyclic order at one vertex of G, then we say that f satisfies the *AS-relation*. If the function f respects the relation in Figure 7.2 below, then we say that f satisfies the *IHX-relation*. In Figure 7.2 the cyclic ordering of the edges incident with a vertex is clockwise and we assume that the graph remains unchanged outside of the drawing.

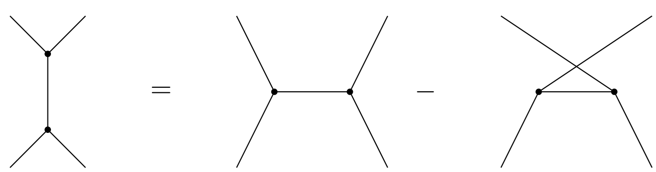


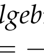
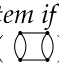
Figure 7.2: The IHX relation

A function $f : \mathcal{T} \to \mathbb{R}$ is a called a (real-valued) *weight system* if it satisfies both the AS-relation and IHX-relation. Key instances of weight systems are the *Lie algebra weight systems*: the partition functions f_c of the structure tensor c of a finite-dimensional Lie algebra $\mathfrak{g}$, expressed in a basis that is orthonormal with respect to some symmetric ad-invariant bilinear form on $\mathfrak{g}$. For $c \in ((\mathbb{R}^n)^{\otimes 3})^{C_3}$ this amounts to c satisfying the following two properties:

$$(i)\ c_{kij} = -c_{kji} \text{ for all } k, i, j \in [n], \tag{7.3}$$

$$(ii)\ \sum_a c_{ija}c_{akl} + c_{ila}c_{ajk} + c_{ika}c_{alj} = 0 \text{ for all } i, j, k, l \in [n]. \tag{7.4}$$

The first property corresponds to the Lie bracket being antisymmetric, the second to it satisfying the Jacobi identity. This roots in the work of Penrose [25], Murphy [23], Bar-Natan [2] and Kontsevich [19].

Corollary 7.2. *A function $f : \mathcal{T} \to \mathbb{R}$ is a Lie algebra weight system if and only if f is weakly reflection positive and satisfies $f($$) = -f($$)$ and $f($$) = 2f($$)$.*

Proof. This follows from Theorem 7.1, as for any n and any $c \in ((\mathbb{R}^n)^{\otimes 3})^{C_3}$, if $f_c(\;) = -f_c(\;)$, then c is an alternating tensor, as $f_c(\;) = -f_c(\;)$ is equivalent to

$$\sum_{i,j,k}(c_{ijk} + c_{ikj})^2 = 0, \text{ and hence to: } c_{ikj} = -c_{ijk} \text{ for all } i,j,k. \tag{7.5}$$

This shows that (7.3) is satisfied. Moreover, if c is alternating, then $f_c(\;) = 2f_c(\;)$ gives us

$$\sum_{i,j,k,l}\Big(\sum_a (c_{ija}c_{akl} + c_{ila}c_{ajk} + c_{ika}c_{alj})\Big)^2 = 0.$$

This shows that (7.4) is also satisfied. □

The rest of this chapter is devoted to proving Theorem 7.1. In Lemma 7.4 we will see that a weakly reflection positive function $f : \mathcal{T} \to \mathbb{R}$ has $f(\bigcirc) \in \mathbb{N}$ using Theorem 3.3 by Hanlon and Wales. Then, using the invariant theory of the orthogonal group and a theorem by Processi and Schwarz [26], we prove Theorem 7.1. Before deciding on the value of $\bigcirc$, we first prove a lemma on k-joins.

7.2 A lemma on k-joins

In the following lemma, ϑ denotes the 3-graph , and ϑ^i is the i-th power of ϑ, that is, the disjoint union of i copies of .

Lemma 7.3. *For any k and any $G \in \mathcal{T}'$ with n vertices:*

$$\binom{n}{k} G = 2^{-k}k!^{-2}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}(G \overset{k}{\vee} \vartheta^i)\vartheta^{k-i}. \tag{7.6}$$

Proof. For each i, let $G \underline{\overset{k}{\vee}} \vartheta^i$ be equal to the sum describing $G \overset{k}{\vee} \vartheta^i$ in (7.2) (with $H := \vartheta^i$) restricting the summation to those $v_1, \ldots, v_k$ where each connected component of ϑ^i contains at least one vertex among $v_1, \ldots, v_k$. So for each i, $G \overset{k}{\vee} \vartheta^i = \sum_{j=0}^{i}\binom{i}{j}(G \underline{\overset{k}{\vee}} \vartheta^j)\vartheta^{i-j}$. Hence

$$\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}(G \overset{k}{\vee} \vartheta^i)\vartheta^{k-i} = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\sum_{j=0}^{i}\binom{i}{j}(G \underline{\overset{k}{\vee}} \vartheta^j)\vartheta^{k-j} =$$

$$\sum_{j=0}^{k}\binom{k}{j}(G \underline{\overset{k}{\vee}} \vartheta^j)\vartheta^{k-j}\sum_{i=j}^{k}(-1)^{k-i}\binom{k-j}{k-i} = G \underline{\overset{k}{\vee}} \vartheta^k = 2^k k!^2\binom{n}{k}G,$$

the last equality because $u_1, \ldots, u_k$ can be chosen in $\binom{n}{k}k!$ ways and $v_1, \ldots, v_k$ in $2^k k!$ ways, while each term of $G \underline{\overset{k}{\vee}} \vartheta^k$ is equal to G. □

7.3 The value of f on $\bigcirc$

This section is devoted to proving the following lemma.

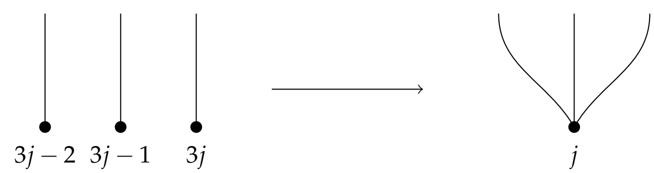
Lemma 7.4. *If $f : \mathcal{T} \to \mathbb{R}$ is weakly reflection positive, then $f(\bigcirc) \in \mathbb{N}$.*

Let $f : \mathcal{T} \to \mathbb{R}$ be weakly reflection positive. A direct computation shows

$$(\vartheta - \vartheta') \overset{2}{\vee} (\vartheta - \vartheta') = \frac{2}{3} \bigcirc (\bigcirc - 1)(\bigcirc - 2). \tag{7.7}$$

By the weak reflection positivity of f this implies $f(\bigcirc)(f(\bigcirc) - 1)(f(\bigcirc) - 2) \geq 0$, hence $f(\bigcirc) \geq 0$. To prove that $f(\bigcirc)$ is integer, define $k := \lceil f(\bigcirc) \rceil + 1$.

Recall that $\mathcal{M}_{6k}$ is the set of perfect matchings on $[6k]$. To each $M \in \mathcal{M}_{6k}$ we can associate a graph $G_M \in \mathcal{T}'$ on $[2k]$ by identifying the vertices $3j - 2, 3j - 1, 3j$ of $([6k], M)$ for $j \in [2k]$, with the cyclic order at j in the following way



. (7.8)

For all $M, N \in \mathcal{M}_{6k}$, $G_M \overset{2k}{\vee} G_N$ is a polynomial in $\bigcirc$, since both G_M and G_N have $2k$ vertices. To describe this polynomial, we recall the natural action of the symmetric group S_{6k} on $\mathcal{M}_{6k}$: for $M \in \mathcal{M}_{6k}$ and $\pi \in S_{6k}$ we define $\pi M = \{\pi(e) \mid e \in M\}$. This induces an action on $\mathbb{R}^{\mathcal{M}_{6k}}$ and makes $\mathbb{R}^{\mathcal{M}_{6k}}$ an S_{6k}-module.

For $j \in [2k]$, let B_j be the group of cyclic permutations of $\{3j-2, 3j-1, 3j\}$, and define $B := B_1 B_2 \cdots B_{2k}$. Let D be the group of permutations $\delta \in S_{6k}$ for which there exists $\pi \in S_{2k}$ such that $\delta(3j - i) = 3\pi(j) - i$ for each $j = 1, \ldots, 2k$ and $i = 0, 1, 2$. Set $Q := BD$, which can be seen to be a group again.

For $M, N \in \mathcal{M}_{6k}$, recall that $c(M \cup N)$ denotes the number of connected components of $([6k], M \cup N)$. Then, by definition of the operation $\overset{2k}{\vee}$, we have

$$G_M \overset{2k}{\vee} G_N = (2k)! 3^{-2k} \sum_{\tau \in Q} \bigcirc^{c(M \cup \tau N)}. \tag{7.9}$$

We briefly recall some concepts from Chapter 3. For $\pi \in S_{6k}$, let P_π be the $\mathcal{M}_{6k} \times \mathcal{M}_{6k}$ permutation matrix corresponding to π; that is, $P_\pi w = \pi w$ for each $w \in \mathbb{R}^{\mathcal{M}_{6k}}$. For any $x \in \mathbb{R}$, let $A(x)$ and $A^Q(x)$ be the $\mathcal{M}_{6k} \times \mathcal{M}_{6k}$ matrices defined by

$$(A(x))_{M,N} := x^{c(M \cup N)} \text{ and } A^Q(x) := \sum_{\tau \in Q} A(x) P_\tau, \tag{7.10}$$

for $M, N \in \mathcal{M}_{6k}$. Note that each P_π commutes with $A(x)$, as for all $M, N \in \mathcal{M}_{6k}$ one has $c(\pi M \cup \pi N) = c(M \cup N)$, implying $A(x) = P_\pi^T A(x) P_\pi = P_\pi^{-1} A(x) P_\pi$.

Define

$$h(x) := \prod_{i=0}^{k-1} (x-i)(x-i+2)(x+2i+4). \tag{7.11}$$

We will show that

$$|Q| h(x) \text{ is an eigenvalue of } A^Q(x). \tag{7.12}$$

This implies the lemma, since $A^Q(f(\bigcirc))_{M,N} = (2k!)^{-1} 3^{2k} f(G_M \overset{2k}{\vee} G_N)$, by (7.9). Hence, by the weak reflection positivity of f, $A^Q(f(\bigcirc))$ is positive semidefinite. So $h(f(\bigcirc)) \geq 0$, hence, as $k-1 = \lceil f(\bigcirc) \rceil$ and as $k-1$ is the largest zero of $h(x)$, with multiplicity 1, we know $f(\bigcirc) = k-1$,

To prove (7.12), we will give an eigenvector u of $A^Q(x)$ belonging to $|Q|h(x)$. We derive u from the eigenvector v of $A(x)$ belonging to $h(x)$ as described by Theorem 3.3. Consider the following Young tableau, associated to the partition $(2k+4, \underbrace{4, \ldots, 4}_{k-1})$ of $6k$:

$$T := \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & \bar{1} & 2 & \bar{2} & 3 & \bar{3} & 6 & \bar{6} & 9 & \bar{9} & \cdots & 3k & \overline{3k} \\
\hline
4 & \bar{4} & 5 & \bar{5} \\
\cline{1-4}
7 & \bar{7} & 8 & \bar{8} \\
\cline{1-4}
\vdots & \vdots & \vdots & \vdots \\
\cline{1-4}
3k-2 & \overline{3k-2} & 3k-1 & \overline{3k-1} \\
\cline{1-4}
\end{array},$$

where $\bar{i} := 3k + i$ for $i \in [3k]$.

Let F be the perfect matching in $\mathcal{M}_{6k}$ with edges $\{i, \bar{i}\}$, for $i \in [3k]$. For $i = 1, \ldots, 4$, let K_i denote the set of elements in the i-th column of T and let C_i be the subgroup of S_{6k} that permutes the elements of K_i. Then C is the group $C_1 C_2 C_3 C_4$. Similarly, for $i = 1, \ldots, k$, let R_i be the subgroup of S_{6k} that permutes the numbers in row i of T and leaves all other numbers fixed, and R is the group $R_1 \cdots R_k$. Define v and u in $\mathbb{R}^{\mathcal{M}_{6k}}$ by

$$v := \sum_{\sigma \in C, \rho \in R} \operatorname{sgn}(\sigma) \sigma \rho F \text{ and } u := \sum_{\tau \in Q} \tau v, \tag{7.13}$$

identifying an element of $\mathcal{M}_{6k}$ with the corresponding basis vector in $\mathbb{R}^{\mathcal{M}_{6k}}$. By Theorem 3.3, v is an eigenvector of $A(x)$ with eigenvalue $h(x)$. Hence

$$A^Q(x) u = \sum_{\tau', \tau \in Q} A P_{\tau'} P_\tau v = \sum_{\tau', \tau \in Q} P_{\tau'} P_\tau A v = h(x) \sum_{\tau', \tau \in Q} P_{\tau'} P_\tau v = |Q| h(x) u. \tag{7.14}$$

So to prove (7.12), and hence the lemma, it suffices to show that u is non-zero. To this end we show that the coefficient u_F of F in u is non-zero. Note that

$$u_F = \sum_{\tau \in Q} (\tau v)_F = \sum_{\tau \in Q} \sum_{\sigma \in C, \rho \in R} \operatorname{sgn}(\sigma)(\tau\sigma\rho F)_F = \sum_{\substack{\tau \in Q, \sigma \in C, \rho \in R \\ \tau\sigma\rho F = F}} \operatorname{sgn}(\sigma). \quad (7.15)$$

So it suffices to show that for any $\tau \in Q$, $\sigma \in C$, and $\rho \in R$, if $\tau\sigma\rho F = F$ then $\operatorname{sgn}(\sigma) = 1$. As Q is a group, equivalently it suffices to show for any $\tau \in Q$, $\sigma \in C$, $\rho \in R$:

$$\text{if } \tau F = \sigma\rho F, \text{ then } \operatorname{sgn}(\sigma) = 1. \quad (7.16)$$

Choose $\tau \in Q$, $\sigma \in C$, and $\rho \in R$ with $\tau F = \sigma\rho F$. Let $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4$ with $\sigma_i \in C_i$ ($i = 1, \ldots, 4$) and define $M := \tau F$. Let $\zeta \in S_{6k}$ be defined by $\zeta(i) := i + 1$ if 3 does not divide i and $\zeta(i) := i - 2$ if 3 divides i. So $\zeta^3 = \text{id}$ and $\zeta F = F$. Moreover, $\zeta\tau = \tau\zeta$ (since ζ commutes with B and with D). Hence $\zeta M = M$.

Let $\phi(i) := \bar{i}$ for $i \in [3k]$. We show that for each $a \in K_1$:

$$\sigma_2\phi\sigma_1^{-1}(a) = \zeta^{-1}\sigma_4\phi\sigma_3^{-1}\zeta(a). \quad (7.17)$$

This implies $\operatorname{sgn}(\sigma_2\sigma_1^{-1}) = \operatorname{sgn}(\sigma_4\sigma_3^{-1})$, and hence $\operatorname{sgn}(\sigma) = 1$.

As both $\sigma_2\phi\sigma_1^{-1}$ and $\zeta^{-1}\sigma_4\phi\sigma_3^{-1}\zeta$ are bijections $K_1 \to K_2$, it suffices to show (7.17) for all $a \in K_1 \setminus \{\sigma_1(1)\}$. Therefore, choose $a \in K_1$ with $i := \sigma_1^{-1}(a) \neq 1$. Let $b := \sigma_2(\bar{i}) = \sigma_2\phi\sigma_1^{-1}(a)$. Note that $b \in K_2$, $\zeta(a) \in K_3$, and $\zeta(b) \in K_4$. We must show that $\sigma_4^{-1}\zeta(b) = \phi\sigma_3^{-1}\zeta(a)$, that is, $\sigma_3^{-1}\zeta(a)$ and $\sigma_4^{-1}\zeta(b)$ belong to the same row of T.

First assume that $\{i, \bar{i}\} \in \rho F$. Then $\{a, b\} \in \sigma\rho F = M$, hence, by the ζ-invariance of M, $\{\zeta(a), \zeta(b)\} \in M$. So $\{\sigma_3^{-1}\zeta(a), \sigma_4^{-1}\zeta(b)\}$ belongs to ρF, and hence it is contained in a single row of T.

Second assume that $\{i, \bar{i}\} \notin \rho F$. Since $i \neq 1$, this implies that i and $\bar{i}$ are matched in ρF with elements of $K_3 \cup K_4$. So a and b are matched in M with elements of $K_3 \cup K_4$. Hence, by the ζ-invariance of M, $\zeta(a)$ and $\zeta(b)$ are matched in M with elements of $\zeta(K_3 \cup K_4)$, which is the first row of T outside $K_1 \cup K_2 \cup K_3 \cup K_4$. So $\sigma_3^{-1}\zeta(a)$ and $\sigma_4^{-1}\zeta(b)$ are matched in ρF with elements of the first row of T, and hence they both also belong to the first row of T.

7.4 The map p_n

Choose $n \in \mathbb{N}$ and let W be the linear space

$$W := ((\mathbb{R}^n)^{\otimes 3})^{C_3}. \quad (7.18)$$

As usual, $\mathcal{O}(W)$ denotes the algebra of polynomials on W. For each 3-graph G, define the polynomial $p_n(G) \in \mathcal{O}(W)$ by $p_n(G)(c) := f_c(G)$ for any $c \in W$ (defined in (7.1)). This can be extended uniquely to an algebra homomorphism $p_n : \mathbb{R}[\mathcal{T}] \to \mathcal{O}(W)$.

For any $q \in \mathcal{O}(W)$, let dq be its derivative, being an element of $\mathcal{O}(W) \otimes W^*$. So $d^k q \in \mathcal{O}(W) \otimes (W^*)^{\otimes k}$. Note that the standard inner product on $\mathbb{R}^n$ induces an inner product on W, hence on W^*, and hence it induces a product $\langle .,. \rangle : (\mathcal{O}(W) \otimes (W^*)^{\otimes k}) \times (\mathcal{O}(W) \otimes (W^*)^{\otimes k}) \to \mathcal{O}(W)$.

The following lemma will be used several times in our proof.

Lemma 7.5. *For all $G, H \in \mathcal{T}'$ and all $k, n \in \mathbb{N}$:*

$$p_n(G \overset{k}{\vee} H) = \langle d^k p_n(G), d^k p_n(H) \rangle. \tag{7.19}$$

Proof. Let $\{b_1, \dots, b_n\}$ be the standard basis of $\mathbb{R}^n$, with dual basis $\{b_1^*, \dots, b_n^*\}$. For $i, j, k = 1, \dots, n$, let y_{ijk} be the element $b_i^* \otimes b_j^* \otimes b_k^*|_W$ of W^*.

Consider some $G \in \mathcal{T}'$. For $\phi : E(G) \to [n]$ and $v \in V(G)$, denote

$$\widehat{\phi}_v := y_{\phi(e_1)\phi(e_2)\phi(e_3)}, \tag{7.20}$$

where e_1, e_2, e_3 are the edges incident with v, in order. Then

$$p_n(G) = \sum_{\phi : E(G) \to [n]} \prod_{v \in V(G)} \widehat{\phi}_v. \tag{7.21}$$

Hence $d^k p_n(G)$ expands as:

$$d^k p_n(G) = \sum_{\phi : E(G) \to [n]} \sum_{u_1, \dots, u_k \in V(G)} \Big(\prod_{v \in V(G) \setminus \{u_1, \dots, u_k\}} \widehat{\phi}_v \Big) \otimes \widehat{\phi}_{u_1} \otimes \cdots \otimes \widehat{\phi}_{u_k}, \tag{7.22}$$

with $u_1, \dots, u_k$ taken distinct. Now for all functions $i, j : [3] \to [n]$,

$$\langle y_{i(1)i(2)i(3)}, y_{j(1)j(2)j(3)} \rangle = \frac{1}{3} |\{\pi \in C_3 \mid j(s) = i(\pi(s)) \text{ for } s \in [3]\}|, \tag{7.23}$$

since for each $i : [3] \to [n]$ and $x \in W$, by the C_3-invariance of x:

$$y_{i(1)i(2)i(3)}(x) = \langle b_{i(1)} \otimes b_{i(2)} \otimes b_{i(3)}, x \rangle = \langle \frac{1}{3} \sum_{\pi \in C_3} b_{i(\pi(1))} \otimes b_{i(\pi(2))} \otimes b_{i(\pi(3))}, x \rangle. \tag{7.24}$$

Hence, as $\frac{1}{3} \sum_{\pi \in C_3} b_{i(\pi(1))} \otimes b_{i(\pi(2))} \otimes b_{i(\pi(3))}$ belongs to W, the left-hand side of (7.23) is equal to

$$\langle \frac{1}{3} \sum_{\pi \in C_3} b_{i(\pi(1))} \otimes b_{i(\pi(2))} \otimes b_{i(\pi(3))}, \frac{1}{3} \sum_{\rho \in C_3} b_{j(\rho(1))} \otimes b_{j(\rho(2))} \otimes b_{j(\rho(3))} \rangle, \tag{7.25}$$

which is equal to the right-hand side of (7.23), as the b_i form an orthonormal basis.

So for any $\phi : E(G) \to [n]$ and $\psi : E(H) \to [n]$ and any $u \in V(G)$ and $v \in V(H)$, $\langle \widehat{\phi}_u, \widehat{\psi}_v \rangle$ is equal to 1/3 of the number of bijections $\eta : \delta(u) \to \delta(v)$ such that $\psi \circ \eta = \phi|_{\delta(u)}$ that preserve the cyclic order. ($\delta(w)$ is the set of edges incident with a vertex w.) This being in conformity with (7.1), we have (7.19). □

Similar to what we have seen in Proposition 5.5 we now find that

$$p_n(\mathbb{R}[\mathcal{T}]) = \mathcal{O}(W)^{\mathrm{O}_n}, \tag{7.26}$$

the latter denoting the space of O_n-invariant elements of $\mathcal{O}(W)$. A direct proof of this was first given by Szegedy [39].

7.5 Proof of Theorem 7.1

To see necessity in the theorem, let $n \in \mathbb{N}$ and let $c = (c_{ijk})_{i,j,k=1}^{n} \in W$ $(= ((\mathbb{R}^n)^{\otimes 3})^{C_3})$. Then the positive semidefiniteness of $M_{f_c,k}$ follows from

$$f_c(G \overset{k}{\vee} H) = p_n(G \overset{k}{\vee} H)(c) = \langle d^k p_n(G)(c), d^k p_n(H)(c) \rangle, \tag{7.27}$$

using Lemma 7.5.

To prove sufficiency, let $f : \mathcal{T} \to \mathbb{R}$ be weakly reflection positive. By Lemma 7.4, $f(\bigcirc)$ belongs to $\mathbb{N}$. Set $n := f(\bigcirc)$. We show that $f = f_c$ for some $c \in ((\mathbb{R}^n)^{\otimes 3})^{C_3}$. First:

$$\text{there is an algebra homomorphism } F : p_n(\mathbb{R}[\mathcal{T}]) \to \mathbb{R} \text{ such that } f = F \circ p_n. \tag{7.28}$$

Otherwise, as p_n and f are algebra homomorphisms, there is a $\gamma \in \mathbb{R}[\mathcal{T}]$ with $p_n(\gamma) = 0$ and $f(\gamma) \neq 0$. We can assume that $p_n(\gamma)$ is homogeneous, that is, all graphs in γ have the same number of vertices, k say. So $\gamma \overset{k}{\vee} \gamma$ has no vertices, that is, it is a polynomial in $\bigcirc$. As moreover $f(\bigcirc) = n = p_n(\bigcirc)$, we have $f(\gamma \overset{k}{\vee} \gamma) = p_n(\gamma \overset{k}{\vee} \gamma) = 0$, the latter equality because of Lemma 7.5. By the weak reflection positivity of f this implies that $f(\gamma \overset{k}{\vee} H) = 0$ for each $H \in \mathcal{T}'$. Hence, by the linearization of (7.6) (substituting γ for G), $f(\gamma) = 0$. This proves (7.28).

As in the proof of Theorem 4.4, (7.28) with (7.26) implies the existence of c in the complex extension of W satisfying $F(q) = q(c)$ for each $q \in \mathcal{O}(W)^{\mathrm{O}_n} = p_n(\mathbb{R}[\mathcal{T}])$. To prove that we can take c real, we apply the Procesi-Schwarz theorem [26]. For all $G, H \in \mathcal{T}$, using Lemma 7.5:

$$F(\langle dp_n(G), dp_n(H) \rangle) = F(p_n(G \overset{1}{\vee} H)) = f(G \overset{1}{\vee} H) = (M_{f,1})_{G,H}. \tag{7.29}$$

Since $M_{f,1}$ is positive semidefinite, (7.29) implies that for each $q \in p_n(\mathbb{R}[\mathcal{T}])$: $F(\langle dq, dq \rangle) \geq 0$, and hence by [26] we can take c real.

Concluding, $f(G) = F(p_n(G)) = p_n(G)(c) = f_c(G)$ for each $G \in \mathcal{T}$, as required. $\square$

We finally observe that if f is the partition function of a 3-graph edge coloring model, then $f = f_c$ for some unique c, up to the natural action of O_n on c (which action leaves f_c invariant (cf. (7.26))). To see this, let $b \in$

$((\mathbb{R}^m)^{\otimes 3})^{C_3}$ and $c \in ((\mathbb{R}^n)^{\otimes 3})^{C_3}$ with $f_b = f_c$. Then $m = f_b(\bigcirc) = f_c(\bigcirc) = n$. We show that there exists $U \in \mathrm{O}_n$ such that $b = c^U$ (where $x \mapsto x^U$ is the natural action of U on $x \in W$).

Suppose to the contrary that $b \neq c^U$ for each $U \in \mathrm{O}_n$. Then the sets $\{b^U \mid U \in \mathrm{O}_n\}$ and $\{c^U \mid U \in \mathrm{O}_n\}$ are disjoint compact subsets of W. So, by the Stone-Weierstrass theorem, there exists a polynomial $q \in \mathcal{O}(W)$ such that $q(b^U) \leq 0$ and $q(c^U) \geq 1$ for each $U \in \mathrm{O}_n$. As O_n is compact, we can average q to make it O_n-invariant. Hence by (7.26), $q \in p_n(\mathbb{R}[\mathcal{T}])$, say $q = p_n(\gamma)$ with $\gamma \in \mathbb{R}[\mathcal{T}]$. Then $f_b(\gamma) = p_n(\gamma)(b) = q(b) \leq 0$ and $f_c(\gamma) = p_n(\gamma)(c) = q(c) \geq 1$. This contradicts $f_b = f_c$.

Chapter 8

Reflection positivity for virtual links

In this chapter we extend Theorem 2.2 by Szegedy [39] to virtual link diagrams. The proof follows the same line as the proof we gave in the previous chapter for 3-graphs. We only provide the necessary background on virtual links. For more information on virtual links, see the book by Chmutov, Duzhin and Mostovoy [9] or the paper by Kauffman [18]. This chapter is based on [30].

8.1 Virtual link diagrams

Virtual link diagrams were introduced by Kauffman [18]. We first give a purely combinatorial description. A *virtual link diagram* is an undirected 4-regular graph G such that at each vertex v a cyclic order of the edges incident with v is specified, together with one pair of edges opposite at v that is labeled *over crossing*. The set of virtual link diagrams is denoted by $\mathcal{V}$.

(a) A link diagram. (b) A virtual link diagram.

Figure 8.1: Two link diagrams.

When we draw a virtual link diagram in the plane, we draw it in such a way that the cyclic ordering at each vertex is clockwise. In doing so we sometimes see crossings that are artefacts of the drawing and not vertices of the diagram. We mark such a crossing by a circle, see Figure 8.1.

Two virtual link diagrams D_1 and D_2 are equivalent if we can go from D_1 to D_2 by a sequence of *Reidemeister moves*, depicted in Figure 8.2. When performing such a move the rest of the diagram remains unchanged. A *virtual link invariant* is a function on $\mathcal{V}$ that is invariant under the Reidemeister moves.

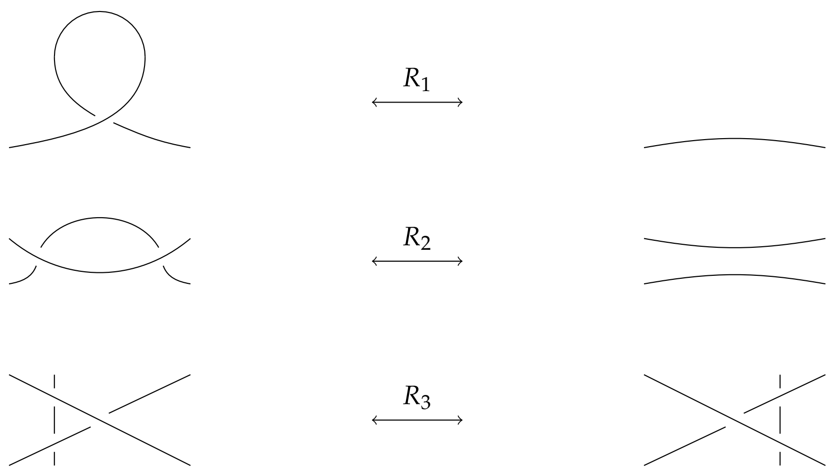


Figure 8.2: The three Reidemeister moves.

Let us briefly recall some concepts from knot theory to digest this definition. A *link* is a smooth embedding of a finite disjoint union of circles into $\mathbb{R}^3$. If we project a link to a plane and keep track of the over and underlying crossings, we get a *link diagram*. Note that a link diagram inherits a cyclic ordering of the edges incident with a vertex at each vertex from the plane. Reidemeister showed that two links are ambient isotopic if and only if their corresponding link diagrams can be obtained from one another by a sequence of Reidemeister moves [27].

Kauffman introduced virtual links as a generalization of links. A virtual link is a smooth embedding of a disjoint union of circles into $\mathbb{R} \times M$, where M is some oriented surface. If we project the virtual link to M and keep track of the over and under crossing edges, we obtain a virtual link diagram. If we apply the Reidemeister moves to the virtual link diagram, we might have to add a handle to the surface M. So the surface M is not stable under the Reidemeister moves. For more information we refer to the paper by Kauffman [18].

8.2 Partition functions and k-joins for virtual links

We work over the real numbers, but most concepts defined below can be defined over any field. Let $n \in \mathbb{N}$. Let $\sigma \in S_2$ be the non-identity element of

S_2. For $x_1, x_2, x_3, x_4 \in \mathbb{R}^n$, we define

$$\sigma(x_1 \otimes x_2 \otimes x_3 \otimes x_4) = x_3 \otimes x_4 \otimes x_1 \otimes x_2$$

and we extend this linearly to an action of S_2 on $(\mathbb{R}^n)^{\otimes 4}$. Let the space of S_2-invariant elements of $(\mathbb{R}^n)^{\otimes 4}$ be denoted by $\mathcal{R}_n$. For $R \in \mathcal{R}_n$, we express R in the standard basis $\{b_1, \ldots, b_n\}$ of $\mathbb{R}^n$, i.e., R_{ijkl} is the coefficient of $b_i \otimes b_j \otimes b_k \otimes b_l$ in R. An element of $\mathcal{R}_n$ will be referred to as a *virtual link diagram edge coloring model over* $\mathbb{R}$.

Let $G = (V, E)$ be a virtual link diagram and let e_1, e_2, e_3, e_4 be the edges incident with a vertex v in cyclic order such that e_1, e_3 is the over crossing pair. For $\phi : E \to [n]$, let $\phi(\delta(v)) = (\phi(e_1), \phi(e_2), \phi(e_3), \phi(e_4))$. Then f_R, the *partition function* of an element $R \in \mathcal{R}_n$, is defined by

$$f_R(G) = \sum_{\phi:E\to[n]} \prod_{v\in V} R_{\phi(\delta(v))}. \tag{8.1}$$

By the S_2-invariance of R this is well-defined. It is straightforward to check using Figure 8.2 that the following conditions on $R \in \mathcal{R}_n$ make f_R into a virtual link invariant:

$$\sum_a R_{iaaj} = \delta_{ij} \text{ for all } i, j, \tag{8.2}$$

$$\sum_{a,b} R_{ijab} R_{alkb} = \delta_{ik}\delta_{jl} \text{ for all } i, j, k, l, \tag{8.3}$$

$$\sum_{a,b,c} R_{iabh} R_{jkca} R_{bclm} = \sum_{a,b,c} R_{ijbc} R_{bkla} R_{camh} \text{ for all } i, j, k, l, m, h, \tag{8.4}$$

where δ_{ij} is the Kronecker delta and all indices run over $[n]$. Condition (8.4) is called the Yang-Baxter equation, which has its roots in statistical physics [3, 41]. An element $R \in \mathcal{R}_n$ that satisfies all three conditions above is called an *R-matrix*.

Let $\mathbb{R}\mathcal{V}$ be the space of formal linear combinations of elements in $\mathcal{V}$. An element of $\mathbb{R}\mathcal{V}$ is called a *quantum virtual link diagram*. Any virtual link diagram invariant can be extended uniquely to a linear function on $\mathbb{R}\mathcal{V}$.

Let $k \in \mathbb{N}$. For $G, H \in \mathcal{V}$ we define the k-join $G \overset{k}{\vee} H \in \mathbb{R}\mathcal{V}$ as the sum over all distinct $u_1, \ldots, u_k \subseteq V(G)$ and distinct $v_1, \ldots, v_k \subseteq V(H)$, where for $i = 1, \ldots, k$ we apply the transformation given in Figure 8.3. Note that for v_i and u_i these are the two ways to identify the over crossing edges at u_i with the over crossing edges at v_i that respect the cyclic ordering.

The k-th connection matrix of a function $f : \mathcal{V} \to \mathbb{R}$ is the $\mathcal{V} \times \mathcal{V}$ matrix with $M_{f,k}(G, H) = f(G \overset{k}{\vee} H)$. If f is real valued and the matrix $M_{f,k}$ is positive semidefinite for each $k \in \mathbb{N}$, then we say that f is *weakly reflection positive*. A function $f : \mathcal{V} \to \mathbb{R}$ is called multiplicative if $f(G \cup H) = f(G)f(H)$ for $G, H \in \mathcal{V}$. We can now state our theorem on virtual link invariants.

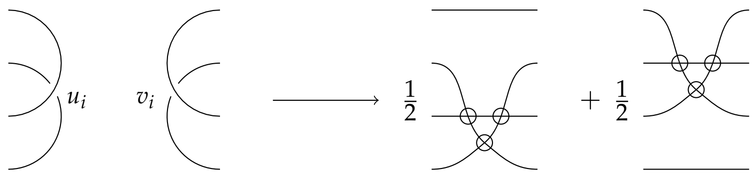


Figure 8.3: The gluing operation on virtual link diagrams.

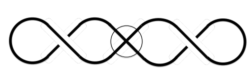
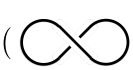
Theorem 8.1. *A function $f : \mathcal{V} \to \mathbb{R}$ is the partition function of some virtual link diagram edge coloring model over $\mathbb{R}$ if and only if f is multiplicative, f is weakly reflection positive, $f(\emptyset) = 1$ and $f(\bigcirc) \geq 0$.*

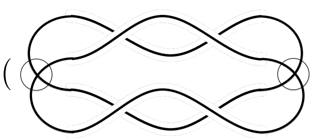
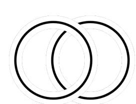
Just as we have seen for 3-graphs, the k-join is a weaker operation than gluing labeled vertices of degree 1 as in Szegedy's characterization for edge coloring models. Hence it gives a stronger characterization.

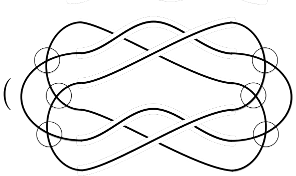
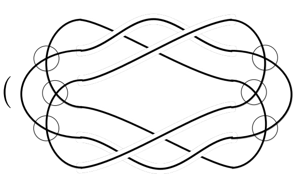
Note the assumption that $f(\bigcirc) \geq 0$ in the theorem. In Lemma 8.3 we will see that for a multiplicative, weakly reflection positive $f : \mathcal{V} \to \mathbb{R}$ we have $f(\bigcirc) \in \{\dots, -6, -4, -2, 0, 1, 2, 3, \dots\}$. This makes us wonder if there is a way to define skew partition functions for virtual link diagrams that are weakly reflection positive.

We will prove the theorem in the next sections. Working over the real numbers, we can detect when a function on virtual link diagrams actually comes from an R-matrix.

Corollary 8.2. *Let $f : \mathcal{V} \to \mathbb{R}$. Then there exists an R-matrix R with $f = p_R$ if and only if f is multiplicative, f is weakly reflection positive, $f(\emptyset) = 1$, $f(\bigcirc) \geq 0$ and f satisfies*

(i) $f($  $) + f(\bigcirc) = 2f($  $)$,

(ii) $f($  $) + f(\bigcirc)^2 = 2f($  $)$,

(iii) $f($  $) = f($  $)$.

Proof of Corollary 8.2. Let $f : \mathcal{V} \to \mathbb{R}$ be a function that satisfies the conditions in the statement of Corollary 8.2. By Theorem 8.1 there is some $R \in \mathcal{R}_n$ such that $f := f_R$. Condition (i) is equivalent to

$$\sum_{i,j} \big(\sum_a R_{iaaj} - \delta_{ij}\big)^2 = 0, \tag{8.5}$$

and hence to (8.2); condition (ii) is equivalent to

$$\sum_{i,j,k,l}\big(\sum_{a,b} R_{ijab}R_{alkb} - \delta_{ik}\delta_{jl}\big)^2 = 0, \tag{8.6}$$

and hence to (8.3); and condition (iii) is equivalent to

$$\sum_{i,j,k,l,m,h}\big(\sum_{a,b,c} R_{iabh}R_{jkca}R_{bclm} - \sum_{a,b,c} R_{ijbc}R_{bkla}R_{camh}\big)^2 = 0, \tag{8.7}$$

and hence to (8.4). So R is an R-matrix, as required. □

The rest of this chapter is devoted to proving Theorem 8.1. The proof follows the same line as the proof of Theorem 7.1. Occasionally we will refer to the proof of Theorem 7.1 if the proofs, *mutatis mutandis*, are equivalent. We first consider the value of f on the vertexless loop.

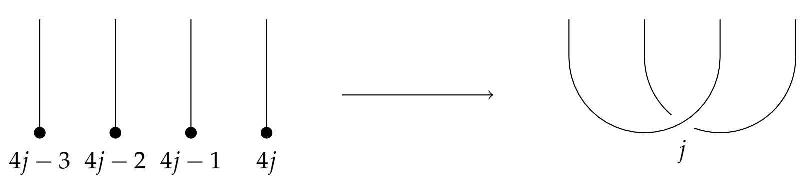
8.3 The value of f on $\bigcirc$

The proof of the following lemma will rely heavily on Theorem 3.3 just like the proof of Lemma 7.4.

Lemma 8.3. *If $f : \mathcal{V} \to \mathbb{R}$ is multiplicative and weakly reflection positive, then $f(\bigcirc)$ belongs to $\{\ldots, -6, -4, -2, 0, 1, 2, 3, \ldots\}$.*

Proof. I. We first describe some tools, using Theorem 3.3. Consider any $k \in \mathbb{N}$. Recall that $\mathcal{M}_{8k}$ is the set of perfect matchings on $[8k]$. For $M \in \mathcal{M}_{8k}$ and $\pi \in S_{8k}$, we defined $\pi M = \{\pi(e) \mid e \in \mathcal{M}\}$. So the group S_{8k} acts on $\mathcal{M}_{8k}$, which induces an action of S_{8k} on $\mathbb{R}^{\mathcal{M}_{8k}}$.

To each $M \in \mathcal{M}_{8k}$ we can associate a virtual link diagram G_M on $[2k]$ by identifying, for each $j \in [2k]$, the vertices $4j-3, 4j-2, 4j-1, 4j$ of $([8k], M)$ to one crossing called j in the following way



. (8.8)

To describe $G_M \overset{2k}{\vee} G_N$ for $M, N \in \mathcal{M}_{8k}$, we define the following subgroups of S_{8k}. For $j \in [2k]$, let B_j be the group consisting of the identity and of $(4j-3, 4j-1)(4j-2, 4j)$. Define $B := B_1B_2\cdots B_{2k}$. Let D be the group of permutations $\delta \in S_{8k}$ for which there exists $\pi \in S_{2k}$ such that $\delta(4j-i) = 4\pi(j) - i$ for each $j = 1, \ldots, 2k$ and $i = 0, \ldots, 3$. Set $Q := BD$, which is a group.

As before, for $M, N \in \mathcal{M}_{8k}$, let $c(M \cup N)$ denote the number of connected components of the graph $([8k], M \cup N)$. Then, by definition of the operation $\overset{2k}{\vee}$, we have

$$G_M \overset{2k}{\vee} G_N = 2^{-2k}(2k)! \sum_{\tau \in Q} \bigcirc^{c(M \cup \tau N)}. \tag{8.9}$$

For $\pi \in S_{8k}$, let P_π be the $\mathcal{M}_{8k} \times \mathcal{M}_{8k}$ permutation matrix corresponding to π; then $P_\pi w = \pi w$ for each $w \in \mathbb{R}^{\mathcal{M}_{8k}}$. For any $x \in \mathbb{R}$, let $A(x)$ and $A^Q(x)$ be the $\mathcal{M}_{8k} \times \mathcal{M}_{8k}$ matrices defined by

$$(A(x))_{M,N} := x^{c(M \cup N)} \text{ and } A^Q(x) := \sum_{\tau \in Q} A(x) P_\tau, \tag{8.10}$$

for $M, N \in \mathcal{M}_{8k}$. So, by the weak reflection positivity of f, (8.9) implies that $A^Q(f(\bigcirc))$ is positive semidefinite. Note that each P_π commutes with $A(x)$, as for all $M, N \in \mathcal{M}_{8k}$ one has $c(\pi M \cup \pi N) = c(M \cup N)$, implying $A(x) = P_\pi^Q A(x) P_\pi = P_\pi^{-1} A(x) P_\pi$.

In Theorem 3.3 we have seen that the irreducible S_{8k}-module of $\mathbb{R}\mathcal{M}_{8k}$ corresponding to any partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ consists of eigenvectors with eigenvalue

$$h_\lambda(x) := \prod_{a=1}^{m} \prod_{b=1}^{\frac{1}{2}\lambda_a} (x - a + 2b - 1). \tag{8.11}$$

It will be convenient to describe the eigenvectors in the following way. Make a Young tableau T associated to λ such that each row of T has the form

$$\boxed{i_1}\boxed{\bar{i}_1}\boxed{i_2}\boxed{\bar{i}_2}\boxed{\;\cdots\;}\boxed{i_t}\boxed{\bar{i}_t} \tag{8.12}$$

for some $i_1, \ldots, i_t \in [4k]$, where $\bar{i} := 4k + i$ for each $i \in [4k]$. For $i = 1, \ldots, \lambda_1$, let K_i denote the set of numbers in column i of T and let C_i be the subgroup of S_{8k} that permutes the elements of K_i. Then $C := C_1 \cdots C_{t_1}$. Similarly, for $i = 1, \ldots, m$, let R_i be the subgroup of S_{8k} that permutes the numbers in row i of T, and $R := R_1 \ldots R_m$.

Let F be the perfect matching on $[8k]$ with edges $\{i, \bar{i}\}$ for $i \in [4k]$. Then

$$v := \sum_{\sigma \in C, \rho \in R} \operatorname{sgn}(\sigma) \sigma \rho F \tag{8.13}$$

is an eigenvector of $A(x)$ belonging to $h_\lambda(x)$. Then for $u := \sum_{\tau \in Q} \tau v$ one has

$$\begin{aligned} A^Q(x) u &= \sum_{\tau', \tau \in Q} A(x) P_{\tau'} P_\tau v = \sum_{\tau', \tau \in Q} P_{\tau'} P_\tau A(x) v \\ &= h_\lambda(x) \sum_{\tau', \tau \in Q} P_{\tau'} P_\tau v = |Q| h_\lambda(x) u. \end{aligned}$$

So u is an eigenvector of $A^Q(x)$ belonging to $|Q|h_\lambda(x)$, *provided that* u is non-zero. For this it suffices that the coefficient u_F of u in F is non-zero. Note that

$$u_F = \sum_{\tau\in Q}(\tau v)_F = \sum_{\tau\in Q}\sum_{\sigma\in C,\rho\in R}\operatorname{sgn}(\sigma)(\tau\sigma\rho F)_F = \sum_{\substack{\tau\in Q,\sigma\in C,\rho\in R\\ \tau\sigma\rho F=F}}\operatorname{sgn}(\sigma). \tag{8.14}$$

So $u \neq 0$ if for any $\tau \in Q$, $\sigma \in C$, and $\rho \in R$, if $\tau\sigma\rho F = F$ then $\operatorname{sgn}(\sigma) = 1$; that is (as Q is a group), if for any $\tau \in Q$, $\sigma \in C$, $\rho \in R$:

$$\text{if } \tau F = \sigma\rho F, \text{ then } \operatorname{sgn}(\sigma) = 1. \tag{8.15}$$

II. We first apply part I to the case where $f(\bigcirc) \geq 0$. Let $k := \lceil f(\bigcirc)\rceil + 1$, and consider the partition $\lambda := (8, 8, \ldots, 8)$ of $8k$. Then, by (8.11),

$$h_\lambda(x) = \prod_{i=0}^{k-1}(x-i)(x-i+2)(x-i+4)(x-i+6). \tag{8.16}$$

We give a Young tableau associated to λ that will yield (8.15). This implies that $|Q|h_\lambda(x)$ is an eigenvalue of $A^Q(x)$. So $h_\lambda(f(\bigcirc)) \geq 0$. Hence, as the polynomial $h_\lambda(x)$ has largest zero $k-1$, with multiplicity 1, and as $k-1 = \lceil f(\bigcirc)\rceil$, we know $f(\bigcirc) = k-1$.

Consider the following Young tableau associated to λ:

$$T := \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & \overline{1} & 2 & \overline{2} & 3 & \overline{3} & 4 & \overline{4} \\ \hline 5 & \overline{5} & 6 & \overline{6} & 7 & \overline{7} & 8 & \overline{8} \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 4k-3 & \overline{4k-3} & 4k-2 & \overline{4k-2} & 4k-1 & \overline{4k-1} & 4k & \overline{4k} \\ \hline \end{array}. \tag{8.17}$$

To prove (8.15), choose $\tau \in Q$, $\sigma \in C$, and $\rho \in R$ with $\tau F = \sigma\rho F$. Let $\sigma = \sigma_1 \cdots \sigma_8$ with $\sigma_i \in C_i$ ($i = 1, \ldots, 8$) and define $M := \tau F$. Since F has no edges between $X := K_1 \cup K_2 \cup K_5 \cup K_6$ (the set of odd numbers in T) and $Y := K_3 \cup K_4 \cup K_7 \cup K_8$ (the set of even numbers in T) and since $QX = X$ and $QY = Y$, we know that M has no edges between X and Y. For any $N \in \mathcal{M}_{8k}$ and $Z \subseteq [8k]$, let N_Z be the set of edges of N contained in Z.

Let $\zeta \in S_{8k}$ be defined by $\zeta(i) := i+1$ if 4 does not divide i and $\zeta(i) := i-3$ if 4 divides i. So ζ^4 is the identity element, $\zeta(X) = Y$, and $\zeta F = F$. Moreover, $\zeta\tau = \tau\zeta$ (since ζ commutes with B and D). So $\zeta M = M$. Hence $\zeta M_X = M_Y$.

Let $N := \rho F$. So $M = \sigma N$. As no edge of M connects X and Y, also no edge in N connects X and Y. Moreover, as $\zeta M_X = M_Y$, for each two columns K_i and K_j in X, we have $|M_{K_i\cup K_j}| = |M_{K_{i+2}\cup K_{j+2}}|$, and hence $|N_{K_i\cup K_j}| = |N_{K_{i+2}\cup K_{j+2}}|$. Moreover, if an edge $e \in N$ connects K_i and K_j, then N has an edge in the same row as e connecting the other two columns in X; similarly for Y.

This implies that there exists a permutation $\sigma' \in C_1C_2C_5C_6$ that permutes complete rows in X in such a way that $\sigma' N_X$ is a shift of N_Y; that is, $\zeta\sigma' N_X = N_Y$. As σ' maintains rows in X, there exists $\rho' \in R$ with $\sigma' N = \rho' F$; so $\sigma(\sigma')^{-1}\rho' F = \sigma\rho F$. Moreover, $\operatorname{sgn}(\sigma') = 1$, and, setting $N' := \rho' F$ we have $\zeta N'_X = \zeta(\rho' F)_X = \zeta(\sigma' N)_X = \zeta\sigma'(N_X) = N_Y = N'_Y$. Therefore, by replacing ρ by ρ' and σ by $\sigma(\sigma')^{-1}$ we can assume that $\zeta N_X = N_Y$.

Next consider any two columns K_i and K_j in X. Let $X' := K_i \cup K_j$ and $Y' := K_{i+2} \cup K_{j+2}$. So $Y' = \zeta(X')$ and $\zeta N_{X'} = N_{Y'}$. Then $e \mapsto \zeta^{-1}\sigma^{-1}\zeta\sigma(e)$ is a permutation σ of the edges e in $N_{X'}$, since $e \in N_{X'} \Rightarrow \sigma(e) \in M_{X'} \Rightarrow \zeta\sigma(e) \in M_{Y'} \Rightarrow \sigma^{-1}\zeta\sigma(e) \in N_{Y'} \Rightarrow \zeta^{-1}\sigma^{-1}\zeta\sigma(e) \in \zeta^{-1}N_{Y'} = N_{X'}$. As σ permutes edges in X', there exists a permutation $\sigma' \in C_iC_j$ such that $\sigma'(e) = \zeta^{-1}\sigma^{-1}\zeta\sigma(e)$ for all $e \in N_{X'}$ and such that σ' only permutes elements covered by $N_{X'}$. Then $\operatorname{sgn}(\sigma') = 1$. By replacing σ by $\sigma(\sigma')^{-1}$ we attain that $e = \zeta^{-1}\sigma^{-1}\zeta\sigma(e)$ for all edges $e \in N_{X'}$. So $\sigma\zeta(e) = \zeta\sigma(e)$ for all $e \in N_{X'}$.

Doing this for all K_i and K_j in X, we finally achieve that $\sigma\zeta(e) = \zeta\sigma(e)$ for all $e \in N_X$. As N_X is a perfect matching on X, this implies $\sigma\zeta(i) = \zeta\sigma(i)$ for all $i \in X$. Equivalently, $\sigma_3\sigma_4\sigma_7\sigma_8\zeta(i) = \zeta\sigma_1\sigma_2\sigma_5\sigma_6(i)$ for all $i \in X$. Hence $\operatorname{sgn}(\sigma_3\sigma_4\sigma_7\sigma_8) = \operatorname{sgn}(\sigma_1\sigma_2\sigma_5\sigma_6)$, implying $\operatorname{sgn}(\sigma) = 1$.

III. Next we apply part I of this proof to the case where $f(\bigcirc) \leq 0$. Choose $k \in \mathbb{N}$, and consider the partition $\lambda := (8k)$ of $8k$ and the following Young tableau

$$T := \boxed{1}\boxed{\overline{1}}\boxed{2}\boxed{\overline{2}}\boxed{\cdots}\boxed{4k-1}\boxed{\overline{4k-1}}\boxed{4k}\boxed{\overline{4k}}\,. \tag{8.18}$$

Then by (8.11),

$$h_\lambda(x) = \prod_{b=1}^{4k}(x - 2 + 2b). \tag{8.19}$$

Moreover, (8.15) trivially holds, as C only consists of the identity. The zeros of h_λ are $-8k+2, -8k+4, -8k+6, \ldots, -2, 0$, all with multiplicity 1, so that $h_\lambda(f(\bigcirc)) \geq 0$ implies that $f(\bigcirc)$ does not belong to any interval $(-4t-2, -4t)$ for any $t \in \mathbb{N}$ with $t < 2k$. As k can be chosen arbitrarily large, we know that $f(\bigcirc) \notin (-4t-2, -4t)$ for all $t \in \mathbb{N}$.

To exclude the intervals $(-4t-4, -4t-2)$, consider the partition $\lambda := (8k-2, 2)$ of $8k$ and the Young tableau

$$T := \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & \overline{1} & 3 & \overline{3} & 4 & \overline{4} & \cdots & 4k-1 & \overline{4k-1} & 4k & \overline{4k} \\ \hline 2 & \overline{2} \\ \cline{1-2}\end{array}\,. \tag{8.20}$$

In this case, again by (8.11),

$$h_\lambda(x) = (x-1)\prod_{b=1}^{4k-1}(x - 2 + 2b). \tag{8.21}$$

To show (8.15), let $\sigma = \sigma_1\sigma_2$ with $\sigma_1 \in C_1$, $\sigma_2 \in C_2$. Observe that $M := \tau F$ contains no edges connecting an odd number with an even number (as F does not, and as Q maintains the sets of odd and even numbers).

If $\{2, \overline{2}\}$ belongs to M, then either σ_1 and σ_2 both are the identity permutation, or σ_1 and σ_2 both are transpositions. In either case, $\text{sgn}(\sigma) = 1$ follows.

If $\{2, \overline{2}\}$ does not belong to M, then 2 and $\overline{2}$ are matched in M to even numbers in the first row of T. In this case, both σ_1 and σ_2 are transpositions, and again $\text{sgn}(\sigma) = 1$ follows. This proves (8.15).

Now the zeros of h_λ are $-8k+4, -8k+6, \ldots, -2, 0, 1$, all with multiplicity 1, so that, like above, $f(\bigcirc) \notin (-4t-4, -4t-2)$ for all $t \in \mathbb{N}$. □

8.4 Proof of Theorem 8.1

The space $\mathbb{R}\mathcal{V}$ of formal linear combinations of elements of $\mathcal{V}$, is in fact an algebra, by taking the disjoint union $G \sqcup H$ of two virtual link diagrams G and H as multiplication GH. Choose $n \in \mathbb{N}$ and recall that $\mathcal{R}_n$ denotes the linear space

$$\mathcal{R}_n := ((\mathbb{R}^n)^{\otimes 4})^{S_2}. \tag{8.22}$$

As usual, $\mathcal{O}(\mathcal{R}_n)$ denotes the algebra of polynomials on $\mathcal{R}_n$. Define an algebra homomorphism $p_n : \mathbb{R}\mathcal{V} \to \mathcal{O}(\mathcal{R}_n)$ by

$$p_n(G)(R) := f_R(G) \tag{8.23}$$

for $G \in \mathcal{V}$ and $R \in \mathcal{R}_n$. So the element R in the theorem can be described as a common zero of the polynomials $p_n(G) - f(G)$ for all $G \in \mathcal{V}$.

For any $q \in \mathcal{O}(\mathcal{R}_n)$, let dq be its derivative, being an element of $\mathcal{O}(\mathcal{R}_n) \otimes \mathcal{R}_n^*$. So $d^k q \in \mathcal{O}(\mathcal{R}_n) \otimes (\mathcal{R}_n^*)^{\otimes k}$. Note that the standard inner product on $\mathbb{R}^n$ induces an inner product on $(\mathbb{R}^n)^{\otimes 4}$, hence on $\mathcal{R}_n$ and $\mathcal{R}_n^*$, and therefore it induces a product $\langle ., . \rangle : (\mathcal{O}(\mathcal{R}_n) \otimes (\mathcal{R}_n^*)^{\otimes k}) \times (\mathcal{O}(\mathcal{R}_n) \otimes (\mathcal{R}_n^*)^{\otimes k}) \to \mathcal{O}(\mathcal{R}_n)$. Then, for all $G, H \in \mathcal{V}$ and all $k, n \in \mathbb{N}$:

$$p_n(G \overset{k}{\vee} H) = \langle d^k p_n(G), d^k p_n(H) \rangle. \tag{8.24}$$

This is similar to Lemma 7.5 and can be proved by a word for word translation of the method. This connection between k-joins and k-th derivatives will be used a number of times in our proof of the theorem.

Similar to what we have seen in Proposition 5.5 we now find that

$$p_n(\mathbb{R}\mathcal{V}) = \mathcal{O}(\mathcal{R}_n)^{\mathrm{O}_n}, \tag{8.25}$$

the latter denoting the space of O_n-invariant elements of $\mathcal{O}(\mathcal{R}_n)$. The proof is similar to that given in [39].

Proof of Theorem 8.1. To see necessity in the theorem, let R be a virtual link diagram edge coloring model over $\mathbb{R}$. Then f_R is trivially multiplicative. Positive semidefiniteness of $M_{f_R,k}$ follows from

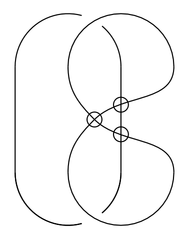
$$f_R(G \overset{k}{\vee} H) = p_n(G \overset{k}{\vee} H)(R) = \langle d^k p_n(G)(R), d^k p_n(H)(R) \rangle, \tag{8.26}$$

using (8.24).

To prove sufficiency, let f satisfy the conditions of the theorem. As $f(\bigcirc) \geq 0$ by assumption, the lemma implies that $n := f(\bigcirc)$ is a nonnegative integer. Then

$$\text{there exists an algebra homomorphism } F : p_n(\mathbb{R}\mathcal{V}) \to \mathbb{R} \text{ such that } f = F \circ p_n. \tag{8.27}$$

Otherwise, as f and p_n are algebra homomorphisms, there exists a quantum virtual link diagram γ with $p_n(\gamma) = 0$ and $f(\gamma) \neq 0$. We can assume that $p_n(\gamma)$ is homogeneous, that is, all virtual link diagrams in γ have the same number of crossings, k say. So $\gamma \overset{k}{\vee} \gamma$ has no crossings, that is, it is a polynomial in $\bigcirc$. As moreover $f(\bigcirc) = n = p_n(\bigcirc)$, we have $f(\gamma \overset{k}{\vee} \gamma) = p_n(\gamma \overset{k}{\vee} \gamma) = 0$, the latter equality because of (8.24). Similarly to Lemma 7.3, γ belongs to the ideal in $\mathbb{R}\mathcal{V}$ generated by $\gamma \overset{k}{\vee} \beta^i$ ($i = 0, \ldots, k$), where β is the virtual link diagram

$\beta :=$ 

(8.28)

Note that $G \overset{1}{\vee} \beta = 2|V(G)|G$ for each virtual link diagram G. As $f(\gamma \overset{k}{\vee} \gamma) = 0$ implies that $f(\gamma \overset{k}{\vee} \beta^i) = 0$ for each i (by the weak reflection positivity of f), we know $f(\gamma) = 0$, proving (8.27).

Now, by (8.25), $p_n(\mathbb{R}\mathcal{V}) = \mathcal{O}(\mathcal{R}_n)^{\mathrm{O}_n}$. Basic invariant theory then gives the existence of an R in the complex extension of $\mathcal{R}_n$ such that $F(q) = q(R)$ for each $q \in \mathcal{O}(\mathcal{R}_n)^{\mathrm{O}_n}$, similar to the proof of Theorem 7.1. To prove that we can take R real, we apply the Procesi-Schwarz theorem [26].

For all $G, H \in \mathcal{V}$, using (8.24):

$$F(\langle dp_n(G), dp_n(H)\rangle) = F(p_n(G \overset{1}{\vee} H)) = f(G \overset{1}{\vee} H) = (M_{f,1})_{G,H}. \tag{8.29}$$

Since $M_{f,1}$ is positive semidefinite, (8.29) implies $F(\langle dq, dq\rangle) \geq 0$ for each $q \in p_n(\mathbb{R}\mathcal{V}) = \mathcal{O}(\mathcal{R}_n)^{\mathrm{O}_n}$. Then by [26] there exists a (real) $R \in \mathcal{R}_n$ such that $F(q) = q(R)$ for each $q \in \mathcal{O}(\mathcal{R}_n)^{\mathrm{O}_n} = p_n(\mathbb{R}\mathcal{V})$. Then $f = f_R$, as $f(G) = F(p_n(G)) = p_n(G)(R) = f_R(G)$ for each $G \in \mathcal{V}$. $\square$

One can also prove that if f is the partition function of a virtual link diagram edge coloring model, then $f = f_R$ for some unique $R \in \mathcal{R}_n$, up to the natural action of O_n on R, by a similar proof as that for 3-graphs.

Summary

New Characterizations of Partition Functions Using Connection Matrices

In this thesis we expand upon a line of research pioneered by Freedman, Lovász and Schrijver [12] and Szegedy [39], that uses algebraic methods to characterize families of partition functions. Before we summarize the contributions of this thesis, we first recall the definition of an ordinary partition function.

If W is a vector space, then SW denotes the *symmetric algebra* on W and $\bigwedge W$ denotes the *exterior algebra* on W. For $x, y \in SW$, we denote their product in SW by $x \odot y$. Let $\mathbb{F}$ be a field of characteristic 0. Let $k \in \mathbb{N}$ and let $\{e_1, \ldots, e_k\}$ be the standard basis of the vector space $\mathbb{F}^k$. If $h \in (S\mathbb{F}^k)^*$, then the *partition function of* h is the $\mathbb{F}$-valued graph parameter p_h, defined, for a graph $G = (V, E)$, by

$$p_h(G) := \sum_{\phi: E \to [k]} \prod_{v \in V} h\Big(\bigodot_{a \in \delta(v)} e_{\phi(a)}\Big), \tag{8.30}$$

where $\delta(v)$ is the multiset consisting of edges incident with v with multiplicities. If a graph parameter f is equal to p_h for some $h \in (S\mathbb{F}^k)^*$ and $k \in \mathbb{N}$, then we say that f is an *ordinary partition function over* $\mathbb{F}$.

We introduce two new types of partition functions: *skew partition functions* and *mixed partition functions*. A skew partition function can be seen as the partition function of an element $h \in (\bigwedge \mathbb{C}^{2\ell})^*$ for some $\ell \in \mathbb{N}$. The definition is slightly more involved than the definition of an ordinary partition function and therefore we do not give it here. Using the invariant theory of the symplectic group and Hilbert's Nullstellensatz, we give a characterization of skew partition functions in terms of identities related to the Second Fundamental Theorem of invariant theory for the symplectic group. This characterization is close in spirit to the characterization of ordinary partition functions given by Draisma, Gijswijt, Lovász, Regts and Schrijver [10].

Mixed partition functions are a common generalization of both ordinary partition functions and skew partition functions, and we show that they satisfy certain identities related to the Second Fundamental Theorem of invariant theory for the orthosymplectic supergroup.

We also give a characterization of skew partition functions in terms of properties of their associated connection matrices, which are defined as follows.

For $t \in \mathbb{N}$, a *t-fragment* is a graph with t labeled vertices of degree 1 labeled $1, 2, \ldots, t$. For two t-fragments F_1 and F_2, we define $F_1 * F_2$ to be the graph obtained as follows: we take the disjoint union of F_1 and F_2, and for each pair of equally labeled vertices, we identify the two vertices, remove the new vertex and join its two incident edges into one edge. See Figure 8.4. Note that if F is the 2-fragment on two vertices, labeled 1 and 2, with one edge between those two vertices, then $F * F = \bigcirc$, the *vertexless loop*, which we also consider to be a graph.

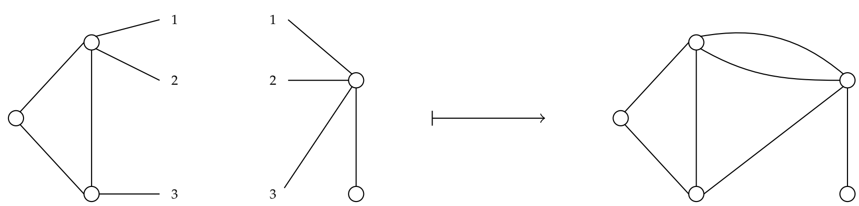


Figure 8.4: An example of the gluing operation.

The *t-th connection matrix* of a graph parameter f is the symmetric matrix $M_{f,t}$ whose rows and columns are indexed by t-fragments such that the entry at the (F_1, F_2) position is $f(F_1 * F_2)$.

We characterize skew partition functions as those $\mathbb{C}$-valued graph parameters f such that $f(\bigcirc) \leq 0$, $f(\emptyset) = 1$ and $\text{rk}\,(M_{f,2t}) \leq f(\bigcirc)^{2t}$ for all $t \in \mathbb{N}$. The proof of this characterization makes use of a framework developed by Schrijver [38]. We also show that for a mixed partition function f there is a constant $r \in \mathbb{R}$ such that $\text{rk}\,(M_{f,t}) \leq r^t$ for each $t \in \mathbb{N}$. An open problem is in how much this characterizes mixed partition functions.

Szegedy [39] showed that an $\mathbb{R}$-valued graph parameter f is an ordinary partition function over $\mathbb{R}$ if and only if $f(\emptyset) = 1$, $f(G \cup H) = f(G)f(H)$ for any two graphs G and H, and $M_{f,t}$ is positive semidefinite for each $t \in \mathbb{N}$. We give similar characterizations of partition functions for the following two types of graphs that are related to knot theory.

A 3-*graph* is a non-empty connected graph such that each vertex has degree 3 and such that each vertex has a cyclic order of the edges incident with it. For two 3-graphs G and H we define their *k-join*, an operation that results in a formal linear combination of disjoint unions of 3-graphs. Using the k-join we define a new type of connection matrix for 3-graphs. We give a characterization of $\mathbb{R}$-valued partition functions for 3-graphs in terms of positive semidefiniteness of the associated connection matrices. The proof makes use of the invariant theory of the orthogonal group, a theorem by Procesi and Schwarz [26] and a theorem by Hanlon and Wales [14]. From this characterization we derive a characterization of real Lie algebra weight systems.

The techniques we use in proving our results on 3-graphs can also be applied to *virtual link diagrams*. We define a k-join for virtual link diagrams and give a characterization of $\mathbb{R}$-valued partition functions on the set of virtual link diagrams in terms of positive semidefiniteness of the associated connection matrices. From this characterization we derive a characterization of partition functions for virtual link diagrams coming from real *R-matrices*.

Samenvatting

Nieuwe Karakteriseringen van Partitiefuncties met Behulp van Connectiematrices

In dit proefschrift bouwen we voort op een onderzoeksprogramma opgezet door Freedman, Lovász en Schrijver [12] en Szegedy [39] dat gebruikmaakt van algebraïsche technieken om families van partitiefuncties te karakteriseren. Voordat we de bevindingen in dit proefschrift samenvatten, herhalen we eerst de definitie van een gewone partitiefunctie.

Als W een vectorruimte is, dan is SW de *symmetrische algebra* op W en $\bigwedge W$ de *uitwendige algebra* op W. Voor $x, y \in SW$ noteren we hun product in SW als $x \odot y$. Laat $\mathbb{F}$ een lichaam van karakteristiek 0 zijn. Laat $k \in \mathbb{N}$ en laat $\{e_1, \ldots, e_k\}$ de standaardbasis van de vectorruimte $\mathbb{F}^k$ zijn. Als $h \in (S\mathbb{F}^k)^*$, dan is de *partitiefunctie van h* de $\mathbb{F}$-waardige graafparameter p_h gedefinieerd, voor een graaf $G = (V, E)$, door

$$p_h(G) := \sum_{\phi: E \to [k]} \prod_{v \in V} h\Big(\bigodot_{a \in \delta(v)} e_{\phi(a)} \Big), \tag{8.31}$$

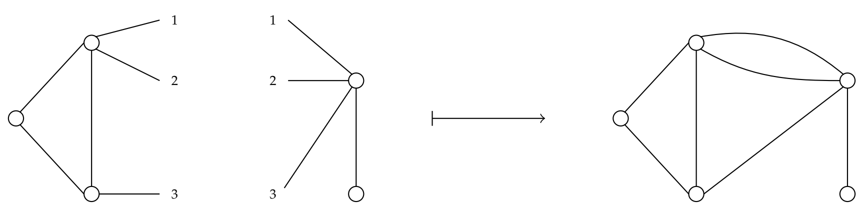
waar $\delta(v)$ de collectie van kanten is die v bevatten (met multipliciteiten). Als een graafparameter f gelijk is aan p_h voor een zekere $h \in (S\mathbb{F}^k)^*$ en $k \in \mathbb{N}$, dan zeggen we dat f een *gewone partitiefunctie over* $\mathbb{F}$ is.

We introduceren twee nieuwe soorten partitiefuncties: *scheve partitiefuncties* en *gemengde partitiefuncties*. Een scheve partitiefunctie kan gezien worden als de partitiefunctie van een element $h \in (\bigwedge \mathbb{C}^{2\ell})^*$ voor een zekere $\ell \in \mathbb{N}$. De definitie is wat ingewikkelder dan die van een gewone partitiefunctie en daarom geven we die hier niet. Gebruikmakend van de invariantentheorie van de symplectische groep en Hilbert's Nullstellensatz, geven we een karakterisering van scheve partitiefuncties aan de hand van identiteiten die gerelateerd zijn aan de Tweede Fundamentele Stelling van de invariantentheorie van de symplectische groep. Deze karakterisering heeft veel weg van de karakterisering van gewone partitiefuncties gegeven door Draisma, Gijswijt, Lovász, Regts en Schrijver [10].

Gemengde partitiefuncties zijn een veralgemenisering van zowel gewone partitiefuncties als scheve partitiefunctes, en we laten zien dat zij voldoen aan bepaalde identiteiten die gerelateerd zijn aan de Tweede Fundamentele Stelling van de invariantentheorie van de orthosymplectische supergroep.

We geven ook een karakterisering van scheve partitiefuncties aan de hand van eigenschappen van hun connectiematrices, die als volgt zijn gedefinieerd.

Voor $t \in \mathbb{N}$ is een *t-fragment* een graaf met t gemarkeerde vertices van graad 1 gemarkeerd met $1, 2, \ldots, t$. Voor twee t-fragmenten F_1 en F_2 definiëren we $F_1 * F_2$ als de graaf die als volgt wordt verkregen: we nemen de disjuncte vereniging van F_1 en F_2, en voor elk paar gelijk gemarkeerde vertices identificeren we de twee vertices, verwijderen we de nieuwe vertex en smelten de twee kanten die met deze vertex verbonden waren samen tot een kant. Zie Figuur 8.5. Als F het 2-fragment is met twee vertices, gemarkeerd 1 en 2, met een kant tussen deze twee vertices, dan $F * F = \bigcirc$, de *vertexloze lus*, die we ook als graaf zien.



Figuur 8.5: Een voorbeeld van het plakken van fragmenten.

De *t-de connectiematrix* van een graafparameter f is de symmetrische matrix $M_{f,t}$ waarvan de rijen en de kolommen geïndiceerd worden door t-fragmenten en waarbij $f(F_1 * F_2)$ in de (F_1, F_2)-positie van de matrix staat.

We karakteriseren scheve partitiefuncties als die $\mathbb{C}$-waardige graafparameters f zodat $f(\bigcirc) \leq 0$, $f(\emptyset) = 1$ en $\operatorname{rk}(M_{f,t}) \leq f(\bigcirc)^{2t}$ voor alle $t \in \mathbb{N}$. Het bewijs van deze karakterisering maakt gebruik van een raamwerk dat ontwikkeld is door Schrijver [38]. We tonen ook aan dat er voor een gemengde partitiefunctie f een constante $r \in \mathbb{R}$ bestaat zodat $\operatorname{rk}(M_{f,t}) \leq r^t$ voor alle $t \in \mathbb{N}$. Een open probleem is in hoeverre dit gemengde partitiefuncties karakteriseert.

Szegedy [39] heeft bewezen dat een $\mathbb{R}$-waardige graafparameter f een gewone partitiefunctie over $\mathbb{R}$ is dan en slechts dan als $f(\emptyset) = 1$, $f(G \cup H) = f(G)f(H)$ voor elke twee grafen G en H, en $M_{f,t}$ positief semi-definiet is voor alle $t \in \mathbb{N}$. We geven gelijksoortige karakteriseringen van partitiefuncties voor de volgende twee soorten grafen die gerelateerd zijn aan knopentheorie.

Een 3-*graaf* is een niet-lege samenhangende graaf zodat elke vertex graad 3 heeft en zodat voor iedere vertex de kanten die de vertex bevatten cyclisch geordend zijn. Voor twee 3-grafen G en H definiëren we hun *k-koppeling*. Dit is een operatie die resulteert in een lineaire combinatie van disjuncte verenigingen van 3-grafen. Aan de hand van de k-koppeling definiëren we een nieuw type connectiematrix voor 3-grafen. We geven een karakterisering van reëelwaardige partitiefuncties voor 3-grafen in termen van positief semi-definietheid van de connectiematrices. Het bewijs van deze stelling maakt gebruik van de invariantentheorie van de orthogonale groep, een stelling van Procesi en Schwarz [26] en een stelling van Hanlon en Wales [14]. Uit deze karakterisering leiden we een karakterisering van reële Lie algebra gewichtssystemen af.

De technieken die we gebruiken om onze resultaten over 3-grafen af te leiden, gebruiken we ook om resultaten over *virtuele-linkdiagrammen* af te leiden. We definiëren een *k-koppeling* voor virtuele-linkdiagrammen en we geven een karakterisering van reëelwaardige partitiefuncties op de verzameling van virtuele-linkdiagrammen in termen van positief semi-definietheid van de connectiematrices. Uit deze karakterisering leiden we een karakterisering van partitiefuncties van virtuele-linkdiagrammen die afkomstig zijn van reële *R-matrices* af.

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